THE PROOF OF TCHAKALOFF’S THEOREM

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Abstract. We provide a simple proof of Tchakaloff’s Theorem on the existence of cubature formulas of degree \( m \) for Borel measures with moments up to order \( m \). The result improves known results for non-compact support, since we do not need conditions on \((m + 1)\)st moments.

We consider the question of existence of cubature formulas of degree \( m \) for Borel measures \( \mu \), i.e. a measure defined on the Borel \( \sigma \)-algebra, where moments up to degree \( m \) exist:

**Definition 1.** Let \( \mu \) be a positive Borel measure on \( \mathbb{R}^N \) and \( m \geq 1 \) such that

\[
\int_{\mathbb{R}^N} \|x\|^k \, \mu(dx) < \infty
\]

for \( 0 \leq k \leq m \) holds true. A cubature formula of degree \( m \) for \( \mu \) is given by an integer \( k \geq 1 \), points \( x_1, \ldots, x_k \in \text{supp} \mu \), weights \( \lambda_1, \ldots, \lambda_k > 0 \) such that

\[
\int_{\mathbb{R}^N} P(x) \mu(dx) = \sum_{i=1}^{k} \lambda_i P(x_i)
\]

for all polynomials on \( \mathbb{R}^N \) with degree less or equal \( m \), where \( \text{supp} \mu \) denotes the closed support of the measure \( \mu \), i.e. the complement of the biggest open set \( O \subset \mathbb{R}^N \) with \( \mu(O) = 0 \).

Cubature formulas of degree \( m \) have been proved to exist for Borel measures \( \mu \), where the \((m + 1)\)st moments exist, see [1] and [6]. The result in the case of compact \( \text{supp} \mu \) is classical, and due to Tchakaloff (see [10]), hence we refer to the assertion as Tchakaloff’s Theorem.

We collect some basic notions and results from convex analysis, see for instance [9]: fix \( N \geq 1 \), for some set \( S \subset \mathbb{R}^N \) the convex hull of \( S \), i.e. the smallest convex set in \( \mathbb{R}^N \) containing \( S \), is denoted by \( \text{conv}(S) \), the (topological) closure of \( \text{conv}(S) \) is denoted by \( \overline{\text{conv}(S)} \). The closure of a convex set is convex. Note that the convex hull of a compact set is always closed, but there are closed sets whose convex hull is no longer closed (see [9]).

Closed convex sets can also be described by their **supporting hyperplanes**. Given a convex set \( C \). Let \( y \in \partial C := \overline{C} \setminus \text{int}(C) \) be a boundary point. There is a linear functional \( l_y \) and a real number \( \beta_y \) such that the hyperplane defined by \( l_y = \beta_y \)
contains \( y \), and \( C \) is contained in the closed half-space \( l_y \leq \beta_y \). Hyperplanes and half-spaces with this property are called supporting hyperplanes and supporting half-spaces, respectively. Moreover, \( \overline{C} \) is the intersection of all its supporting half-spaces. Furthermore, if \( C \) is not contained in any hyperplane of \( \mathbb{R}^N \) (i.e. it has non-empty interior), then a point \( x \in C \) is contained in a supporting hyperplane if and only if \( x \in C \) is not an interior point of \( C \) (see [9], Th. 11.6). This means that we can characterize the boundary of \( C \) as those points, which lie at least in one supporting hyperplane of \( C \).

In the case of a convex cone \( C \) the supporting hyperplanes can be chosen to be homogeneous, i.e. to be of the form \( l_y = 0 \). We denote the convex cone generated by some set \( A \subset \mathbb{R}^N \) by \( \text{cone}(A) \) and its closure by \( \overline{\text{cone}}(A) \).

We also introduce the notion of the relative interior of a convex set \( C \): a point \( x \in C \) lies in the relative interior \( \text{ri}(C) \) if for every \( y \in C \) there is \( \epsilon > 0 \) such that \( x - \epsilon(y - x) \in C \). In particular we have that the relative interior of a convex set \( C \) coincides with the relative interior of its closure \( \overline{C} \). Interior points of \( C \) lie in the relative interior (see [9]), this remains true even if \( C \) lies in an affine subspace of \( \mathbb{R}^N \), and a point of \( C \) lies in the interior with respect to the subspace topology.

Given a measure \( \mu \) on some measurable space \((\Omega, \mathcal{F})\) and a Borel measurable map \( \phi : \Omega \to \mathbb{R}^N \), we denote by \( \phi_* \mu \) the push-forward Borel measure on \( \mathbb{R}^N \), which is defined via

\[
\phi_* \mu(A) := \mu(\phi^{-1}(A)),
\]

for all Borel sets \( A \subset \mathbb{R}^N \).

**Theorem 1.** Let \( \mu \) be a positive Borel measure on \( \mathbb{R}^N \), such that the first moments exist, i.e.

\[
\int_{\mathbb{R}^N} ||x|| \mu(dx) < \infty,
\]

and let \( A \subset \mathbb{R}^N \) be a measurable set with \( \mu(\mathbb{R}^N \setminus A) = 0 \). Then the first moment \( E = \int_{\mathbb{R}^N} x \mu(dx) \), where \( x \) denotes the vector \((x_1, \ldots, x_N)\), is contained in \( \text{cone}(A) \).

**Proof.** We first assume that there is no \( B \subset A \) with \( \mu(A \setminus B) = 0 \) such that \( B \) is contained in a hyperplane, since otherwise we could work in a lower-dimensional space instead (with \( A \) replaced by \( B \)). Fix some \( y \in \overline{K} \setminus \text{int}(K) \) in the boundary of \( K = \text{cone}(A) \). Then all linear functionals \( l_y : \mathbb{R}^N \to \mathbb{R} \) corresponding to the supporting half-spaces \( l_y \leq 0 \) at \( y \) are certainly integrable and we have

\[
l_y(E) = \int_{\mathbb{R}^N} l_y(x) \mu(dx) \leq 0,
\]

consequently \( E \in \overline{\text{cone}}(A) \).

By existence of the first moments, for each \( \delta > 0 \) we have \( \mu(\mathbb{R}^N \setminus B(0, \delta)) < \infty \), where \( B(0, \delta) \) denotes the centered ball with radius \( \delta \). Given \( l_y \) as above, we may conclude that \( \mu(\{x \in A|l_y(x) < 0\}) > 0 \), since otherwise the complement in \( A \) of the intersection of \( A \) with the hyperplane \( l_y = 0 \) would have measure 0, a contradiction to the assumption above. Then we can find \( \epsilon > 0 \) such that \( 0 < \mu(\{x \in A|l_y(x) \leq -\epsilon\}) < \infty \) and get

\[
l_y(E) = \int_{\mathbb{R}^N} l_y(x) \mu(dx) \leq -\epsilon \mu(\{x \in A|l_y(x) \leq -\epsilon\}) < 0.
\]

Hence \( E \in \overline{\text{cone}}(A) \) is an interior point of \( \overline{\text{cone}}(A) \). In particular \( E \in \text{cone}(A) \), since the interior lies in the convex cone hull of \( A \). If the first condition is not satisfied,
we obtain that $E$ is an interior point of cone$(A)$ in an affine subspace of $\mathbb{R}^N$ (where
the first condition is satisfied), but then $E$ lies in the relative interior of cone$(A)$ in
$\mathbb{R}^N$, which is the desired result. □

Corollary 1. Let $\mu$ be a positive Borel measure on $\mathbb{R}^N$ concentrated in $A \subset \mathbb{R}^N$,
i.e. $\mu(\mathbb{R}^N \setminus A) = 0$, such that the first moments exist, i.e.
$$\int_{\mathbb{R}^N} \|x\| \mu(dx) < \infty.$$ Then there exist an integer $1 \leq k \leq N$, points $x_1, \ldots, x_k \in A$ and weights $\lambda_1, \ldots, \lambda_k > 0$ such that
$$\int_{\mathbb{R}^N} f(x) \mu(dx) = \sum_{i=1}^{k} \lambda_i f(x_i)$$ for any monomial $f$ on $\mathbb{R}^N$ of degree 1.

Proof. The corollary follows immediately from Theorem 1 and Caratheodory’s Theorem (see [9], Th. 17.1 and Cor. 17.1.2).

Corollary 2. Let $\mu$ be a positive measure on the measurable space $(\Omega, F)$ concentrated in $A \in F$, i.e. $\mu(\Omega \setminus A) = 0$, and $\phi : \Omega \to \mathbb{R}^N$ a Borel measurable map. Assume that the first moments of $\phi_* \mu$ exist, i.e.
$$\int_{\mathbb{R}^N} \|x\| \phi_* \mu(dx) < \infty.$$ Then there exist an integer $1 \leq k \leq N$, points $\omega_1, \ldots, \omega_k \in A$ and weights $\lambda_1, \ldots, \lambda_k > 0$ such that
$$\int_{\Omega} \phi_j(\omega) \mu(d\omega) = \sum_{i=1}^{k} \lambda_i \phi_j(\omega_i)$$ for $1 \leq j \leq N$, where $\phi_j$ denotes the $j$-th component of $\phi$.

Remark 1. In other words, $A \in F$ such that $\mu(\Omega \setminus A) = 0$ correspond to $B \subset \phi(\Omega)$ such that $\phi_* \mu(\mathbb{R}^N \setminus B) = 0$.

Remark 2. Note that $\mu(\Omega) = \infty$ is also possible, since we only speak about integrability of $N$ measurable functions $\phi_1, \ldots, \phi_N$. If we have $\mu(\Omega) < \infty$, we could add $\phi_{N+1} = 1$, and we obtain in particular $\sum_{i=1}^{k} \lambda_i = \mu(\Omega)$ (with possibly different number $1 \leq k \leq N + 1$ of points $x_i$ and weights $\lambda_i$).

In the setting of Theorem 1 assume that $\mu$ is a probability measure on $\mathbb{R}^N$. Then
by the previous consideration $E = \int_{\mathbb{R}^N} x \mu(dx)$ lies in the convex hull $\text{conv}(A)$. This fact is well-known in financial mathematics, since it means that the price range of forward contracts is given by the relative interior of the convex hull of the no-arbitrage bounds of the (discounted) price process (see for instance [2], Th. 1.40).

The result is also well-known in the field of geometry of the moment problem, see for instance [4]. As mentioned therein, the result for compactly supported measures essentially even goes back to Riesz, see [8].
Proof. We solve the problem with respect to $\phi_*\mu$ on $\mathbb{R}^N$ and obtain $1 \leq k \leq N$, $y_1, \ldots, y_k \in \phi(A)$ and $\lambda_1, \ldots, \lambda_k > 0$ such that
\[
\int_{\mathbb{R}^N} f(y)(\phi_*\mu)(dy) = \sum_{i=1}^{k} \lambda_i f(y_i)
\]
for all polynomials $f$ of degree 1. Thus we obtain points $\omega_1, \ldots, \omega_k$ with $\phi(\omega_i) = y_i$ for $1 \leq i \leq k$, furthermore
\[
\int_{\mathbb{R}^N} f(y)(\phi_*\mu)(dy) = \int_{\Omega} (f \circ \phi)(\omega)\mu(d\omega)
\]
by definition, hence the result.

In an adequate algebraic framework the previous Theorem 1 yields all cubature results in full generality, and even generalizes those results (see [1], [6] and [7] for related theory and interesting extensive references).

For this purpose we consider polynomials in $N$ (commuting) variables $e_1, \ldots, e_N$ with degree function $\deg(e_i) := k_i$ for $1 \leq i \leq N$ and integers $k_i \geq 1$. Hence, we can associate a degree to monomials $e_{i_1} \cdots e_{i_l}$ with $(i_1, \ldots, i_l) \in \{1, \ldots, N\}^l$ for $l \geq 0$ (note that the monomial associated to the empty sequence is by convenience 1), namely
\[
\deg(e_{i_1} \cdots e_{i_l}) = \sum_{r=1}^{l} k_{i_r}.
\]
We denote by $A_{\deg \leq m}^N$ the vector space of polynomials generated by monomials of degree less or equal $m$, for some integer $m \geq 1$. We define a continuous map $\phi : \mathbb{R}^N \rightarrow A_{\deg \leq m}^N$, via
\[
\phi(x_1, \ldots, x_N) = \sum_{l \geq 0} \sum_{(i_1, \ldots, i_l) \in \{1, \ldots, N\}^l, \sum_{r=1}^{l} k_{i_r} \leq m} x_{i_1} \cdots x_{i_l} e_{i_1} \cdots e_{i_l}.
\]
Continuity is obvious, since we are given monomials in each coordinate. $\phi$ is even an embedding and a closed map.

The following example shows the relevant idea in coordinates, since for $N = 1$ and $\deg(e_1) = 1$ we obtain $A_{\deg \leq m}^1 = \mathbb{R}^{m+1}$.

**Example 1.** Fix $m \geq 1$. Then $\phi(x) = (1, x, x^2, \ldots, x^m)$ is a continuous map $\phi : \mathbb{R}^1 \rightarrow \mathbb{R}^{m+1}$. Given a positive Borel measure $\mu$ on $\mathbb{R}^1$ such that moments up to degree $m$ exist, i.e.
\[
\int_{\mathbb{R}} |x|^k \mu(dx) < \infty
\]
for $0 \leq k \leq m$, then $\phi_*\mu$ admits moments up to degree 1. Hence we conclude that there exist $1 \leq k \leq m+1$, points $x_1, \ldots, x_k$ and weights $\lambda_1, \ldots, \lambda_k > 0$ such that
\[
\int_{\mathbb{R}^N} P(x)\mu(dx) = \sum_{i=1}^{k} \lambda_i P(x_i)
\]
for all polynomials $P$ of degree less or equal $m$. 
Theorem 2. Given $N \geq 1$ and degree function $\deg$ and $m \geq 1$. Fix a finite, positive Borel measure $\mu$ on $\mathbb{R}^N$, i.e. $\mu(\mathbb{R}^N \setminus A) = 0$, such that
\[
\int_{\mathbb{R}^N} |x_{i_1} \cdots x_{i_l}| \mu(dx) < \infty
\]
for $(i_1, \ldots, i_l) \in \{1, \ldots, N\}^l$ with $\sum_{r=1}^l k_{i_r} \leq m$. Then there exist an integer $1 \leq k \leq \dim A_{\deg \leq m}$, points $x_1, \ldots, x_k \in A$ and weights $\lambda_1, \ldots, \lambda_k > 0$ such that
\[
\int_{\mathbb{R}^N} P(x) \mu(dx) = \sum_{i=1}^k \lambda_i P(x_i)
\]
for $P \in A_{\deg \leq m}$.

Proof. The measure $\phi_* \mu$ admits first moments by assumption, hence we conclude by Corollary 2. \qed

Remark 3. Tchakaloff’s Theorem is a special case of the above theorem with $A = \text{supp } \mu$.

Remark 4. Fix a non-empty, closed set $K \subset \mathbb{R}^N$. We note that a finite sequence of real numbers $m_{i_1, \ldots, i_l}$ for $(i_1, \ldots, i_l) \in \{1, \ldots, N\}^l$ with $\sum_{r=1}^l k_{i_r} \leq m$ represents the sequence of moments of a Borel probability measure $\mu$ with support $\text{supp } \mu \subset K$, where moments of degree less or equal $m$ exist, if and only if
\[
\sum_{l \geq 0} \sum_{(i_1, \ldots, i_l) \in \{1, \ldots, N\}^l, \sum_{r=1}^l k_{i_r} \leq m} m_{i_1, \ldots, i_l} e_{i_1} \cdots e_{i_l} \in \text{conv } \phi(K).
\]

The argument in one direction is that any element of $\text{conv } \phi(K)$ is represented as expectation with respect to some probability measure with support in $K$, for instance the given convex combination. The other direction is Tchakaloff’s Theorem in the general form of Theorem 2. Consequently we have a precise geometric characterization of solvability of the Truncated Moment Problem for measures with support in $K$. Notice that one can often describe $\text{conv } \phi(K)$ by finitely many inequalities.

REFERENCES
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