Flexibility of OU-Interest Rate Models

Martin Keller-Ressel, Thomas Steiner, Josef Teichmann

TU Wien, Institute of mathematical methods in Economics

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- Today's yield curve is given in this terminology through $T \mapsto Y(0, T)$.

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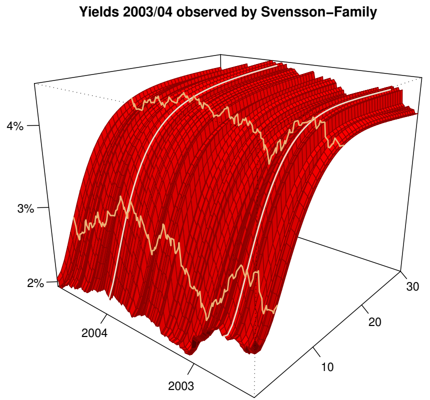
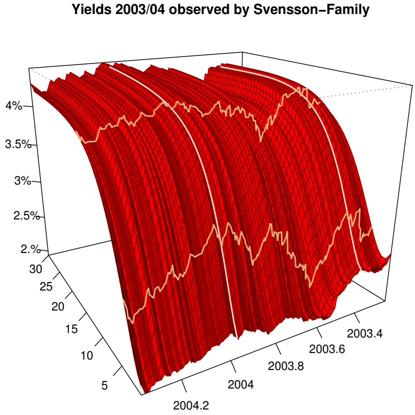
- Statistics and Prediction (only here the physical measure plays a decisice role).

Data


Yields 2003/04 observed by Svensson−Family
4%
3%
2%
2004
2003
10
20
30

Data


Yields 2003/04 observed by Svensson–Family
4%
3.5%
3%
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30
25
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15
10
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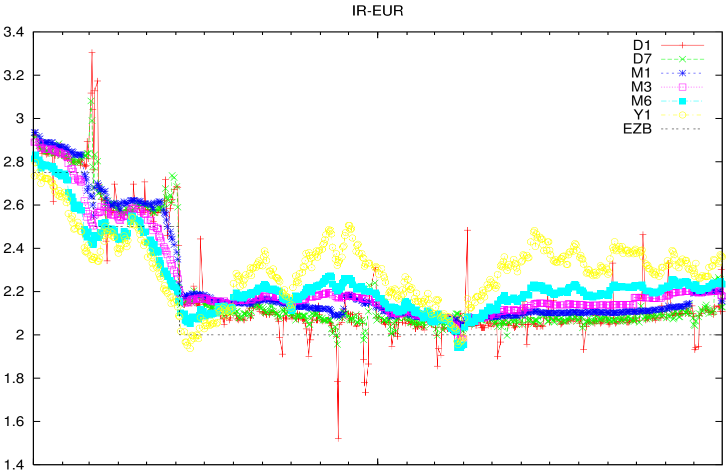

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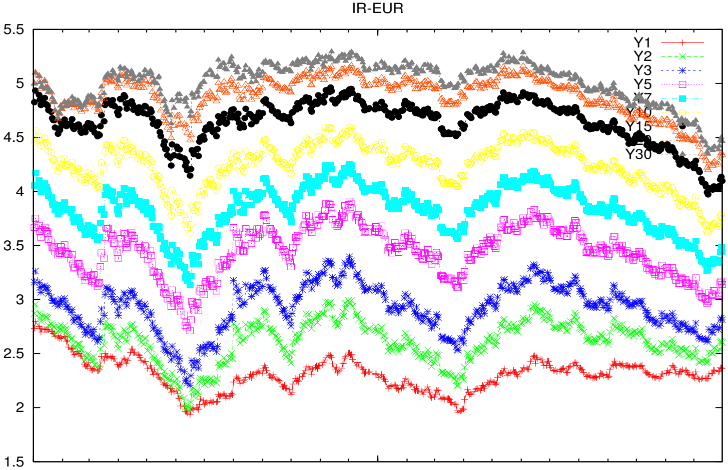

Basic Approaches

- Short Rate Approach: provide the short rate process with respect to the martingale measure and use

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- HJM Approach: write a model directly for the processes $(P(t,T))_{0\leq t\leq T}$ in the historical measure such that no arbitrage can appear.
- Generalized Approaches: weaken existence of a short rate process and/or the existence of martingale measures.

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- HJM-approach allows to (re-)calibrate easily, but pricing might be numerically very challenging.
- Consistency appears in both contexts as a problems: the market data can be fit by many curves, hence there is an a priori choice how to fit. Is today's method consistent with tomorrow's with respect to the model, i.e. does the chosen model evolve within the chosen class of curves? For instance the Svensson-Family of curves is poor in this respect.

The Vasiček Model

The short rate is given through

$$dR_t = \lambda(\mu(t) - R_t)dt + \sigma dW_t \quad ,$$

which yields the following formulas for calibration:

$$Y(t,T) = A(t,T) + B(t,T)R_t,$$

where both terms A, B are explicit in the parameters of the process λ, μ, σ and today's Yield Curve (sic!). The second term B depends only on λ in the typical quasi-exponential form.

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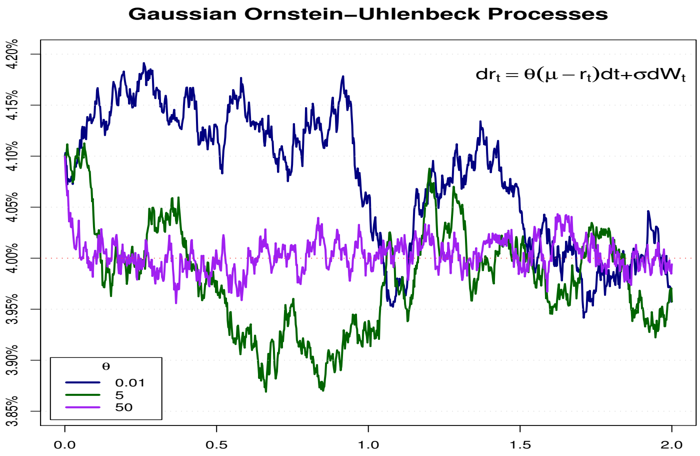
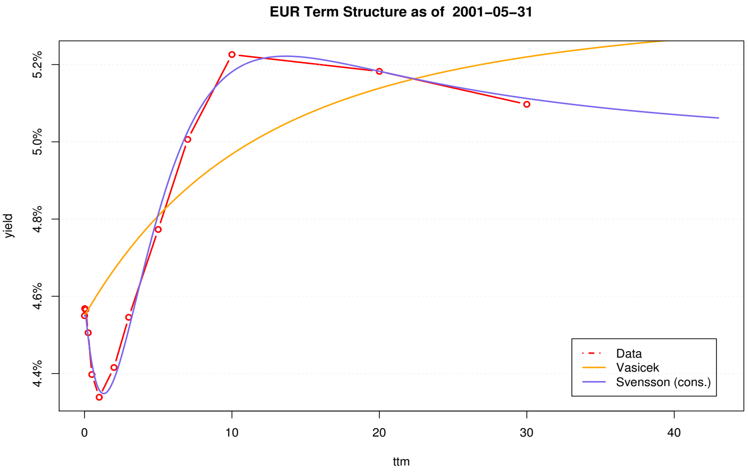
- A HJM equation can be formulated. On the space of Yield Curves this evolution is a time-homogenous Markov process!

Typical trajectories



Today's Yield Curve



Affine OU-Models

The appropriate generalization of the Vasiček Model in a diffusion setting are OU-Models

$$dZ_t = (b(t) - \Lambda Z_t)dt + \sum_{i=1}^{d} \sigma_i dW_t^i$$

in $\mathbb{R}^N$ with vectors $b(t)$, σ_i and a matrix Λ. The short rate appears as one component of the vector process $(Z_t)_{t \geq 0}$. The affine solution structure is analogous, in particular

$$Y(t, T) = A(t, T) + \sum_{i=1}^{N} B_i(t, T) Z_t^i,$$

where completely similar assertions to the Vasiček Model hold:

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- Calibration can be done perfectly today, but recalibraton might soon pose problems.
- HJM-equation can be formulated, which is time-homogenous.
- More parameters will in principle bring better results, but also less stability of fitting procedures. We do not (sic!) obtain more flexibility in the terms A, B.

Just for fun...

One starts to obtain new phenomena by choosing the OU-process

$$dZ_t = (b(t) - \Lambda Z_t)dt + \sum_{i=1}^{d} \sigma_i dW_t^i$$

in an infinite dimensional Banach space X with vectors $b(t)$, σ_i and an unbounded operator $-\Lambda$, which generates a strongly continuous semigroup. The short rate then appears as application of a linear functional $l : X \to \mathbb{R}$ on the process $(Z_t)_{t \geq 0}$. Here we can obtain new types of functions B_i in order to improve the (re-)calibration. However, the price to pay is an infinite number of B_i to fit.

Non-Gaussian OU-Models

One also starts to obtain new phenomena by choosing the OU-process non-Gaussian, i.e. through replacing the Wiener process by a Lévy process (such that the resulting strong solution admits an invariant measure at the end of the day).

$$dZ_t = (b(t) - \Lambda Z_t)dt + \sum_{i=1}^{d} \sigma_i dL_t^i$$

The structure is again similar to the previous ones, with the fine difference that the evolution over time of A is more flexible!

The one-dimensional factor

The model is conveniently written as

$$dR_t = -\lambda R_t \, dt + dZ_{\lambda t},$$

where the mean-reversion level is absorbed by the drift of the Levy process $(Z_{\lambda t})_{t \geq 0}$. We will only consider the case where Z_t is a Levy subordinator, i.e. an almost surely increasing Levy process. This property will imply that Z_t has no diffusion part, positive jumps only and non-negative drift. Apart from that it also implies that R_t is positive a.s., which is not true for the Vasiček model.

Tractability

One can write the solution explicitly as

$$R_t = e^{-\lambda t} R_0 + \int_0^t e^{-\lambda(t-s)}\, dZ_{\lambda s}$$

and

$$\int_t^T r_s\, ds = \alpha(T-t,\lambda) r_t + \int_t^T \alpha(T-s,\lambda)\, dZ_{\lambda s}$$

where we define

$$\alpha(T,\lambda) = \frac{1-e^{-\lambda T}}{\lambda}\,.$$

Under certain conditions there are exponential formulas, which allow to calculate yields explicitly,

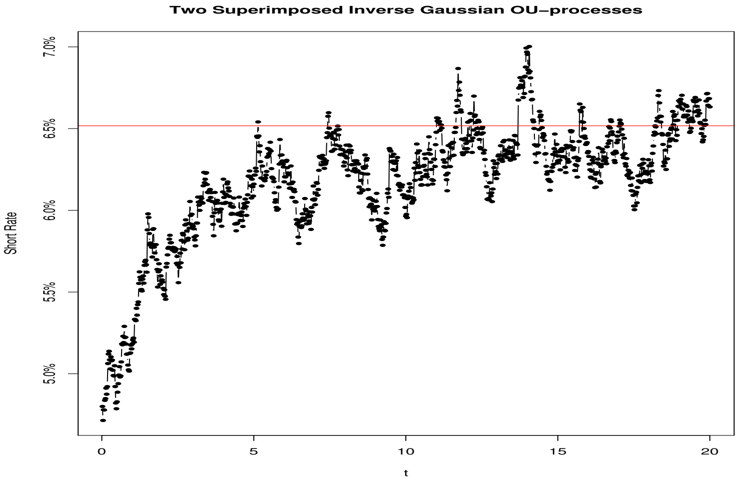
$$Y(t,T) = \frac{1}{T-t}\alpha(T-t,\lambda)R_t - \frac{\lambda}{T-t}\int_t^T \kappa\left(-\alpha(T-s,\lambda)\right)\, ds.$$

Additonal Freedom

Obviously here the function κ, which comes from the invariant distribution of the OU-process creates more flexibility in the model. In particular time-homogenous short-rate models admit quite general functions $A(t, T)$, in contrast to Vasiček-like models. This additional, in principle infinite-dimensional degree of freedom, can be applied to produce typical shapes of curves and to observe their behavior over time.

The following example shows a super-position of two independent OU-processes with infinite jump intensity and the resulting yield curves along a sample trajectory. The typical double-humped structure of the yield curve is preserved and depends on the relative levels of the factors.

Typical Shapes of Yield Curves



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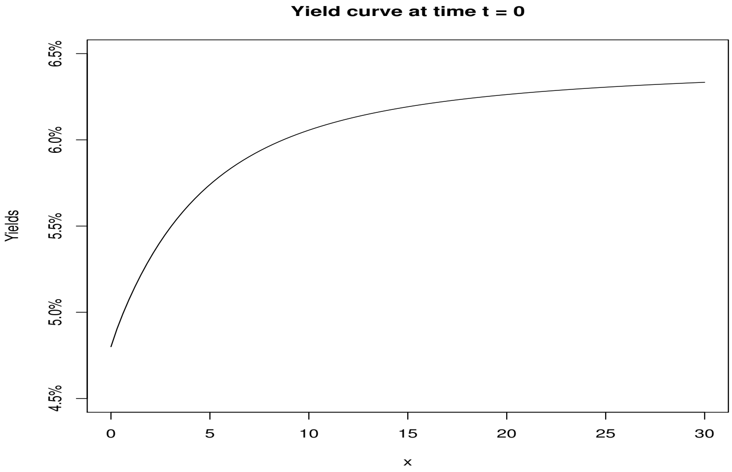

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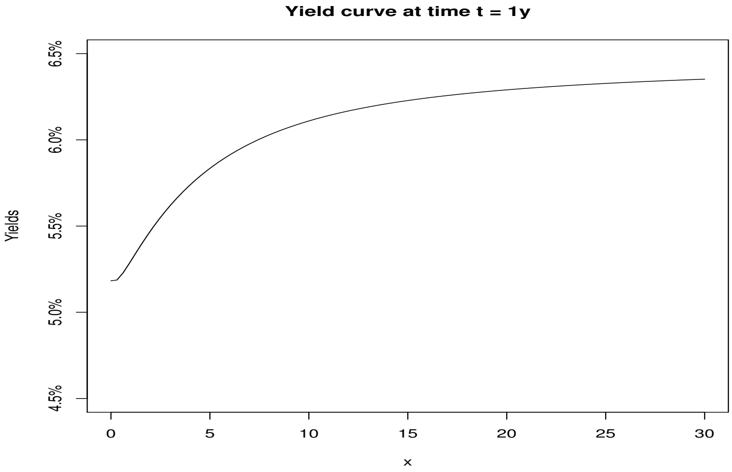

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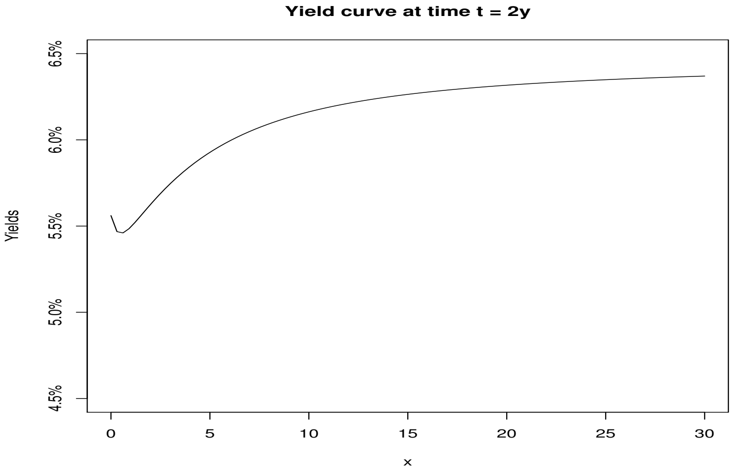

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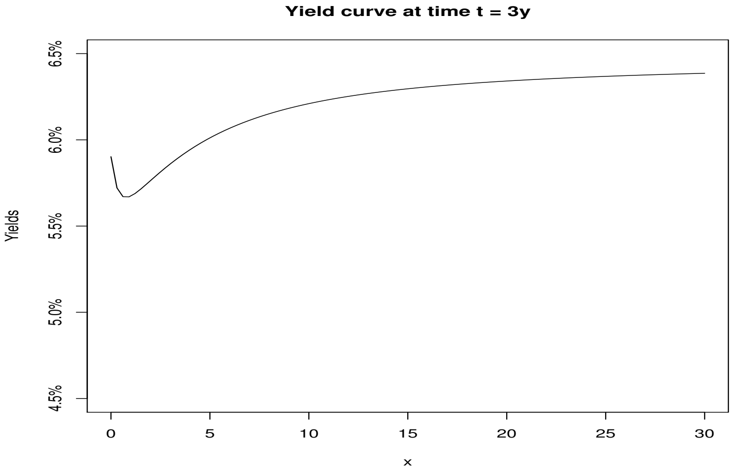

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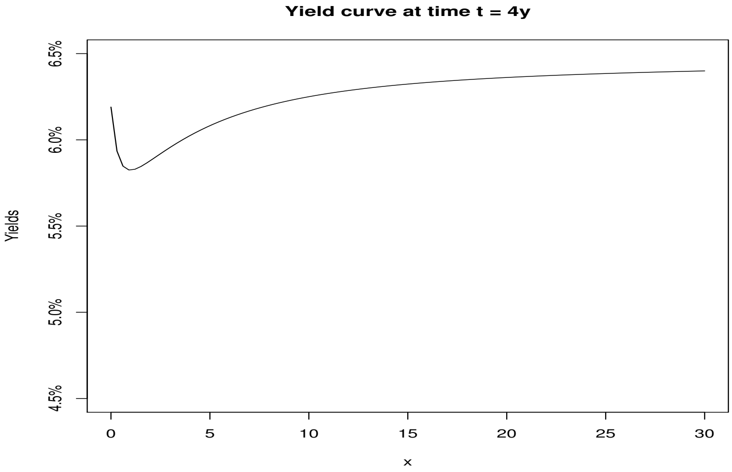

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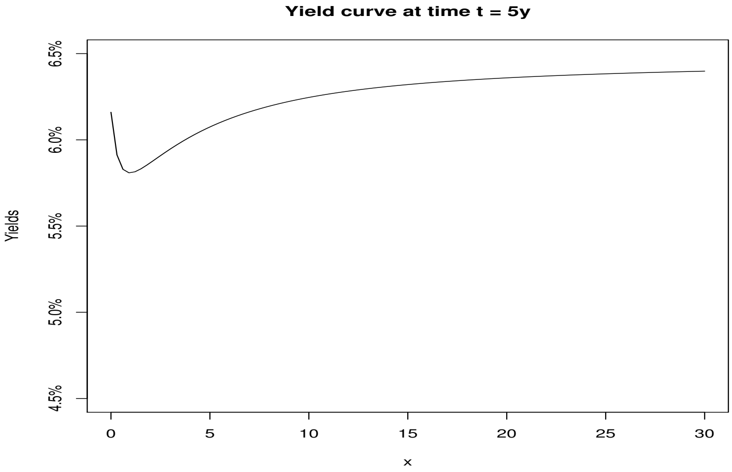

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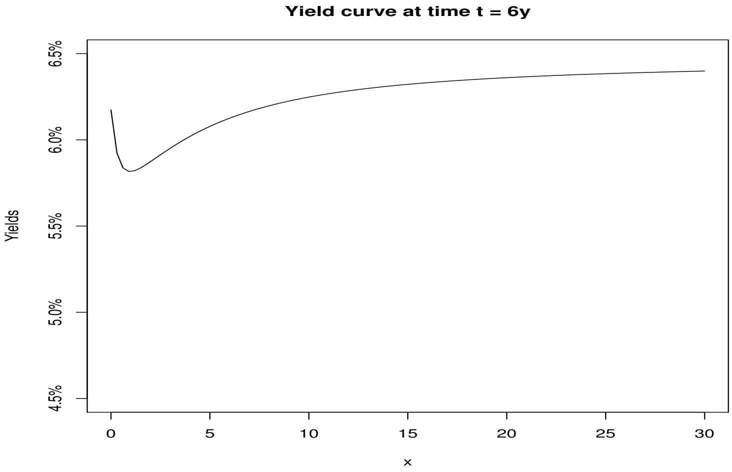

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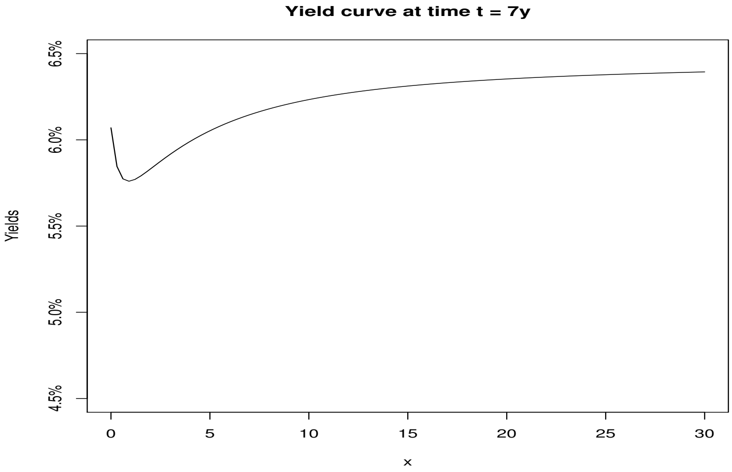

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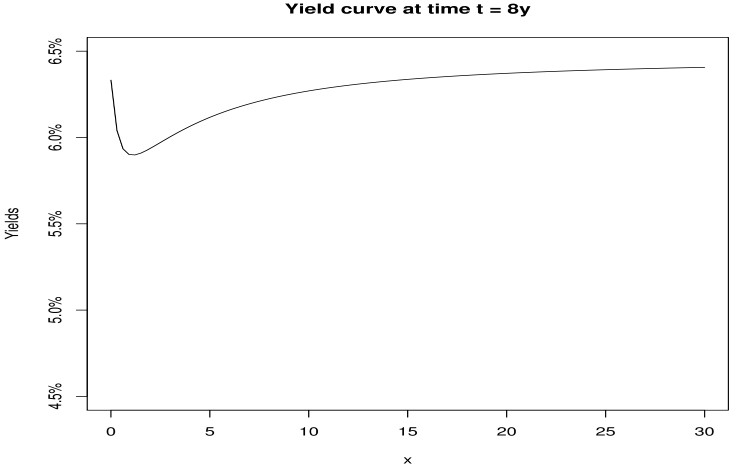

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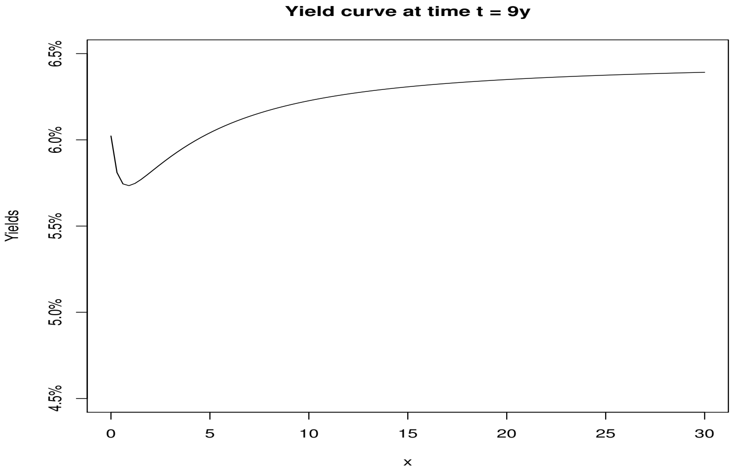

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