Numerical modelling of core-collapse supernovae

Roger Käppeli
Department of Physics

Collaborators:
Simon Scheidegger
Christian Winteler
Albino Perego
Stuart C. Whitehouse
Matthias Liebendörfer
Thomas Rauscher
F.-K. Thielemann
John Biddiscombe
Outline

i. Core-collapse Supernova
   • A quick introduction of the problem

ii. Numerical models and methods
   • Radiation-MHD equations
   • Overview of solution algorithm
   • Source terms & Godunov type schemes

iii. Simulation of magneto-rotational core collapse
   • Influence of strong magnetic fields and rotation
Stellar life cycle

Evolution as a function of mass

- Birth
- Life (Evolution as a function of mass)
- Death
**Stellar evolution**

- Where do the elements come from?

Adapted from Asplund 2005

From primordial abundances of roughly H (75%), He (25%), (very) small amount of Li

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i) Core-Collapse Supernova

Stellar life cycle

Evolution as a function of mass

Core-Collapse Supernova

Birth

Life

Time

Death
Core-collapse supernova

- Huge energy scales
  - $\sim 1 \times 10^{53}$ erg neutrinos
  - $\sim 1 \times 10^{51}$ erg mechanical
  - $\sim 1 \times 10^{48}$ erg elm
  - $\sim 1 \times 10^{41}$ erg visible elm

  Total worldwide energy consumption $\sim 4.7 \times 10^{27}$ erg/y
  Sun radiation energy $\sim 1.2 \times 10^{41}$ erg/y

- Observables
  - Elm
  - Neutrinos
  - Gravitational waves

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Elm = ELectroMagnetic

1 erg = 1e-7 Joule
Core-collapse supernova

• General idea:
  • Implosion of iron core of massive $M \gtrsim 8M_\odot$ at the end of thermonuclear evolution
  • Explosion powered by gravitational binding energy of forming compact remnant:

$$E_b \approx 3 \times 10^{53} \left( \frac{M}{M_\odot} \right)^2 \left( \frac{R}{10\text{km}} \right)^{-1} \text{erg}$$

GRAVITY BOMB!

$M$ Mass of remnant
$R$ Radius of remnant
Core-collapse supernova

General idea:
- Implosion of iron core of massive stars at the end of thermonuclear evolution
- Explosion powered by gravitational binding energy of forming compact remnant

From S. Scheidegger

From M. Liebendoerfer
CCSN Explosion Mechanism?

• Discussed explosion mechanisms:

  • “Enhanced” neutrino-driven explosion mechanism
  
  • MHD mechanism
    Rapid rotation + Magnetic field amplification (Flux compression, winding, MRI, dynamos) e.g. Akiyama et al. 2003, Wilson et al. 2005, Kotake et al. 2006, Burrows et al. 2007, ...
  
  • Acoustic mechanism
    Excitation of ProtoNeutron Star (PNS) oscillations by accretion/SASI generating acoustic power to reheat the stalled shock Burrows et al. 2006,2007
  
  • Phase transition induced explosion mechanism
    Additional compactification of PNS due to phase transition from hadronic matter to quark matter Migdal et al. 1971, … Sagert, Fischer et al. 2009
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CCSN model

Model's ingredients wish list:

1) Multi-D hydro. (no explosions generally in 1D, e.g. Thompson et al. (2003), Rampp & Janka (2002), Liebendoerfer et al. (2002/2005))

2) Plasma physics

(iii) Numerical models & methods

3) Weak interactions

Except phase transition scenario!

4) Neutrino transport

Stars have magnetic fields, e.g. Sun!

5) Nuclear physics

Most of the released gravitational binding energy “available” in form of neutrinos!

6) General relativity

Equation of state describing matter at extreme conditions

Very compact and very massive objects!

7) “Accurate” initial conditions
The Radiation-MHD equations

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0 \]

\[ \frac{\partial (\rho v_i)}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_i v_j + P_* I_{ij} - b_i b_j) = -\rho \frac{\partial \phi}{\partial x_i} + (\rho v_i)_\nu \]

\[ \frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j} [(E + P_*) v_j - v_i b_i b_j] = -\rho v_i \frac{\partial \phi}{\partial x_i} + (\rho e)_\nu \]

\[ \frac{\partial \rho Y_e}{\partial t} + \frac{\partial Y_e \rho v_i}{\partial x_i} = (\rho Y_e)_\nu \]

\( b = \frac{B}{\sqrt{4\pi}} \)

No monopoles \( \nabla \cdot b = 0 \)

\( E = \frac{1}{2} \rho v^2 + \rho e + \frac{b^2}{2} \)

\( P_* = p + \frac{b^2}{2} \)

EoS: \( p = p(\rho, e, \ldots) \)
The Radiation-MHD equations (2)

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0 \]

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EoS: \( p = p(\rho, e, ...) \)

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The Radiation-MHD equations (2)

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0 \]

\[ \nabla^2 \phi = 4\pi G \rho \]

\[ \left( \frac{dx^\alpha}{d\tau} \right) \frac{\partial f}{\partial x^\alpha} + \left( \frac{dp^\alpha}{d\tau} \right) \frac{\partial f}{\partial p^\alpha} = \left( \frac{\delta f}{\delta \tau} \right)_{\text{coll}} \]

Relativistic Boltzmann eq. For each \( \nu \) species

(Neutrinos massless! Propagate at speed of light \( c \).)
Solution Algorithm: An Overview


- Split hydro. and magnetic variables update
- Dimensional splitting: solves eqs in 1D
- Uses dim.-split constrained transport for $\nabla \cdot b = 0$
- Correction for grav. source term for steady state

ii. Radiative transfer (ELEPHANT)  Liebendörfer et al. 2009

- Full transfer NOT feasible in 3D ($3+3+1=7$ dim. problem!)
- Approximation: Isotropic Diffusion Source Approx. (IDSA)

iii. Coupling: Radiation-MHD (FISH+ELEPHANT)
Solution Algorithm: An Overview

i. MHD (FISH)  
   
   • Split hydro. and magnetic variables update
   • Dimensional splitting: solves eqs in 1D
   • Uses dim.-split constrained transport for \( \nabla \cdot b = 0 \)
   • Correction for grav. source term for steady state

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iii. Coupling: Radiation-MHD (FISH+ELEPHANT)  

- Numerical models & methods
  - MPI + OpenMP
  - FISH
  - Strong scaling for 600^3 cells

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Source terms & Godunov schemes

• The problem: (in our simulations)

Ability to maintain near hydrostatic equilibrium for a long time!
Source terms & Godunov schemes (2)

- Consider 1D hydrodynamics eqs with gravity

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} &= S \\
F &= \begin{bmatrix}
\rho v \\
\rho v^2 + p \\
(E + p)v
\end{bmatrix} \\
S &= -\begin{bmatrix}
0 \\
\rho \\
\rho v
\end{bmatrix} \frac{\partial \phi}{\partial x}
\end{align*}
\]

- Classical solution algorithm:
  - Solve homogeneous eqs with Godunov type method (i.e. solve Riemann problem)
  - Account for source term in second step (split/unsplit)
Source terms & Godunov schemes (3)

• Classical solution algorithm:

\[
\frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} S(u(x, t)) \, dx
\]

\[
u_{i}^{n+1} = u_{i}^{n} - \frac{\Delta t}{\Delta x} \left( F_{i+1/2}^{n} - F_{i-1/2}^{n} \right) + \Delta t S_{i}^{n}
\]

• Numerical flux \( F_{i+1/2}^{n} \) from (approximate) Riemann solver, e.g.

  • (Local) Lax-Friedrichs  Lax (1954), Rusanov (1961)
  • HLL (C)  Harten, Lax and van Leer (1983), Toro et al. (1994)
  • Roe  Roe (1981)
Source terms & Godunov schemes (4)

Interested in hydrostatic equilibrium:

\[ \frac{\partial F}{\partial x} = S \quad \iff \quad \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x} \]

EoS: \( p = p(\rho, e) \)
Interested in hydrostatic equilibrium:
\[ \frac{\partial F}{\partial x} = S \implies \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x} \]

EoS: \( p = p(\rho, e) \)

Discretise in cells \([x_{i-1/2}, x_{i+1/2}]\)
ii) Numerical models & methods

Source terms & Godunov schemes (4)

Interested in hydrostatic equilibrium:

\[ \frac{\partial F}{\partial x} = S \quad \Rightarrow \quad \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x} \]

\[ \text{EoS: } p = p(\rho, e) \]

Discretise in cells \([x_{i-1/2}, x_{i+1/2}]\)

Define cell averages

\[ u_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t^n) \, dx \]

\[ S_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} S(u(x, t)) \, dx \]
Source terms & Godunov schemes (4)

Interested in hydrostatic equilibrium:

$$\frac{1}{\Delta x} \left( F_{i+1/2}^n - F_{i-1/2}^n \right) \equiv S_i^n = - \begin{bmatrix} 0 \\ \rho_i \\ 0 \end{bmatrix} \frac{\phi_{i+1} - \phi_{i-1}}{\Delta x}$$

Contains also gravity induced gradient!

$$F_{i+1/2}^{LxF} = \frac{1}{2} (F_i + F_{i+1}) - \frac{S_{\text{max}}}{2} \left( u_{i+1} - u_i \right)$$

$$F_{i-1/2}^{LxF} = \frac{1}{2} (F_{i-1} + F_i) - \frac{S_{\text{max}}}{2} \left( u_i - u_{i-1} \right)$$

\(\Delta p_{i-1/2} = -(\rho \Delta \phi)_{i-1/2}\)

\(\Delta p_{i+1/2} = -(\rho \Delta \phi)_{i+1/2}\)
ii) Numerical models & methods

**Source terms & Godunov schemes (4)**

Hydrostatic atmosphere in a constant gravitational field

\[
\phi(x) = gx, \quad \rho(x) = \left[ \rho_0^{\gamma-1} - \frac{g}{K} \frac{\gamma - 1}{\gamma} x \right]^{\frac{1}{\gamma-1}} \quad p = \frac{\rho_0^\gamma}{\rho_0^\gamma} \rho^\gamma = K \rho^\gamma
\]

<table>
<thead>
<tr>
<th>N</th>
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**Error**

\[
Err = \frac{1}{N} \sum_{i} \left| \rho_i - \rho_i^0 \right|
\]
Source terms & Godunov schemes (4)

- Proposed solution methods in literature, e.g.:
  - Upwinding the source terms
    e.g. Bermudez & Vazquez (1994), Jenny & Mueller (1998), Bale et al. (2003, ...)
  - Generalised Riemann problem (ADER methods)
    e.g. Toro & Titarev (2002), Ben-Artzi & Li (2007), Dumbser et al. (2008), Castro & Toro (2008), ...
  - Steady state preserving reconstructions, well-balanced schemes
    e.g. LeVeque (1998), LeVeque & Bale (1998), Botta et al. (2004), Fuchs et al. (2010)

Note: there are many, many more... especially for shallow-water eqs!!!
Source terms & Godunov schemes (4)

- Proposed solution methods in literature, e.g.:
  - Requirements
    - Equilibrium not known in advance (self-gravity)
    - Extensible for general EoS
    - (At least) second order accuracy
  - ... other methods and schemes

Note: there are many, many more... especially for shallow-water eqs!!!
Source terms & Godunov schemes (4)

Interested in (numerical) hydrostatic equilibrium:

\[
\frac{1}{2} \left( \frac{p_{i+1} - p_i}{\Delta x} + \frac{p_i - p_{i-1}}{\Delta x} \right) = \frac{1}{2} \left( \frac{\rho_i + \rho_{i+1}}{2} \frac{\phi_{i+1} - \phi_i}{\Delta x} + \frac{\rho_{i-1} + \rho_i}{2} \frac{\phi_i - \phi_{i-1}}{\Delta x} \right) = -\rho \frac{\partial \phi}{\partial x} + O(\Delta x^2)
\]

(Analytical) gradients in HSE

\[
\begin{align*}
\left( \frac{\partial u}{\partial x} \right)_{\text{HSE}} &= -\frac{1}{c^2} \begin{bmatrix} \rho \\ 0 \\ \rho e + p \end{bmatrix} \frac{\partial \phi}{\partial x} \\
\left( \frac{\partial w}{\partial x} \right)_{\text{HSE}} &= - \begin{bmatrix} \rho / c^2 \\ 0 \\ \rho \end{bmatrix} \frac{\partial \phi}{\partial x}
\end{align*}
\]

\[
\begin{bmatrix} u \\ \rho v \\ E \end{bmatrix} = \begin{bmatrix} \rho \\ \rho v \\ p \end{bmatrix}
\]

\[
\begin{bmatrix} \rho \\ v \\ p \end{bmatrix}
\]
Interested in (numerical) hydrostatic equilibrium:

\[
\frac{1}{2} \left( \frac{p_{i+1} - p_i}{\Delta x} + \frac{p_i - p_{i-1}}{\Delta x} \right) = -\frac{1}{2} \left( \frac{\rho_i + \rho_{i+1}}{2} \frac{\phi_{i+1} - \phi_i}{\Delta x} + \frac{\rho_{i-1} + \rho_i}{2} \frac{\phi_i - \phi_{i-1}}{\Delta x} \right) = -\rho \frac{\partial \phi}{\partial x} + O(\Delta x^2)
\]

Numerical gradients in HSE

\[
\mathbf{\Delta u}^{\text{HSE}}_{i+1/2} = -\frac{1}{2} \begin{bmatrix} (\rho/c^2)_{i+1/2} \\ 0 \\ (\rho e)_{i+1/2} + p_{i+1/2} \end{bmatrix} \Delta \phi_{i+1/2}
\]

\[
\mathbf{\Delta w}^{\text{HSE}}_{i+1/2} = -\frac{1}{2} \begin{bmatrix} (\rho/c^2)_{i+1/2} \\ 0 \\ \rho_{i+1/2} \end{bmatrix} \Delta \phi_{i+1/2}
\]

\[
q_{i+1/2} = \frac{q_i + q_{i+1}}{2} \quad \Delta q_{i+1/2} = q_{i+1} + q_i
\]
Interested in (numerical) hydrostatic equilibrium:

\[
\frac{\text{d}u}{\text{d}t} + \frac{1}{\Delta x} \left( F_{i+1/2}^n - F_{i-1/2}^n \right) = S_{i-1/2}^n + S_{i+1/2}^n
\]

\[ u_{i+1/2}^L = u_i + \frac{\Delta u_{i+1/2}^\text{HSE}}{2} \quad u_{i+1/2}^R = u_{i+1} - \frac{\Delta u_{i+1/2}^\text{HSE}}{2} \]

\[ w_{i+1/2}^L = w_i + \frac{\Delta w_{i+1/2}^\text{HSE}}{2} \quad w_{i+1/2}^R = w_{i+1} - \frac{\Delta w_{i+1/2}^\text{HSE}}{2} \]

\[ q_{i+1/2} = \frac{q_i + q_{i+1}}{2} \quad \Delta q_{i+1/2} = q_{i+1} + q_i \]

\[ \rho, p \quad \phi \]

\[ x_{i-1} \quad x_{i-1/2} \quad x_i \quad x_{i+1/2} \quad x_{i+1} \]

\[ \rho \text{ sound speed} \quad \rho e \text{ int. energy density} \]
Source terms & Godunov schemes (4)

Hydrostatic atmosphere in a constant gravitational field

\[ \phi(x) = gx \quad \rho(x) = \left[ \rho_0^{\gamma-1} - \frac{g}{K} \frac{\gamma - 1}{\gamma} x \right]^{\frac{1}{\gamma-1}} \quad p = \frac{p_0}{\rho_0} \rho^{\gamma} = K \rho^{\gamma} \]

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\[ Err = \frac{1}{N} \sum_i |\rho_i - \rho_0^0| \]

\[ \Delta u_{i+1/2}^{\text{HSE}} = \frac{2}{\Delta x^2} \]

\[ \Delta w_{i+1/2}^{\text{HSE}} = \frac{2}{\Delta x^2} \]

\[ q_{i+1/2} = \frac{q_i + q_{i+1}}{2} \quad \Delta q_{i+1/2} = q_{i+1} - q_i \]

\( c \) sound speed \quad \( \rho e \) int. energy density

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Source terms & Godunov schemes (4)

- Second order extension

\[ \Delta u_{i+1/2} = u_{i+1} - u_i - \Delta u_{i+1/2} \text{ slopes} \]

\[ \Delta F^+_{i+1/2} = F_{i+1} - F_{i+1/2} - \Delta F_{i+1/2}^{HSE} \]

\[ \Delta F^-_{i+1/2} = F_{i+1/2} - F_i - \Delta F_{i+1/2}^{HSE} \]

Reconstruction with HSE (flux) differences

- Multi-dimensional straight forward
- Higher order extension for HSE reconstruction?
- Non-zero velocity steady state?
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Stars have magnetic fields, e.g. Sun!

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Most of the released gravitational binding energy “available” in form of neutrinos!

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Equation of state describing matter at extreme conditions

6) General relativity
Very compact and very massive objects!

7) “Accurate” initial conditions
CCSN Explosion Mechanism?

- Discussed explosion mechanisms:
  - “Enhanced” neutrino-driven explosion mechanism
    - Hydro. instabilities: convection, Standing Accretion Shock Instabilities (SASI)
  - MHD mechanism
    - Rapid rotation + Magnetic field amplification (Flux compression, winding, MRI, dynamos)
      - e.g. Akiyama et al. 2003, Wilson et al. 2005, Kotake et al. 2006, Burrows et al. 2007, ...
  - Acoustic mechanism
    - Excitation of ProtoNeutron Star (PNS) oscillations by accretion/SASI generating acoustic power to reheat the stalled shock
  - Phase transition induced explosion mechanism
    - Additional compactification of PNS due to phase transition from hadronic matter to quark matter
MHD CCSN model

Actual model's ingredients list:

1) Multi-D hydro.  
2) Plasma physics  
3) Weak interactions  
4) Neutrino transport  
5) Nuclear physics  
6) General relativity  
7) “Accurate” initial conditions

- Assume infinite conductivity
- Parallel 3D ideal MHD code
- Parametrised
  - Liebendörfer 2005
  - “Not so bad”... 2D simulations shown that \( \nu \) contribute only 10-25% to explosion energy
- EoS
  - Lattimer & Swesty 1991
- Spherical effective GR potential
  - Marek et al. 2006
+ 2D axisymmetric Newton potential
Role of Rotation & Magnetic Field

Pre-collapse

- Rotation (???)
- B (???)

Distribution in Fe core ???

Heger et al. 2005
Hirschi et al. 2004 & 2005

Successful Explosion...

Post-collapse

- Pulsar Magnetar
  Rotation (???)
  B (???)
  Taylor et al. 1993
  Kouveliotou et al. 1998

- Observable Asymmetries
  Wang & Wheeler 2008
  Kjaer et al. 2010
Role of Rotation & Magnetic Field

Pre-collapse

- Rotation (???)
- B (???)

Post-collapse

- Pulsar
- Magnetar

Rotation & Magnetic fields present before and after explosion

Influence of Rotation & B on explosion???

If strong effects, is it common or only (very) rare

iii) Simulation of MHD CCSN

Heeger et al. 2005
Hirschi et al. 2004 & 2005
Wang & Wheeler 2008
Kjaer et al. 2010
Taylor et al. 1993
Kouveliotou et al. 1998

Observable Asymmetries
Wang & Wheeler 2008
Kjaer et al. 2010
MHD CCSN Mechanism

- Rotational energy of Proto-Neutron Star (PNS)

\[
T_{\text{rot}} = \frac{1}{2} I_{\text{PNS}} \Omega_{\text{PNS}}^2 \\
\approx 1 \times 10^{51} \text{ ergs} \times \left( \frac{M}{1.5M_\odot} \right) \left( \frac{P}{2 \text{ ms}} \right)^{-2} \left( \frac{R}{10 \text{ km}} \right)^2
\]

- Idea: Extract “free” energy stored in differential rotation with Magnetic Field

Typical explosion energy \( E_{\text{expl}} \sim 10^{51} \text{ erg} \)
MHD CCSN Mechanism

- Rotational energy of Proto-Neutron Star (PNS)
  - "Free" energy in differential rotation?
    \[ E_{\text{rot,free}} = T_{\text{rot}} (L) - T_{\text{rot, solid}} (L) \]
  - Appreciable fraction of energy can be extracted by magnetic field and maybe trigger an explosion

iii) Simulation of MHD CCSN
MHD CCSN Mechanism (2)

- How big must the magnetic field be to make a difference?
  - Gas pressure: \( P_{\text{gas}} \sim 10^{30} \text{ dyn/cm}^2 \)
  - Magnetic pressure: \( P_B = \frac{B^2}{2} \implies \sim 10^{15} \text{ G} \)

- How to amplify the magnetic field?
MHD CCSN Mechanism (2)

- How big must the magnetic field be to make a difference?

How to amplify the magnetic field?

- **Flux compression** Works well during collapse!
- **Winding** With differential rotation, linear growth with time
- **Magneto-Rotational Instability (MRI)**
  - With differential rotation, exponential growth with time, VERY small wavelengths...
- **Dynamo?**
iii) Simulation of MHD CCSN

A simulation example

- 3D MHD inner 600km cube
  Resolution ~ 1 km ⇒ No MRI!!!

- Outside followed by 1D spherical symmetric code

- Progenitor: 15 $M_{\odot}$
  Heger et al. 2002

Computations @ CSCS

Swiss National Supercomputing Centre
A simulation example

- Rotation: shellular type
  \[ \Omega(r) = \Omega_0 \frac{R_0^2}{R_0^2 + r^2} \]
  \[ \Omega_0 = 2\pi \]
  \[ R_0 = 500 \text{ km} \]
  \[ r = \sqrt{x^2 + y^2 + z^2} \]
  \[ \beta_{\text{initial}} \equiv \frac{E_{\text{rot}}}{E_{\text{pot}}} \approx 0.48\% \]

- Magnetic field: vector potential
  \[ B = \nabla \times A \]
  \[ \nabla \cdot B = 0 \]
  Initially

\[ A = B_0 \left( ye_x - xe_y \right) \times \frac{\rho(r)}{\rho_0} \]
\[ B_0 = 10^{12} \text{ G} \]
\[ \rho_0 = 5 \times 10^7 \text{ g/cm}^3 \]
A simulation example

- 3D MHD inner 600km cube
- Outside followed by 1D spherical symmetric code
- Progenitor: 15 Heger et al. 2002
- Resolution ~ 1 km
- Computations @ No MRI!!!
- Heger et al. 2002
- Käppeli et al. 2009
- Resolution ~ 1 km
- Rotation: shellular type
- Magnetic field: vector potential

Computations @ CSCS
Swiss National Supercomputing Centre

Time: 0.017300 s
MHD CCSN mechanism

Extracting “free” rotational energy by field winding

Entropy

[kB/baryon]

log $\Omega$

Angular velocity [rad/s]

iii) Simulation of MHD CCSN
MHD CCSN mechanism

Extracting “free” rotational energy by field winding

r-process?

iii) Simulation of MHD CCSN
Conclusions

- Simulation of core-collapse supernova a challenging field for physics/computational science/numerics

- HSE reconstruction improving accuracy
  - Steady flow? Higher order?

- MHD core-collapse simulation
  - Very high magnetic field initially?
  - Third dimension crucial for dynamics of the toroidal field

The End, Thanks!
Energy production in stars

- Energy generated via nuclear fusion and gravitational contraction
- Energy continuously transported away (lost!) by photon and neutrino emission
Supernovae classification

- **Taxonomical/Morphological approach**
  
  Like botanists and zoologists, find observable characteristics that eventually provide a deeper physical understanding. However, not all necessarily meaning full...

<table>
<thead>
<tr>
<th>thermonuclear</th>
<th>core collapse</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>II</td>
</tr>
<tr>
<td>SiII</td>
<td>IIb</td>
</tr>
<tr>
<td>HeI</td>
<td>III</td>
</tr>
<tr>
<td>Ia</td>
<td>II In</td>
</tr>
<tr>
<td>Ic</td>
<td>Ib/c pec</td>
</tr>
<tr>
<td>Ib</td>
<td>II P</td>
</tr>
</tbody>
</table>

Tarutto 2003
Conditions at onset of core-collapse

- Onion like structure: history of nuclear burning
- Central density $\rho_c \approx 10^{10} \text{g cm}^{-3}$
- Central temperature $T_c \approx (8 - 10) \times 10^9 \text{K}$
- Core mainly iron group nuclei $\implies$ “Iron core”
  Radius $\sim 3000 \text{ km}$
- Dynamical or free fall time scale
  $$\tau_{dyn} \propto (G\bar{\rho})^{-1/2} \approx 1\text{ ms}$$

$\bar{\rho}$ Average density
Source terms & Godunov schemes (4)

Interested in (numerical) hydrostatic equilibrium:
Source terms & Godunov schemes (3)

- Classical solution algorithm:
  \[
  \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} S(u(x, t)) \, dx
  \]
  \[
  u_{i}^{n+1} = u_{i}^{n} - \frac{\Delta t}{\Delta x} \left( F_{i+1/2}^{n} - F_{i-1/2}^{n} \right) + \Delta t S_{i}^{n}
  \]

- Numerical Methods
  - Steady state
    \[
    \frac{\partial F}{\partial x} = S \quad \text{Exact balance}
    \]
    \[
    \frac{1}{\Delta x} \left( F_{i+1/2}^{n} - F_{i-1/2}^{n} \right) = S_{i}^{n}
    \]
  - Classical solution algorithm:
    \[
    \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} S(u(x, t)) \, dx
    \]
  - (Local) Lax-Friedrichs
  - HLL (C)
  - Roe
  - Roe (1981)

R. Käppeli, ETH Zürich
Radiative transfer: IDSA

• Goal:
  • Algorithm that captures dominant features of radiative transfer efficiently

\[
\left( \frac{dx^\alpha}{d\tau} \right) \frac{\partial f}{\partial x^\alpha} + \left( \frac{dp^\alpha}{d\tau} \right) \frac{\partial f}{\partial p^\alpha} = \left( \frac{\delta f}{\delta \tau} \right)_{\text{coll}}
\]

Currently we can NOT solve

Regimes: Diffusive    Semi-Transparent    Transparent
Radiative transfer: IDSA

- Goal:
  - Algorithm that captures dominant features of radiative transfer efficiently

\[ \left( \frac{dx^\alpha}{d\tau} \right)_{\text{coll}} \]

Regimes:
- Diffusive: \( \lambda \ll 1 \)
- Semi-Transparent
- Transparent: \( \lambda \gg 1 \)

Mean free path
Radiative transfer: IDSA (2)

- Decompose distribution function of transported particles into both extreme cases

\[ f = f^t + f^s \]

- Trapped particles: Diffusive regime
- Streaming particles: Transparent regime

Boltzmann eq.

\[ D(f = f^t + f^s) = C = C^t + C^s \]

\[ D(f^t) = C^t - \Sigma \]
\[ D(f^s) = C^s + \Sigma \]

\[ \Sigma \text{ Diffusion Source: chosen to reproduce diffusion limit} \]
Radiative transfer: IDSA (2)

- Decompose distribution function of transported particles into both extreme cases

**Key point:** Use different/adapted approx. for trapped and streaming components!

- e.g. for trapped particles use thermal spectrum
- for streaming particles use stationary-state approximation

**Application dependent!**
Radiative transfer: IDSA (3)

150 ms after bounce

- Comparison reference Boltzmann solution vs IDSA approx. In spherical symmetry

\[ f^t \] dominates!

- At 40 km radius (trapped regime)

- At 80 km radius (semi-transparent)

\[ f^t + f^S \]

Good Agreement!

- At 160 km radius (free streaming)

R. Käppeli, ETH Zürich
Radiative transfer: IDSA (4)

Shock position as function of time

Good Agreement!
Simulation setup

i. Successful explosion determined in the very central region

ii. Simulate the innermost region with:
   - Gravity (Newtonian/General relativity)
   - Multi-D fluid dynamics + plasma physics
   - Neutrino transport
   - Nuclear physics

iii. Outer region followed by spherical symmetric assumption

iii) Simulation of MHD CCSN