

$$\left[\sim \frac{1}{2} (x a a x - 2 x a b + b b) = \frac{1}{2} (a^2 x^2 - 2 a b x + b^2) \right] \quad \&$$

At an extremum:

$$\vec{\nabla} \phi(\vec{x}) = A^T A \vec{x} - A^T \vec{b} \stackrel{!}{=} 0$$

These are the so-called normal equations and their solution is the sought least-squares solution

Rem.: (i) The solution to the normal equations is indeed a minimum of $\phi(\vec{x})$ and it is unique:

① $A^T A$ is a positive definite matrix

$$\|A\vec{x}\|_2^2 = \underbrace{\vec{x}^T A^T A \vec{x}}_{\text{Def. of positive definite}} > 0 \text{ for all } x \in \mathbb{R}^n, \vec{x} \neq 0$$

$$\textcircled{2} \vec{x}^T A^T A \vec{x} = 0 \Leftrightarrow \vec{x} = 0$$

" \Leftarrow ": clear

$$\text{"} \Rightarrow \text{"}: \underbrace{\vec{x}^T}_{(A\vec{x})^T} \underbrace{A^T A}_{=0} \vec{x} = 0$$

$= 0 \Leftrightarrow \vec{x} = 0$ because $\text{rank}(A) = n$!
 $(A\vec{x})^T \nearrow \checkmark$

This is just a generalization of the well known fact $\frac{1}{2}(a a x^2 - 2 a b x + b^2) \dots a \cdot a > 0$ for a minimum!

$$(ii) A^T A \vec{x} = A^T \vec{b}: \underbrace{\begin{pmatrix} A^T \\ n \times m \end{pmatrix}}_{\begin{pmatrix} A^T A \\ n \times n \end{pmatrix}} \underbrace{\begin{pmatrix} A \\ m \times n \end{pmatrix}}_{\begin{pmatrix} \vec{x} \\ n \times 1 \end{pmatrix}} = \underbrace{\begin{pmatrix} A^T \\ n \times m \end{pmatrix}}_{\begin{pmatrix} A^T \vec{b} \\ n \times 1 \end{pmatrix}} \underbrace{\begin{pmatrix} \vec{b} \\ m \times 1 \end{pmatrix}}_{\begin{pmatrix} \vec{x} \\ n \times 1 \end{pmatrix}} = \begin{pmatrix} A^T \vec{b} \\ n \times 1 \end{pmatrix}$$