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For f $(n+1)$-times continuously differentiable, one can show that for every $x \in I = [a,b]$ there is a $\xi(x) \in I$ such that

↑ depends on x!

(n+1)-th derivative

$$e(x) = f(x) - p[f|x_0,\dots,x_n](x) = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}}_{\text{depends on } f} \cdot \underbrace{\prod_{j=0}^{n}(x-x_j)}_{\text{nodes}}$$

Here $e(x)$ is a function over the whole interpolation interval I. Often, one is just interested in the biggest/maximum error over I:

$$\|e\|_\infty = \max_{x \in I} |e(x)| \qquad (\underline{\text{Maximum norm}})$$

$$= \max_{x \in I} \left| \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \cdot \prod_{j=0}^{n}(x-x_j) \right|$$

$$\leq \max_{x \in I} \left| \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \right| \cdot \max_{x \in I} \left| \prod_{j=0}^{n}(x-x_j) \right|$$

$$= \frac{\|f^{(n+1)}\|_\infty}{(n+1)!} \cdot \| \prod_{j=0}^{n} \underbrace{(x-x_j)}_{\leq b-a} \|_\infty$$

Estimates

$$\leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!} (b-a)^{n+1}$$