$$\left[\sim \frac{1}{2}\left(x\,a\,a\,x - 2\,x\,a\,b + b\,b\right) = \frac{1}{2}\left(a^2x^2 - 2abx + b^2\right) \right] \checkmark$$

At an extremum:

$$\vec{\nabla}\phi(\vec{x}) = A^T A \vec{x} - A^T \vec{b} \stackrel{!}{=} 0$$

These are the so-called <u>normal equations</u> and their solution is the seeked least-squares solution

<u>Rem.:</u> (i) The solution to the normal equations is indeed a minimum of $\phi(\vec{x})$ and it is unique:

① $A^T A$ is a positive definit matrix

$$\|A\vec{x}\|_2^2 = \underbrace{\vec{x}^T A^T A \vec{x} > 0}_{\text{Def. of positive definite}} \quad \text{for all } x \in \mathbb{R}^n, \vec{x} \neq 0$$

② $\vec{x}^T A^T A \vec{x} = 0 \Leftrightarrow \vec{x} = 0$

"$\Leftarrow$": clear

"$\Rightarrow$": $\underbrace{\vec{x}^T A^T}_{(A\vec{x})^T} \underbrace{A\vec{x}}_{} = 0$

$= 0 \Leftrightarrow \vec{x}$ because $\text{rank}(A) = n$!

$\checkmark$.

This is just a generalization of the well known fact $\frac{1}{2}\left(a\,a\,x^2 - 2abx + b^2\right) \dots a \cdot a > 0$ for a minimum!

(ii) $A^T A \vec{x} = A^T \vec{b}$: $\underbrace{\begin{pmatrix} A^T \\ n\times m \end{pmatrix} \begin{pmatrix} A \\ m\times n \end{pmatrix}}_{\begin{pmatrix} A^T A \\ n\times n \end{pmatrix}} \begin{pmatrix} \vec{x} \\ n\times 1 \end{pmatrix} = \underbrace{\begin{pmatrix} A^T \\ n\times m \end{pmatrix} \begin{pmatrix} \vec{b} \\ m\times 1 \end{pmatrix}}_{\begin{pmatrix} A^T b \\ n\times 1 \end{pmatrix}}$

$$\begin{pmatrix} A^T A \\ n\times n \end{pmatrix} \begin{pmatrix} \vec{x} \\ n\times 1 \end{pmatrix} = \begin{pmatrix} A^T b \\ n\times 1 \end{pmatrix}$$