V.1.2 The orthogonal decomposition method

Def.: an orthogonal matrix Q is a real square matrix whose columns are orthonormal:

$$Q^T Q = Q Q^T = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Rem.: – $Q^{-1} = Q^T$

An important property of orthogonal matrices is that they don't change the Euclidean norm (i.e. length):

$$\|Q\vec{x}\|_2^2 = \vec{x}^T \underbrace{Q^T Q}_{I} \vec{x} = \vec{x}^T \vec{x} = \|\vec{x}\|_2^2$$

Fact: every matrix $A \in \mathbb{R}^{m \times n}$ with full column rank n (linearly independent columns) has a so-called QR-decomposition

$$A = QR$$

where $Q \in \mathbb{R}^{m \times m}$ is an orthogonal matrix and $R \in \mathbb{R}^{m \times n}$ an upper triangular matrix with $r_{ii} \neq 0$ $(i = 1, \dots, n)$