On the Stability of the Unitary Group

by Maria Saliani

1. Introduction

Let U_{2n} be the group of isometries of the hyperbolic space H^{2n} over a unitary ring $A^{\#} = (A, \Lambda, \varepsilon)$ and let EU_{2n} be the elementary unitary subgroup. The first stability results for U_{2n} are due to Bak [1969] and Vaserstein [1970]. They say that

$$(*) \qquad U_{2n} = U_{2(n-1)} \cdot EU_{2n}$$

if A is a finite algebra over a commutative ring R with $\operatorname{Dim} \operatorname{Max}(R) = d$ and $n > d + 1$. A comprehensive proof is in Bass [1973]. In his paper, Vaserstein also claims that

$$(**) \qquad U_{2(n-1)} \cap EU_{2n} = EU_{2(n-1)}$$

for $n > d + 2$. However its proof is not correct: the property $(2.2)_n^0$ in Theorem 2.2 holds for all $n \geq 2$, not, as claimed, for all $n \geq 1$ (this was noticed by B. Keller). Furthermore there is an error in the proof that Theorem 2.7 implies property $(2.6)_n$ (p. 313 of the english translation). The condition (3), given there, cannot be fullfilled in general. The proof of (**) for $n > d + 2$ given in a subsequent paper (Vaserstein [1974]) also has gaps: the proof of Lemma 3.1 contains a mistake. It is not true that $Lx + L(1-x)(C^{-1}u', w)_r = L$ (see p. 293 of the english translation). Furthermore we were not able to understand the proofs of Proposition 4.1 and Corollary 4.2. A proof of (**) for unitary rings satisfying a so-called *unitary stable range condition* is in Kolster [1975[a]] (see Kolster [1975[b]] for a sketch). However these two papers were never published. In 1976 Suslin and Tulenbaev published an ingenious simple proof of the formula $Gl_{n-1} \cap E_n = E_{n-1}$ for the linear group under a (linear) stable range condition. In 1980 Mustafa-Zade gave, in a short communication without proofs, indications for a unitary version of Suslin–Tulenbaev's proof. The main aim of this report is to fill in the details of such a proof. But we use in the proof a unitary stable range condition which is different from the condition of Kolster or Mustafe-Zade. If A is a finite algebra over a commutative ring R with $\dim \operatorname{Max}(R) = d$, our unitary stable range condition is satisfied if $n > d + 2$ and, if A is semilocal, the condition is satisfied if $n > 2$. In contrast, the condition required by Kolster or Mustafe-Zade needs $n > d + 3$, resp. $n > 3$. Thus (**) holds, as claimed by Vaserstein, for all $n > d + 2$ or, if A is semilocal, for all $n > 2$. The result for semilocal rings is applied in Ojanguren [1984].

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2. Some Notations

Let A be an associative ring with an involution $\sigma : a \mapsto \bar{a}$. The set of $(n\times k)$- matrices with elements in A is denoted by $M_{n,k}(A)$ and we put $M_{n,n}(A) = M_n(A)$. The subset of $M_{n,k}(A)$ consisting of left-invertible matrices is denoted by $Um_{n,k}(A)$. Elements of $Um_{n,1}(A)$ are called *unimodular columns* and elements of $Um_{1,n}(A)$ *unimodular rows*. We write α^t for the transpose of $\alpha \in M_{n,k}(A)$, $\bar{\alpha}$ for the matrix $(\bar{a}_{ij})$ if $\alpha = (a_{ij})$ and α^* for $\bar{\alpha}^t$. The row matrix having its i^{th}-element equal to 1 and all other elements equal to zero is denoted by e_i. The canonical basis of $M_{n,k}(A)$ is denoted by $\{E_{ij},\ i = 1,\dots,n,\ j = 1,\dots k\}$ and the identity matrix of $M_n(A)$ by 1_n. Let

$$e_{ij}(r) = 1_n + E_{ij}r,\ r\in A,\ i,j = 1,\dots,n,\ i\neq j,$$

then $e_{ij}(r)\in GL_n(A)$ and the subgroup generated by all $e_{ij}(r)$ is the *elementary subgroup* $E_n(A)$ of $GL_n(A)$. We have $e_{ij}(r)e_{ij}(s) = e_{ij}(r+s)$ and the commutator $[a,b] = aba^{-1}b^{-1}$ of two elementary matrices can be computed as follows

$$[e_{ij}(r), e_{k\ell}(s)] = \begin{cases} 1 & \text{if } j\neq k, i\neq \ell \\ e_{i\ell}(rs) & \text{if } j = k, i\neq \ell \\ e_{kj}(-sr) & \text{if } j\neq k, i = \ell \end{cases}$$

3. The Hyperbolic Space

Let ε be an element of the center of A such that $\varepsilon\bar{\varepsilon} = 1$. A *form parameter* is an additive subgroup Λ of A such that

1) $\Lambda_{\min} = \{r = a - \varepsilon\,\bar{a},\ a\in A\} \subset \Lambda \subset \Lambda_{\max} = \{r \mid r + \varepsilon\,\bar{r} = 0\}$
2) For all $a\in A$, $\bar{a}\lambda a \subset \Lambda$

A triple $A^{\#} = (A,\Lambda,\varepsilon)$ is called a *unitary ring* (or a *form ring*). This notion is due to Bak [1969] (see also Bak [1981]). Let $A^{2n} = M_{2n,1}(A)$. We write the elements of A^{2n} as pairs $x = \binom{u}{v}$ with $u,v\in M_{n,1}(A)$. Let $x = \binom{u}{v}$ and $y = \binom{w}{z}$. We define an ε–hermitian form on A^{2n} by putting

$$\langle x,y\rangle = x^*\left(\begin{smallmatrix}0 & 1\\ \varepsilon & 0\end{smallmatrix}\right)y = u^*z + \varepsilon\, v^*w$$

and we define the *length* $[q](x)$ of $x = \binom{u}{v}$ as the class $[u^*v] = [\varepsilon\, v^*u]$ of u^*v in the additive group A/Λ. Elements of length 0 are called *isotropic*. A pair $\{x,y\}$ of unimodular isotropic elements of A^{2n} such that $\langle x,y\rangle = 1$ is called *hyperbolic*. The pairs $\{e_i^t, e_{i+n}^t\}$ of A^{2n} are hyperbolic. The triple $(A^{2n}, \langle,\rangle, [q])$ is called the *hyperbolic space (of rank 2n)* over $A^{\#}$ and is denoted by $H^{2n} = H^{2n}(A^{\#})$. An *isometry* of H^{2n} is an element $\alpha\in GL_{2n}(A)$ such that $\langle\alpha(x),\alpha(y)\rangle = \langle x,y\rangle$ and $[q](\alpha(x)) = [q](x)$ for all $x,y\in A^{2n}$. The *unitary group* $U_{2n} = U_{2n}(A^{\#})$ is the group of all isometries of H^{2n}. There is a canonical involution on the group sending α to α^*. Let

$$\Lambda_n = \{a = (a_{ij})\in M_n(A) \mid a = -\,\varepsilon\, a^* \text{ and } a_{ii}\in\Lambda,\ 1\leq i\leq n\}$$

and

$$\Lambda_n^* = \{a \in M_n(A) \mid a^* \in \Lambda_n\}$$

We have

$$\begin{aligned} U_{2n} &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_{2n}(A) \mid ad^* + \varepsilon\, bc^* = 1_n,\ ab^* \in \Lambda_n,\ cd^* \in \Lambda_n \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_{2n}(A) \mid d^*a + \bar{\varepsilon} b^*c = 1_n,\ a^*c \in \Lambda_n,\ b^*d \in \Lambda_n \right\} \end{aligned}$$

4. The Elementary Unitary Group

The *elementary unitary group* $EU_{2n} = EU_{2n}(A^\#)$ is the subgroup of U_{2n} generated by all elements $x^+(b) = \left(\begin{smallmatrix} 1_n & b \\ 0 & 1_n \end{smallmatrix}\right)$, $b \in \Lambda_n^*$, $x^-(c) = \left(\begin{smallmatrix} 1_n & 0 \\ c & 1_n \end{smallmatrix}\right)$, $c \in \Lambda_n$ and $H(d) = \left(\begin{smallmatrix} d & 0 \\ 0 & d^{*-1} \end{smallmatrix}\right)$, $d \in E_n(A)$. Since $x^+(b)^* = x^-(b^*)$ and $H(d)^* = H(d^*)$, the canonical involution of U_{2n} maps EU_{2n} to itself. We put

$$\begin{aligned} x_{ij}^+(a) &= x^+(E_{ij}a - \bar{\varepsilon} E_{ji}\bar{a}) \\ x_{ij}^-(a) &= x^-(E_{ij}a - \varepsilon\, E_{ji}\bar{a}) \\ x_{ii}^+(r) &= x^+(E_{ii}r) \\ x_{ii}^-(s) &= x^-(E_{ii}s) \\ h_{ij}(b) &= H(e_{ij}(b)) \end{aligned}$$

for $a, b \in A$ and $s, \bar{r} \in \Lambda$. These elements generate EU_{2n}. A list of identities satisfied by these elements is in Klein–Mikhalev [1971].

We embed U_{2n} into $U_{2(n+1)}$ through the map

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \alpha \perp 1 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Through the same map we can embed EU_{2n} into $EU_{2(n+1)}$.

For further computations it is useful to introduce special elements of EU_{2n}. We first need a definition: we say that a row $r = (r_1, \ldots, r_{2n}) \in M_{1,2n}(A)$ satisfies *condition* $R(i)$ if

$$R(i) \qquad r_i = 0 \text{ and } \bar{r}_{n+i} + \varepsilon \sum_{j=1}^{n} r_{n+j}\bar{r}_j \in \Lambda$$

and that a column $s = (s_1, \ldots, s_{2n})^t$ satisfies *condition* $C(i)$ if

$$C(i) \qquad s_i = 0 \text{ and } s_{n+i} + \varepsilon \sum_{j=1}^{n} \bar{s}_{n+j}s_j \in \Lambda$$

Observe that $R(i)$ is a necessary condition for $(r_1, \ldots, 1, \ldots, r_{2n})^t$, 1 in i^{th}-position, to be a row of a unitary matrix. Similarly, $C(i)$ is a necessary condition for $(s_1, \dots, 1, \dots, s_{2n})$ to be a column of a unitary matrix. For a row r satisfying $R(i)$, we put

$$x_i(r) = \prod_{j=1}^{n} h_{ij}(r_j) \prod_{j=1}^{n} y_{ij}^{+}(r_{n+j}) y_{ii}^{+}(r_{n+i} + \bar{\varepsilon} \sum_{j=1}^{n} r_j \bar{r}_{n+j})$$

and for a column s satisfying $C(i)$, we put

$$x^i(s) = y_{ii}^{-}(s_{n+i} + \varepsilon \sum_{j=1}^{n} \bar{s}_{n+j} s_j) \prod_{j=1}^{n} y_{ji}^{-}(s_{n+j}) \prod_{j=1}^{n} h_{ji}(s_j)$$

We have (only the non-zero entries are marked):

$$x_i(r) = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & 0 & \dots & -\bar{\varepsilon}\bar{r}_{n+1} & \dots & 0 \\ \vdots & \ddots & & & \vdots & \vdots & & \vdots & & \vdots \\ r_1 & \dots & 1 & \dots & r_n & r_{n+1} & \dots & r_{n+i} & \dots & r_{2n} \\ \vdots & & \vdots & \ddots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 1 & 0 & \dots & -\bar{\varepsilon}\bar{r}_{2n} & \dots & 0 \\ & & & & & 1 & \dots & -\bar{r}_1 & \dots & 0 \\ & & & & & \vdots & \ddots & \vdots & & \vdots \\ & & & & & 0 & \dots & 1 & \dots & 0 \\ & & & & & \vdots & & \vdots & \ddots & \vdots \\ & & & & & 0 & \dots & -\bar{r}_n & \dots & 1 \end{pmatrix}$$

and

$$x^i(s) = \begin{pmatrix} 1 & \dots & s_1 & \dots & 0 & & & & & \\ \vdots & \ddots & \vdots & & \vdots & & & & & \\ 0 & \dots & 1 & \dots & 0 & & & & & \\ \vdots & & \vdots & \ddots & \vdots & & & & & \\ 0 & \dots & s_n & \dots & 1 & & & & & \\ 0 & \dots & s_{n+1} & \dots & 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & \vdots & \ddots & \vdots & & \vdots \\ -\varepsilon\,\bar{s}_{n+1} & \dots & s_{n+i} & \dots & -\varepsilon\,\bar{s}_{2n} & -\bar{s}_1 & \dots & 1 & \dots & -\bar{s}_n \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & s_{2n} & \dots & 0 & 0 & \dots & 0 & \dots & 1 \end{pmatrix}.$$

5. The Stable Range Condition

We say that a ring A satisfies the *linear stable range condition* SR_m if for any $n \geq m$ and all unimodular columns $(a_1, \dots, a_n)^t \in Um_{n,1}(A)$ there exist $(r_1, \dots, r_{n-1})^t \in M_{n-1,1}(A)$ such that $(a_1 + r_1 a_n, \dots, a_{n-1} + r_{n-1} a_n)^t$ is unimodular. Obviously SR_m implies SR_n for every $n \geq m$. This definition coincides with that of Bass [1968]. By Bass [1968], a semilocal ring satisfies SR_2 (Proposition (3.4), p. 238) and a finite algebra over a commutative ring R with dim Max$(R) = d$ satisfies SR_{d+2} (Theorem (3.5), p. 239).

Remark. If $n \geq m+1$, we can choose $(r_1, \ldots, r_{n-1})$ with $r_i = 0$ for $i \geq m$. We first take $(r'_1, \ldots, r'_{n-1})$ such that $(a_1 + r'_1 a_n, \ldots, a_{n-1} + r'_{n-1} a_n)$ is unimodular and then $(r''_1, \ldots, r''_{n-2})$ such that $(a_i + r'_i a_n + (a_{n-1} + r'_{n-1} a_n) r''_i)$, $1 \leq i \leq n-2$, is unimodular. Then $(a_i + r_i a_n)$, $1 \leq i \leq n-1$ with $r_i = r'_i + r''_i r'_{n-1}$ for $i = 1, \ldots, n-2$ and $r_{n-1} = 0$ is unimodular.

We say that a unitary ring $A^{\#} = (A, \Lambda, \varepsilon)$ satisfies the *unitary stable range condition* USR_m if A satisfies SR_m and, for all $n \geq m$ and all unimodular columns $x = (p_1^t, p_0, q_1^t, q_0)^t$ in H^{2n} with $p_0, q_0 \in A$ and $p_1, q_1 \in M_{n-1,1}(A)$, there exist $u_1 \in M_{1,n-1}(A)$ and $u_2 \in A$ such that

i) $r = (0, 0, u_1, u_2)$ satisfies $R(n)$ and

ii) $x_n(r)x = (p_1'^t, p'_0, q_1'^t, q'_0)^t$ is such that $(p_1'^t, p'_0)^t$ is unimodular.

The following result is a consequence of a result of Bak (see Bass [1973], p. 241), more precisely of the proof of Theorem (3.3) in Bass [1973]:

(5.1) Theorem. *Let $A^{\#} = (A, \Lambda, \varepsilon)$ be a unitary ring.*

1) If A is semilocal, $A^{\#}$ satisfies USR_2.

2) If A is a finite algebra over a commutative ring R with dim Max$(R) = d$, *$A^{\#}$ satisfies USR_{d+2}.*

6. The Stability Theorem

We now come to the main result of this note:

(6.1) Theorem. *Assume that $A^{\#} = (A, \Lambda, \varepsilon)$ satisfies the unitary stable range condition USR_m. Then, for all $n \geq m$,*

1) EU_{2n} acts transitively on the set of unimodular elements of a given length and on the set of hyperbolic pairs of H^{2n}.

2) EU_{2n} is a normal subgroup of U_{2n}.

3) $U_{2n} = U_{2(n-1)} EU_{2n} = U_{2(m-1)} EU_{2n}$.

Further, if $n \geq m+1$,

4) $EU_{2(n-1)} = EU_{2n} \cap U_{2(n-1)}$.

Proof. 1) Let $x = (a_1, \ldots, a_n, b_1, \ldots, b_n) \in H^{2n}$ be a unimodular element. Since $n \geq m$ and USR_m holds, we can assume that $(a_1, \ldots, a_n)^t$ is unimodular and, since SR_m holds, there exist $s_1, \ldots, s_{n-1} \in A$ such that the column $(a'_1, \ldots, a'_{n-1})^t$, $a'_i = a_i + s_i a_n$, is unimodular (see the remark in section 5). For $s = (s_1, \ldots, s_{n-1}, 0, \ldots, 0)$ we get

$$x^n(s)\,x = (a'_1,\dots,a'_{n-1},a_n,b_1,\dots,b_{n-1},b'_n)^t$$

with $b'_n = b_n - \sum \bar{s}_i b_i$. Thus we can assume that $(a_1,\dots,a_{n-1})^t$ is unimodular. Let $a = (a_1,\dots,a_{n-1})^t$ and let c_1 be such that $c_1^t a = 1$. Putting $c = (1-a_n)c_1$, we have $c^t a = 1 - a_n$ and

$$x_n(c^t,0,0,0)(a^t,a_n,b^t,b_n)^t = (a'^t,1,b'^t,b'_n)^t$$

Therefore we can now assume that $a_n = 1$. We reduce $(a^t,1,b^t,b_n)^t$ to the form $(0,1,b,b_n)^t$ by applying $x^n(-a^t,0,\dots,0)$ and $(0,1,b,b_n)$ to the form $(0,1,0,b_n)$ by applying $x^n(0,0,-b,0)$. By construction we have $b_n \equiv u^*v$ modulo Λ. Let now $u^*v = b_n + \lambda$, $\lambda \in \Lambda$. Then $x^n(0,0,0,0,0,\lambda)(0,1,0,b_n) = (0,1,0,u^*v)$. This gives the first claim of 1). For the last claim of 1), we show that any hyperbolic pair $\{x,y\}$ can be mapped onto $\{e_n^t, e_{2n}^t\}$. By the first part of 1), we can assume that $y = e_{2n}^t$. Then x has the form $(a^t,1,b^t,b_n)$ and the elementary isometry $x^n(-a,0,-b,\bar{a}b - b_n)$, which leaves y fixed, maps x to $(0,1,0,0)$. The claim 2) is a consequence of 1) and Theorem 3.4, p. 234, in Bass [1973]. For 3) it suffices to prove the first equality. Let $\alpha \in U_{2n}$. By 1) there is $\beta \in EU_{2n}$ such that $\alpha\beta$ fixes the hyperbolic pair $\{e_n^t, e_{2n}^t\}$. Since $U_{2(n-1)}$ is the subgroup of U_{2n} which fixes $\{e_n^t, e_{2n}^t\}$, $\alpha = \alpha\beta\beta^{-1}$ is the required factorization. For the proof of 4) we need more properties of EU_{2n}.

Let M_i be the subgroup of EU_{2n} generated by the elements $h_{k\ell}(a)$, $x^-_{k\ell}(b)$, $x^+_{k\ell}(c)$, $x^-_{kk}(r)$ and $x^+_{kk}(s)$ for all $k,\ell \neq i, k \neq \ell, a,b,c \in A$ and $r,\bar{s} \in \Lambda$. Let N_i (resp. N^i) be the subgroup of EU_{2n} generated by all elements $h_{m\ell}(a)$, $x^-_{k\ell}(b)$, $x^+_{mp}(c)$, $x^-_{kk}(r)$, $x^+_{mm}(s)$ for all $k,\ell \neq i, k \neq \ell$, all $m,p,m \neq p$ and $a,b,c \in A$, $r,\bar{s} \in \Lambda$ (resp. $h_{km}(a)$, $x^+_{k\ell}(b)$, $x^-_{mp}(c)$, $x^+_{kk}(s)$, $x^-_{mm}(r)$ for all $k,\ell \neq i$, $k \neq \ell$, all m,p, $m \neq p$). An element of N_i has $\binom{e_i^t}{0}$ as i^{th}- column and $(0,e_i)$ as $(n+i)^{th}$- row. Accordingly, the i^{th}- row of an element of N^i is $(e_i,0)$ and the $(n+i)^{th}$- column is $\binom{0}{e_i^t}$ (the shape of matrices in N_n and N^1 is shown after Lemma (6.7)). For $x \in M_{2n}(A)$, let $x(\hat{k}) \in M_{2(n-1)}(A)$ be the matrix obtained from x by cancelling the k^{th}- and $(n+k)^{th}$- rows and columns. In the next lemma we summarize some results needed for further computations.

(6.2) Lemma.

1) *For any $x \in N_i$ (or $x \in N^i$) and any k, $1 \leq k \leq n$, $x(\hat{k})$ lies in $EU_{2(n-1)}$.*
2) *N_n and all $h_{1n}(\nu)$, $\nu \in A$, generate EU_{2n}.*
3) *Let $y \in N^i$ and $z \in N_i$. If r satisfies $R(i)$, then $zx_i(r)z^{-1} = x_i(rz^{-1})$ and if s satisfies $C(i)$, then $yx^i(s)y^{-1} = x^i(ys)$.*
4) *If $y \in N^i$, $z \in N_i$, then $s = ye_i^t - e_i^t$ satisfies $C(i)$, $r = e_iz - e_i$ satisfies $R(i)$ and $y = x^i(s)\,y_1, z = z_1 x_i(r)$ with y_1, $z_1 \in M_i$.*

Proof. 1) The claim is obvious for the generators of N_i (resp. N^i).
2) It suffices to check that for all $i \neq n$ the elements $h_{in}(a)$, $x^-_{in}(b)$, $x^-_{ni}(c)$ and $x_{nn}(s)$, $a,b,c \in A$, $s \in \Lambda$ lie in the subgroup generated by N_n and the $h_{1n}(\nu)$. This follows from the relations

$$\begin{aligned}
[h_{i1}(a), h_{1n}(a)] &= h_{in}(ab),\ i \neq 1 \\
[x^-_{i1}(a), h_{1n}(b)] &= x^-_{in}(ab),\ i \neq 1, i \neq n \\
[x^-_{12}(a), h_{2n}(b)] &= x^-_{1n}(ab) \\
[x^-_{1i}(a), h_{1n}(b)] &= x^-_{ni}(\bar{b}a),\ i \neq 1, i \neq n \\
[x^-_{21}(a), h_{2n}(b)] &= x^-_{n1}(\bar{b}\bar{a}) \\
[x^+_{11}(r), x^-_{n1}(b)] &= h_{1n}(\bar{a}\bar{b})x^-_{nn}(b\bar{r}\bar{b})
\end{aligned}$$

3) Let $e_i z = (r_1, \ldots, 1, \ldots r_{2n})$ (1 is in i^{th}- position), so that $r = e_i z - e_i = (r_1, \ldots, 0, \ldots, r_{2n})$. Since $z \in N_n \subset U_{2n}$, we have

$$r_i = 0 \text{ and } \bar{r}_{n+i} + \sum_j r_j \bar{r}_{n+j} \in \Lambda$$

or

$$\bar{r}_{n+i} + \varepsilon \sum_j r_{n+j}\bar{r}_j + \sum_j r_j \bar{r}_{n+j} - \varepsilon \sum_j r_{n+j}\bar{r}_j \in \Lambda$$

Thus $r = e_i z - e_i$ satisfies $R(i)$. Putting $z_1 = z\, x_i(r)^{-1}$, we get a matrix $z_1 \in N_i$ with $(e_i, 0)$ as i^{th}- row. The fact that the $(n+i)^{th}$- column is $(0, e_i^t)$ (i.e. $z_1 \in M_i$) follows from the relations satisfied by the elements of U_{2n} (see section 3). A similar argument applies to $s = ye_i^t - e_i^t$. □

Let now V be the subset of EU_{2n} consisting of the elements x which can be expressed as $x = y\, h_{1n}(\alpha) z$, $y \in N^1$, $z \in N_n$ and $\alpha \in A$. We shall show that $EU_{2n} = V$. Toward proving this, we first check that

(6.3) Lemma. *$Vh_{1n}(\nu) \subset V$ for all $\nu \in A$.*

Proof. It suffices to check that $h_{1n}(\alpha) z h_{1n}(\nu) \in V$ for all $\alpha, \nu \in A$ and $z \in N_n$. We first assume that z is such that $e_n z e_1^t = 0$.

(6.4) Lemma. *Let $z \in N_n$ be such that $e_n z e_1^t = 0$. Then $h_{1n}(\alpha) z h_{1n}(\nu) \in V$ for all $\alpha, \nu \in A$.*

Proof. By (6.2), 2), we can write $z = z_1 x_n(u)$ with $z_1 \in M_n$ and $u = e_n z - e_n$, so that $ue_1^t = 0$. We write

$$\begin{aligned}
h_{1n}(\alpha) z h_{1n}(\nu) &= h_{1n}(\alpha)\, z_1\, x_n(u)\, h_{1n}(\nu) \\
&= h_{1n}(\alpha) z_1 h_{1n}(\nu) z_1^{-1} z_1\, h_{1n}(-\nu) x_n(u) h_{1n}(\nu)
\end{aligned}$$

A direct computation shows that

$$h_{1n}(-\nu) x_n(u)\, h_{1n}(\nu) \in N_n$$

Further we have $h_{1n}(\nu) = x^n(e_1^t \nu)$ and by (7.2), 3)

$$z_1 h_{1n}(\nu) z_1^{-1} = x^n(z_1 e_1^t \nu)$$

Thus we are reduced to show that

$$h_{1n}(\alpha)x^n(z_1e_1^t\nu)\in V$$

Let $z_1e_1^t = \binom{\mu}{v}$, $\mu\in A$ and $v\in M_{2(n-1)}(A)$. We have $x^n(z_1e_1^t\nu) = x^n\binom{0}{v\nu}h_{1n}(\mu\nu)$. Since $= x^n\binom{0}{v\nu}\in N^1$ the claim follows from

$$h_{1n}(\alpha)x^n(z_1e_1^t\nu) = x^n\binom{0}{v\nu}h_{1n}(\mu\nu+\alpha)$$

□

Assume now that $e_nze_1^t \neq 0$ and let

$$\zeta = (a_1, p_1^t, b_1, c_1, q_1^t, 0)^t$$

with $a_1, b_1, c_1 \in A$ and $p_1, q_1 \in M_{n-2,1}(A)$, be the first column of z. It follows from $z \in N_n$ that $(a_1, p_1^t, 0, c_1, q_1^t)^t$ is unimodular. Since $n > m+1$ and $A^\#$ satisfies USR_m, there is $\tau_1 = x_n(0,0,0,u_1,u_2,0)$ such that

$$\tau_1\zeta = (a_2, p_2^t, b_1, c_2, q_2^t, 0)^t$$

with $(a_2, p_2^t)^t$ unimodular. Let $\tau_2 = x^1(0, u_4, 0, 0, 0, 0)$ be such that

$$\tau_2\tau_1\zeta = (a_2, p_3^t, b_1, c_3, q_2^t, 0)^t$$

and p_3 is unimodular. Let $c \in M_{1,n-1}(A)$ be such that $cp_3 = 1 - b_1$. Putting

$$\tau_3 = x_n(0, c, 0, 0, 0, 0)\, x_1(0, -\alpha c, 0, 0, 0)$$

we get

$$\tau_3\tau_2\tau_1\zeta = (*, *, 0, *, *, 0)^t$$

so that $e_n(\tau_3\tau_2\tau_1\zeta)e_1^t = 0$. We now write

$$\begin{aligned} &h_{1n}(\alpha)zh_{1n}(\nu) \\ &\quad = h_{1n}(\alpha)\tau_1^{-1}\tau_2^{-1}h_{1n}(-\alpha)h_{1n}(\alpha)\tau_3^{-1}h_{1n}(-\alpha)h_{1n}(\alpha)\tau_3\tau_2\tau_1zh_{1n}(\nu)\end{aligned}$$

Since

$$h_{1n}(\alpha)\tau_1^{-1}\tau_2^{-1}h_{1n}(-\alpha) = x^n(0, -u_4\,\alpha, 0, 0, 0, 0)\tau_1^{-1}\tau_2^{-1}$$

and $x^n(0, -p_1\alpha, 0, 0, 0, 0) \in N^1$, it suffices to check that

$$\tau_1^{-1}\tau_2^{-1}h_{1n}(\alpha)\tau_3^{-1}h_{1n}(-\alpha)h_{1n}(\alpha)\tau_3\tau_2\tau_1zh_{1n}(\nu) \in V$$

Let $\tilde z = \tau_3\tau_2\tau_1 z$. Since $e_n\tilde z e_1^t = 0$, the proof of (7.4) shows that we are reduced to check that

$$\tau_1^{-1}\tau_2^{-1}h_{1n}(\alpha)\tau_3^{-1}h_{1n}(-\alpha)h_{1n}(\alpha)x^n(w)\in V$$

where $w = \tilde z_1e_1^t\nu$ and $\tilde z_1$ is obtained from the decomposition $\tilde z = \tilde z_1x_n(\tilde u)$ (see (6.2), 4)). We have

$$h_{1n}(\alpha)\tau_3^{-1}h_{1n}(-\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_{n-2} & 0 & 0 & 0 & 0 \\ 0 & -c & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{n-2} & c^* \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\tau_2\tau_1$ is of the form

$$\tau_2\tau_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & -\bar{\varepsilon}v_2^* & 0 \\ v_1 & 1_{n-2} & 0 & v_2 & v_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -v_1^* & 0 \\ 0 & 0 & 0 & 0 & 1_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It follows that

$$\tau_1^{-1}\tau_2^{-1}h_{1n}(\alpha)\tau_3^{-1}h_{1n}(-\alpha)\tau_2\tau_1x_1(0,0,0,0,0,-\bar{\varepsilon}v_2^*c^*)\in N^1$$

We have $x_1(0,0,0,0,0,-\bar{\varepsilon}v_2^*c^*) = x_{n1}^+(cv_2)$. Therefore we have to check that

$$x_{n1}^+(-cv_2)\tau_1^{-1}\tau_2^{-1}h_{1n}(\alpha)x^n(w)\in V$$

Let $w = (a_1, p_1^t, 0, b_1, q_1^t, 0)^t$ and $w_1 = (0, p_1^t, 0, b_1, q_1^t, 0)^t$. We have

$$h_{1n}(\alpha)x^n(w) = x^n(w_1)h_{1n}(\alpha + a_1)$$

and

$$\tau_1^{-1}\tau_2^{-1}x^n(w_1)\tau_2\tau_1h_{1n}(-\bar{\varepsilon}v_2^*q_1^t) = x^n(w_2)$$

with $w_2 = (0, p_2^t, 0, b_2, q_1^t, 0)^t$ of the same form as w_1. Thus (6.3) is a consequence of the following lemma:

(6.5) Lemma. *For all* $u \in M_{1,n-2}(A)$, $\alpha, a \in A$ *and* $w = (0, p^t, 0, b, q^t, 0)^t$ *in* $M_{n,1}(A)$ *satisfying* $C(n)$, *we have* $x_{n1}^+(a)x^n(w)h_{1n}(\alpha)\in V$.

Proof. We write $x^n(w) = x^n(w_3)x_{n1}^-(-\varepsilon\,\bar{b})$ with $w_3 = (0, p^t, 0, 0, q^t, 0)^t$. We have

$$x_{n1}^+(a)x^n(w_3)x_1(0, -\bar{a}q^*, 0, 0, \bar{\varepsilon}\bar{a}p^*, \bar{\varepsilon}\bar{a})\in N^1$$

Thus putting

$$x_1(0, -\bar{a}q^*, 0, 0, \bar{\varepsilon}\bar{a}p^*, \bar{\varepsilon}\bar{a}) = \tau x_{n1}^+(-a),\ \tau = x_1(0, -\bar{a}q^*, 0, 0, \bar{\varepsilon}\bar{a}p^*, 0)$$

we are reduced to check that

$$x_{n1}^+(a)\tau^{-1}x_{n1}^-(-\varepsilon\,\bar{b})h_{1n}(\alpha)\in V$$

Since τ^{-1} commutes with $x_{n1}^-(-\varepsilon\,\bar{b})$ and $h_{1n}(\alpha)$ and $\tau\in N_n$, the claim follows from

(6.6) Lemma. *$x_{n1}^+(a)x_{n1}^-(b)h_{1n}(\alpha)\in V$ for all $a,b,\alpha\in A$.*

Proof. The following relations will be used:

$$\begin{aligned}
[x_{n1}^+(a), x^-_{n2}(b)] &= 1 \\
[x_{n1}^+(a), h_{21}(b)] &= x_{n2}^+(-a\bar{b})
\end{aligned}$$

We put $x' \equiv x$ if x and x' are congruent modulo N^1, that is if $x' = yx$ with $y \in N^1$. A brace under two factors means that the factors will be switched in the next step of the computation. Replacing $x_{n1}^-(b)$ by $[x_{n2}^-(b), h_{21}(1)]$ we get

$$\begin{aligned}
x_{n1}^+(a)x_{n1}^-(b)h_{1n}(\alpha) &= \underbrace{x_{n1}^+(a)x_{n2}^-(b)}\,h_{21}(1)x_{n2}^-(-b)h_{21}(-1)h_{1n}(\alpha) \\
&= x_{n2}^-(b)\,\underbrace{x_{n1}^+(a)h_{21}(1)}\,x_{n2}^-(-b)h_{21}(-1)h_{1n}(\alpha) \\
&\equiv x_{n2}^+(-a)h_{21}(1)\,\underbrace{x_{n1}^+(a)x_{n2}^-(-b)}\,h_{21}(-1)h_{1n}(\alpha) \\
&\equiv x_{n2}^-(-b)x_{n1}^+(a)h_{21}(-1)h_{1n}(\alpha) \\
&\equiv x_{n1}^+(a)h_{21}(-1)h_{1n}(\alpha)
\end{aligned}$$

Since $z = x_{n1}^+(a)h_{21}(-1)\in N_n$ is such that $e_1 z e_n^t = 0$, the claim now follows from (6.4). □

(6.7) Proposition. *If $n \geq m+1$ and A satisfies USR_m, then every element $x \in EU_{2n}$ can be written as a product $x = y\,h_{1n}(\alpha)z$ with $y\in N^1$, $\alpha\in A$ and $z\in N_n$.*

Proof. Obviously $VN_n \subset V$ and by Lemma (6.3), $Vh_{1n}(\nu) \subset V$. Since N_n and $h_{1n}(\nu)$, for all $\nu\in A$, generate EU_{2n}, we have $Vx \subset V$ for all $x\in EU_{2n}$. Thus we get $x\in Vx \subset V$ and $x\in V$ as claimed. □

We now prove the last claim of Theorem (6.1), i.e.

$$EU_{2n}\cap U_{2(n-1)} = EU_{2(n-1)} \text{ if } n\geq m+1$$

Let $\phi\in U_{2(n-1)}$ be such that $\phi \perp 1 = y\,h_{1n}(\alpha)z$ for $z\in N^1$, $\alpha\in A$ and $y\in N_n$. We put

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \beta_{21} & \beta_{22} & \beta_{23} & 0 & \beta_{25} & \beta_{26} \\ \beta_{31} & \beta_{32} & \beta_{33} & 0 & \beta_{35} & \beta_{36} \\ \beta_{41} & \beta_{42} & \beta_{43} & 1 & \beta_{45} & \beta_{46} \\ \beta_{51} & \beta_{52} & \beta_{53} & 0 & \beta_{55} & \beta_{56} \\ \beta_{61} & \beta_{62} & \beta_{63} & 0 & \beta_{65} & \beta_{66} \end{pmatrix}$$

and

$$z = \begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 & \gamma_{14} & \gamma_{15} & \gamma_{16} \\ \gamma_{21} & \gamma_{22} & 0 & \gamma_{24} & \gamma_{25} & \gamma_{26} \\ \gamma_{31} & \gamma_{32} & 1 & \gamma_{34} & \gamma_{35} & \gamma_{36} \\ \gamma_{41} & \gamma_{42} & 0 & \gamma_{44} & \gamma_{45} & \gamma_{46} \\ \gamma_{51} & \gamma_{52} & 0 & \gamma_{54} & \gamma_{55} & \gamma_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In the following computations we systematically use the matrix identities satisfied by the elements of U_{2n} (see section 3). Let

$$\beta = \begin{pmatrix} \beta_{22} & \beta_{23} & \beta_{25} & \beta_{26} \\ \beta_{32} & \beta_{33} & \beta_{35} & \beta_{36} \\ \beta_{52} & \beta_{53} & \beta_{55} & \beta_{56} \\ \beta_{62} & \beta_{63} & \beta_{65} & \beta_{66} \end{pmatrix} \quad \text{and } \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{14} & \gamma_{15} \\ \gamma_{21} & \gamma_{22} & \gamma_{24} & \gamma_{25} \\ \gamma_{41} & \gamma_{42} & \gamma_{44} & \gamma_{45} \\ \gamma_{51} & \gamma_{52} & \gamma_{54} & \gamma_{55} \end{pmatrix}$$

We have obtained β and γ from $y \in N^1$, resp. $z \in N_n$ by cancelling certain rows and columns. By Lemma (6.2), β and γ must lie in $EU_{2(n-1)}$. Multiplying the matrices y, $h_{1n}(\alpha)$ and z we get that $\alpha = 0$, $\beta_{23} = \beta_{43} = \beta_{53} = \beta_{63} = 0$ and $\beta_{33} = 1$. Further the identities satisfied by β as an element of U_{2n} show that $\beta_{62} = \beta_{65} = 0$ and $\beta_{66} = 1$, thus

$$\beta = \begin{pmatrix} \beta_{22} & 0 & \beta_{25} & \beta_{26} \\ \beta_{32} & 1 & \beta_{35} & \beta_{36} \\ \beta_{52} & 0 & \beta_{55} & \beta_{65} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

On the other hand, we have $\phi = \beta'\gamma$ with

$$\beta' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \beta_{21} & \beta_{22} & 0 & \beta_{25} \\ \beta_{41} & \beta_{42} & 1 & \beta_{45} \\ \beta_{51} & \beta_{52} & 0 & \beta_{55} \end{pmatrix}$$

Thus we are reduced to show that $\beta' \in EU_{2(n-1)}$. We have (using the notations $x_i(u_1, u_2, u_3, u_4)$ and $x^i(v_1, v_2, v_3, v_4)$ also for blocs of matrices and noting that the corresponding matrices are elementary)

$$x^1(0,0,0,-\beta_{51})\, x^1(0,0,-\beta_{41} - \beta_{21}^*\beta_{51}, 0)\, x^1(0, -\beta_{21}, 0, 0)\beta' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta_{22} & 0 & \beta_{25} \\ 0 & 0 & 1 & 0 \\ 0 & \beta_{52} & 0 & \beta_{55} \end{pmatrix}$$

Similarly we get

$$\beta x^2(-\beta_{32}, 0, 0, 0)\, x^2(0,0,0,-\beta_{36} - \beta_{35}\beta_{32}^*)\, x^2(0,0,-\beta_{35},0) = \begin{pmatrix} \beta_{22} & 0 & \beta_{25} & 0 \\ 0 & 1 & 0 & 0 \\ \beta_{52} & 0 & \beta_{55} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the claim is a consequence of the following

(6.8) Lemma. *For any* $\alpha \in U_{2(n-1)}$, $1 \perp \alpha \equiv \alpha \perp 1$ *modulo* EU_{2n}.

Proof. Let $x, y \in GL_{n-1}(A)$ and let

$$h^-(x) = H(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}) \in U_{2n} \text{ and } h^+(x) = H(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}) \in U_{2n}$$

Since $h^-(x)h^-(y) = h^-(x+y)$ and $h^+(x)h^+(y) = h^+(x+y)$, we see that all $h^-(x)$, $h^+(y)$ lie in EU_{2n}. The claim follows by operating on the left (row operations) and on the right (column operations) with elements $h^-(x)$ and $h^+(y)$. □

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