On the discriminant of an involution
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1. Introduction

In recent papers [KPS₁], [KPS₂] and [KPS₃] the authors gave some applications of
the reduced pfaffian of a central simple algebra with involution or, more generally, of
an Azumaya algebra. In [KPS₃], an invariant, called the pfaffian discriminant, was
attached to any involution \( \sigma \) of a central simple 16–dimensional algebra \( A \) and it was
shown that \( A \) has a \( \sigma \)–invariant quaternion subalgebra if and only if the discriminant
of \( \sigma \) is trivial. In this note we use Galois descent to describe the discriminant and the
pfaffian of a central simple algebra with involution of arbitrary dimension.

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2. The Pfaffian

Let \( K \) be a field and let \( A \) be a central simple \( K \)-algebra with an involution \( \sigma \) of
the first kind. We recall that \( \sigma \) is an antiautomorphism of \( A \) of order \( \leq 2 \). Typical
examples are matrix algebras \( M_n(K) \) with transpose \( t : x \mapsto \overline{x} \) or quaternion algebras
(i.e. central simple algebras of dimension 4) with conjugation \( x \mapsto \overline{x} \). If \( n_H \) is the
reduced norm of a quaternion algebra \( H \), the conjugation map \( x \mapsto \overline{x} \) is the unique
map with \( x\overline{x} = \overline{x}x = n_H(x) \).

For any central simple \( K \)-algebra \( A \) there exists a finite Galois extension \( K \subset L \) such that
\[
\alpha : A \otimes L \overset{\sim}{\rightarrow} M_n(L).
\]
We call \( \alpha \) (or \( L \)) a Galois splitting of \( A \). The transport
\[
\tilde{\sigma} = \alpha(\sigma \otimes 1)\alpha^{-1}
\]
of \( \sigma \) to \( M_n(L) \) is, by the Skolem-Noether Theorem, of the form \( \tilde{\sigma} = \sigma_u = \text{Int}(u) \circ t \),
where \( \text{Int}(u) \) is the inner automorphism \( x \mapsto uxu^{-1}, u \in GL_n(L) \), and \( t \) is, as above,
the transpose. It follows from \( \sigma^2 = 1 \) that \( u^t = \varepsilon u \) for some \( \varepsilon = \pm 1 \).

Let \( \beta : A \otimes L' \overset{\sim}{\rightarrow} M_n(L') \) be another splitting. By replacing \( L \) and \( L' \) by a bigger
Galois extension, we can assume that \( L = L' \). If \( \beta(\sigma \otimes 1)\beta^{-1} = \sigma_{u'}, \sigma_{u'} \) and \( \sigma_u \), then

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$\sigma_{u'} = \text{Int}(v)\sigma_u\text{Int}(v)^{-1}$ for some inner automorphism \text{Int}(v) and there is $\lambda \in L^*$ such that

$$\lambda u' = vu^t.$$ 

It follows that the element $\varepsilon$ is independent of the splitting $\alpha$. We call it the type of $\sigma$. In view of the formula $\lambda u' = vu^t$ we can assume that $u$ is either diagonal or of the form

$$u = \text{diag}(S_2, \ldots, S_2) \text{ with } S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

Observe that this is also true in characteristic 2 (see Kaplansky [K] p. 23). We say that $\sigma$ is of orthogonal type if $u$ is diagonalizable and of symplectic type if $u$ is alternating.

We call the set

$$\text{Alt}^\sigma(A) = \{x - \varepsilon \sigma(x), x \in A\},$$

where $\varepsilon$ is the type of $\sigma$, the set of alternating elements of $(A, \sigma)$. Let $\text{Alt}_n(K)$ be the set of alternating $(n \times n)$-matrices. If $\alpha : A \otimes L \iso M_n(L)$ is a splitting of $A$ such that $\tilde{\alpha} = \alpha(\sigma \otimes 1)\alpha^{-1} = \text{Int}(u) \circ t$, we have

$$u^{-1}\alpha(\text{Alt}^\sigma(A) \otimes L) = \alpha(\text{Alt}^\sigma(A) \otimes L)u = \text{Alt}_n(L)$$

since

$$u^{-1}(x - \varepsilon ux^t u^{-1}) = u^{-1}x - (u^{-1}x)^t$$

$$(x - \varepsilon ux^t u^{-1})u = xu - (xu)^t.$$

It follows that $\dim_K \text{Alt}^\sigma(A) = \frac{n(n-1)}{2}$. For $n$ even the determinant of an alternating $(n \times n)$-matrix has a square root, the pfaffian. For example

$$pf(x) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$$

for $n = 4$ and $x = (a_{ij})$. For central simple algebras of dimension $n^2 = 4m^2$, we have, denoting the reduced norm by $n_A$:

**Theorem (2.1)** Let $A$ be a central simple $K$-algebra with involution $\sigma$. There exists an element $d_\sigma \in K^*$ and a map $pf_\sigma : \text{Alt}^\sigma(A) \to K$ such that $pf_\sigma(x)^2 = d_\sigma n_A(x)$ for all $x \in \text{Alt}^\sigma(A)$. The map $pf_\sigma$ is determined up to a unit of $K$ and satisfies the identity

$$pf_\sigma(ax\sigma(a)) = n_A(a)pf_\sigma(x), \ x \in \text{Alt}^\sigma(A), a \in A.$$ 

If $\varphi : (A, \sigma) \to (A', \sigma')$ is an isomorphism of algebras with involution and if $pf_\sigma$ is fixed, then $pf_{\sigma'}$ can be chosen in such a way that $pf_{\sigma'}(\varphi(x)) = pf_\sigma(x), x \in \text{Alt}^\sigma(A)$. 

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Proof Let $\alpha : A \otimes L \rightarrow M_n(L)$ be a Galois splitting and let $G$ be the Galois group of the extension $K \subset L$. Let

$$\alpha(\sigma \otimes 1)\alpha^{-1} = \text{Int}(u) \circ t$$

as above and let, for $g \in G$, $\tilde{g} : M_n(L) \rightarrow M_n(L)$ be the entrywise action, i.e. $\tilde{g}(a_{ij}) = (g(a_{ij}))$ for $a_{ij} \in M_n(L)$. The automorphism $\alpha(1 \otimes g)\alpha^{-1}\tilde{g}^{-1}$ of $M_n(L)$ is $L$-linear. Thus there exists $c(g) \in GL_n(L)$ such that

$$\alpha(1 \otimes g)\alpha^{-1} = \text{Int}(c(g)) \circ \tilde{g}.$$ 

Replacing $g$ by $gh$, $g, h \in G$ we get

$$c(gh) = c(g)\tilde{g}(c(h)) \cdot r(g, h)$$

for elements $r(g, h) \in L^\bullet$. Since $\alpha(1 \otimes g)\alpha^{-1}$ and $\alpha(\sigma \otimes 1)\alpha^{-1}$ commute, we have

$$\text{Int}(c(g)^t u^{-1} c(g)\tilde{g}(u)) = 1.$$ 

Thus there exist elements $\lambda(g) \in L^\bullet$ such that

$$c(g)\tilde{g}(u)c(g)^t = \lambda(g)u$$

and

$$\lambda(gh) = \lambda(g)g(\lambda(h))r(g, h)^2.$$ 

It follows that the elements

$$\chi(g) = \det(c(g))\lambda(g)^{-n/2}$$

satisfy $\chi(gh) = \chi(g)g(\chi(h))$. By Hilbert theorem 90 there exists $a \in L^\bullet$ such that

$$\chi(g) = g(a)a^{-1}$$

for all $g \in G$. Observe that $a$ is determined up to a unit of $K$. We now define

$$pf_\sigma : \text{Alt}^\sigma(A) \otimes L \rightarrow L$$

by $pf_\sigma(x \otimes \ell) = pf(\alpha(x \otimes \ell)u)a$, $x \in \text{Alt}^\sigma(A)$, $\ell \in L$, where $pf$ is the pfaffian $\text{Alt}_n(L) \rightarrow L$. We claim that the restriction of $pf_\sigma$ to $\text{Alt}^\sigma(A)$ takes values in $K$. For this it suffices to check that

$$g(pf_\sigma(x \otimes 1)) = pf_\sigma(x \otimes 1)$$

for all $g \in G$. 

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We have 
\[ pf(\alpha(1 \otimes g)(x \otimes 1)u)a = pf(c(g)\bar{g}(\alpha(x \otimes 1))c(g)^{-1}u)a \]
\[ = pf(c(g)\bar{g}(\alpha(x \otimes 1))\lambda(g)^{-1}\bar{g}(u)c(g)^{t})a \]
\[ = \det(c(g))\lambda(g)^{-n/2}pf(\bar{g}(\alpha(x \otimes 1))\bar{g}(u))a \]
\[ = \lambda(g)g(pf(\alpha(x \otimes 1)u))a = g(pf(\alpha(x \otimes 1)u)a) \]
\[ = g(pf_\sigma(x \otimes 1)) \]

and on the other hand
\[ pf(\alpha(1 \otimes g)(x \otimes 1)u)a = pf_\sigma(x \otimes 1). \]

Thus, as claimed $pf_\sigma$ restricts to a map $pf_\sigma : \text{Alt}^\sigma(A) \to K$. We have
\[ (pf_\sigma(x \otimes 1))^2 = \det(\alpha(x \otimes 1))\det(u)a^2. \]
\[ = n_A(x)\det(u)a^2 \]

and obviously $d_\sigma = \det(u)a^2 \in K^\bullet$ is as wanted. We next check that the construction of $pf_\sigma$ is (up to a unit of $K$) independent of the chosen splitting. Let
\[ \beta : A \otimes L' \xrightarrow{\sim} M_n(L') \]

and let $\beta(\sigma \otimes 1)\beta^{-1} = \text{Int}(u') \circ t$. By replacing $L$ and $L'$ by a common Galois extension, we can assume that $L = L'$. There exists $v \in GL_n(L)$ such that $\beta = \text{Int}(v) \circ \alpha$. Since $\text{Int}(v)$ is an isomorphism of algebras with involution there exists $\rho \in L^\bullet$ such that
\[ u' = \rho vv^t. \]

Let $c'(g), \lambda'(g)$ and $\chi'(g)$ be the data induced by $\beta$. We have a relation
\[ c'(g)\bar{g}(v) = \nu vc(g) \]
for some $\nu \in L^\bullet$ and it follows from $\lambda'(g)u' = c'(g)\bar{g}(u')c'(g)^t$ that
\[ \lambda'(g) = \rho^{-1}\nu^2\lambda(g)g(\rho). \]

Furthermore we get
\[ \det(c'(g)) = \nu^n\det(c(g))\det(v) \cdot g(\det(v))^{-1}. \]

Thus
\[ \chi'(g) = \det(c'(g))\lambda'(g)^{-n/2} \]
\[ = \chi(g)\det(v)\rho^{n/2}g(\det(v)\rho^{n/2})^{-1}. \]
Putting \( a' = a(\det(v)\rho^{n/2})^{-1} \) and defining \( pf_{\sigma'} \) by
\[
 pf_{\sigma'}(x \otimes \ell) = pf(\beta(x \otimes \ell)u')a'
\]
we get, as wanted, \( pf_{\sigma'}(x \otimes 1) = pf_{\sigma}(x \otimes 1) \). The formula \( pf_{\sigma}(ax\sigma(a)) = n_A(a)pf_{\sigma}(x) \) follows from the corresponding formula \( pf(yxy^t) = \det(y)pf(x) \) for the classical pfaffian and the rest of the claims is an easy consequence of the above computations.

**Remark (2.2)** The pfaffian can be constructed in a slightly different way (see Tama-gawa [T] and [KS]). Using the same notations as above, we put \( V = L^n \) and denote the action of \( G \) componentwise by \( gv \). We define an action of \( G \) on \( ^m V ; m = 2, 4, \ldots; n; \) by
\[
g^*(x_1 \wedge \cdots \wedge x_m) = \lambda(g)^{-m/2}c(g)gx_1 \wedge \cdots \wedge c(g)gx_m.
\]
It is easy to check that \((gh)^* = g^*h^* \) and we put
\[
 A^{(m)} = (\wedge^m V)^G = \{ \xi \in \wedge^m V | g^*(\xi) = \xi, \forall g \in G \}.
\]
The space \( A^{(2)} \) has dimension \( \frac{n(n-1)}{2} \) and \( A^{(n)} \) has dimension 1. We claim that there is a canonical isomorphism \( A^{(2)} \simeq \text{Alt}^\sigma(A) \) and that \( pf_{\sigma} \) can be viewed as a map \( A^{(2)} \rightarrow A^{(n)} \). We write \( \tilde{\sigma} (x) = \alpha(\sigma \otimes 1)\alpha(x) = u\alpha \alpha^t u^{-1} \) with \( u : V^* \rightarrow V, V^* = \text{Hom}_L(V, L) \) and \( x^t : V^* \rightarrow V^* \) the transpose. Identifying \( \text{End}_L(V) \) with \( V \otimes L^V \) through the map \((x \otimes f)(y) = xf(y), x, y \in V, f \in V^*, \) we have
\[
 \tilde{\sigma} (x \otimes f) = u(f) \otimes u^{-1}(x).
\]
Through the composite
\[
 A \otimes L \xrightarrow{\sigma} \text{End}_L(V) \xrightarrow{\sim} V \otimes V^* \xrightarrow{1 \otimes u} V \otimes V,
\]
\( \text{Alt}^\sigma(A) \otimes L \) has image the subspace \( \text{Alt}(V \otimes V) \) generated by all tensors \( x \otimes y - y \otimes x \). We further identify \( \text{Alt}(V \otimes V) \) with \( \wedge^2 V \) by sending \( x \otimes y - y \otimes x \) to \( x \wedge y \), so that we get an isomorphism
\[
 \text{Alt}^\sigma(A) \otimes L \xrightarrow{\sim} \wedge^2 V.
\]
This map is \( G \)-equivariant for the action \( 1 \otimes g \) on \( \text{Alt}^\sigma(A) \otimes L \) and \( g^* \) on \( \wedge^2 V \), thus induces an isomorphism \( \text{Alt}^\sigma(A) \xrightarrow{\sim} A^{(2)} \). To construct the pfaffian it now suffices to notice that the classical pfaffian can be viewed as a map \( pf_V : \wedge^2 V \rightarrow \wedge^n V \) and that this map is \( G \)-equivariant. Let \( x = (a_{ij}) \) be an alternating matrix. We define
\[
 pf_V(\sum_{i<j} a_{ij}e_i \wedge e_j) = pf(x)e_1 \wedge \ldots \wedge e_n
\]
for a fixed basis \( \{ e_1, \ldots, e_n \} \) of \( V \). Using the identity \( pf(yx^t) = \det(y)pf(x) \), it is easy to check that \( pf_V \) is independent of the choice of the basis and that it is \( G \)-equivariant.

Let \( d_\sigma \) be as given in (2.1). Its class \( \delta(\sigma) \) in \( K^*/K^{*2} \) is independent of the choice of \( pf_\sigma \). We call it the discriminant of \( \sigma \). Observe that \( \delta(\sigma) \) is the class of \( n_A(x) \) in \( K^*/K^{*2} \) for any invertible \( x \in \text{Alt}^\sigma(A) \). This was already noticed in [CDTW].

**Remark (2.3)** The construction of the \( A^{(m)} \) given above could also be applied to symmetric powers of \( V \) or to a combination of exterior and symmetric powers. It was used by Jacobson in his construction of the even Clifford algebra of a central simple algebra with involution of orthogonal type [J1].

**Remark (2.4)** Theorem (2.1) can be generalized to the setting of 2-torsion data as introduced in [KPS1]. We hope to come back to this at some other place.

**Examples (2.5)**

1. If \( A = M_n(K) \) and \( \sigma = \text{Int}(u) \circ t \), we have
   \[
   x - \varepsilon ux^tu^{-1} = u(u^{-1}x - (u^{-1}x)^t)
   \]
   so that \( \det(x - \varepsilon ux^tu^{-1}) = \det(u)pf(u^{-1}x - (u^{-1}x)^t)^2 \) and \( \delta(\sigma) \) is the class of \( [\det(u)] \in K^*/K^{*2} \).

2. Let \( \sigma \) be of even symplectic type, so that we can choose a splitting of \( A \) such that \( u = \text{diag}\{S_2, \ldots, S_2\} \). With the notations of the proof of (2.1), we have \( \bar{g}(u) = u \) and taking pfaffians on both sides of \( c(g)\bar{g}(u)c(g)^t = \lambda(g)u \), we get \( \det(c(g)) = \lambda(g)^{n/2} \) and \( \chi(g) = 1 \). Thus we can choose \( a = 1 \) and get \( \delta(\sigma) = 1 \). This case was considered by Fröhlich in [F].

3. If \( A = A_1 \otimes A_2 \) is the tensor product of two quaternion algebras and \( \sigma \) is the tensor product \( \sigma_1 \otimes \sigma_2 \) of two involutions, we can split both algebras \( A_1, A_2 \) separately and get \( c(g) = c_1(g) \otimes c_2(g), \lambda(g) = \lambda_1(g)\lambda_2(g) \). We have
   \[
   c_i(g)\bar{g}(u_i)c_i(g)^t = \lambda_i(g)u_i, i = 1, 2
   \]
   and
   \[
   \det(c_i(g))^2 = \lambda_i(g)^2 a_ig(a_i)^{-1}
   \]
   with \( a_i = \det(u_i) \). It follows that
   \[
   \chi(g) = \det(c(g))\lambda(g)^{-2}
   = \det(c_1(g))^2\det(c_2(g))^2\lambda_1(g)^{-2}\lambda_2(g)^{-2}
   = a_1a_2g(a_1a_2)^{-1} = g(a)^{-1}
   \]
with \( a = (a_1a_2)^{-1} \). Thus

\[
\delta(\sigma) = [\det(u)a^2] = [\det(u_1)^2\det(u_2)^2a_1^{-2}a_2^{-2}] = 1.
\]

In this case the pfaffian is the generic norm defined by Jacobson in [J2].

4. If \( K \) is an hilbertian field of characteristic not two and \( A \) is a central simple \( K \)-algebra of order two in the Brauer group \( \text{Br}(K) \), then, by a recent result of Saltman [S], there exists an involution on \( A \) with nontrivial discriminant.

3. The pfaffian adjoint

In this section we generalize the construction of the pfaffian adjoint to central simple algebras of arbitrary rank. It will follow that the discriminant defined in Section 2 and the discriminant of [KPS3] are identical.

Let \( n = 2m \). For any alternating \((n \times n)\)-matrix \( x = (x_{ij}) \) let \( p_0(x) = (X_{ij}) \) be alternating with \( X_{ij} = (-1)^{i+j+1}pf(\xi) \) for \( i > j \) and \( \xi \) is the \(((n - 2) \times (n - 2))\)-matrix obtained from \( x \) by cancelling the \( i \)-th and \( j \)-th rows and columns. The map \( p_0 \) is polynomial and homogeneous of degree \( m - 1 \) satisfies \( xp_0(x) = p_0(x)x = pf(x) \) [B, §5, No 3, Exercice 5] and \( p_0^2(x) = (-1)^mpf(x)^{m-2}x \) for all \( x \in \text{Alt}_{2m}(K) \). For example, if \( m = 2 \), \( p_0 \) is the \( K \)-linear automorphism \( p_0 : \text{Alt}_4(K) \rightarrow \text{Alt}_4(K) \) given by

\[
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
0 & a_{23} & a_{24} & \ \\
0 & a_{34} & 0 & \ \\
\end{pmatrix} \mapsto \begin{pmatrix}
0 & -a_{34} & a_{24} & -a_{23} \\
0 & -a_{14} & a_{13} & \ \\
0 & -a_{12} & 0 & \ \\
\end{pmatrix}
\]

A similar map can be constructed for any central simple \( K \)-algebra with involution \( \sigma \) of dimension \( 2m \).

**Proposition (3.1)** Let \( pf_\sigma : \text{Alt}^\sigma(A) \rightarrow K \) be a pfaffian on \( A \) such that \( pf_\sigma(x)^2 = d_\sigma n_A(x) \) for all \( x \in \text{Alt}^\sigma(A) \). There exists a polynomial map

\[
p_\sigma : \text{Alt}^\sigma(A) \rightarrow \text{Alt}^\sigma(A),
\]

homogeneous of degree \( m - 1 \), such that \( xp_\sigma(x) = p_\sigma(x)x = pf_\sigma(x) \) and

\[
p_\sigma^2(x) = (-1)^mpf_\sigma(x)^{m-2}d_\sigma x.
\]
Proof. Let \( \alpha : A \otimes L \xrightarrow{\sim} M_{2m}(L) \) be a Galois splitting of \( A \) such that

\[
\alpha(\sigma \otimes 1)\alpha^{-1} = \text{Int}(u) \circ t
\]

and, with the notations of Section 1, let \( a \in K^* \) with \( \chi(g) = g(a)a^{-1}, g \in \text{Gal}(L/K) \), so that \( d_\sigma = \det(u)a^2 \). We put

\[
p_\sigma(x) = \alpha^{-1}(up_\sigma(\alpha(x \otimes 1)u)a).
\]

To prove that \( p_\sigma \) has image in \( \text{Alt}^\sigma(A) \), we shall need the formula

\[
zp_\sigma(z^txz)z^t = \det(z)p_\sigma(x)
\]

for \( x \in \text{Alt}_{2m}(K) \) and \( z \in M_{2m}(K) \). This formula is a simple consequence of the formulas \( \text{pf}(z^txz) = \det(z)\text{pf}(x) \) and \( xp_\sigma(x) = p_\sigma(x)x = \text{pf}(x) \). We get

\[
(1 \otimes g)(p_\sigma(x)) = \alpha^{-1}(c(g)\bar{g}(up_\sigma(\alpha(x \otimes 1)u)a)c(g)^{-1})
\]

\[
= \alpha^{-1}(\lambda(g)u(c(g)^{-1})p_\sigma(\bar{g}(\alpha(x \otimes 1))\bar{g}(ua)c(g)^{-1})
\]

\[
= \lambda(g)g(a)\alpha^{-1}(\text{det}(c(g))p_\sigma(c(g)\bar{g}(\alpha(x \otimes 1))\bar{g}(u(c(g)^{-1}))
\]

\[
= \det(c(g))^{-1}\lambda(g)^2g(a)\alpha^{-1}(up_\sigma(\alpha(x \otimes 1)u))
\]

\[
= p_\sigma(x)
\]

so that \( p_\sigma(x) \in \text{Alt}^\sigma(A) \) for \( x \in \text{Alt}^\sigma(A) \). Further we have

\[
xp_\sigma(x) = \alpha^{-1}(\alpha(x \otimes 1)up_\sigma(\alpha(x \otimes 1)u)a)
\]

\[
= \text{pf}(\alpha(x \otimes 1)u)a = \text{pf}_{\sigma}(x)
\]

\[
p_\sigma(x)x = \alpha^{-1}(up_\sigma(\alpha(x \otimes 1)u)\alpha(\alpha(x \otimes 1)uu^{-1})
\]

\[
= \alpha^{-1}(upf(\alpha(x \otimes 1)u)u^{-1}) = \text{pf}_{\sigma}(x).
\]

Finally we get

\[
p_\sigma^2(x) = \alpha^{-1}(up_\sigma(\alpha^{-1}(up_\sigma(\alpha(x \otimes 1)u)a))u)a)
\]

\[
= \alpha^{-1}(u^mup_\sigma(u^m\alpha(\alpha(x \otimes 1)u))
\]

\[
= \alpha^{-1}(u^mup_\sigma(u^m\alpha(x \otimes 1)u)u^mu^{-1}
\]

\[
= \alpha^{-1}(u^2\text{det}(u)a^{-2}p_\sigma^2(\alpha(x \otimes 1)u)u^{-1})
\]

\[
= (-1)^mp_\sigma(x)^{-2}d_\sigma x
\]
using that $p^2_0(x) = (-1)^mpf(x)^{m-2}x$. For algebras of dimension 16, the map $p_\sigma$ is a $K$–linear automorphism of $\text{Alt}^\sigma(A)$ and $p^2_\sigma(x) = d_\sigma x$. The class $\delta(\sigma)$ of the element $d_\sigma$ with the property that $p^2_\sigma(x) = d_\sigma x$ was defined in [KPS$_3$] as the discriminant of $\sigma$. Thus both discriminants coincide. We finally mention the main result of [KPS$_3$], which relies on (3.1) for $n = 16$:

**Theorem 3.2** Let $A$ be a central simple algebra of dimension 16 with an involution $\sigma$ of the first kind. There exists a $\sigma$–invariant quaternion subalgebra $A_1$ of $A$ if and only if the discriminant $\delta(\sigma)$ of $\sigma$ is trivial.

In characteristic not 2, the set $A'_1$ of pure quaternions of $A_1$ can be chosen as

$$A'_1 = \{ x \in \text{Alt}^\sigma(A) \mid p_\sigma(x) = x \},$$

where $p_\sigma$ is taken such that $p^2_\sigma(x) = x$. For this choice of $A_1$ the restriction of $\sigma$ to $A_1$ is the conjugation map of $A_1$. Details are in [KPS$_3$].
References


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