Abstract. Quadratic quaternion forms, introduced by Seip-Hornix (1965), are special cases of generalized quadratic forms over algebras with involutions. We apply the formalism of these generalized quadratic forms to give a characteristic free version of different results related to hermitian forms over quaternions:

1) An exact sequence of Lewis
2) Involutions of central simple algebras of exponent 2.
3) Triality for 4-dimensional quadratic quaternion forms.

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1. Introduction

Let \( F \) be a field of characteristic not 2 and let \( D \) be a quaternion division algebra over \( F \). It is known that a skew-hermitian form over \( D \) determines a symmetric bilinear form over any separable quadratic subfield of \( D \) and that the unitary group of the skew-hermitian form is the subgroup of the orthogonal group of the symmetric bilinear form consisting of elements which commute with a certain semilinear mapping (see for example Dieudonné [3]). Quadratic forms behave nicer than symmetric bilinear forms in characteristic 2 and Seip-Hornix developed in [9] a complete, characteristic-free theory of quadratic quaternion forms, their orthogonal groups and their classical invariants. Her theory was subsequently (and partly independently) generalized to forms over algebras (even rings) with involution (see [11], [10], [1], [8]).

Similitudes of hermitian (or skew-hermitian) forms induce involutions on the endomorphism algebra of the underlying space. To generalize the case where only similitudes of a quadratic form are considered, the notion of a quadratic pair was worked out in [6]. Relations between quadratic pairs and generalized quadratic forms were first discussed by Elomary [4].

The aim of this paper is to apply generalized quadratic forms to give a characteristic free presentation of some results on forms and involutions. After briefly recalling in Section 2 the notion of a generalized quadratic form (which, following the standard literature, we call a \((\varepsilon, \sigma)\)-quadratic form) we give in Section
3 a characteristic-free version of an exact sequence of Lewis (see [7], [8, p. 389] and the appendix to [2]), which connects Witt groups of quadratic and quaternion algebras. The quadratic quaternion forms of Seip-Hornix are the main ingredient. Section 4 describes a canonical bijective correspondence between quadratic pairs and \((\varepsilon, \sigma)\)-quadratic forms and Section 5 discusses the Clifford algebra. In particular we compare the definitions given in [10] and in [6]. In Section 6 we develop triality for 4-dimensional quadratic quaternion forms whose associated forms (over a separable quadratic subfield) are 3-Pfister forms. Any such quadratic quaternion form \(\theta\) is an element in a triple \((\theta_1, \theta_2, \theta_3)\) of forms over 3 quaternion algebras \(D_1, D_2, D_3\) such that \([D_1][D_2][D_3] = 1\) in the Brauer group of \(F\). Triality acts as permutations on such triples.

2. Generalized quadratic forms

Let \(D\) be a division algebra over a field \(F\) with an involution \(\sigma: x \mapsto \overline{x}\). Let \(V\) be a finite dimensional right vector space over \(D\). A \(F\)-bilinear form 

\[ k: V \times V \to D \]

is \textit{sesquilinear} if \(k(ax, yb) = \overline{a}k(x, y)b\) for all \(x, y \in V, a, b \in D\). The additive group of such maps will be denoted by \(\text{Sesq}_\sigma(V, D)\). For any \(k \in \text{Sesq}_\sigma(V, D)\) we write 

\[ k^\ast(x, y) = \overline{k(y, x)}. \]

Let \(\varepsilon \in F^\times\) be such that \(\overline{\varepsilon} = 1\). A sesquilinear form \(k\) such that \(k = \varepsilon k^\ast\) is called \(\varepsilon\)-\textit{hermitian} and the set of such forms on \(V\) will be denoted by \(\text{Herm}_\varepsilon^\sigma(V, D)\). Elements of 

\[ \text{Alt}_\varepsilon^\sigma(V, D) = \{ g = f - \varepsilon f^\ast \mid f \in \text{Sesq}_\varepsilon(V, D) \}. \]

are \(\varepsilon\)-\textit{alternating forms}. We obviously have \(\text{Alt}_\varepsilon^{-\varepsilon}(V, D) \subset \text{Herm}_\varepsilon^\sigma(V, D)\). We set 

\[ Q^\varepsilon_\sigma(V, D) = \text{Sesq}_\varepsilon(V, D) / \text{Alt}_\varepsilon^\sigma(V, D) \]

and refer to elements of \(Q^\varepsilon_\sigma(V, D)\) as \((\varepsilon, \sigma)\)-\textit{quadratic forms}. We recall that \((\varepsilon, \sigma)\)-quadratic forms were introduced by Tits [10], see also Wall [11], Bak [1] or Scharlau [8, Chapter 7]. For any algebra \(A\) with involution \(\tau\), let \(\text{Sym}^\tau(A, \tau) = \{ a \in A \mid a = \varepsilon \tau(a) \}\) and \(\text{Alt}^\tau(A, \tau) = \{ a \in A \mid a = c + \varepsilon \tau(c), c \in A \}\). To any class \(\theta = [k] \in Q^\varepsilon_\sigma(V, D)\), represented by \(k \in \text{Sesq}_\varepsilon(V, D)\), we associate a quadratic map 

\[ q_\theta : V \to D / \text{Alt}^\varepsilon(D, \sigma), \quad q_\theta(x) = [k(x, x)] \]

where \([d]\) denotes the class of \(d\) in \(D / \text{Alt}^\varepsilon(D)\). The \(\varepsilon\)-hermitian form 

\[ b_\theta(x, y) = k(x, y) + \varepsilon k^\ast(x, y) = k(x, y) + \overline{\varepsilon k(y, x)} \]

depends only on the class \(\theta\) of \(k\) in \(Q^\varepsilon_\sigma(V, D)\). We say that \(b_\theta\) is the \textit{polarization} of \(q_\theta\).
Proposition 2.1. The pair $(q_0, b_0)$ satisfies the following formal properties:

\[
\begin{align*}
q_0(x + y) &= q_0(x) + q_0(y) + [b_0(x, y)] \\
q_0(xd) &= d q_0(x) d \\
b_0(x, x) &= q_0(x) + \varepsilon q_0(x)
\end{align*}
\]

for all $x, y \in V$, $d \in D$. Conversely, given any pair $(q, b)$, $q : V \to D/\text{Alt}^\varepsilon(D, \sigma)$, $b \in \text{Herm}_F^F(V, D)$ satisfying (1), there exist a unique $\theta \in Q_2^\varepsilon(V, D)$ such that $q = q_\theta$, $b = b_\theta$.

Proof. The formal properties are straightforward to verify. For the converse see [11, Theorem 1].

Example 2.2. Let $D = F$, $\sigma = \text{Id}_F$ and $\varepsilon = 1$. Then sesquilinear forms are $F$-bilinear forms, $\text{Alt}^\varepsilon(D, \sigma) = 0$ and a $(\sigma, \varepsilon)$-quadratic form is a (classical) quadratic form. We denote the set of bilinear forms on $V$ by $\text{Bil}(V, F)$. Accordingly we speak of $\varepsilon$-symmetric bilinear forms instead of $\varepsilon$-hermitian forms.

Example 2.3. Let $D$ be a division algebra with involution $\sigma$ and let $D$ be a finite dimensional (right) vector space over $D$. We use a basis of $V$ to identify $V$ with $D^n$ and $\text{End}_D(V)$ with the algebra $M_n(D)$ of $(n \times n)$-matrices with entries in $D$. For any $(n \times n)$-matrix $x = (x_{ij})$, let $x^\tau = (x_{ji})$, where $\tau$ is transpose and $x^\tau = (x_{ij})$. In particular the map $a \mapsto a^*$ is an involution of $A = M_n(D)$. If we write elements of $D^n$ as column vectors $x = (x_1, \ldots, x_n)^t$ any sesquilinear form $k$ over $D^n$ can be expressed as $k(x, y) = x^* a y$, with $a \in M_n(D)$, and $k^\varepsilon(x, y) = x^* a^* y$. We write $\text{Alt}_n(D) = \{a = b - \varepsilon b^*\} \subset M_n(D)$, so that $Q_2^\varepsilon(V, D) = M_n(D)/\text{Alt}_n(D)$.

Example 2.4. Let $D$ be a quaternion division algebra, i.e. $D$ is a central division algebra of dimension 4 over $F$. Let $K$ be a maximal subfield of $D$ which is a quadratic Galois extension of $F$ and let $\sigma : x \mapsto \bar{x}$ be the nontrivial automorphism of $K$. Let $j \in K \setminus F$ be an element of trace 1, so that $K = F(j)$ with $j^2 = j + \lambda$, $\lambda \in F$. Let $\ell \in D$ be such that $\ell x \ell^{-1} = \bar{x}$ for $x \in K$, $\ell^2 = \mu \in F^\times$. The elements $\{1, j, \ell, \ell j\}$ form a basis of $D$ and $D = K \oplus \ell K$ is also denoted $[K, \mu]$. The $F$-linear map $\sigma : D \to D$, $\sigma(d) = \text{Tr}_{D/F}(d) - d = d$ is an involution of $D$ (the "conjugation") which extends the automorphism $\sigma$ of $K$. The element $N(d) = d \sigma(d) = \sigma(d)d$ is the reduced norm of $d$. We have $\text{Alt}_n^\varepsilon(D) = F$ and $(\sigma, -1)$-quadratic forms correspond to the quadratic quaternion forms introduced by Seip-Hornix in [9]. Accordingly we call $(\sigma, -1)$-quadratic forms quadratic quaternion forms.

The restriction of the involution $\tau$ to the center $Z$ of $A$ is either the identity (involutions of the first kind) or an automorphism of order 2 (involutions of the second kind). If the characteristic of $F$ is different from 2 or if the involution is of second kind there exists an element $j \in Z$ such that $j + \sigma(j) = 1$. Under such conditions the theory of $(\sigma, \varepsilon)$-quadratic forms reduces to the theory of $\varepsilon$-hermitian forms:
Proposition 2.5. If the center of $D$ contains an element $j$ such that $j + \sigma(j) = 1$, then $\text{Herm}_\sigma^+(V, D) = \text{Alt}_\sigma^+(V, D)$ and a $(\sigma, \varepsilon)$-quadratic form is uniquely determined by its polar form $b_\theta$.

Proof. If $k = -\varepsilon k^* \in \text{Herm}_\sigma^+(V, D)$, then $k = 1k = jk + j^k = jk - j\varepsilon k^* \in \text{Alt}_\sigma^+(V, D)$. The last claim follows from the fact that polarization induces an isomorphism $\text{Sesq}_\sigma(V, D)/\text{Herm}_\sigma^+(V, D) \cong Q^+(V, D)$.

For any left (right) $D$-space $V$ we denote by $^\sigma V$ the space $V$ viewed as right (left) $D$-space through the involution $\sigma$. If $^\sigma x$ is the element $x$ viewed as an element of $V'$, we have $^\sigma xd = ^\sigma(\sigma(d)x)$. Let $V^*$ be the dual $^\sigma\text{Hom}_D(V, D)$ as a right $D$-module, i.e., $(^\sigma f)(x) = ^\sigma(\theta f)(x)$, $x \in V$, $d \in D$. Any sesquilinear form $k \in \text{Sesq}_\sigma(V, D)$ induces a $D$-module homomorphism $\hat{k} : V \rightarrow V^*$, $x \mapsto k(x, -)$. Conversely any homomorphism $g : V \rightarrow V^*$ induces a sesquilinear form $k \in \text{Sesq}_\sigma(V, D)$, $k(x, y) = g(x)(y)$ and the additive groups $\text{Sesq}_\sigma(V, D)$ and $\text{Hom}_D(V, V^*)$ can be identified through the map $h \mapsto \hat{k}$. For any $f : V \rightarrow V'$, let $f^* : V'^* \rightarrow V^*$ be the transpose, viewed as a homomorphisms of right vector spaces. We identify $V$ with $V'^*$ through the map $v \mapsto v^*$. Let for any $f \in \text{Hom}_D(V, V^*)$, $f^*$ is again in $\text{Hom}_D(V, V^*)$ and $\hat{k}^* = \hat{k}^\sigma$. A $(\sigma, \varepsilon)$-quadratic form $q_\theta$ is called nonsingular if its polar form $b_\theta$ induces an isomorphism $\hat{b}_\theta$. A pair $(V, q_\theta)$ with $q_\theta$ nonsingular is called a $(\sigma, \varepsilon)$-quadratic space. For any vector space $W$, the hyperbolic space $V = W \oplus W^*$ equipped with the quadratic form $q_\theta$, $\theta = [k]$ with

$$k((p, q), (p', q')) = q(p'),$$

is nonsingular. There is an obvious notion of orthogonal sum $V \perp V'$ and a quadratic space decomposes whenever its polarization does. Most of the classical theory of quadratic spaces extends to $(\sigma, \varepsilon)$-quadratic spaces. For example Witt cancellation holds and any $(\sigma, \varepsilon)$-quadratic space decomposes uniquely (up to isomorphism) as the orthogonal sum of its anisotropic part with a hyperbolic space. Moreover, if we exclude the case $\sigma = 1$ and $\varepsilon = -1$, any $(\sigma, \varepsilon)$-quadratic space has an orthogonal basis. A similitude of $(\sigma, \varepsilon)$-quadratic spaces $t : (V, q) \cong (V', q')$ is a $D$-linear isomorphism $V \cong V'$ such that $q'(tx) = \mu(t)q(x)$ for some $\mu(t) \in F^\times$. The element $\mu(t)$ is called the multiplier of the similitude. Similitudes with multipliers equal to 1 are isometries. As in the classical case there is a notion of Witt equivalence and corresponding Witt groups are denoted by $W^e(D, \sigma)$.

3. An exact sequence of Lewis

Let $D$ be a quaternion division algebra. We fix a representation $D = [K, \mu] = K \oplus tK$, with $t^2 = \mu$, as in (2.4). Let $V$ be a vector space over $D$. Any sesquilinear form $k : V \times V \rightarrow D$ can be decomposed as

$$k(x, y) = P(x, y) + tR(x, y)$$

with $P : V \times V \rightarrow K$ and $R : V \times V \rightarrow K$. The following properties of $P$ and $R$ are straightforward.
Lemma 3.1. 1) \( P \in \text{Sesq}_\ell(V, K) \), \( R \in \text{Sesq}_0(V, K) = \text{Bil}(V, K) \).
2) \( k^* = P^* - \ell R^* \), where \( P^*(x, y) = P(y, x) \) and \( R^*(x, y) = R(y, x) \).

The sesquilinearity of \( k \) implies the following identities:

\[
\begin{align*}
R(x\ell, y) &= -P(x, y), \quad R(x, y\ell) = \overline{P(x, y)} \\
P(x\ell, y) &= -\mu P(x, y), \quad P(x, y\ell) = \mu \overline{P(x, y)} \\
P(x\ell, y\ell) &= -\mu P(x, y), \quad R(x\ell, y\ell) = -\mu \overline{R(x, y)}
\end{align*}
\] (2)

Let \( V^0 \) be \( V \) considered as a (right) vector space over \( K \) (by restriction of scalars) and let \( T : V^0 \to V^0, x \mapsto x\ell \). The map \( T \) is a \( K \)-semilinear automorphism of \( V^0 \) such that \( T^2 = \mu \). Conversely, given a vector space \( U \) over \( K \), together with a semilinear automorphism \( T \) such that \( T^2 = \mu \in F^\times \), we define the structure of a right \( D \)-module on \( U \), \( D = [K, \mu] \), by putting \( x\ell = T(x) \).

Lemma 3.2. Let \( V \) be a vector space over \( D \). 1) Let \( f_1 : V^0 \times V^0 \to K \) be a sesquilinear form over \( K \). The form

\[ f(x, y) = f_1(x, y) - \ell \mu^{-1} f_1(Tx, y) \]

is sesquilinear over \( D \) if and only if \( f_1(Tx, Ty) = -\mu f_1(x, y) \).
2) Let \( f_2 : V^0 \times V^0 \to K \) be a bilinear form over \( K \). The form

\[ f(x, y) = -f_2(Tx, y) + \ell f_2(x, y) \]

is sesquilinear over \( D \) if and only if \( f_2(Tx, Ty) = -\mu f_2(x, y) \).

Proof. The two claims follow from the identities (2). \( \square \)

Let \( f \) be a bilinear form on a space \( U \) over \( K \) and let \( \lambda \in K^\times \). A semilinear automorphism \( t \) of \( U \) such that \( f(tx, ty) = \lambda f(x, y) \) for all \( x, y \in U \) is a \( \lambda \)-semilinear \( \sigma \)-similitude of \( (U, f) \), with multiplier \( \lambda \). In particular \( Tx = x\ell \) is a \( \lambda \)-semilinear \( \sigma \)-similitude of \( R \) on \( V^0 \), such that \( T^2 = \mu \) and with multiplier \( -\mu \). The following nice observation of Seip-Hornix [9, p. 328] will be used later:

Proposition 3.3. Let \( R \) be a \( K \)-bilinear form over \( U \) and let \( T \) be a \( \lambda \)-semilinear \( \sigma \)-similitude of \( U \) with multiplier \( \lambda \in K^\times \) and such that \( T^2 = \mu \). Then:
1) \( \mu \in F^\times \),
2) For any \( \xi \in K \) and \( x \in U \), let \( \rho \xi(x) = x\xi \). There exists \( \nu \in K^\times \) such that \( T' = \rho \nu \circ T \) satisfies \( T'^2 = \mu' \) and \( R(T'x, T'y) = -\mu' \overline{R(x, y)} \).

Proof. The first claim follows from \( \mu = \lambda \overline{\lambda} \). For the second we may assume that \( \lambda \neq \mu \) (if \( \lambda = \mu \) replace \( T \) by \( T \circ \rho_k \) for an appropriate \( k \)). For \( \nu = (1 - \mu \lambda^{-1}) \) we have \( \mu' = 2\mu - \lambda - \overline{\lambda} \). \( \square \)

Assume that \( k \in \text{Sesq}_\ell(V, D) \) defines a \( (\sigma, \varepsilon) \)-quadratic space \([k]\) on \( V \) over \( D \). It follows from (3.1) that \( P \) defines a \( (\sigma, \varepsilon) \)-quadratic space \([P]\) on \( V \) over \( K \) and \( R \) a \((Id, -\varepsilon)\)-quadratic space \([R]\) on \( V^0 \) over \( K \). Let \( K = F(j) \) with \( j^2 = j + \lambda \). Let \( r(x, y) = R(x, y) - \varepsilon R(y, x) \) be the polar of \( R \).
Proposition 3.4. 1) \( q_B(x) = \pi_j[r(x, Tx)] \)
2) \( q_B(x) = \pi_j[r(x, Tx)] + \epsilon q_B(x) \)
3) The map \( T \) is a semilinear similitude of \( (q_B, V^0) \) with multiplier \(-\mu\).

Proof. It follows from the relations (2) that
\[ (3) \quad P(x, x) + \epsilon P(x, x) = R(x, Tx) - \epsilon R(Tx, x) = r(x, Tx) \]
and obviously this relation determines \( P(x, x) \) up to a function with values in \( \text{Sym}^{-\epsilon}(K, \sigma) \). Since \( \text{Sym}^{-\epsilon}(K, \sigma) = \text{Alt}^{+\epsilon}(K, \sigma) \) by (2.5), \([P]\) is determined by (3). Since \( r(x, Tx) = \pi r(x, Tx) \) by (2), we have \( \pi r(x, Tx) + \epsilon (\pi r(x, Tx)) = r(x, Tx) \) and (1) follows. The second claim follows from 1) and 3) is again a consequence of the identities (2).

Corollary 3.5. Any pair \(([R], T)\) with \([R] \in \mathcal{Q}_K(U, K)\) and \( T \) a semilinear similitude with multiplier \(-\mu \in F^x\) such that \( T^2 = \mu \), determines the structure of a \((\sigma, \epsilon)\)-quadratic space on \( U \) over \( D = [K, \mu] \).

Proposition 3.6. The assignments \( h \mapsto P \) and \( h \mapsto R \) induce homomorphisms of groups \( \pi_1 : W^\epsilon(D, -) \rightarrow W^\epsilon(K, -) \) and \( \pi_2 : W^{-\epsilon}(D, -) \rightarrow W^\epsilon(K, Id) \).

Proof. The assignments are obviously compatible with orthogonal sums and Witt equivalence.

We recall that \( W^\epsilon(K, -) \) can be identified with the corresponding Witt group of \( \epsilon \)-hermitian forms (apply (2.5)). However, it is more convenient for the following computations to view \( \epsilon \)-hermitian forms over \( K \) as \((\sigma, \epsilon)\)-quadratic forms. Let \( i \in K^x \) be such that \( \sigma(i) = -i \) (take \( i = 1 \) if \( \text{Char} F = 2 \)). The map \( k \mapsto ik \) induces an isomorphism \( s : W^\epsilon(K, -) \rightarrow W^{-\epsilon}(K, -) \) (“scaling”). For any space \( U \) over \( K \), let \( U_D = U \otimes_K D \). We identify \( U_D \) with \( U \oplus U^t \) through the map \( u \otimes (x + ty) \mapsto (ux, uy) \) and get a natural \( D \)-module structure on \( U_D = U \oplus U^t \). Any \( K \)-sesquilinear form \( k \) on \( U \) extends to a \( D \)-sesquilinear form \( k_D \) on \( U_D \) through the formula
\[ k_D(x \otimes a, y \otimes b) = \pi k(x, y)b \]
for \( x, y \in U \) and \( a, b \in D \).

Lemma 3.7. The assignment \( k \mapsto (ik)_D \) induces a homomorphism
\[ \beta : W^\epsilon(K, -) \rightarrow W^{-\epsilon}(D, -) \]

Proof. Let \( \tilde{k} = (ik)_D \). We have \( \tilde{k}^* = -\tilde{k}^* \).

Theorem 3.8 (Lewis). With the notations above, the sequence
\[ W^\epsilon(D, -) \xrightarrow{\pi_1} W^\epsilon(K, -) \xrightarrow{\beta} W^{-\epsilon}(D, -) \xrightarrow{\pi_2} W^\epsilon(K, Id) \]
is exact.

Proof. This is essentially the proof given in Appendix 2 of [2] with some changes due to the use of generalized quadratic forms, instead of hermitian forms. We first check that the sequence is a complex. Let \([k] \in \mathcal{Q}_K(V, D)\) and let \( V^0 = U \).
We write elements of $U_D = U ⊕ U \ell$ as pairs $(x, y \ell)$ and decompose $k_D = P + \ell R$.

By definition we have $\beta \pi_1([k]) = [\beta(P)]$ and

$$\beta(P)((x_1, y_1), (x_2, y_2)) = i(P(x_1, x_2) + P(x_2, y_1)\ell + \ell P(y_1, x_2) + \ell P(y_2, x_2))$$

Let $(x \ell, x \ell) \in U \oplus U \ell$. We get $\beta(P)((x \ell, x \ell), (x \ell, x \ell)) = 0$ hence $W = \{(x \ell, x \ell)\} \subset U \oplus U \ell$ is totally isotropic. It is easy to see that $W \subset W^\perp$, so that $[\beta(P)]$ is hyperbolic and $\beta \circ \pi_1 = 0$. Let $[g] \in Q_\sigma^+(U, K)$. The subspace $W = \{(x, 0) \in U \oplus U \ell\}$ is totally isotropic for $\pi_2(\beta([g]))$ and $W \subset W^\perp$. Hence $\pi_2(\beta([g])) = 0$. We now prove exactness at $W^\perp(K, -)$. Since the claim is known if $\text{Char} = 0$, we may assume that $\text{Char} = 2$ and $x = 1$. Let $[g] \in Q_\sigma^+(U, K)$ be anisotropic such that $\beta([g]) = 0 \in W^{-\varepsilon}(D, -)$. In particular $\beta([g]) \in Q_\sigma^+(U_D, D)$ is isotropic. Hence the exist elements $x_1, x_2 \in U$ such that $[\beta([g])(x_1, x_2\ell), (x_1, x_2\ell)] = 0$. This implies (in $\text{Char} = 2$) that

$$(4) \quad g(x_1, x_1) + \mu g(x_1, x_2) = g(x_1, x_2)\ell + \ell g(x_2, x_1) = 0$$

Let $V_1$ be the $K$-subspace of $V$ generated by $x_1$ and $x_2$. Since $[g]$ is anisotropic, $[g] = [g_1] \perp [g_2]$ with $g_1 = g|V_1$. We make $V_1$ into a $D$-space by putting

$$(x_1a_1 + x_2a_2)\ell = \mu x_2a_1 + x_1a_2$$

To see that the action is well-defined, it suffices to show that $\dim_K V_1 = 2$. The elements $x_1$ and $x_2$ cannot be zero since $[g]$ is anisotropic, so assume $x_2 = x_1c, c \in K^\times$. Then (4) implies $g(x_1, x_1) + \mu \overline{g(x_1, x_1)} = 0$, which contradicts the fact that $g$ is anisotropic. Let $g_1(x_1, x_1) + \mu g_1(x_2, x_2) = z \in F$. Let $f \in \text{Ses}_q(V_1, K)$. Replacing $g_1$ by $g_1 + f + f^*$ defines the same class in $Q_\sigma^+(V_1, K)$ (recall that $\text{Char} F = 2$). Choosing $f$ as

$$f(x_1, x_1) = jz, f(x_2, x_2) = 0, f(x_1, x_2) = f(x_2, x_1) = 0,$$

we may assume that

$$(5) \quad g_1(x_1, x_1) + \mu \overline{g_1(x_2, x_2)} = 0, g_1(x_1, x_2)\ell + \ell g_1(x_2, x_1) = 0$$

By (3.2) we may extend $g_1$ to a sesquilinear form

$$g^*(x, y) = g_1(x, y) + \ell \mu^{-1}g_1(x\ell, y)$$

over $D$ if $g_1$ satisfies

$$g_1(x\ell, y\ell) = -\mu g_1(x, y)$$

This can easily be checked using (5) (and the definition of $x\ell$). Then $g_1$ is in the image of $\pi_1$. Exactness at $W^\varepsilon(K, -)$ now follows by induction on the dimension of $U$. We finally check exactness at $W^{-\varepsilon}(D, -)$. Let $[k]$ be anisotropic such that $\pi_2([k]) = 0$ in $W^{-\varepsilon}(K, Id)$. In particular $\pi_2([k])$ is isotropic; let $x \neq 0$ be such that $\pi_2(k(x, x) = 0$ and let $W$ be the $D$-subspace of $V$ generated by $x$. Since $[k]$ is anisotropic, $[k'] = [k]|_W$ is nonsingular and $[k] = [k'] \perp [k']$. The condition $\pi_2(k(x, x) = 0$ implies $k(x, x) \in K$. Let $W_1$ be the $K$-subspace of $W$ generated by $x$. Define $g : W_1 \times W_1 \to K$ by $g(xa, xb) = k(xa, xb)i^{-1}$. for $a,$
b ∈ K. Then clearly [g] defines an element of \( W^\varepsilon(K, -) \) and \( \beta(g) = k' \). Once again exactness follows by induction on the dimension of \( V \).

\section{Involutions on Central Simple Algebras}

Let \( D \) be a central division algebra over \( F \), with involution \( \sigma \) and let \( b : V \times V → D \) be a nonsingular \( \varepsilon \)-hermitian form on a finite dimensional space over \( D \). Let \( A = \text{End}_D(V) \). The map \( \sigma_b : A → A \) such that \( \sigma_b(\lambda) = \sigma(\lambda) \) for all \( \lambda ∈ F \) and

\[ b(\sigma_b(f)(x), y) = b(x, f(y)) \]

for all \( x, y ∈ V \), is an involution of \( A \), called the involution \textit{adjoint to} \( b \). We have \( \sigma_b(f) = \hat{b}^{-1}f^*\hat{b} \), where \( \hat{b} : V → V^* \) is the adjoint of \( b \). Conversely, any involution of \( A \) is adjoint to some nonsingular \( \varepsilon \)-hermitian form \( b \) and \( b \) is uniquely multiplicative up to a \( \sigma \)-invariant element of \( F^* \).

Any automorphism \( \phi \) of \( A \) compatible with \( \sigma \), i.e., \( \sigma_b(\phi(a)) = \phi(\sigma_b(a)) \), is of the form \( \phi(a) = uau^{-1} \) with \( u : V → V \) a similitude of \( b \). We say that an involution \( \tau \) of \( A \) is a \textit{q-involution} if \( \tau \) is adjoint to the polar \( b^q \) of a \( (\sigma, \varepsilon) \)-quadratic form \( \theta \). We write \( \tau = \sigma_\theta \). Two algebras with \( q \)-involutions are isomorphic if the isomorphism is induced by a similitude of the corresponding quadratic forms. Over fields \( q \)-involutions differ from involutions only in characteristic 2 and for symplectic involutions. In view of possible generalizations (for example rings in which \( 2 ≠ 0 \) is not invertible) we keep to the general setting of \( (\sigma, \varepsilon) \)-quadratic forms. Let \( F_0 \) be the subfield of \( F \) of \( \sigma \)-invariant elements and let \( T_{F/F_0} \) be the corresponding trace.

\begin{lemma}
The symmetric bilinear form on \( A \) given by \( \text{Tr}(x, y) = T_{F/F_0}(\text{Trd}_A(xy)) \) is nonsingular and \( \text{Sym}(A, \tau)^{1/2} = \text{Alt}(A, \tau) \).
\end{lemma}

\textit{Proof.} If \( \tau \) is of the first kind \( F_0 = F \) and the claim is (2.3) of \( [6] \). Assume that \( \tau \) is of the second kind. Since the bilinear form \( (x, y) → \text{Trd}_A(xy) \) is nonsingular, \( \text{Tr} \) is also nonsingular and it is straightforward that \( \text{Alt}(A, \tau) ⊂ \text{Sym}(A, \tau)^{1/2} \).

Equality follows from the fact that \( \dim_{F_0} \text{Alt}(A, \tau) = \dim_{F_0} \text{Sym}(A, \tau) = \dim_F A. \)

\begin{proposition}
Let \( (V, \theta) = [k] \) be a \( (\sigma, \varepsilon) \)-quadratic space over \( D \) and let \( h = \hat{k} + \varepsilon\hat{k}^*: V → V^* \). The \( F_0 \)-linear form

\[ f_\theta : \text{Sym}(A, \sigma_\theta) → F_0, \quad f_\theta(s) = \text{Tr}(h^{-1}\hat{k}s), \quad s ∈ \text{Sym}(A, \sigma_\theta) \]

depends only on the class \( \theta \) and satisfies \( f_\theta(x + \sigma_\theta(x)) = \text{Tr}(x) \).
\end{proposition}

\textit{Proof.} The first claim follows from (4.1) and the fact that if \( k ∈ \text{Alt}_\varepsilon^+(V, D) \) then \( h^{-1}\hat{k} ∈ \text{Alt}^+_{\sigma_\theta}(V, D) \). For the last claim we have:

\begin{align*}
f_\theta(x + \sigma_\theta(x)) &= \text{Tr}(h^{-1}\hat{k}(x + \sigma_\theta(x))) \\
&= \text{Tr}(h^{-1}\hat{k}x) + \text{Tr}(h^{-1}\hat{k}hx^*) \\
&= \text{Tr}(h^{-1}\hat{k}x) + \text{Tr}(\hat{k}x^*) \\
&= \text{Tr}(h^{-1}\hat{k}x) + \text{Tr}(x(h^{-1})^*\hat{k}^*) \\
&= \text{Tr}(h^{-1}\hat{k}x) + \text{Tr}(h^{-1}\varepsilon\hat{k}^*x) = \text{Tr}(x).
\end{align*}
Lemma 4.3. Let \( \tau \) be an involution of \( A = \text{End}_D(V) \) and let \( f \) be a \( \mathbb{F}_0 \)-linear form on \( \text{Sym}(A, \tau) \) such that \( f(x + \tau(x)) = \text{Tr}(x) \) for all \( x \in A \). There exists an element \( u \in A \) such that \( f(s) = \text{Tr}(us) \) and \( u + \tau(u) = 1 \). The element \( u \) is uniquely determined up to additivity by an element of \( \text{Alt}(A, \tau) \). We take \( u = 1/2 \) if \( \text{Char} \mathbb{F} \neq 2 \).

Proof. The proof of (5.8) of [6] can be adapted. We prove 1) for completeness.

Proposition 4.4. Let \( \tau \) be an involution of \( A = \text{End}_D(V) \) and let \( f \) be a \( \mathbb{F}_0 \)-linear form on \( \text{Sym}(A, \tau) \) such that \( f(x + \tau(x)) = \text{Tr}(x) \) for all \( x \in A \).

1) There exists a nonsingular \((\sigma, \varepsilon)\)-quadratic form \( \theta \) on \( V \) such that \( \tau = \sigma \theta \) and \( f = f_\theta \).

2) \((\sigma_\theta, f_\theta) = (\sigma_\psi, f_\psi)\) if and only if \( \theta' = \lambda \theta \) for \( \lambda \in \mathbb{F}_0 \).

3) If \( \tau = \sigma_\theta \) and \( f = f_\theta \) with \( f_\theta(s) = \text{Tr}(us) \), the class of \( u \) in \( A/\text{Alt}(A, \sigma_\theta) \) is uniquely determined by \( \theta \).

Proof. Here the proof of (5.8) of [6] can be adapted. We prove 1) for completeness. Let \( \tau(x) = h^{-1}x^*h, h = \varepsilon^(x) : V \mapsto V^* \). Let \( f(s) = \text{Tr}(us) \) with \( u + \tau(u) = 1 \) and let \( k \in \text{Sesq}(V, D) \) be such that \( \hat{k} = hu : V \mapsto V^* \). We set \( \theta = [k] \). It is then straightforward to check that \( h = k + \varepsilon k^* \).

Proposition 4.5. Let \( \phi : (\text{End}_D(V), \sigma_\theta) \mapsto (\text{End}_D(V'), \sigma_\psi) \) be an isomorphism of algebras with involution. Let \( f_\theta(s) = \text{Tr}(us) \) and \( f_\psi(s') = \text{Tr}(u's') \). The following conditions are equivalent:

1) \( \phi \) is an isomorphism of algebras with \( q \)-involutions.

2) \( f_\theta(\phi(s)) = f_\psi(s') \) for all \( s \in \text{Sym}(\text{End}_D(V), \sigma_\theta) \).

3) \( [\phi(u)] = [u'] \in \text{End}_D(V')/\text{Alt}(\text{End}_D(V'), \sigma_\psi) \).

Proof. The implication 1) \( \Rightarrow \) 2) is clear. We check that 2) \( \Rightarrow \) 3). Let \( \phi \) be induced by a similitude \( t : (V, b_0) \mapsto (V', b_\psi) \). Since \( f_\theta(\phi(s)) = f_\psi(s') \), we have \( \text{Tr}(t^{-1}us) = \text{Tr}(u'ts^{-1}) = \text{Tr}(us) \) for all \( s \in \text{Sym}(\text{End}_D(V), \sigma_\theta) \), hence \([\phi(u)] = [u'] \). The implication 3) \( \Rightarrow \) 1) follows from the fact that \( u \) can be chosen as \( h^{-1}k, h = \hat{k} + \varepsilon k^* \).

Remark 4.6. We call the pair \((\sigma_\theta, f_\theta)\) a \((\sigma, \varepsilon)\)-quadratic pair or simply a quadratic pair. It determines \( \theta \) up to the multiplication by a \( \sigma \)-invariant scalar \( \lambda \in \mathbb{F}^\times \). In fact \( \sigma_\theta \) determines the polar \( b_\theta \) up to \( \lambda \) and \( f_\theta \) determines \( u \). We have \( \theta = [b_\theta u] \).

Example 4.7. Let \( q : V \rightarrow \mathbb{F} \) be a nonsingular quadratic form. The polar \( b_q \) induces an isomorphism \( \psi : V \otimes_D V \rightarrow \text{End}_D(V) \) such that \( \sigma_q(\psi(x \otimes y)) = \psi(y \otimes x) \). Thus \( \psi(x \otimes x) \) is symmetric and \( f_q(\psi(x \otimes x)) = q(x) \) (see [6, (5.11)]).

More generally, if \( V \) is a right vector space over \( D \), we denote by \( V \) the space \( V \) viewed as a left \( D \)-space through the involution \( \sigma \) of \( D \). The adjoint \( b_\theta \) of \((\sigma, \varepsilon)\)-quadratic space \((V; \theta)\) induces an isomorphism \( \psi_\theta : V \otimes_D V \rightarrow \text{End}_D(V) \) and \( \psi_\theta(xd \otimes x) \) is a symmetric element of \((\text{End}_D(V), \sigma_\theta)\) for all \( x \in V \) and all.
\varepsilon\text{-symmetric } d \in D. \text{ One has } f_\theta(\psi(xd \otimes x)) = [dk(x, x)], \text{ where } \theta = [k] \text{ (see [4, Theorem 7]).}

5. Clifford algebras

Let \( \sigma \) be an involution of the first kind on \( D \) and let \( \theta \) be a nonsingular \((\varepsilon, \varepsilon)\)-quadratic form on \( V \). Let \( \sigma_\theta \) be the corresponding \( \theta \)-involution on \( A = \text{End}_D(V) \). We assume in this section that over a splitting \( A \otimes_F \tilde{F} \overset{\sim}{\to} \text{End}_{\tilde{F}}(M) \) of \( A, \theta_\tilde{F} = \theta \otimes 1_{\tilde{F}} \) is a \((1d, 1)\)-quadratic form \( \tilde{q} \) over \( \tilde{F} \), i.e. \( \theta_\tilde{F} \) is a (classical) quadratic form. In the terminology of [6] this means that \( \sigma_\theta \) is orthogonal if \( \text{Char} \neq 2 \) and symplectic if \( \text{Char} = 2 \). From now on we call such forms over \( D \) quadratic forms over \( D \), resp. quadratic spaces over \( D \) if the forms are non-singular.

Classical invariants of quadratic spaces \((V, \theta)\) are the dimension \( \text{dim}_D V \) and the discriminant \( \text{disc}(\theta) \) and the Clifford invariant associated with the Clifford algebra. We refer to [6, §7] for the definition of the discriminant. We recall the definition of the Clifford algebra \( \mathrm{Cl}(V, \theta) \); following [10, 4.1]. Given \( (V, \theta) \) as above, let \( \theta = [k], k \in \text{Sesq}_\theta(V, D), b_\theta = k + \varepsilon k^* \) and \( h = b_\theta \in \text{Hom}_D(V, V^*) \). Let \( A = \text{End}_D(V), B = \text{Sesq}_\theta(V, D) \) and \( B' = V \otimes_D \sigma V \). We identify \( A \) with \( V \otimes_D \sigma V \) through the canonical isomorphism \((x \otimes \sigma f)(v) = xf(v)\) and \( B \) with \( V^* \otimes_D \sigma V^* \) through \((f \otimes \sigma y)(x, y) = g(x)f(y)\). The isomorphism \( h \) can be used to define further isomorphisms:
\[
\varphi_\theta : B' = V \otimes_D \sigma V \overset{\sim}{\to} A = \text{End}_D(M), \quad \varphi_\theta : x \otimes y \mapsto x \otimes h(y)
\]
and the isomorphism \( \psi_\theta \) already considered in (4.7):
\[
\psi_\theta : A \overset{\sim}{\to} B, \quad \psi_\theta : x \otimes \sigma f \mapsto h(x) \otimes \sigma f.
\]
We use \( \varphi_\theta \) and \( \psi_\theta \) to define maps \( B' \times B \to A, (b', b) \mapsto b'b \) and \( A \times B' \to B', (a, b') \mapsto ab' :\)
\[
(x \otimes \sigma y)(h(u) \otimes g) = xb(y, u) \otimes \sigma f \text{ and } (x \otimes \sigma f)(u \otimes \sigma v) = xf(u) \otimes \sigma h(v)
\]
Furthermore, let \( \tau_\theta = \varphi_\theta^{-1} \sigma_\theta \varphi_\theta : B' \to B' \) be the transport of the involution \( \sigma_\theta \) on \( A \). We have \( \tau_\theta(x \otimes \sigma y) = \varepsilon y \otimes \sigma x \). Let \( S_1 = \{s_1 \in B' \mid \tau_\theta(s_1) = s_1\} \). We have \( S_1 = (\text{Alt}^\varepsilon(V, D))^+ \) for the pairing \( B' \times B \to F, (b', b) \mapsto \text{Trd}_A(b'b). \)

Let \( \text{Sand} \) be the bilinear map \( B' \otimes B' \times B \to B' \) defined by \( \text{Sand}(b'_1 \otimes b'_2, b) = b'_1bb'_2. \)

The Clifford algebra \( \mathrm{Cl}(V, \theta) \) of the quadratic space \((V, \theta)\) is the quotient of the tensor algebra of the \( F \)-module \( B' \) by the ideal \( I \) generated by the sets
\[
I_1 = \{s_1 - \text{Trd}_A(s_1k)1, s_1 \in S_1\}
I_2 = \{c - \text{Sand}(c, k) \mid \text{Sand}(c, \text{Alt}^\varepsilon(V, D)) = 0\}.
\]

The Clifford algebra \( \mathrm{Cl}(V, \theta) \) has a canonical involution \( \sigma_\theta \) induced by the map \( \tau \). We have \( \mathrm{Cl}(V, \theta) \otimes_F \tilde{F} = \mathrm{Cl}(V \otimes_F \tilde{F}, \theta \otimes 1_{\tilde{F}}) \) for any field extension \( \tilde{F} \) of \( F \) and \( \mathrm{Cl}(V, \theta) \) is the even Clifford algebra \( C_0(V, q) \) of \((V, q)\) if \( D = F \) ([10, Théorème 2]). The reduction is through Morita theory for hermitian spaces (see for example [5, Chapter I, §9] for a description of Morita theory). In [6, §8] the Clifford algebra \( C(A, \sigma_\theta, f_\theta) \) of the triple \((A, \sigma_\theta, f_\theta)\) is defined as the
Proposition 5.1. The isomorphism \( \varphi_\theta : V \otimes_D \mathbb{Q} V \cong \text{End}_D(V) \) induces an isomorphism \( \text{Cl}(V, \theta) \cong C(A, \sigma_\theta, f_\theta) \).

Proof. We only check that \( \varphi_\theta \) maps \( I \) to \( J \). By definition of \( \tau \) and \( S_1 \), \( s = \varphi_\theta(s_1) \) is a symmetric element of \( A \). On the other hand we have by definition of the pairing \( B' \times B \to A \),

\[
\text{Tr}_A(s k) = \text{Tr}_A(\varphi_\theta(s_1) \psi_\theta^{-1}(k)) = \text{Tr}_A(s h^{-1} k) = \text{Tr}_A(s u) = \text{Tr}_A(s u),
\]

hence the claim. \( \square \)

In particular we have \( C(\text{End}_D(V), \sigma_q, f_q) = C_0(V, q) \) for a quadratic space \( (V, q) \) over \( F \). It is convenient to use both definitions of the Clifford algebra of a generalized quadratic space.

Let \( D = [K, \mu] = K \otimes \ell K \) be a quaternion algebra with conjugation \( \sigma \). Let \( V \) be a \( D \)-module and let \( V^0 \) be \( V \) as a right vector space over \( K \) (through restriction of scalars). Let \( T : V^0 \to V^0 \), \( T x = x \ell \). We have \( \text{End}_D(V) \subset \text{End}_K(V^0) \) and

\[
\text{End}_D(V) = \{ f \in \text{End}_K(V^0) \mid f T = T f \}.
\]

Let \( \theta = [k] \) be a \((\sigma, -1)\)-quadratic space and let \( k(x, y) = P(x, y) + \ell R(x, y) \) as in Section 3. It follows from (3.1) that \( R \) defines a quadratic space \([R]\) on \( V^0 \) over \( K \).

Proposition 5.2. We have \( \sigma_{[R]}|_{\text{End}_D(V)} = \sigma_\theta \) and \( f_\theta = f_{[R]}|_{\text{End}_D(V)} \).

Proof. We have an embedding \( D \hookrightarrow M_2(K), a + \ell b \mapsto \begin{pmatrix} a & \mu b \\ b & \pi \end{pmatrix} \) and conjugation given by \( x \mapsto x^* = c^{-1} x^t c \), \( c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). The choice of a basis of \( V \) over \( D \) identifies \( V \) with \( D^n, V^0 \) with \( K^{2n}, \text{End}_D(V) \) with \( M_n(D) \) and \( \text{End}_K(V^0) \) with \( M_{2n}(K) \), where \( n = \dim_D V \). We further identify \( V \) and \( V^* \) through the choice of the dual basis. We embed any element \( x = x_1 + \ell x_2 \in M_{k,t}(D) \), \( x_t \in M_{k,t}(K) \) in \( M_{2k,2t}(K) \) through the map \( \iota : x \mapsto \xi = \begin{pmatrix} x_1 & \mu x_2 \\ x_2 & \pi t \end{pmatrix} \). In particular \( D^n \) is identified with a subspace of the space of \((2n \times 2n)\)-matrices over \( K \). Then \( D \subset M_2(K) \) operates on the right through \((2 \times 2)\)-matrices and \( M_n(D) \subset M_{2n}(K) \) operates on the left through \((2n \times 2n)\)-matrices. With the notations of Example (2.3) we have \( \iota(x^*) = \text{Int}(c^{-1})(x^t) \). Any \( D \)-sesquilinear
form $k$ on $D^\alpha$ can be written as $k(x, y) = x^* ay$, where $a \in M_\alpha(D)$, as in (2.3).

Let $a = a_1 + \ell a_2$, $a_i \in M_\alpha(K)$ and let

$$\alpha = \iota(a) = \begin{pmatrix} a_1 & \mu a_2 \\ a_2 & \overline{\mu a_1} \end{pmatrix}.$$ 

Let $\eta = \iota(y)$, $y = y_1 + \ell y_2$. We have

$$k(x, y) = x^* ay = \xi^* \alpha \eta = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} \mu a_2 \\ \overline{\mu a_1} \end{pmatrix}^* \begin{pmatrix} a_1 & \mu a_2 \\ a_2 & \overline{\mu a_1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$ 

On the other side it follows from $h = P + \ell R$ that $R(x, y) = \xi^* \rho \eta$ with

$$\rho = \begin{pmatrix} a_2 & -\overline{\mu a_1} \\ -\overline{a_1} & -\overline{\mu a_2} \end{pmatrix}.$$ 

Assume that $\theta = [k]$, so that $\sigma_\theta$ corresponds to the involution $\text{Int}(\gamma^{-1}) \circ \gamma$, where $\gamma = \alpha - \alpha^*$. Similarly $\sigma_{[R]}$ corresponds to the involution $\text{Int}(\tilde{\rho}^{-1}) \circ \tilde{t}$ where $\rho = c\overline{a}$ with $c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so that $\rho^t = \alpha^t c^t = -\alpha^t c = -c\alpha^*$ and $\rho + \rho^t = c(\alpha - \alpha^*)$ or $c\gamma = \tilde{\rho}$. Now $^* = \text{Int}(c^{-1}) \circ t$ implies $\sigma_{[R]}|_{M_\alpha(D)} = \sigma_\theta$. We finally check that $f_\theta = f_{[R]}|_{\text{Sym}(M_\alpha(D), \sigma_\theta)}$. We have $f_\theta(s) = \text{Trd}_{M_\alpha(D)}(\gamma^{-1} \alpha s)$ and $f_{[R]}(s) = \text{Trd}_{M_{2n}(K)}(\tilde{\rho}^{-1} \rho s)$, hence the claim, since $\rho = c\alpha$ and $\tilde{\rho} = c\gamma$ implies $\gamma^{-1} \alpha = \tilde{\rho}^{-1} \rho$. \hfill $\Box$

**Corollary 5.3.** The embedding $\text{End}_D(V) \hookrightarrow \text{End}_K(V^0)$ induces

1) an isomorphism $(\text{End}_D(V), \sigma_\theta, f_\theta) \otimes K \sim (\text{End}_K(V^0), \sigma_{[R]}, f_{[R]})$,

2) an isomorphism $C(\text{End}_D(V), \sigma_\theta, f_\theta) \otimes K \sim C_0(V^0, [R]).$

In view of (2) the semilinear automorphism $T : V^0 \sim V^0$, $Tx = xf$, is a semilinear similitude with multiplier $-\mu$ of the quadratic form $[R]$, such that $T^2 = \mu$.

**Lemma 5.4.** The map $T$ induces a semilinear automorphism $C_0(T)$ of $C_0(V^0, R)$ such that

$$C_0(T)(xy) = (-\mu)^{-1}T(x)T(y) \text{ for } x, y \in V^0$$

and $C_0(T)^2 = \text{Id}$.

**Proof.** This follows (for example) as in [6, (13.1)] \hfill $\Box$

**Proposition 5.5.**

$$C(\text{End}_D(V), \sigma_\theta, f_\theta) = \{ c \in C_0(V^0, R) \mid C_0(T)(c) = c \}.$$

**Proof.** The claim follows from the defining relations of $C(\text{End}_D(V), \sigma_\theta, f_\theta)$ and the fact that

$$\text{End}_D(V) = \{ f \in \text{End}_K(V^0) \mid T^{-1} f T = f \}.$$

\hfill $\Box$
We call $\text{Cliff}(V, \theta)$ the Clifford algebra of the quadratic quaternion space $(V, \theta)$.

Let $t$ be a semilinear similitude of a quadratic space $(U, q)$ of even dimension over $K$. Assume that $\text{disc}(q)$ is trivial, so that $C_0(U, q)$ decomposes as product of two $K$-algebras $C^+(U, q)$ and $C^-(U, q)$. We say that $t$ is proper if $C_0(t)(C^+(U, q)) \subset C^+(U, q)$ and we say that $t$ is improper if $C_0(t)(C^+(U, q)) \subset C^-(U, q)$. In general we say that $t$ is proper if $t$ is proper over some field extension of $F$ which trivializes $\text{disc}(q)$. For any semilinear similitude $t$, let $d(t) = 1$ is $t$ if proper and $d(t) = -1$ if $t$ is improper.

**Lemma 5.6.** Let $t_i$ be a semilinear similitude of $(U_i, q_i)$, $i = 1, 2$. We have $d(t_1 \perp t_2) = d(t_1)d(t_2)$.

**Proof.** We assume that $\text{disc}(q_i)$, $i = 1, 2$, is trivial. Let $e_i$ be an idempotent generating the center $Z_i$ of $C_0(q_i)$. We have $t_i(e_i) = e_i$ if $t_i$ is proper and $t_i(e_i) = 1 - e_i$ if $t_i$ is improper. The idempotent $e = e_1 + e_2 - 2e_1e_2 \in C_0(q_1 \perp q_2)$ generates the center of $C_0(q_1 \perp q_2)$ (see for example [5, (2.3)], Chap. IV) and the claim follows by case checking.

**Lemma 5.7.** Let $V, \theta, V^0, R$ and $T$ be as above. Let $\dim_K V^0 = 2m$. Then $T$ is proper if $m$ is even and is improper if $m$ is odd.

**Proof.** The quadratic space $(V, \theta)$ is the orthogonal sum of 1-dimensional spaces and we get a corresponding orthogonal decomposition of $(V^0, (R))$ into subspaces $(U_i, q_i)$ of dimension 2. In view of (5.6) it suffices to check the case $m = 1$. Let $\alpha = a = a_1 + \ell a_2 \in D$ and $\rho = \begin{pmatrix} a_2 & \overline{\alpha} \\ -\overline{\alpha} & -a_1 - \mu a_2 \end{pmatrix}$. We choose $\mu = 1$, $a_1 = J$ (as in (2.4)), put $i = 1 - 2j$, so that $\overline{J} = -i$ and choose $a_2 = 0$. Let $x = x_1e_1 + x_2e_2 \in V^0$, so $R(x_1, x_2) = ix_1x_2$ and $C((R))$ is generated by $e_1, e_2$ with the relations $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 + e_2e_1 = i$. The element $e = i^{-1}e_1e_2$ is an idempotent generating the center. Since $T(x_1e_1 + x_2e_2) = \overline{x}_1e_1 + \overline{x}_2e_2$, we have $C_0(T)(e) = -e_2e_1$ and $C_0(T)(e) = 1 - e$. Thus $T$ is not proper.

Of special interest for the next section are quadratic quaternion forms $[k]$ such that the induced quadratic forms $\pi_2([k])$ are Pfister forms. For convenience we call such forms Pfister quadratic quaternion forms. Hyperbolic spaces of dimension $2^n$ are Pfister forms, hence spaces of the form $\beta([b])$, $b$ a hermitian form over $K$, are Pfister, in view of the exactness of the sequence of Lewis [7]. It is in fact easy to give explicit examples of Pfister forms using the following constructions:

**Example 5.8 (Char $F \neq 2$).** Let $q = \langle \lambda_1, \ldots, \lambda_n \rangle$ be a diagonal quadratic form on $F^n$, i.e., $q(x) = \sum \lambda_i x_i^2$. Let $[k]$ on $D^n$ be given by the diagonal form $\ell q$. Then the corresponding quadratic form $[R]$ on $K^{2n}$ is given by the diagonal form $\langle 1, -\mu \rangle \otimes q$. In particular we get the 3-Pfister form $\langle a, b, \mu \rangle$ choosing for $q$ the norm form of a quaternion algebra $(a, b)_F$. 
Example 5.9 (Char $F = 2$). Let $b = \langle \lambda_1, \ldots, \lambda_n \rangle$ be a bilinear diagonal form on $F^n$, i.e., $b(x, y) = \sum \lambda_i x_i y_i$. Let $k = (j + \ell)b$ on $D^n$. Then the corresponding quadratic form $[R]$ over $K = R(f)$, $j^2 = j + \lambda$, is given by the form $[R] = b \otimes [1, \lambda]$ where $[\xi, \eta] = \xi x_1^2 + x_1 x_2 + \eta x_2^2$. In particular, for $b = \langle 1, a, c, ac \rangle$, we get the 3-Pfister form $\langle a, c, \lambda \rangle$ with the notations of [6], p. xxi.

6. Triality for semilinear similitudes

Let $\mathcal{C}$ be a Cayley algebra over $F$ with conjugation $\pi : x \mapsto \overline{x}$ and norm $n : x \mapsto x\overline{x}$. The new multiplication $x \ast y = \overline{x}y$ satisfies

$$x \ast (y \ast x) = (x \ast y) \ast x = n(x)y$$

for $x, y \in \mathcal{C}$. Further, the polar form $b_n$ is associative with respect to $\ast$, in the sense that

$$b_n(x \ast y, z) = b_n(x, y \ast z).$$

Proposition 6.1. For $x, y \in \mathcal{C}$, let $r_x(y) = y \ast x$ and $\ell_x(y) = x \ast y$. The map

$$\mathcal{C} \to \text{End}_F(\mathcal{C} \oplus \mathcal{C})$$

given by

$$x \mapsto \begin{pmatrix} 0 & \ell_x \\ r_x & 0 \end{pmatrix}$$

induces isomorphisms $\alpha : (C(\mathcal{C}, n), \tau) \xrightarrow{\sim} (\text{End}_F(\mathcal{C} \oplus \mathcal{C}), \sigma_{n \perp n})$ and

$$\alpha_0 : (C_0(\mathcal{C}, n), \tau_0) \xrightarrow{\sim} (\text{End}_F(\mathcal{C}), \sigma_n) \times (\text{End}_F(\mathcal{C}), \sigma_n),$$

of algebras with involution.

Proof. We have $r_x(\ell_x(y)) = \ell_x(r_x(y)) = n(x) \cdot y$ by (6). Thus the existence of the map $\alpha$ follows from the universal property of the Clifford algebra. The fact that $\alpha$ is compatible with involutions is equivalent to

$$b_n(x \ast (z \ast y), u) = b_n(z, y \ast (u \ast x))$$

for all $x, y, z, u \in \mathcal{C}$. This formula follows from the associativity of $b_n$. Since $C(\mathcal{C}, n)$ is central simple, the map $\alpha$ is an isomorphism by a dimension count. $\square$

Assume from now on that $\mathcal{C}$ is defined over a field $K$ which is quadratic Galois over $F$. Any proper semilinear similitude $t$ of $n$ induces a semilinear automorphism $C(t)$ of the even Clifford algebra $(C_0(\mathcal{C}, n), \tau_0)$, which does not permute the two components of the center of $C_0(\mathcal{C}, n)$. Thus $\alpha_0 \circ C_0(t) \circ \alpha_0^{-1}$ is a pair of semilinear automorphisms of $(\text{End}_K(\mathcal{C}), \sigma_n)$. It follows as in (4.5) that, for any quadratic space $(V, q)$, semilinear automorphisms of $(\text{End}_K(V), \sigma_q, f_q)$ are of the form $\text{Int}(f)$, where $f$ is a semilinear similitude of $q$. The following result is due to Wonenburger [12] in characteristic different from 2:
Proposition 6.2. For any proper semilinear similitude $t_1$ of $\mathfrak{n}$ with multiplier $\mu_1$, there exist proper semilinear similitudes $t_2$, $t_3$ such that

$$\alpha_0 \circ C_0(t_1) \circ \alpha_0^{-1} = (\text{Int}(t_2), \text{Int}(t_3))$$

and

$$\begin{align*}
\mu_3^{-1} t_3(x \star y) &= t_1(x) \star t_2(y), \\
\mu_1^{-1} t_1(x \star y) &= t_2(x) \star t_3(y), \\
\mu_2^{-1} t_2(x \star y) &= t_3(x) \star t_1(y).
\end{align*}$$

(8)

Let $t_1$ be an improper similitude with multiplier $\mu_1$. There exist improper similudates $t_2$, $t_3$ such that

$$\begin{align*}
\mu_3^{-1} t_3(x \star y) &= t_1(y) \star t_2(x), \\
\mu_1^{-1} t_1(x \star y) &= t_2(y) \star t_3(x), \\
\mu_2^{-1} t_2(x \star y) &= t_3(y) \star t_1(x).
\end{align*}$$

The pair $(t_2, t_3)$ is determined by $t_1$ up to a factor $(\lambda, \lambda^{-1})$, $\lambda \in K^\times$, and we have $\mu_1 \mu_2 \mu_3 = 1$.

Furthermore, any of the formulas in (8) implies the two others.

Proof. The proof given in [6, (35.4)] for similitudes can also be used for semilinear similitudes. $\square$

Remark 6.3. The class of two of the $t_i$, $i = 1, 2, 3$, modulo $K^\times$ is uniquely determined by the class of the third $t_i$.

Corollary 6.4. Let $T_1$ be a proper semilinear similitude of $(\mathfrak{C}, \mathfrak{n})$ such that $T_1^2 = \mu_1$, $\mu_1 \in K^\times$ and with multiplier $-\mu_1$. There exist elements $a_i \in K^\times$, $i = 1, 2, 3$, and proper semilinear similitudes $T_i$ of $(\mathfrak{C}, \mathfrak{n})$, with $T_i^2 = \mu_i$, $\mu_i \in K^\times$ and with multiplier $-\mu_i$, $i = 2, 3$, such that $a_i \omega_i \mu_i = \mu_{i+1} \mu_{i+2}$ and

$$\begin{align*}
a_3 T_3(x \star y) &= T_1(x) \star T_2(y), \\
a_1 T_1(x \star y) &= T_2(x) \star T_3(y), \\
a_2 T_2(x \star y) &= T_3(x) \star T_1(y).
\end{align*}$$

The class of any $T_i$ modulo $K^\times$ determines the two other classes and the $\mu_i$’s are determined up to norms from $K^\times$. Furthermore any of the three formulas determines the two others.

Proof. Counting indices modulo 3, we have relations

$$T_i(x) \star T_{i+1}(y) = b_{i+2} T_{i+2}, \quad b_i \in K^\times$$

in view of (6.2). If we replace all $T_j$ by $T_j \circ \rho_{\nu_j}$, $\nu_j \in K^\times$, we get new constants $a_i$. The claim then follows from (3.3). $\square$
7. TRIALITY FOR QUADRATIC QUATERNION FORMS

Let $D_1 = K \oplus \ell_1 K = [K, \mu_1]$ be a quaternion algebra over $F$ and let $(V_1, q_{\theta_1})$ be a quaternion quadratic space of dimension 4 over $D_1$. Let $\theta_1 = [h_1], h_1(x,y) = P_1(x,y) + \ell R_1(x,y)$, so that $[R_1] = \pi_2(\theta_1)$ corresponds to a 8-dimensional (classical) quadratic form on $V_1^0$ over $K$. The map $T_1 : V_1^0 \to V_1^0$, $T_1(x) = x\ell_1$, is a semilinear similitude of $(V_1^0, [R_1])$ with multiplier $-\mu_1$ and such that $T_1^2 = \mu_1$. We recall that by (3.5) it is equivalent to have a quaternion quadratic space $(V_1, q_{\theta_1})$ or a pair $(V_1^0, [T_1])$. We assume from now on that the quadratic form $q_{\theta_1}$ is a 3-Pfister form, i.e., the norm form of a Cayley algebra $C$ over $K$. In view of (6.4) $T_1$ induces two semilinear similitudes $T_2$, resp. $T_3$, with multipliers $\mu_2$, resp. $\mu_3$, which in turn define a quaternion quadratic space $(V_2, \theta_2)$ of dimension 4 over $D_2 = [K, \mu_2]$, resp. a quaternion quadratic space $(V_3, \theta_3)$ of dimension 4 over $D_3 = [K, \mu_3]$. Let $Br(F)$ be the Brauer group of $F$.

PROPOSITION 7.1. 1) $[D_1][D_2][D_3] = 1 \in Br(F)$,
2) The restriction of $\alpha : C_0(\mathcal{C}, n)$ to $\text{End}_K(C) \times \text{End}_K(C)$ to $C(V_i, D_i, \theta_i)$ induces isomorphisms

$$
\alpha_i : (C(V_i, D_i, \theta_i), \tau) \xrightarrow{\sim} (\text{End}_{D_i+1}(V_i+1), \sigma_{\theta_i+1}) \times (\text{End}_{D_i+2}(V_i+2), \sigma_{\theta_i+2})
$$

Proof. The first claim follows from the fact that $\mu_1 \mu_2 = \mu_3 \text{Nrd}_{D_3}(a_3)$ and the second is a consequence of (5.5), (3.5) and the definition of $\alpha$. \hfill \Box

EXAMPLE 7.2. Let $\mathcal{C}_0$ be a Cayley algebra over $F$ and let $\mathcal{C} = \mathcal{C}_0 \otimes_F K$. For any $c \in \mathcal{C}$ such that $c^2 = \mu_1 \in F^\times$, $T_1 : \mathcal{C} \to \mathcal{C}$ given by $T_1(k \otimes x) = k \otimes xc$ is a semilinear similitude with multiplier $-\mu_1$ such that $T_1^2 = \mu_1$. The Moufang identity $(cx)(ye) = c(xy)c$ in $\mathcal{C}$ implies that

$$
(xc) \star (cy) = \overline{c}(x \star y)\overline{c}.
$$

Thus $T_3(k \otimes y) = \overline{k} \otimes cy$ and $T_3(k \otimes z) = i\overline{k} \otimes z\overline{c}$ (where $i \in K^\times$ is such that $7 = -i$) satisfy (6.4). The corresponding triple of quaternion algebras is $([K, \mu_1], [K, \mu_1], [K, \overline{\mu_1}])$, the third algebra being split.

EXAMPLE 7.3. Let $D_i$, $i = 1, 2, 3$, be quaternion algebras over $F$ such that $[D_1][D_2][D_3] = 1 \in Br(F)$. We may assume that the $D_i$ contain a common separable quadratic field $K$ and that $D_i = [K, \mu_i], \mu_i \in F^\times$ such that $\mu_1 \mu_2 \mu_3 \in F^\times 2$. In [6, (43.12)] similitudes $S_i$ with multiplier $\mu_i, i = 1, 2, 3$, of the split Cayley algebra $\mathcal{C}_s$ over $F$ are given, such that 1) $\mu_3^{-1}S_3(x \otimes y) = S_1(x) \star S_2(y)$ and 2) $S_2^2 = \mu_2$. Let $\mathcal{C} = K \otimes \mathcal{C}_s$. Let $u \in K^\times$ be such that $\overline{u} = -u$. The semilinear similitudes $T_i(k \otimes x) = uk \otimes S_i(x), i = 1, 2, 3$, satisfy

$$
a_3T_3(x \star y) = T_1(x) \star T_2(y)
$$

with $a_3 = u\mu_3^{-1}$ (we use the same notation $\star$ in $\mathcal{C}_s$ and in $\mathcal{C}$). Thus there exist a triple of quadratic quaternion forms $(\theta_1, \theta_2, \theta_3)$ corresponding to the three given quaternion algebras. We hope to describe the corresponding quadratic quaternion forms in a subsequent paper.
References