BOUNDS FOR DEGREES AND SUMS OF DEGREES OF IRREDUCIBLE CHARACTERS OF SOME CLASSICAL GROUPS OVER FINITE FIELDS

E. KOWALSKI

The goal of this note (which is incorporated and expanded in Chapter 5 of the author's book "The large sieve and its applications") is to bound from above in a suitable manner the degree of irreducible representations, and the sum of the degrees of irreducible representations, of a group G_ℓ, which in applications is either between $SL(r, \mathbf{F}_\ell)$ and $GL(r, \mathbf{F}_\ell)$, or between $Sp(2g, \mathbf{F}_\ell)$ and $CSp(2g, \mathbf{F}_\ell)$.

More generally, given a finite group G and $p \in [1, +\infty]$, we denote

$$A_p(G) = \Bigl(\sum_{\rho} \dim(\rho)^p\Bigr)^{1/p}, \quad \text{if } p \neq +\infty \text{ and } \quad A_\infty(G) = \max\{\dim(\rho)\}$$

where ρ runs over irreducible linear representations of G (in characteristic zero). For example, we have $A_2(G) = \sqrt{|G|}$ for all G and if G is abelian, then $A_p(G) = |G|^{1/p}$ for all p. We are primarily interested in $A_1(G)$ and $A_\infty(G)$, but other cases may turn out to be useful.

We start with an easy monotonicity lemma.

Lemma 1. *Let G be a finite group and $H \subset G$ a subgroup, $p \in [1, +\infty]$. We have*

$$A_p(H) \leqslant A_p(G).$$

Proof. For any irreducible representation ρ of H, choose (arbitrarily) an irreducible representation $\pi(\rho)$ of G that occurs with positive multiplicity in the induced representation $\operatorname{Ind}_H^G \rho$.

Let π be a representation of G in the image of $\rho \mapsto \pi(\rho)$. For any ρ where $\pi(\rho) = \pi$, we have

$$\langle \rho, \operatorname{Res}_H^G \pi\rangle_H = \langle \operatorname{Ind}_H^G \rho, \pi\rangle_G > 0,$$

by Frobenius reciprocity, (i.e., all ρ with $\pi(\rho) = \pi$ occur in the restriction of π to H. Hence

$$\sum_{\substack{\rho \\ \pi(\rho)=\pi}} \dim(\rho)^p \leqslant \Bigl(\sum_{\substack{\rho \\ \pi(\rho)=\pi}} \dim(\rho)\Bigr)^p \leqslant \dim(\pi)^p,$$

and summing over all possible $\pi(\rho)$ gives the inequality

$$A_p(H)^p \leqslant A_g(G)^p$$

by positivity. This settles the case $p \neq +\infty$, and the other case only requires noticing that $\dim(\rho) \leqslant \dim(\pi(\rho)) \leqslant A_\infty(G)$. □

We come to the main result of this note. The terminology is partly explained by examples after the proof. The argument we will give was suggested by J. Michel.

Proposition 2. (1) *Let $\mathbf{G}/\mathbf{F}_q$ be a split connected reductive linear algebraic group of dimension d and rank r over a finite field, with connected center. Let W be its Weyl group and $G = \mathbf{G}(\mathbf{F}_q)$ the finite group of rational points of $\mathbf{G}$.*

For any subgroup $H \subset G$ and $p \in [1, +\infty]$, we have

$$A_p(H) \leqslant (q + 1)^{(d-r)/2+r/p}\Bigl(1 + \frac{2r|W|}{q-1}\Bigr),$$

with the convention $r/p = 0$ if $p = +\infty$.

(2) *If $\mathbf{G}$ is a product of groups of type A or C, i.e., of linear and symplectic groups, then*

$$A_p(H) \leqslant (q + 1)^{(d-r)/2+r/p}.$$

The proof is based on a simple interpolation argument from the extreme cases $p = 1$, $p = +\infty$. Indeed by Lemma 1 we can clearly assume $H = G$ and by writing the obvious inequality

$$A_p(G)^p = \sum_{\rho} \dim(\rho)^p \leqslant A_\infty(G)^{p-1} A_1(G),$$

we see that it suffices to prove the following:

Proposition 3. *Let $\mathbf{G}/\mathbf{F}_q$ be a split connected reductive linear algebraic group of dimension d with connected center, and let $G = \mathbf{G}(\mathbf{F}_q)$ be the finite group of its rational points. Let r be the rank of $\mathbf{G}$. Then we have*

$$(1)\qquad A_\infty(G) \leqslant \frac{|G|_{p'}}{(q-1)^r} \leqslant (q+1)^{(d-r)/2}, \qquad \textit{and} \qquad A_1(G) \leqslant (q+1)^{(d+r)/2}\Big(1 + \frac{2r|W|}{q-1}\Big),$$

where $n_{p'}$ denotes the prime-to-p part of a rational number n, p being the characteristic of $\mathbf{F}_q$. Moreover, if the principal series of G is not empty[1]*, there is equality*

$$A_\infty(G) = \frac{|G|_{p'}}{(q-1)^r}$$

and $\dim \rho = A_\infty(G)$ if and only if ρ is in the principal series.

Finally if $\mathbf{G}$ is a product of groups of type A or C, then the factor $(1 + 2r|W|/(q-1))$ may be removed in the bound for $A_1(G)$.

Remark 4. Although we were not aware of this when first writing this note (and Chapter 5 of the book already mentioned), the case $p = +\infty$ was proved earlier by G. Seitz [S, Th. 2.1], in fact in greater generality (e.g., also for non-split groups with non-connected centers, or for representations in arbitrary characteristic). His argument is also based on Deligne-Lusztig characters, but is nevertheless slightly different, as it uses the so-called "Jordan decomposition" of characters, due to Lusztig, with a certain amount of case-by-case analysis of unipotent characters.

The bounds in these results are not optimal when q is fixed and (say) we consider $GL(n, \mathbf{F}_q)$ as $n \to +\infty$. The problem in such cases is studied by Larsen, Malle and Tiep in [LMT].

Remark 5. It seems quite possible that the factor $(1 + 2r|W|/(q-1))$ can always be removed, but we haven't been able to figure this out using Deligne-Lusztig characters, and in fact for groups of type A or C, we simply quote *exact formulas* for $A_1(G)$ due to Gow, Klyachko and Vinroot, which are proved in completely different ways. The "right" upper bound for the case of groups of type A may also be recovered using the structure of unipotent representations of such groups.

Note that the extra factor is not likely to be a problem in applications where $q \to +\infty$, but is more questionable for uniformity with respect to the rank. Hence it is useful to note that Vinroot [V2, Th. 6.1] has recently shown that one can remove the factor $(1 + 2r|W|/(q-1))$ for any connected classical group, including non-split ones.

Proof. This is based on properties of the Deligne-Lusztig generalized characters. We will mostly refer to [DM], [Ca] and [L] for all facts which are needed (using notation from [DM], except for writing simply G for what is denoted G^F there). We identify irreducible representations of G (up to isomorphism) with their characters seen as complex-valued functions on G.

First, for a connected reductive group $\mathbf{G}/\mathbf{F}_q$ over a finite field, Deligne and Lusztig have constructed (see e.g. [DM, 11.14]) a family $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ of generalized representations of $G = \mathbf{G}(\mathbf{F}_q)$ (i.e., linear combinations with integer coefficients of "genuine" representations of G), parameterized by pairs $(\mathbf{T}, \theta)$ consisting of a maximal rational torus (i.e., defined over $\mathbf{F}_q$) $\mathbf{T} \subset \mathbf{G}$ and a (one-dimensional) character θ of the finite abelian group $T = \mathbf{T}(\mathbf{F}_q)$. The $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ are not all irreducible, but any irreducible character occurs (with positive or negative multiplicity) in the decomposition of at least one such character. Moreover, $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ only depends (up to isomorphism) on the G-conjugacy class of the pair $(\mathbf{T}, \theta)$.

[1] In particular if q is large enough given $\mathbf{G}$.

We quote here a useful classical fact: for any $\mathbf{T}$ we have

(2) $$(q-1)^r \leqslant |T| \leqslant (q+1)^r$$

(see e.g. [DM, 13.7 (ii)]), and moreover $|T| = (q-1)^r$ if and only if $\mathbf{T}$ is a split torus (i.e., $\mathbf{T} \simeq \mathbf{G}_m^r$ over $\mathbf{F}_q$). Indeed, we have

$$|T| = |\det(q^n - w \mid Y_0)|$$

where $w \in W$ is such that $\mathbf{T}$ is obtained from a split torus $\mathbf{T}_0$ by "twisting with w" (see e.g. [Ca, Prop. 3.3.5]), and $Y_0 \simeq \mathbf{Z}^r$ is the group of cocharacters of $\mathbf{T}$. If $\lambda_1, \ldots, \lambda_i$ are the eigenvalues of w acting on Y_0, which are roots of unity, then we have

$$|T| = \prod_{i=1}^{r} (q - \lambda_i),$$

and so $|T| = (q-1)^r$ if and only if each λ_i is equal to 1, if and only if w acts trivially on Y_0, if and only if $w = 1$ and $\mathbf{T}$ is split.

As in [DM, 12.12], we denote by $\rho \mapsto p(\rho)$ the orthogonal projection of the space $\mathcal{C}(G)$ of real-valued conjugacy-invariant functions on G to the subspace generated by Deligne-Lusztig characters, where $\mathcal{C}(G)$ is given the standard scalar product

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x)g(x),$$

and for a representation ρ, we of course denote $p(\rho) = p(\operatorname{Tr}\rho)$ the projection of its character.

For any representation ρ, we have $\dim(\rho) = \dim(p(\rho))$, where $\dim(f)$, for an arbitrary function $f \in \mathcal{C}(G)$ is obtained by linearity from the degree of characters. Indeed, for any f standard character theory shows that

$$\dim(f) = \langle f, \mathrm{reg}_G \rangle$$

where reg_G is the regular representation of G. From [DM, 12.14], the regular representation is in the subspace spanned by the Deligne-Lusztig characters, so by definition of an orthogonal projector we have

$$\dim(\rho) = \langle \rho, \mathrm{reg}_G \rangle = \langle p(\rho), \mathrm{reg}_G \rangle = \dim(p(\rho)).$$

Now because the characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ for distinct conjugacy classes of $(\mathbf{T}, \theta)$ are orthogonal (see e.g. [DM, 11.15]), we can write

$$p(\rho) = \sum_{(\mathbf{T}, \theta)} \beta(\mathbf{T}, \theta) R_{\mathbf{T}}^{\mathbf{G}}(\theta)$$

where

$$\beta(\mathbf{T}, \theta) = \frac{\langle \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta) \rangle}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta) \rangle},$$

and so

$$\dim(p(\rho)) = \sum_{(\mathbf{T}, \theta)} \beta(\mathbf{T}, \theta) \dim(R_{\mathbf{T}}^{\mathbf{G}}(\theta)).$$

By [DM, 12.9] we have

(3) $$\dim(R_{\mathbf{T}}^{\mathbf{G}}(\theta)) = \varepsilon_{\mathbf{G}} \varepsilon_{\mathbf{T}} |G|_{p'} |T|^{-1},$$

where $\varepsilon_{\mathbf{G}} = (-1)^r$ and $\varepsilon_{\mathbf{T}} = (-1)^{r(\mathbf{T})}$, $r(\mathbf{T})$ being the $\mathbf{F}_q$-rank of $\mathbf{T}$ (see [DM, p. 66] for the definition). This yields the formula

(4) $$\dim(p(\rho)) = |G|_{p'} \sum_{(\mathbf{T}, \theta)} \frac{1}{|T|} \frac{\langle \rho, \varepsilon_{\mathbf{G}} \varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\theta) \rangle}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta) \rangle}.$$

Now we use the fact that pairs $(\mathbf{T}, \theta)$ are partitioned in *geometric conjugacy classes*, defined as follows: two pairs $(\mathbf{T}, \theta)$ and $(\mathbf{T}', \theta')$ are geometrically conjugate if and only if the generalized characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ and $R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ have a common irreducible component (see e.g. [DM, 13.2]). In particular, for a given ρ, if $\langle \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle$ is non-zero for some $(\mathbf{T}, \theta)$, then by definition only pairs $(\mathbf{T}', \theta')$ geometrically conjugate to $(\mathbf{T}, \theta)$ may satisfy $\langle \rho, R_{\mathbf{T}'}^{\mathbf{G}}(\theta)\rangle \neq 0$. So we have

$$\dim(p(\rho)) = |G|_{p'} \sum_{(\mathbf{T},\theta)\in\kappa} \frac{1}{|T|} \frac{\langle \rho, \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle},$$

for some geometric conjugacy class κ, depending on ρ. By Cauchy-Schwarz, we have

$$\dim(p(\rho)) \leqslant |G|_{p'} \Big(\sum_{(\mathbf{T},\theta)\in\kappa} \frac{1}{|T|^2} \frac{1}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle} \Big)^{1/2} \Big(\sum_{(\mathbf{T},\theta)\in\kappa} \frac{|\langle \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle|^2}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle} \Big)^{1/2}. \tag{5}$$

The second term on the right is simply $\langle p(\rho), p(\rho)\rangle \leqslant \langle \rho, \rho\rangle = 1$. As for the first term we have

$$\sum_{(\mathbf{T},\theta)\in\kappa} \frac{1}{|T|^2} \frac{1}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle} \leqslant \frac{1}{(q-1)^{2r}} \sum_{(\mathbf{T},\theta)\in\kappa} \frac{1}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}$$

by (2). Now it is known that for each class κ, the assumption that $\mathbf{G}$ has connected center implies that the generalized characters

$$\chi(\kappa) = \sum_{(\mathbf{T},\theta)\in\kappa} \frac{\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} R_{\mathbf{T}}^{\mathbf{G}}(\theta)}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}$$

is in fact an irreducible character of G (such characters are called *regular* characters). This implies that

$$\sum_{(\mathbf{T},\theta)\in\kappa} \frac{1}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle} = \langle \chi(\kappa), \chi(\kappa)\rangle = 1,$$

and so we have

$$\dim p(\rho) \leqslant \frac{|G|_{p'}}{(q-1)^r}. \tag{6}$$

Now observe that we will have equality in this argument if ρ is itself of the form $\pm R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, and if $|T| = (q-1)^r$. Those conditions hold for representations of the principal series, i.e., characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ for an $\mathbf{F}_q$-split torus $\mathbf{T}$ and a character θ "in general position" (see e.g. [Ca, Cor. 7.3.5]). Such characters are also, more elementarily, induced characters $\mathrm{Ind}_B^G(\theta)$, where $B = \mathbf{B}(\mathbf{F}_q)$ is a Borel subgroup containing T, for some Borel subgroup $\mathbf{B}$ defined over $\mathbf{F}_q$ containing $\mathbf{T}$ (which exist for a split torus $\mathbf{T}$); there θ is extended to B by setting $\theta(u) = 1$ for unipotent elements $u \in B$. For this, see e.g. [L, Prop.2.6].

Conversely, let ρ be such that

$$\dim \rho = \frac{|G|_{p'}}{(q-1)^r}$$

and let κ be the associated geometric conjugacy class. From the above, for any $(\mathbf{T}, \theta)$ in κ, we have $|T| = (q-1)^r$, i.e., $\mathbf{T}$ is $\mathbf{F}_q$-split. Now it follows from Lemma 6 (probably well-known) that this implies that the geometric conjugacy class κ contains a single pair $(\mathbf{T}, \theta)$, and then $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is an irreducible representation (e.g. from the definition of $\chi(\kappa)$), so must be equal to ρ.

We now come to $A_1(G)$. To deal with the fact that in (4), $|T|$ depends on $(\mathbf{T}, \theta) \in \kappa$, we write

$$\dim(p(\rho)) = \frac{|G|_{p'}}{(q-1)^r} \sum_{\kappa} \langle \rho, \chi(\kappa)\rangle + |G|_{p'} \sum_{(\mathbf{T},\theta)} \Big(\frac{1}{|T|} - \frac{1}{(q-1)^r}\Big) \frac{\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}\langle \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}$$

(since by (2), the dependency is rather weak).

Consider the first term's contribution. Since $\chi(\kappa)$ is an irreducible character, the sum

$$\sum_{\rho} \sum_{\kappa} \langle \rho, \chi(\kappa)\rangle$$

is simply the number of geometric conjugacy classes. This is given by $q^{r'}|Z|$ by [DM, 14.42] or [Ca, Th. 4.4.6 (ii)], where r' is the semisimple rank of $\mathbf{G}$ and $Z = Z(\mathbf{G})(\mathbf{F}_q)$ is the group of rational points of the center of $\mathbf{G}$. For this quantity, note that the center of $\mathbf{G}$ being connected implies that $Z(\mathbf{G})$ is the radical of $\mathbf{G}$ (see e.g. [Sp, Pr. 7.3.1]) so $Z(\mathbf{G})$ is a torus and $r = r' + \dim Z(\mathbf{G})$. So using again the bounds (2) for the cardinality of the group of rational points of a torus, we obtain

$$|Z|q^{r'} \leqslant (q+1)^r. \tag{7}$$

To estimate the sum of the contributions in the second term, say $\sum t(\rho)$, we write

$$\sum_\rho t(\rho) = |G|_{p'} \sum_{(\mathbf{T},\theta)} \Big(\frac{1}{|T|} - \frac{1}{(q-1)^r}\Big) \frac{\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}\langle \sum_\rho \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle}{\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle},$$

and we bound

$$\Big|\langle \sum_\rho \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle\Big| \leqslant \langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle$$

for any $(\mathbf{T},\theta)$, since we can write

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta) = \sum_\rho a(\rho)\rho \quad \text{with } a(\rho) \in \mathbf{Z},$$

and therefore

$$\Big|\langle \sum_\rho \rho, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle\Big| = \Big|\sum_\rho a(\rho)\Big| \leqslant \sum_\rho |a(\rho)|^2 = \langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle.$$

Thus

$$\sum_\rho t(\rho) \leqslant \frac{|G|_{p'}}{(q-1)^r}\frac{2r}{q-1}|\{(\mathbf{T},\theta)\}|.$$

There are at most $|W|$ different choices of $\mathbf{T}$ up to G-conjugacy, and for each there are at most $|T| \leqslant (q+1)^r$ different characters, and so we have

$$\sum_\rho t(\rho) \leqslant \frac{|G|_{p'}}{(q-1)^r}\frac{2r|W|}{q-1}(q+1)^r$$

and

$$\sum_\rho \dim\rho \leqslant (q+1)^r \frac{|G|_{p'}}{(q-1)^r}\Big(1 + \frac{2r|W|}{q-1}\Big). \tag{8}$$

To conclude, we use the classical formula

$$|G| = q^N \prod_{1\leqslant i\leqslant r} (q^{d_i} - 1),$$

where N is the number of positive roots of $\mathbf{G}$, and the d_i are the degrees of reflections of the Weyl group (this is because G is split; see e.g. [Ca, 2.4.1 (iv); 2.9, p. 75]). So

$$|G|_{p'} = \prod_{1\leqslant i\leqslant r} (q^{d_i} - 1,$$

and

$$\frac{|G|_{p'}}{(q-1)^r} = \prod_{1\leqslant i\leqslant r} \frac{q^{d_i}-1}{q-1} \leqslant \prod_{1\leqslant i\leqslant r} (q+1)^{d_i-1} = (q+1)^{\sum (d_i-1)} = (q+1)^{(d-r)/2}, \tag{9}$$

since $\sum (d_i - 1) = N$ and $N = (d-r)/2$ (see e.g.[Ca, 2.4.1], [Sp, 8.1.3]).

Inserting this in (6) we derive the first inequality in (1), and with (8), we get

$$A_1(G) \leqslant (q+1)^{(d+r)/2}\Big(1 + \frac{2r|W|}{q-1}\Big),$$

which is the second part of (1).

Now we explain why the extra factor involving the Weyl group can be removed for products of groups of type A and C. Clearly it suffices to work with $\mathbf{G} = GL(n)$ and $\mathbf{G} = CSp(2g)$.

For $\mathbf{G} = GL(n)$, with $d = n^2$ and $r = n$, Gow [Go] and Klyachko [K] have proved independently that $A_1(G)$ is equal to the number of symmetric matrices in G. The bound

$$A_1(G) \leqslant (q+1)^{(n^2+n)/2}$$

follows immediately.

For $\mathbf{G} = CSp(2g)$, with $d = 2g^2 + g + 1$ and $r = g + 1$, the exact analog of Gow's theorem is due to Vinroot [V1]. Again, Vinroot's result implies $A_1(G) \leqslant (q+1)^{(d+r)/2}$ in this case (see [V1, Cor 6.1], and use the formulas for the order of unitary and linear groups to check the final bound). □

Here is the lemma used in the determination of $A_\infty(G)$ when there is a character in general position of a split torus:

Lemma 6. *Let $\mathbf{G}/\mathbf{F}_q$ be a split connected reductive linear algebraic group of dimension d and let $G = \mathbf{G}(\mathbf{F}_q)$ be the finite group of its rational points. Let $\mathbf{T}$ be a split torus in $\mathbf{G}$, θ a character of T. If $\mathbf{T}'$ is also a split torus for any pair $(\mathbf{T}', \theta')$ geometrically conjugate to $(\mathbf{T}, \theta)$, then the geometric conjugacy class of $(\mathbf{T}, \theta)$ is the singleton $\{(\mathbf{T}, \theta)\}$.*

Proof. Consider $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$. If it is irreducible, then clearly we are done. Otherwise, by the scalar product formula for Deligne-Lusztig characters, there exists $w \in W$, $w \neq 1$, such that ${}^w\theta = \theta$ (see e.g. [DM, Cor. 11.15]). Let $\mathbf{T}'$ be a torus obtained from $\mathbf{T}$ by "twisting by w", i.e., $\mathbf{T}' = g\mathbf{T}g^{-1}$ where $g \in \mathbf{G}$ is such that $g^{-1}\operatorname{Fr}(g) = w$ (see e.g. [Ca, 3.3]). Let $Y = \operatorname{Hom}(\mathbf{G}_m, \mathbf{T}) \simeq \mathbf{Z}^r$ (resp Y') be the abelian group of cocharacters of $\mathbf{T}$ (resp.$\mathbf{T}$;); the conjugation isomorphism $\mathbf{T} \to \mathbf{T}'$ gives rise to a conjugation isomorphism $Y \to Y'$ (loc. cit.). Moreover, there is an action of the Frobenius Fr on Y and a canonical isomorphism $T \simeq Y/(\operatorname{Fr} - 1)Y$ (see e.g. [DM, Prop. 13.7]), hence canonical isomorphisms of the character groups $\hat{T}$ and $\hat{T}'$ as subgroups of the characters groups of Y and Y':

$$\hat{T} \simeq \{\chi : Y \to \mathbf{C}^\times \mid (\operatorname{Fr} - 1)Y \subset \ker \chi\}, \quad \hat{T}' \simeq \{\chi : Y' \to \mathbf{C}^\times \mid (\operatorname{Fr} - 1)Y' \subset \ker \chi\}.$$

Unravelling the definitions, a simple calculation shows that the condition ${}^w\theta = \theta$ is precisely what is needed to prove that the character χ of Y associated to θ, when "transported" to a character χ' of Y' by the conjugation isomorphism, still satisfies $\ker \chi' \supset (\operatorname{Fr} - 1)Y'$ (see in particular [Ca, Prop. 3.3.4]), so is associated with a character $\theta' \in \hat{T}'$.

It is then clear (see the characterization of geometric conjugacy in [DM, Prop. 13.8]) that $(\mathbf{T}, \theta)$ is geometrically conjugate to $(\mathbf{T}', \theta')$, and since $w \neq 1$, the torus $\mathbf{T}'$ is not split, this means that the geometric conjugacy class of $(\mathbf{T}, \theta)$ contains two elements at least. □

Example 7. (1) Let ℓ be prime, $r \geqslant 1$ and let $\mathbf{G} = GL(r)/\mathbf{F}_\ell$. Then $G = GL(r, \mathbf{F}_\ell)$, $\mathbf{G}$ is a split connected reductive of rank r and dimension r^2, with connected center of dimension 1. So from Lemma 1 and Proposition 2, we get

$$A_p(H) \leqslant (\ell+1)^{r(r-1)/2 + r/p}$$

for $p \in [1, +\infty]$ for any subgroup H of G, and in particular

$$A_\infty(H) \leqslant (\ell+1)^{r(r-1)/2} \quad \text{and} \quad A_1(H) \leqslant (\ell+1)^{r(r+1)/2}$$

It would be interesting to know if there are other values of p besides $p = 1$, 2 and $+\infty$ (the latter when q is large enough) for which $A_p(GL(n, \mathbf{F}_q))$ can be computed exactly.

(2) Let $\ell \neq 2$ be prime, $g \geqslant 1$ and let $\mathbf{G} = CSp(2g)/\mathbf{F}_\ell$. Then $G = CSp(2g, \mathbf{F}_\ell)$ and G is a split connected reductive group of rank $g + 1$ and dimension $2g^2 + g + 1$, with connected center. So from Lemma 1 and Proposition 2, we get

$$A_p(H) \leqslant (\ell+1)^{g^2 + (g+1)/p}$$

for $p \in [1, +\infty]$ for any subgroup H of G, and in particular

$$A_\infty(H) \leqslant (\ell+1)^{g^2} \quad \text{and} \quad A_1(H) \leqslant (\ell+1)^{g^2+g+1}.$$

Remark 8. Here is a mnemotechnical way to remember the bounds for $A_\infty(G)$ in (1)[2]: among the representatons of G, we have the principal series $R(\theta)$, a family parameterized by the characters of a maximal split torus, of which there are about q^r, and those share a common maximal dimension Δ. Hence

$$q^r\Delta^2 = \sum_\theta \dim(R(\theta))^2 \simeq |G| \sim q^d,$$

so Δ is of order $q^{(d-r)/2}$. In other words: in the formula $\sum \dim(\rho)^2 = |G|$, the principal series contributes a positive proportion.

The bound for $A_1(G)$ is also intuitive : there are roughly q^r conjugacy classes, and for a "positive proportion" of them, the degree of the representation is of the maximal size given by $A_\infty(G)$.

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E-mail address: `kowalski@math.ethz.ch`

[2] Which explains why it seemed to the author to be a reasonable statement to look for...