Geometry and probability of exponential sums

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What are exponential sums?

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where \( \alpha_n \in \mathbb{C} \) satisfies \( |\alpha_n| \leq 1 \).
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$$|S| \leq N.$$

But in applications, this is not usually the case, and we can hope to prove

$$|S| \leq \frac{N}{\theta(N)}$$

where $\theta(N) > 1$ is “large”.
For \( n \geq 1 \), define
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\lambda(n) = (-1)^{\Omega(n)}
\]
where \( \Omega(n) \) is the total number of prime divisors of \( n \). (E.g., \( \Omega(12) = 3 \)).
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\frac{1}{N} \left| \sum_{1 \leq n \leq N} \lambda(n) \right| \rightarrow 0
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\left| \sum_{1 \leq n \leq N} \lambda(n) \right| \leq C \sqrt{N} (\log N)^2,
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but this is equivalent to the Riemann Hypothesis for the Riemann zeta function.
Digression: why is that so?

Using the Euler product

\[ \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \]

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Then by summation by parts, note that

$$\frac{\zeta(2s)}{\zeta(s)} = s \int_{1}^{+\infty} \left( \sum_{n \leq x} \lambda(n) \right) x^{-s-1} \, dx,$$
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for all \( N \geq 2 \), the right-hand side is holomorphic for \( \text{Re}(s) > 1/2 \).
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for all $N \geq 2$, the right-hand side is holomorphic for $\text{Re}(s) > 1/2$. So $\zeta(s) \neq 0$ for $\text{Re}(s) > 1/2$. 
Next examples

Let $p$ be a prime number. For integers $a$, $b$, not divisible by $p$, define

$$K(a, b; p) = \sum_{1 \leq x \leq p-1} e\left( \frac{ax + b\bar{x}}{p} \right), \quad B(a; p) = \sum_{0 \leq x \leq p-1} e\left( \frac{ax + x^3}{p} \right),$$

where $e(z) = e^{2i\pi z}$, and $x\bar{x} \equiv 1 \pmod{p}$ if $p$ does not divide $x$. (E.g., for $p = 11$, $3 = 4$).
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These are called *Kloosterman sums* and *Birch sums* respectively. They are classical examples of exponential sums over finite fields.
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**Question:** How large can $|K(a, b; p)|$ or $|B(a; p)|$ be, in terms of $p$?
Digression: why “geometry” and “probability”? 

Geometry:

- Many properties of sums like $K(a, b; p)$ and $B(a; p)$ turn out to be best studied using methods from algebraic geometry;
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Probability:
- Heuristic reasoning about these sums is often phrased in probabilistic terms;
- And they satisfy probabilistic limit theorems that justify these heuristics.
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Kloosterman sums were first written down by Poincaré around 1911 as coefficients in Fourier expansions of Poincaré series. They are discrete analogues of Bessel functions.

Nous allons maintenant grouper ensemble les termes qui correspondent aux diverses valeurs de $\delta$ non congrues entre elles suivant le module $\gamma$. Si nous appelons $\omega(\gamma)$ la somme de ces termes, le coefficient de $q^j$ dans $\omega(\gamma)$ sera

$$\mu_j J(m, G) \sum E.$$ 

Il faut donc calculer $\sum E$, c'est-à-dire

$$\sum e^{\frac{2i\pi}{\gamma} (j\delta - px)}.$$ 

Les entiers $j$, $p$ et $\gamma$ sont donnés; mais on donne à $\alpha$ toutes les valeurs entières premières avec $\gamma$ et incongrues entre elles par rapport au module $\gamma$, et à $\delta$ les valeurs correspondantes, de telle façon que

$$\alpha \delta \equiv 1 \pmod{\gamma}.$$ 

Je me bornerai à constater que $\sum E$ n'est pas nul en général. Il reste à sommer...
Kloosterman re-defined them in 1925 and used them to establish the solubility of equations

\[ a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = n \]

for fixed positive integers \((a_1, \ldots, a_4)\) and \(x_i \in \mathbb{Z}\) and suitable \(n \geq 1\).

2. 4. The sum \(S(u, v; \lambda, A; q)\).

We shall show afterwards, that the approximation for large values of \(q\) of the sum occurring on the right hand side of the formula of lemma 3*, can be reduced to the calculation for large values of \(q\) of the sum

\[ S(u, v; \lambda, A; q) = \sum' \exp \left( \frac{2\pi iup}{q} + \frac{2\pi ivp'}{q} \right) . \]

But before performing the reduction, we shall first consider this sum \(S\). The object of this section is the proof of lemma 4. The lemmas 4b—4e are special cases of lemma 4, from which the general lemma 4 will be deduced.
"Square-root cancellation" philosophy

The standard heuristic for guessing the size of a sum

\[ S = \sum_{n \leq N} \alpha_n, \quad |\alpha_n| \leq 1, \]

is that if the arguments of the complex numbers \( \alpha_n \) vary "randomly", then the sum should have size about \( \sqrt{N} \).
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\[ P\left( \left| \sum_{n \leq N} \alpha_n \right| \geq t\sqrt{N} \right) \rightarrow e^{-t^2} \]

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$$P\left( \left| \sum_{n \leq N} \alpha_n \right| \geq t \sqrt{N} \right) \longrightarrow e^{-t^2}$$

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The problem is to show that this heuristic applies to deterministic sums, like Kloosterman sums, or to the Möbius function.
First bounds

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H. Weyl introduced a general technique for exponential sums that leads to

$$|B(a; p)| \leq C_\varepsilon p^{7/8+\varepsilon}$$

for any $\varepsilon > 0$. 
The Weil bounds

As an application of the *Riemann Hypothesis for curves over finite fields*, Weil proved in the 1940’s quite general bounds for one-variable exponential sums that show that they behave according to the square-root cancellation philosophy.

- For all primes $p$ and $1 \leq a, b \leq p - 1$, we have $|K(a, b; p)| \leq 2\sqrt{p}$.
- For all primes $p$ and $0 \leq a \leq p - 1$, we have $|B(a; p)| \leq 2\sqrt{p}$. 
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where \((x, y)\) belong to an algebraic closure of the finite field \(F_p = \mathbb{Z}/p\mathbb{Z}\). The geometry of algebraic curves is key to the proof. Later, Stepanov found a proof which is elementary; as interpreted by Bombieri, the key point is the Riemann-Roch theorem.
Equidistribution

So, for some deep geometric reason, the summands $e((ax + b\bar{x})/p)$ behave extremely randomly as $x$ varies over the interval $1 \leq x \leq p - 1$. 

Deligne proved in the 1980's a general equidistribution theorem that gives some hint of the probabilistic nature of these exponential sums. 

Theorem (Deligne; Katz) As $p \to +\infty$, the normalized Kloosterman sums $K(a, b; p)/p^{1/2}$ for $1 \leq a, b \leq p - 1$ become equidistributed with respect to the measure $\mu_{ST} = \frac{1}{\pi} \sqrt{1 - x^2} \, dx$ on $[-2, 2]$. The same holds for Birch sums $B(a; p)/p^{1/2}$. 


Equidistribution

So, for some deep geometric reason, the summands $e((ax + b\bar{x})/p)$ behave extremely randomly as $x$ varies over the interval $1 \leq x \leq p - 1$. But randomly in a subtle way that leads to $K(a, b; p)/\sqrt{p}$ lying always in $[-2, 2]$, instead of being (rarely) unbounded, as the Central Limit Theorem suggests.
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**Theorem (Deligne; Katz)**

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$$\mu_{ST} = \frac{1}{\pi} \sqrt{1 - \frac{x^2}{4}} \, dx$$

on $[-2, 2]$. The same holds for Birch sums $B(a; p)/p^{1/2}$. 
What does this mean?

(1) For any continuous function $f : [-2, 2] \to \mathbb{C}$, we have

$$\lim_{p \to +\infty} \frac{1}{(p - 1)^2} \sum_{1 \leq a, b \leq p - 1} f \left( \frac{K(a, b; p)}{\sqrt{p}} \right) = \int_{-2}^{2} f(x) d\mu_{ST}(x).$$
What does this mean?

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(2) Or equivalently: the sequences of random variables

$$(a, b) \mapsto \frac{K(a, b; p)}{\sqrt{p}}$$

on $\{1 \leq a, b \leq p - 1\}$ with uniform probability measure converges weakly to $\mu_{ST}$. 
The shape of exponential sums

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$$\sum_{1 \leq x \leq j} e\left(\frac{ax + b\overline{x}}{p}\right)$$

for $0 \leq j \leq p - 1$, 

![Graph of exponential sums](image-url)
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and join these points by line segments, to obtain a polygonal curve in the plane.
D.H. Lehmer and J.H. Loxton (1970's–1980's) looked at and studied similar graphs for more regular exponential sums, especially quadratic Gauss sums

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\[ \sum_{0 \leq x \leq j} e\left( \frac{x^2}{p} \right). \]

These behave more regularly, staying close to Cornu spirals

\[ \int_{0}^{j} e^{2i\pi x^2/p} dx \]

up to \( j \) about \( p/2 \).
Loxton mentions in a paper the case of Kloosterman sums:

The other extreme may be exemplified by the incomplete Kloosterman sum

$$K(h) = \sum_{a \leq x < a + h \atop (x, q) = 1} e_q(mx + n\bar{x}),$$

where $\bar{x}$ denotes the solution of $x\bar{x} \equiv 1 \pmod{q}$. The graph of $K(h)$ seems to be absolutely chaotic and it is natural to think of it as a random walk in the plane.
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Is he right, or wrong?
Right...
and wrong...
For \( p \) prime and \( 1 \leq a, b \leq p - 1 \), we define a continuous map

\[
K_{\ell_p}(a, b) : [0, 1] \rightarrow \mathbb{C}
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by linear interpolation between the normalized partial sums

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For each $p$, we view $(K\ell_p(\cdot, \cdot)(t))_{t \in [0, 1]}$ as a stochastic process, defined on the finite probability space

$$\Omega_p = \{1 \leq a, b \leq p - 1\}$$

with uniform probability.
We can also view this as a $C([0, 1])$-valued random variable on this space.
Theorem (K.–Sawin, 2014)

- The sequence \((K\ell_p)\) converges in law, as random variables with values in \(C([0, 1])\), to a limiting process \(V\).
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- This limiting process is the random Fourier series

\[ V(t) = \sum_{h \in \mathbb{Z}} \left( \frac{e^{2i\pi ht} - 1}{2i\pi h} \right) X_h, \]

where \((X_h)\) is a sequence of independent random variables, identically distributed according to \(\mu_{ST}\).

(Note: the term \(h = 0\) should be interpreted as \(tX_0\)).
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Different look...
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Define $C([0,1])$-valued random variables $B_p$ from normalized partial sums of Birch sums on $\{1 \leq a \leq p - 1\}$. 
Limit for Birch sums

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Define $C([0, 1])$-valued random variables $B_p$ from normalized partial sums of Birch sums on $\{1 \leq a \leq p - 1\}$.

The sequence $(B_p)$ converges in law, as random variables with values in $C([0, 1])$, to the same limiting process $V$. 
Limit for Birch sums

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The different appearance between these graphs and those of Kloosterman sums is only at smaller scales than those that are retained in the limit.
Ideas of the proof

There are two parts, following Prokhorov’s Theorem:

▶ Step 1: convergence of finite distributions:

for any \( k \geq 1 \), and \( 0 \leq t_1 < t_2 < \cdots < t_k \leq 1 \), the vectors
\(
( K_{\ell p}(t_1), \ldots, K_{\ell p}(t_k) )
\)
converge in law to
\(
( V(t_1), \ldots, V(t_k) )
\).

▶ Step 2: tightness / weak-compactness in \( C([0,1]) \):

by Kolmogorov's criterion, it is enough to prove that
\[
E\left( |K_{\ell p}(t) - K_{\ell p}(s)|^\alpha \right) \leq C|t - s|^{1+\delta}
\]
for \( 0 \leq s, t \leq 1 \) and \( C \geq 0, \alpha > 0 \) and \( \delta > 0 \) independent of \((p, t, s)\)
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  for $0 \leq s, t \leq 1$ and $C \geq 0$, $\alpha > 0$ and $\delta > 0$ independent of $(p, t, s)$. 
Finite distributions

- One can deal with the actual partial sums (no linear interpolation);
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- Properties of $V(t)$ show that one can use the method of moments;

$$1 \sqrt{p} \sum_{1 \leq x \leq j} e^{(ax + b \bar{x})p} = 1 \sqrt{p} \sum_{-p/2 < h < p/2} \alpha p (h, j) K(a + h_1, b; p) \cdots K(a + h_k, b; p) \frac{p}{2}$$

Deligne's Riemann Hypothesis in very strong form gives asymptotic formulas for $S$:

$$S = E(X_{h_1} \cdots X_{h_k}) + O(p^{-1/2})$$

Then unwind...
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  - Deligne’s Riemann Hypothesis *in very strong form* gives asymptotic formulas for $S$:
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  - Then unwind…
Tightness

The goal is

\[ E(|K\ell_p(t) - K\ell_p(s)|^\alpha) \leq C|t - s|^{1+\delta}. \]

Write \(|t - s| = p^{-\gamma}\) where \(\gamma \geq 0\).
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- If \(\gamma \geq 1\): the linear interpolation gives the result;
- If \(1/2 + \varepsilon_1 \leq \gamma \leq 1\) (where \(\varepsilon_1 > 0\)): use trivial bound by number of terms;
- If \(0 \leq \gamma \leq 1/2 - \varepsilon_1\): use equidistribution as for Step 1;
- If \(\gamma\) is close to \(1/2\): take \(\alpha = 4\), and apply Kloosterman’s method!
First application

Using some relatively basic probability in Banach spaces, we get a limiting distribution $\mu$ for

$$\max_{1 \leq j \leq p-1} \frac{1}{\sqrt{p}} \left| \sum_{1 \leq x \leq j} e\left( \frac{ax + b\bar{x}}{p} \right) \right|$$

and doubly-exponential tail bounds

$$c^{-1} \exp(-\exp(ct)) \leq \mu([t, +\infty[) \leq c \exp(-\exp(c^{-1}t)).$$
Similar results

- Bober, Goldmakher, Granville, Koukoulopoulos, Soundararajan: “classical” character sums (functional limit theorem in progress, with very different limiting random Fourier series, much work on tail bounds);
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Jurkat and van Horne; Marklof, Akarsu, Cellarosi: quadratic Gauss sums with arbitrary real coefficients (functional limit theorem in progress, again different limiting process);
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- Jurkat and van Horne; Marklof, Akarsu, Cellarosi: quadratic Gauss sums with arbitrary real coefficients (functional limit theorem in progress, again different limiting process);
- Others?
Questions

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- What are further properties of $V(t)$ that would have nice consequences for Kloosterman sums?