ON SOME EXPONENTIAL SUMS OF CONREY AND IWANIEC

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Let $\mathbf{F}_q$ be a finite field with q elements, and χ_1, χ_2 non-trivial multiplicative characters of $\mathbf{F}_q^\times$. Define the exponential sum

$$S(\chi_1,\chi_2)=\sum_{u,v}\chi_1(uv(u+1)(v+1))\chi_2(uv-1).$$

These sums occur in the paper [1] of Conrey and Iwaniec on the third moment of central values of twisted automorphic L-functions. A crucial part of their argument requires the proof of a (best-possible) square-root cancellation estimate for $S(\chi_1,\chi_2)$:

Theorem 1 (Conrey–Iwaniec)**.** *We have*

$$S(\chi_1,\chi_2)\ll q,$$

where the implied constant is absolute.

This is proved in [1, §13, 14] (see also [5, Th. 11.42] for an outline of the proof). In this note, we sketch a different proof, based on the general philosophy of reduction to one-variable sums and on the use of the powerful form of Deligne's proof of the Riemann Hypothesis over finite fields involving sums of trace functions of general sheaves [3] (see [4] for other recent systematic applications of this principle).

Fix an auxiliary prime ℓ different from the characteristic of $\mathbf{F}_q$, and a field-isomorphism $\iota:\bar{\mathbf{Q}}_\ell\simeq\mathbf{C}$, which we use as an identification. For an ℓ-adic sheaf $\mathcal{F}$ on some algebraic variety $X/\mathbf{F}_q$, and some $x\in X(\mathbf{F}_q)$, we denote by $t_{\mathcal{F},\mathbf{F}_q}(x)$ the value, under ι, of the trace function of $\mathcal{F}$ at the geometric Frobenius of $\mathbf{F}_q$ acting on the stalk at x.

We have

$$S(\chi_1,\chi_2)=\sum_{u\in\mathbf{F}_q-\{0,-1\}}\chi_1(u(u+1))\overline{T(u)}$$

where

$$T(u)=\sum_{v\in\mathbf{F}_q-\{0,-1,1/u\}}\overline{\chi_1(v(v+1))\chi_2(uv-1)}$$

(we omit the dependency on χ_1 and χ_2 in the notation).

We can express this as an inner-product of trace functions of sheaves. Indeed, we have first

$$\chi_1(u(u+1))=t_{\mathcal{F}_1,\mathbf{F}_q}(u)$$

where $\mathcal{F}_1=\mathcal{L}_{\chi_1(X(X+1)}$ is a Kummer sheaf on the open curve $X=\mathbf{A}^1-\{0,-1\}$. Further, let

$$Y=\{(x,y)\in X\times X\mid xy\neq 1\}\subset\mathbf{A}^2,$$

and let $\pi:Y\to X$, $m:Y\to\mathbf{A}^1$ be the maps defined on Y by

$$\pi:(x,y)\mapsto x,\qquad m:(x,y)\mapsto xy-1.$$

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Define

$$\mathcal{F}_2 = R^1\pi_!(\pi^*\mathcal{L}_{\bar{\chi}_1(X(X+1))} \otimes m^*\mathcal{L}_{\bar{\chi}_2(X)}).$$

By proper base change and the Grothendieck-Lefschetz trace formula, this sheaf has the crucial property that

$$T(u) = -t_{\mathrm{D}(\mathcal{F}_2),\mathbf{F}_q}(u)$$

for $u \in X(\mathbf{F}_q)$, where $\mathrm{D}(\cdot)$ denotes the dual lisse sheaf.

As a rank 1 Kummer sheaf, $\mathcal{F}_1$ is pointwise pure of weight 0 on X, and is geometrically irreducible. As for $\mathcal{F}_2$, we first observe that, by Weil's theory of exponential sums in one variable, $\mathcal{F}_2$ is pointwise pure of weight 1 on X, and furthermore that the stalks are of rank 2 at all points of X (see [2, Sommes Trig., §3]: each $T(u)$ is the sum of the trace function of a lisse rank 1 Kummer sheaf on $X_u = X - \{1/u\} \simeq \mathbf{P}^1 - \{\text{four points}\}$, which has Euler-Poincaré characteristic $\chi(X_u) = -2$, i.e., $T(u)$ behaves like the character sum giving the correction term for the number of points on an elliptic curve). Moreover, using the Leray spectral sequence of π and the fact that

$$H^1_c(Y \times \bar{\mathbf{F}}_q, \pi^*\mathcal{L}_{\bar{\chi}_1(X(X+1))} \otimes m^*\mathcal{L}_{\bar{\chi}_2(X)}) = 0$$

(because Y is affine of dimension 2), we see that

$$H^0_c(X \times \bar{\mathbf{F}}_q, \mathcal{F}_2) = 0$$

(compare [8, Lemma 10.1.6 (2)]). Using this, the equality of generic rank and rank of stalks implies that $\mathcal{F}_2$ is also lisse on X.

Now, we claim that $\mathcal{F}_2$ is also geometrically irreducible on X. If that is the case, then it follows that

$$H^2_c(X \times \bar{\mathbf{F}}_q, \mathcal{F}_1 \otimes \mathrm{D}(\mathcal{F}_2)) = 0$$

by the co-invariant formula (the two geometrically irreducible sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ do not have the same rank, and are therefore certainly not geometrically isomorphic) and by the trace formula and the Riemann Hypothesis, that

$$|S(\chi_1, \chi_2)| \leqslant C(\bar{\mathbf{F}}_q)q \qquad (1)$$

where

$$C(\bar{\mathbf{F}}_q) = \dim H^1_c(X \times \bar{\mathbf{F}}_q, \mathcal{F}_1 \otimes \mathrm{D}(\mathcal{F}_2)).$$

Now, the conductor $\mathrm{c}(\mathcal{F}_1)$ of $\mathcal{F}_1$ (as defined in [4, Def. 1.10], i.e., the sum of the rank, the number of points at infinity of X, and the Swan conductors at the points at infinity) is bounded independently of q, in fact

$$\mathrm{c}(\mathcal{F}_1) = 1 + 3 = 4$$

since Kummer sheaves are everywhere tame. Similarly, $\mathcal{F}_2$ is of rank 2 and lisse on X. Moreover, for $p \geqslant 5$ at least, $\mathcal{F}_2$ is also everywhere tame (because it is part of a K-compatible system $(\mathcal{F}_{2,\ell})_\ell$ for $\ell \neq p$, where K is a fixed number field, and one can argue as in [7, Lemma 7.5.1]). It follows now that $C(\bar{\mathbf{F}}_q)$ is also bounded independently of q (see, e.g., [4, Prop. 7.2 (2)]), and the bound (1) above therefore proves Theorem 1.

Remark. An alternative approach to bounding $C(\bar{\mathbf{F}}_q)$ is to go back to the two-variable sum, and use the fact that the sum of dimensions of the cohomology groups for this character sum is bounded independently of q (from work of Bombieri and Adolphson–Sperber); this argument is used in [1].

To prove the claim concerning $\mathcal{F}_2$, we apply the diophantine criterion for irreducibility [6, 7.0.3] (see also [8, proof of Lemma 10.1.15]). Let $k/\mathbf{F}_q$ be a finite extension, and let $\chi_{1,k}$, $\chi_{2,k}$ denote the characters $\chi_i \circ N_{k/\mathbf{F}_q}$ of k. Since $\mathcal{F}_2$ is of weight 1, it is enough to prove that

$$\lim_{[k:\mathbf{F}_q]\to+\infty} \frac{1}{|k|^2} \sum_{u\in X(k)} |t_{\mathcal{F}_2,k}(u)|^2 = 1. \tag{2}$$

We expand the sum over u and find that

$$\sum_{u\in X(k)} |t_{\mathcal{F}_2,k}(u)|^2 = \sum_{v_1,v_2} \chi_{1,k}\Bigl(\frac{v_2(v_2+1)}{v_1(v_1+1)}\Bigr) \sum_{u\in X(k)} \chi_{2,k}\Bigl(\frac{uv_2-1}{uv_1-1}\Bigr).$$

The contribution of the diagonal terms $v_1 = v_2$ is equal to $|X(k)|(|X(k)|-1) \sim |k|^2$. On the other hand, if $v_1 \neq v_2$, the map

$$u \mapsto \frac{uv_2-1}{uv_1-1}$$

is an automorphism of $\mathbf{P}^1$, and thus, by orthogonality of characters, we get

$$\sum_{u\in X(k)} \chi_{2,k}\Bigl(\frac{uv_2-1}{uv_1-1}\Bigr) = -1 - \chi_{2,k}\Bigl(\frac{v_2+1}{v_1+1}\Bigr)$$

(the two terms being the missing values of this map at 0 and -1). Hence the contribution (say R) of the non-diagonal terms is

$$\begin{aligned} R &= -\sum_{v_1\neq v_2} \chi_{1,k}\Bigl(\frac{v_2(v_2+1)}{v_1(v_1+1)}\Bigr) - \sum_{v_1\neq v_2} \chi_{1,k}\Bigl(\frac{v_2(v_2+1)}{v_1(v_1+1)}\Bigr)\chi_{2,k}\Bigl(\frac{v_2+1}{v_1+1}\Bigr) \\ &= 2|k| - \Bigl|\sum_v \chi_{1,k}(v(v+1))\Bigr|^2 - \Bigl|\sum_v \chi_{1,k}(v(v+1))\chi_{2,k}(v+1)\Bigr|^2 \leqslant 2|k|, \end{aligned}$$

and (2) follows.

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