Arithmetic Fourier transforms over finite fields

Generic vanishing, convolution, and equidistribution

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Abstract. We study the arithmetic Fourier transforms of trace functions on general connected commutative algebraic groups. To do so, we first prove a generic vanishing theorem for twists of perverse sheaves by characters, and using this tool, we construct a tannakian category with convolution as tensor operation. Using Deligne's Riemann hypothesis, we show how this leads to a general equidistribution theorem for the discrete Fourier transforms of trace functions of perverse sheaves, generalizing the work of Katz in the case of the multiplicative group. We then give some concrete examples of applications of these results and raise a number of questions.

Dedicated to Nick Katz, with the greatest admiration

Contents

Preface

The Fourier transform, and the whole collection of its variants whose study is summarized under the heading of "harmonic analysis", is one of the most important tools of mathematics. In its many forms, its applications cover the whole range not only of mathematics, but also physics, computer science, chemistry and indeed of all sciences where quantitative tools are applied.

In 1976, P. Deligne observed in a letter to D. Kazhdan (which is reproduced in Appendix D) that the formalism of algebraic geometry, and especially of ℓ-adic cohomology and the derived category of ℓ-adic sheaves, provided a new "geometric" form of the Fourier transform. Instead of the familiar integral formula

$$\widehat{f}(y) = \int_{\mathbf{R}} f(x) e^{-2i\pi xy} dx$$

associating to a function f (say $f\colon \mathbf{R} \to \mathbf{C}$ in the Schwartz space) its Fourier transform $\widehat{f}$, Deligne's version takes as input an ℓ-adic constructible sheaf M, or a complex of those, on the one-dimensional affine space over a finite field k of characteristic p, and outputs a Fourier transform $\widehat{\mathrm{M}}$ which is of the same kind.

We note that although the most general and convenient category of input objects M, which we will also call "coefficients", is given by the formalism of derived categories of ℓ-adic complexes with ℓ prime different from p, there is a simpler definition in the case considered here, where M can (in almost all cases) be thought of as being a continuous finite-dimensional representation

$$\varrho\colon\ \mathrm{Gal}(k(\mathrm{T})^{\mathrm{sep}}/k(\mathrm{T})) \to \mathbf{GL}_r(\overline{\mathbf{Q}}_\ell)$$

of the absolute Galois group of the field $k(\mathrm{T})$ of rational functions with coefficients in k.

The crucial point for the interpretation of this construction as a Fourier transform is that to each object M is associated classically a sequence of "trace functions", which are functions

$$t_{\mathrm{M}}(\cdot\,; k_n)\colon\, k_n \to \mathbf{C} \simeq \overline{\mathbf{Q}}_\ell$$

defined on the finite extensions k_n of k of degree n, for all integers $n \geqslant 1$, and Deligne's Fourier transform then satisfies

$$t_{\widehat{\mathrm{M}}}(y; k_n) = \sum_{x \in k_n} t_{\mathrm{M}}(x; k_n) e^{2i\pi \operatorname{Tr}_{k_n/\mathbf{F}_p}(xy)/p}.$$

Thus, the trace functions of $\widehat{\mathrm{M}}$ coincide with the discrete Fourier transforms of those of M.

Deligne's Fourier transform shares many features with the classical euclidean Fourier transform, once properly interpreted in terms of the coefficients M. For instance:

- It satisfies a form of the Fourier inversion formula

$$f(x) = \int_{\mathbf{R}} \widehat{f}(y) e^{2i\pi xy} dy,$$

  in the sense that applying the (similarly defined) analogue of the inverse Fourier transform to $\widehat{\mathrm{M}}$ recovers M.

– It satisfies analogues of the Plancherel formula, which are however less obvious: one interpretation is that if the representation ϱ above is irreducible, then so is the representation associated to $\widehat{\mathrm{M}}$.

– It satisfies a geometric analogue of the fundamental algebraic relation $\widehat{f * g} = \widehat{f}\,\widehat{g}$, which relates the Fourier transform and the convolution product

$$(f * g)(x) = \int_{\mathbf{R}} f(y)g(x-y)dy$$

of functions (this property is often taken as the key feature of Fourier analysis and especially Pontryagin duality [14]). Indeed, to two coefficients M_1 and M_2, another geometric construction associates a third one M_3, such that the trace function of M_3 is given by

$$t_{\mathrm{M}_3}(x; k_n) = \sum_{y \in k_n} t_{\mathrm{M}_1}(y; k_n) t_{\mathrm{M}_2}(x - y; k_n),$$

the discrete convolution of those of M_1 and M_2.

– And there is a subtle analogue, due to Laumon, of the stationary phase principle for estimating oscillatory integrals.

There are however also special features related to the geometric nature of trace functions:

– Deligne's Fourier transform preserves a particularly important subcategory of coefficients, that of *perverse sheaves* – this extremely important fact has no obvious classical analogue.

– If a coefficient object M is a perverse sheaf, and hence also its transform $\widehat{\mathrm{M}}$, then one can associate to it a natural intrinsic *symmetry group*, also called its *monodromy group*, which is an algebraic group over $\overline{\mathbf{Q}}_\ell$ (or over $\mathbf{C}$). The definition of this group can be seen as a wide-ranging generalization of that of the Galois group of a polynomial. (In the one-dimensional case, where $\widehat{\mathrm{M}}$ can be identified, in most cases, with a Galois representation $\varrho\colon \mathrm{Gal}(k(\mathrm{T})^{\mathrm{sep}}/k(\mathrm{T})) \to \mathbf{GL}_r(\overline{\mathbf{Q}}_\ell)$ as above, this symmetry group is nothing but the Zariski-closure of the image of ϱ.)

Deligne's Fourier transform has found a number of very important applications in arithmetic and algebraic geometry, as well as number theory. In the former direction, Laumon [98] used it to obtain a product formula for the epsilon factors of Artin-type L-functions on curves over finite fields. In number theory, Katz used it extensively to study in depth the distribution properties of families of exponential sums, which are obtained as discrete Fourier transforms of simple trace functions (see, for instance, [68] and [69]); the symmetry group of the Fourier transform $\widehat{\mathrm{M}}$ plays an essential role here. A prominent example of such sums are the Kloosterman sums

$$\mathrm{Kl}_2(a; p) = \frac{1}{\sqrt{p}} \sum_{x \in \mathbf{F}_p^\times} e\Big(\frac{\bar{x} + ax}{p}\Big)$$

which are the values of the trace function of the Fourier transforms of a one-dimensional Galois representation, and are omnipresent in modern analytic number theory (here and below, we use the notation $e(z) = \exp(2i\pi z)$, and $\bar{x}$ is the inverse of x modulo p). Results about these and similar sums, which often rely on properties of the ℓ-adic Fourier transform, have by now become essential in many fundamental results of analytic number theory – some concrete examples appear in Zhang's famous work on bounded gaps between primes [128, Lemma 12], and systematic use of the Fourier transform begins in various papers of Fouvry, Kowalski and Michel (see, for instance, [41]).

Deligne's transform is the geometric analogue of the classical euclidean Fourier transform on $\mathbf{R}$ and can be generalized to n variables. But, in recent years, a number of applications have led to questions concerning similar properties of other discrete Fourier transforms, for instance those related to the multiplicative group $k_n^\times$, which are functions on the group of multiplicative characters $\chi\colon k_n^\times \to \overline{\mathbf{Q}}_\ell$. The study of the distribution, or average properties, of these sums is *outside* of the realm of applications of Deligne's Fourier transform, and these functions *cannot* be expressed as trace functions of complexes of ℓ-adic sheaves on an algebraic variety over k.

The fundamental motivation for this book is the search for a definition of the analogue of Deligne's Fourier transform on an arbitrary commutative algebraic group over a finite field, and for the general theory and applications of this form of harmonic analysis. In particular, we believe that these *arithmetic Fourier transforms* can be interpreted in the context of much more general arithmetic or geometric avatars of harmonic and functional analysis.

The basic examples of commutative algebraic groups are the multiplicative groups (or tori), and abelian varieties, and these can be combined (together with additive groups) in various ways. The choice of an input object M on such a group G leads to its arithmetic Fourier transforms, which are the functions of the form

$$\widehat{t}_{\mathrm{M}}(\chi;k_n) = \sum_{x\in \mathrm{G}(k_n)} \chi(x) t_{\mathrm{M}}(x;k_n),$$

defined for any $n \geqslant 1$, where the parameter χ ranges over characters of the finite group $\mathrm{G}(k_n)$.

The simplest example beyond the additive case is that of $\mathrm{G}(k_n) = k_n^\times$, in which case the characters are multiplicative characters of k_n, and $\widehat{t}_{\mathrm{M}}$ is called an *arithmetic Mellin transform.* N. Katz, in a striking breakthrough, succeeded a few years ago in finding an interpretation of these functions in his book [74]. He exploited the formalism of tannakian categories, and the fact that the *convolution product* extends to any commutative algebraic group: given coefficients M_1 and M_2 on G, there exists a geometrically-defined object M_3 such that, for all $n \geqslant 1$ and $x \in \mathrm{G}(k_n)$, their respective trace functions satisfy

$$t_{\mathrm{M}_3}(x;k_n) = \sum_{y\in \mathrm{G}(k_n)} t_{\mathrm{M}_1}(y;k_n) t_{\mathrm{M}_2}(xy^{-1};k_n).$$

Although Katz's interpretation of the arithmetic Mellin transforms is *not* fully geometric (there is no analogue of the object $\widehat{\mathrm{M}}$ which "is" Deligne's Fourier transform for the additive group), Katz shows that it is enough to define a *symmetry group* for the arithmetic Mellin transform. In combination with another fundamental tool, Deligne's general form of the Riemann hypothesis over finite fields [28], this allowed Katz to prove an equidistribution theorem which controls the distributions of arithmetic Mellin transforms. A number of significant applications followed, including the work of Keating and Rudnick [80] and Hall, Keating and Roditty-Gershon [56].

One of the main theoretical achievements of this book is the extension of these ideas of Katz to *any* connected commutative algebraic group. This is far from routine, since certain necessary tools, such as generic cohomological vanishing, or estimates for Betti numbers, which are very elementary in the case considered by Katz, were not known previously for groups of dimension at least 2. Indeed, we rely in an essential way on the very recent *quantitative sheaf theory* due to Sawin [115] (which was partly motivated by this work and drafted in final form jointly with the authors).

For any suitable coefficient object on the group G, our construction provides the fundamental invariant of its arithmetic Fourier transform, its *intrinsic symmetry group.* Combined again with other tools such as Deligne's Riemann hypothesis over finite fields, this is already sufficient to prove a very general form of equidistribution theorem, which encompasses the previously known cases of Deligne and Katz (and in fact sharpens these in certain aspects). In turn, we can use this equidistribution theorem for a number of first applications, including strengthening and simplifying the results of [56]. But there remain also many open questions and problems, both on the theoretical side and on that of applications – we will discuss briefly some of these at the end of this book.

After this preface, the book will continue with a more technical introduction, which contains precise statements of some of the key results and a quick description of some of the crucial points which are involved in the proofs. We then split the remainder of the book in two parts, one containing the main theoretical results, and the other devoted to a variety of applications. These are complemented by appendices recalling important material, and Deligne's letter to Kazhdan. A more precise outline of each chapter will be found at the end of the introduction.

Readers with a background in analytic number theory who are not familiar with the theory of trace functions and the underlying geometric objects are invited to first read Appendix E, where we attempt to present them in a concrete and intuitive way.

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Introduction

1. Statement of results

Since Deligne's proof of his equidistribution theorem for traces of Frobenius of ℓ-adic local systems on varieties over finite fields [28], it has been known that any family of exponential sums parameterized by an algebraic variety satisfies some form of equidistribution, and that the concrete expression of this equidistribution statement depends on the determination of the geometric monodromy group of the ℓ-adic sheaf that underlies the family of exponential sums.

The best known result of this kind is probably the computation by Katz [68] of these monodromy groups in the case of Kloosterman sums in several variables over finite fields, which are defined for some fixed non-trivial additive character $\psi\colon \mathbf{F}_q \to \mathbf{C}^\times$ and $a \in \mathbf{F}_q^\times$ as

$$\mathrm{Kl}_m(a;q) = \frac{1}{q^{(m-1)/2}} \sum_{\substack{(x_1,\ldots,x_m)\in(\mathbf{F}_q^\times)^m \\ x_1\cdots x_m=a}} \psi(x_1+\cdots+x_m).$$

This computation led him in particular to the proof of the average version of the Sato–Tate law for classical Kloosterman sums, namely the equidistribution of the sets $\{\mathrm{Kl}_2(a;q) \,|\, a \in \mathbf{F}_q^\times\}$ with respect to the Sato–Tate measure on the interval $[-2,2]$ as $q \to +\infty$ among prime powers. Further deep investigations by Katz, especially in his monograph [69], provide a cornucopia of examples of equidistribution statements.

Among other things, this framework allows for the study of exponential sums of the form

$$\mathrm{S}(\mathrm{M},\psi) = \sum_{x\in\mathbf{F}_{q^n}} t_{\mathrm{M}}(x;\mathbf{F}_{q^n})\psi(x),$$

where t_{M} is the trace function of a perverse sheaf M on the additive group $\mathbf{G}_a$ over $\mathbf{F}_q$ and ψ ranges over characters of $\mathbf{F}_{q^n}$. These sums are the discrete Fourier transform $\psi \mapsto \mathrm{S}(\mathrm{M},\psi)$ of the function $x \mapsto t_{\mathrm{M}}(x,\mathbf{F}_{q^n})$ on the finite group $\mathbf{F}_{q^n} = \mathbf{G}_a(\mathbf{F}_{q^n})$, and the key point is that they are themselves the trace functions of another perverse sheaf on the dual group parameterizing additive characters, namely Deligne's Fourier transform of M.

In a more recent conceptual breakthrough, Katz [74] succeeded in proving equidistribution results for families of exponential sums parameterized by multiplicative characters, despite the fact that the set of multiplicative characters of a finite field $\mathbf{F}_q$ does not naturally arise as the set of $\mathbf{F}_q$-points of an algebraic variety. In analogy with the above, such sums are of the form

$$\mathrm{S}(\mathrm{M},\chi) = \sum_{x\in\mathbf{F}_{q^n}^\times} t_{\mathrm{M}}(x;\mathbf{F}_{q^n})\chi(x),$$

except that M is now a perverse sheaf on the multiplicative group $\mathbf{G}_m$ over $\mathbf{F}_q$ and χ ranges over characters of $\mathbf{F}_{q^n}^\times$. Katz's beautiful insight was to replace points of algebraic varieties by fiber functors of tannakian categories as parameter spaces, and produce the groups governing equidistribution by means of the tannakian formalism (see [43] for an accessible survey). Further work of Katz generalized this to elliptic curves [76] and certain abelian varieties (unpublished).

The primary goal of this book is to extend these ideas to exponential sums (arithmetic Fourier transforms) parameterized by the characters of the points of any connected commutative algebraic group over a finite field.

More precisely, let k be a finite field and $\bar{k}$ an algebraic closure of k. For each $n \geqslant 1$, we denote by k_n the extension of k of degree n inside $\bar{k}$. Let ℓ be a prime number different from the characteristic of k and $\overline{\mathbf{Q}}_\ell$ an algebraic closure of the field of ℓ-adic numbers. Let G be a connected commutative algebraic group over k. We denote by $\widehat{\mathrm{G}}(k_n)$ the group of $\overline{\mathbf{Q}}_\ell$-valued characters of $\mathrm{G}(k_n)$ and, for each $\chi \in \widehat{\mathrm{G}}(k_n)$, by $\mathscr{L}_\chi$ the ℓ-adic lisse character sheaf of rank one associated to χ by means of the Lang torsor construction, as briefly recalled in Section 1.6. By perverse sheaves, we always understand $\overline{\mathbf{Q}}_\ell$-perverse sheaves.

In rough outline, we establish the following types of theoretical results:

– We prove generic and stratified vanishing theorems for the cohomology of twists of perverse sheaves on G by the sheaves $\mathscr{L}_\chi$ associated to characters $\chi \in \widehat{\mathrm{G}}(k_n)$.

– Using the stratified vanishing theorems, we construct a tannakian category of perverse sheaves on G with the convolution coming from the group law as tensor product.

– We prove that the tannakian group of a semisimple object M of this category that is pure of weight zero controls the distribution properties of the sums

$$\mathrm{S}(\mathrm{M}, \chi) = \sum_{x \in \mathrm{G}(k_n)} t_{\mathrm{M}}(x; k_n)\chi(x),$$

where χ ranges over the set $\widehat{\mathrm{G}}(k_n)$. Under some assumptions on G (e.g., for tori and abelian varieties), we prove the stronger result that the unitary conjugacy classes of which these sums are traces become equidistributed in a maximal compact subgroup of the tannakian group as $n \to +\infty$, as is customary since Deligne's work.

Once this is done, we provide a number of applications, both of a general nature and for concrete groups and perverse sheaves.

The following statements are special cases of our main results, which we formulate in simplified form in order to make it possible to present self-contained statements at this stage.

Theorem 1. *Let* M *be a perverse sheaf on a connected commutative algebraic group* G *of dimension* d *over a finite field* k.

(1) (Generic vanishing) *The sets*

$$\mathscr{X}(k_n) = \big\{\chi \in \widehat{\mathrm{G}}(k_n) \mid \mathrm{H}^i_c(\mathrm{G}_{\bar{k}}, \mathrm{M} \otimes \mathscr{L}_\chi) = \mathrm{H}^i(\mathrm{G}_{\bar{k}}, \mathrm{M} \otimes \mathscr{L}_\chi) = 0 \textit{ for all } i \neq 0 \textit{ and } \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M} \otimes \mathscr{L}_\chi) \textit{ is isomorphic to } \mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M} \otimes \mathscr{L}_\chi)\big\}$$

are generic, *in the sense that the estimate*

$$\Big|\widehat{\mathrm{G}}(k_n) - \mathscr{X}(k_n)\Big| \ll |k_n|^{d-1}$$

holds for $n \geqslant 1$, *with an implied constant that only depends on* M.

(2) (Stratified vanishing) *For* $-d \leqslant i \leqslant d$ *and* $n \geqslant 1$, *the estimate*

$$\Big|\big\{\chi \in \widehat{\mathrm{G}}(k_n) \mid \mathrm{H}^i_c(\mathrm{G}_{\bar{k}}, \mathrm{M} \otimes \mathscr{L}_\chi) \neq 0 \textit{ or } \mathrm{H}^i(\mathrm{G}_{\bar{k}}, \mathrm{M} \otimes \mathscr{L}_\chi) \neq 0\big\}\Big| \ll |k_n|^{d-i}$$

holds, with an implied constant that only depends on M.

The most general vanishing statements that we prove appear as Theorems 2.1 and 2.3. Applications to "stratification" estimates for exponential sums are then given in Chapter 6.

REMARK 1. (1) With variations in the definition of generic set of characters, such statements were proved by Katz–Laumon [77] for powers of the additive group, Saibi [112] for unipotent groups, Gabber–Loeser [50] for tori, Krämer-Weissauer [93], Weissauer [127] for abelian varieties and Krämer [90] for semiabelian varieties (see Remark 2.2 for more precise references).

(2) In characteristic zero, and especially over the complex numbers, theorems of this type have also been proved for abelian and semiabelian varieties by Schnell [116], Bhatt–Scholze–Schnell [9] and Liu–Maxim–Wang [102] (see also [101] for a survey of some applications of such results). Over arbitrary algebraically closed fields, there has also been recent works of Esnault and Kerz [36].

(3) In contrast with the case of abelian varieties, lack of properness and wild ramification phenomena are the reason one must formulate conditions on cohomology groups both with and without compact support.

Using the vanishing theorems, and ideas going back to Gabber–Loeser and Katz, we can construct tannakian categories with the convolution on G as tensor operation. Using these, and Deligne's Riemann hypothesis over finite fields, we obtain the following equidistribution theorem for the Fourier transforms of trace functions on G, i.e., for families of exponential sums parameterized by characters of G.

THEOREM 2 (Equidistribution on average for arithmetic Fourier transforms). *Let* G *be a connected commutative algebraic group over* k. *Let* M *be a geometrically simple* ℓ*-adic perverse sheaf on* G *that is pure of weight zero, with complex-valued trace functions* $t_{\mathrm{M}}(\cdot; k_n)\colon \mathrm{G}(k_n) \to \mathbf{C}$ *for* $n \geqslant 1$. *There exists an integer* $r \geqslant 0$ *and a compact subgroup* $\mathrm{K} \subset \mathbf{U}_r(\mathbf{C})$ *of the unitary group such that the sums*

$$\mathrm{S}(\mathrm{M}, \chi) = \sum_{x \in \mathrm{G}(k_n)} t_{\mathrm{M}}(x; k_n)\chi(x)$$

for complex-valued characters χ *of* $\mathrm{G}(k_n)$ *become* equidistributed on average *in* $\mathbf{C}$ *with respect to the image by the trace of the Haar probability measure* μ *on* K. *That is, for any bounded continuous function* $f\colon \mathbf{C} \to \mathbf{C}$, *the following equality holds:*

$$\lim_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}} \sum_{1\leqslant n\leqslant \mathrm{N}} \frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi} f(\mathrm{S}(\mathrm{M}, \chi)) = \int_{\mathrm{K}} f(\mathrm{Tr}(x)) d\mu(x), \tag{1}$$

where χ *runs over all characters of* $\mathrm{G}(k_n)$.

The general version of this theorem appears as Theorem 4.8. Under an additional assumption (which holds for tori, abelian varieties and $\mathbf{G}_a$, at least), we can also deduce it from Theorem 4.11, which is a more precise equidistribution result for unitary conjugacy classes of Frobenius in the compact group K. (The difference between these two statements is similar to that between the Frobenius equidistribution theorem for cycle types of Frobenius classes in the Galois group of a polynomial, viewed as a permutation group, and the more precise Chebotarev density theorem.)

REMARK 2. (1) In the classical setting of $\mathbf{G}_a$ and the Fourier transform, the group K is a maximal compact subgroup of the *arithmetic* monodromy group of the (lisse sheaf underlying the) ℓ-adic Fourier transform of M (see Proposition 3.33).

Note that this is in contrast with more usual versions of Deligne's equidistribution theorem, without the extra Cesàro average over n, where the focus is on the *geometric* monodromy group (see, e.g., the versions of Katz [68, Ch. 3] and Katz–Sarnak [78, Ch. 9]). This slight change of emphasis extends to the general situation, and means that we can avoid additional (necessary) assumptions such as the equality of the geometric and arithmetic monodromy groups, which occur frequently otherwise (see, e.g., [68, § 3.3]), and are not always easy to check.

The Cesàro average can of course be interpreted as a form of "smoothing" (a "summation method" in the classical terminology). Although it is quite natural, it can be replaced by many others (see Remark 4.7).

(3) We will also discuss a "horizontal" version, where we consider suitable families $(\mathrm{M}_p)_p$ of perverse sheaves over $\mathbf{F}_p$ for primes $p \to +\infty$. However, such results depend on a more quantitative version of the stratified vanishing theorem, which we have not established in full generality yet.

(4) As already mentioned, this equidistribution theorem is essentially Deligne's equidistribution theorem on average for the ℓ-adic Fourier transform of M when $\mathrm{G} = \mathbf{G}_a$. When G is the multiplicative group (or its non-split form), one obtains (an average version of) Katz's equidistribution theorem [74]. In [76], Katz proves a similar theorem for elliptic curves.

(5) The assumption that G is connected arises from the fact that the Lang torsor construction is only applicable in this case. For the purpose of equidistribution results, however, one can easily handle a non-connected algebraic group by considering one by one the restrictions to the neutral component of G of the objects $([x \mapsto c^{-1}x]^*\mathrm{M})$, where c runs over representatives of the connected components of G. (Note that different connected components might give rise to exponential sums with different distributions.)

Example 1. A simple concrete class of examples where we obtain equidistribution statements is the following (in the case when G is not an abelian variety): assume that $k = \mathbf{F}_p$, and let d be the dimension of G; then for any non-constant function $f\colon \mathrm{G} \to \mathbf{A}^1$, there exists a perverse sheaf M_f on G with trace functions

$$t_{\mathrm{M}_f}(x; \mathbf{F}_{p^n}) = \frac{(-1)^d}{p^{nd/2}} e\Big(\frac{\mathrm{Tr}_{\mathbf{F}_{p^n}/\mathbf{F}_p}(f(x))}{p}\Big)$$

for all $n \geqslant 1$ and $x \in \mathrm{G}(\mathbf{F}_{p^n})$ (where $e(z) = \exp(2i\pi z)$), so that Theorem 2 shows that the exponential sums

$$\frac{1}{p^{nd/2}} \sum_{x \in \mathrm{G}(\mathbf{F}_{p^n})} \chi(x) e\Big(\frac{\mathrm{Tr}_{\mathbf{F}_{p^n}/\mathbf{F}_p}(f(x))}{p}\Big)$$

(which are intuitively sums over d variables) become equidistributed on average, with limiting measure of a very specific kind. Specializing even more to $\mathrm{G} = \mathbf{G}_m^d$, the function f is a Laurent polynomial in $x_1, \ldots, x_d$ and their inverses, and these exponential sums become the sums

$$\frac{1}{p^{nd/2}} \sum_{x_1,\ldots,x_d \in \mathbf{F}_{p^n}^\times} \chi_1(x_1) \cdots \chi_d(x_d) e\Big(\frac{\mathrm{Tr}_{\mathbf{F}_{p^n}/\mathbf{F}_p}(f(x))}{p}\Big)$$

parameterized by a tuple $(\chi_1, \ldots, \chi_d)$ of characters of $\mathbf{F}_{p^n}^\times$.

As a further concrete application, we will see how to deduce statements like the following, which considerably strengthens earlier work of Hall, Keating and Roddity-Gershon [56].

Theorem 3 (Variance of the von Mangoldt function of the Legendre elliptic curve). *Let k be a finite field of characteristic $\geqslant 3$. Let $\mathscr{E}$ be the Legendre elliptic curve with affine model*

$$y^2 = x(x-1)(x-t)$$

over the field $k(t)$. Let $\Lambda_{\mathscr{E}/k(t)}$ be the von Mangoldt function of $\mathscr{E}$, defined by the generating series

$$-\mathrm{T}\frac{\mathrm{L}'(\mathscr{E}/k(t), \mathrm{T})}{\mathrm{L}(\mathscr{E}/k(t), \mathrm{T})} = \sum_g \Lambda_{\mathscr{E}/k(t)}(g)\mathrm{T}^{\deg(g)}$$

over monic polynomials $g \in k[t]$.

Let $f \in k[t]$ be a square-free polynomial of degree $\geqslant 4$ and set $\mathrm{B} = k[t]/fk[t]$. Let $m \geqslant 1$ be an integer. For any $a \in \mathrm{B}^\times$, consider the sum

$$\psi_{\mathscr{E}}(m; f, a) = \sum_{\substack{\deg(g)=m \\ g \equiv a \,(\mathrm{mod}\, f)}} \Lambda_{\mathscr{E}/k(t)}(g)$$

over monic polynomials $g \in k[t]$ of degree m. Then the following equality holds:

$$\lim_{|k|\to+\infty} \frac{1}{|k|^2} \frac{1}{|\mathrm{B}^\times|} \sum_{a\in\mathrm{B}^\times} \Big|\psi_{\mathscr{E}}(m; f, a) - \frac{1}{|\mathrm{B}^\times|} \sum_{b\in\mathrm{B}^\times} \psi_{\mathscr{E}}(m; f, b)\Big|^2$$
$$= \min\big(m, 2\deg(f) - 2 + \deg\gcd(t(t-1), f)\big).$$

The meaning of the above limit is that we replace k by its extensions k_n of degree $n \geqslant 1$, compute the variance for the base change of $\mathscr{E}$ to k_n (note that B depends on k, so it is also replaced by $k_n[t]/fk_n[t]$), and let $n \to +\infty$. This theorem is proved at the end of Chapter 10.

Remark 3. The version in [56] requires the assumptions $\deg(f) > 900$ and $\gcd(t(t-1), f) = t$. We have greatly relaxed the former condition and fully removed the latter, which was recognized as being quite artificial (see [56, Rem. 11.0.2]). These improvements are due to the consideration of the problem in its natural setting, involving characters of a torus of dimension $\deg(f)$, whereas the authors of [56] used cosets of a one-dimensional torus together with Katz's work on $\mathbf{G}_m$.

We also give a proof of an unpublished theorem of Katz [73] answering a question of Tsimerman about equidistribution of Artin L-functions on curves over finite fields.

Theorem 4 (Katz). *Let C be a smooth projective geometrically connected curve of genus $g \geqslant 2$ over a finite field k and let $\mathrm{D} = \sum n_i x_i$ be a divisor of degree one on C. For each geometrically non-trivial character $\varrho\colon \pi_1(\mathrm{C})^{\mathrm{ab}} \to \mathbf{C}^\times$ of finite order satisfying $\prod \varrho(\mathrm{Fr}_{k(x_i),x_i})^{n_i} = 1$, we write its normalized Artin L-function as*

$$\mathrm{L}(\varrho, \mathrm{T}/\sqrt{|k|}) = \det(1 - \mathrm{T}\Theta_{\mathrm{C}/k,\varrho})$$

for a conjugacy class $\Theta_{\mathrm{C}/k,\varrho}$ in the unitary group $\mathbf{U}_{2g-2}(\mathbf{C})$.

1. *If C is non-hyperelliptic and $(2g-2)\mathrm{D}$ is a canonical divisor on C, then the classes $\Theta_{\mathrm{C}/k,\varrho}$ lie in $\mathbf{SU}_{2g-2}(\mathbf{C})$ and become equidistributed with respect to the image on the space of conjugacy classes of the Haar probability measure of $\mathbf{SU}_{2g-2}(\mathbf{C})$.*
2. *If C is hyperelliptic, the hyperelliptic involution has a fixed point $\mathrm{O} \in \mathrm{C}(k)$ and $\mathrm{D} = \mathrm{O}$, then the classes $\Theta_{\mathrm{C}/k,\varrho}$ lie in $\mathbf{USp}_{2g-2}(\mathbf{C})$ and become equidistributed with respect to the image on the space of conjugacy classes of the Haar probability measure on $\mathbf{USp}_{2g-2}(\mathbf{C})$.*

See Chapter 11 for the proof of this result, as well as some more general statements (including, in Theorem 11.5, a result where the algebraic group G occurring may involve abelian, toric and unipotent parts).

2. Outline

In this section, we present the plan of the book, and we sketch one of the main ideas of the proof of Theorem 2, in order to point out the key difficulties for groups of dimension bigger than one, which are solved using Sawin's quantitative sheaf theory [115].

The book is organized as follows:

– In Chapter 1, we state some preliminary results; these include a survey of the formalism of quantitative sheaf theory [115], as well as basic structural results concerning commutative algebraic groups and character sheaves.

– In Chapter 2, we prove the generic and stratified vanishing theorems for commutative algebraic groups over finite fields. The very rough idea is to prove a *relative version* of the vanishing theorems for the various basic types of commutative groups, with a good control of the implicit constants. These relative statements are of independent interest. For example, in the case of tori, Gabber–Loeser [50] prove the stratified vanishing theorem as stated above only under the assumption that resolution of singularities over k holds for varieties of dimension up to that of the torus. We remove this assumption using alterations. For abelian varieties, we extend Weissauer's work [127] by proving a relative version of the theorem, which relies on Orgogozo's work [109] on constructibility and moderation.

– In Chapter 3, we construct a suitable tannakian category of perverse sheaves on a commutative group over a finite field with convolution as tensor operation, and establish its basic properties, as well as those of the corresponding tannakian monodromy group. We will see that some subtleties arise when defining "Frobenius conjugacy classes" corresponding to characters of G.

– In Chapter 4, we combine these two ingredients to establish a number of "vertical" equidistribution theorems; there are some issues when we want to refine the statements at the level of conjugacy classes (related to those of the previous sections), which we are not currently able to solve in full generality, although we can always establish equidistribution for the characteristic polynomials.

– The beginning of Part 2 introduces a selection of first applications of a general nature. These include the following:

  (1) the definition of the analogue of the L-function for arithmetic Fourier transforms, which is used to study finite tannakian groups over abelian varieties (Chapter 5);

  (2) a stratification result for exponential sums, similar to those of Katz, Laumon and Fouvry, although currently often restricted to the "vertical" direction (Chapter 6);

  (3) a "generic Fourier invertibility" result (Chapter 6);

  (4) some preliminary results of independence of ℓ for the tannakian group when working with perverse sheaves which are part of a compatible system (Chapter 7);

  (5) various results of "Diophantine group theory", where averages of exponential sums are related to invariants of the tannakian group; this includes in particular Larsen's alternative, but also some criteria to recognize the exceptional group $\mathbf{E}_6$ (Chapter 8).

– Chapters 9, 10 and 11 contain applications to concrete cases. The algebraic groups involved are, respectively, the product $\mathbf{G}_a \times \mathbf{G}_m$, higher-dimensional tori, and jacobians of curves, as well as the intermediate jacobian of a smooth cubic threefold (where the relevant tannakian group is $\mathbf{E}_6$, as first shown in the complex setting by Krämer).

– In Chapter 12, we list some open questions and problems. The title "Much remains to be done" paraphrases Katz ([74, p. 18]).

– Finally, we include appendices to survey the basic theory of perverse sheaves (Appendix A), as well as to recall the most important results of Katz concerning the arithmetic Mellin transform on $\mathbf{G}_m$ (Appendix B) and the product formula of Laumon for the epsilon factor of L-functions over finite fields (Appendix C). In Appendix D, we reproduce, with Deligne's permission, the letter to Kazhdan in which the ℓ-adic Fourier transform was first discussed. To conclude, we attempt to sketch the intuitive nature of the theory of general trace functions, to provide some intuition for analytic number theorists in Appendix E.

We now survey the key analytic step in the proof of Theorem 2 (see Proposition 4.12).

By fixing an isomorphism $\iota_0\colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$, we can work with trace functions and characters with values in $\overline{\mathbf{Q}}_\ell$. The first step, following from the generic vanishing theorem, will be to prove that there exist subsets $\mathscr{Y}(k_n) \subset \widehat{\mathrm{G}}(k_n)$ of characters and conjugacy classes $\Theta_{\mathrm{M},k_n}(\chi)$ in some unitary group $\mathbf{U}_r(\mathbf{C})$ such that $\mathrm{Tr}(\Theta_{\mathrm{M},k_n}(\chi)) = \mathrm{S}(\mathrm{M},\chi)$ holds for all $\chi \in \mathscr{Y}(k_n)$ and

$$|\mathscr{Y}(k_n)| \sim |\mathrm{G}(k_n)|$$

as $n \to +\infty$. The second step (an application of the theory of tannakian categories) will be an intrinsic a priori definition of the compact group K for which equidistribution should hold.

By (essentially) the Weyl criterion for equidistribution, Theorem 2 will follow from the proof that, for every non-trivial irreducible representation ϱ of the unitary group $\mathbf{U}_r(\mathbf{C})$, the limit

$$\lim_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}} \sum_{1\leqslant n\leqslant \mathrm{N}} \frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi\in\mathscr{Y}(k_n)} \mathrm{Tr}\,\varrho(\Theta_{\mathrm{M},k_n}(\chi))$$

exists and is equal to the multiplicity of the trivial representation in the restriction of ϱ to K. Now, the tannakian formalism associates to each ϱ a perverse sheaf $\varrho(\mathrm{M})$ on G such that the equality

$$\mathrm{Tr}\,\varrho(\Theta_{\mathrm{M},k_n}(\chi)) = \sum_{x\in\mathrm{G}(k_n)} \chi(x) t_{\varrho(\mathrm{M})}(x;k_n)$$

holds for $n \geqslant 1$ and $\chi \in \mathscr{Y}(k_n)$. The Grothendieck–Lefschetz trace formula yields then the equality

$$\sum_{x\in\mathrm{G}(k_n)} \chi(x) t_{\varrho(\mathrm{M})}(x;k_n) = \sum_{|j|\leqslant d} (-1)^j\, \mathrm{Tr}\left(\mathrm{Fr}_{k_n} \mid \mathrm{H}^j_c(\mathrm{G}_{\bar{k}}, \varrho(\mathrm{M})\otimes\mathscr{L}_\chi)\right) \tag{2}$$

for $n \geqslant 1$ and *any* character χ of $\mathrm{G}(k_n)$, where Fr_{k_n} is the geometric Frobenius automorphism.

The definition of the set $\mathscr{Y}(k_n)$ implies the property that for $\chi \in \mathscr{Y}(k_n)$, the only possibly non-zero term in the right-hand side of (2) is the one with $j=0$. Thus we have

$$\sum_{\chi\in\mathscr{Y}(k_n)} \mathrm{Tr}\,\varrho(\Theta_{\mathrm{M},k_n}(\chi)) = \sum_{\chi\in\mathscr{Y}(k_n)} \mathrm{Tr}\left(\mathrm{Fr}_{k_n} \mid \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \varrho(\mathrm{M})\otimes\mathscr{L}_\chi)\right).$$

If we add to the right-hand side of this last expression the two sums

$$\mathrm{S}_1 = \sum_{\chi\notin\mathscr{Y}(k_n)} \mathrm{Tr}\left(\mathrm{Fr}_{k_n} \mid \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \varrho(\mathrm{M})\otimes\mathscr{L}_\chi)\right),$$

$$\mathrm{S}_2 = \sum_{1\leqslant|j|\leqslant d} \sum_{\chi\notin\mathscr{Y}(k_n)} (-1)^j\, \mathrm{Tr}\left(\mathrm{Fr}_{k_n} \mid \mathrm{H}^j_c(\mathrm{G}_{\bar{k}}, \varrho(\mathrm{M})\otimes\mathscr{L}_\chi)\right),$$

then the resulting quantity is

$$\begin{aligned}\sum_{|j|\leqslant d} \sum_{\chi\in\widehat{\mathrm{G}}(k_n)} (-1)^j\, \mathrm{Tr}\left(\mathrm{Fr}_{k_n} \mid \mathrm{H}^j_c(\mathrm{G}_{\bar{k}}, \varrho(\mathrm{M})\otimes\mathscr{L}_\chi)\right) &= \sum_{\chi\in\widehat{\mathrm{G}}(k_n)} \sum_{x\in\mathrm{G}(k_n)} \chi(x) t_{\varrho(\mathrm{M})}(x;k_n) \\ &= \sum_{x\in\mathrm{G}(k_n)} t_{\varrho(\mathrm{M})}(x;k_n) \sum_{\chi\in\widehat{\mathrm{G}}(k_n)} \chi(x) \\ &= |\mathrm{G}(k_n)|\; t_{\varrho(\mathrm{M})}(1;k_n)\end{aligned}$$

by the trace formula again, followed by an exchange of the sums and an application of the orthogonality of characters of finite abelian groups. This is a single value of the trace function, and it is relatively straightforward to show that it gives the desired multiplicity as limit. So *the key difficulty is to control the two auxiliary sums* S_1 *and* S_2.

This can be done if:

(1) We have some bound on the size of the individual traces $\mathrm{Tr}\left(\mathrm{Fr}_{k_n} \mid \mathrm{H}^j_c(\mathrm{G}_{\bar{k}}, \varrho(\mathrm{M}) \otimes \mathscr{L}_\chi)\right)$;

(2) We have some bound on the number of characters χ such that $\mathrm{H}^j_c(\mathrm{G}_{\bar{k}}, \varrho(\mathrm{M}) \otimes \mathscr{L}_\chi)$ can be non-zero in a given degree j.

The second bound is given by the stratified vanishing theorem for $\varrho(\mathrm{M})$. For the first, Deligne's Riemann hypothesis (see Theorem A.19) implies the inequality

$$\left|\mathrm{Tr}\left(\mathrm{Fr}_{k_n} \mid \mathrm{H}^j_c(\mathrm{G}_{\bar{k}}, \varrho(\mathrm{M}) \otimes \mathscr{L}_\chi)\right)\right| \leqslant |k_n|^{(j-d)/2} \dim \mathrm{H}^j_c(\mathrm{G}_{\bar{k}}, \varrho(\mathrm{M}) \otimes \mathscr{L}_\chi),$$

and we see that we require a bound on the dimension of the cohomology spaces, which should be independent of χ. We obtain such bounds as special cases of Sawin's quantitative sheaf theory [115], which is a quantitative form of the finiteness theorems for the six operations on the derived category of ℓ-adic sheaves on quasi-projective algebraic varieties.

Remark 4. If G is one-dimensional, then the Euler–Poincaré characteristic formula (see Theorem C.2) easily implies precise bounds on the dimension of the cohomology spaces that arise, and hence this critical issue does not arise for the additive or multiplicative groups, or for elliptic curves (for such groups, Theorem 1 is also straightforward). It also does not arise if the set of "good" characters $\mathscr{Y}(k_n)$ is the whole group $\widehat{\mathrm{G}}(k_n)$, which is the case in some instances considered by Katz for higher-dimensional abelian varieties.

3. Conventions and notation

We summarize the notation that we use, as well as some typographical conventions that we follow consistently unless otherwise specified.

Given complex-valued functions f and g defined on a set S, we write $f \ll g$ if there exists a real number $\mathrm{C} \geqslant 0$ (called an "implicit constant") such that the inequality $|f(s)| \leqslant \mathrm{C}g(s)$ holds for all $s \in \mathrm{S}$. We write $f \asymp g$ if $f \ll g$ and $g \ll f$. If f and g are defined on a topological space X, and $\mathfrak{F}$ is a filter on X, then we say that $f \sim g$ along $\mathfrak{F}$ if $\lim_{\mathfrak{F}} f(x)/g(x) = 1$.

For any complex number z, we write $e(z) = \exp(2i\pi z)$; the value $e(a/q)$ is well-defined for $q \geqslant 1$ and $a \in \mathbf{Z}/q\mathbf{Z}$.

By a *variety over a field* k, we mean a reduced separated k-scheme of finite type. In particular, an *algebraic group*, as opposed to a group scheme, is always supposed to be reduced, and hence smooth if the field k is perfect.

Let S be a scheme. We say that a pair (X, u) is *a quasi-projective scheme over* S if X is a scheme over S and u is a locally-closed immersion $u \colon \mathrm{X} \to \mathbf{P}^n_{\mathrm{S}}$ for some integer $n \geqslant 0$. We call n the embedding dimension of (X, u), or simply of u, and we say that u is a *quasi-projective embedding* of X. When S is the spectrum of a field k and X is a variety over k, we will speak of quasi-projective varieties over k. In some cases, we omit the mention of u, when it is clear from the context which locally-closed immersion is used. By a morphism $f \colon (\mathrm{X}, u) \to (\mathrm{Y}, v)$ of quasi-projective schemes over S, we mean an S-morphism of the underlying schemes.

An algebraic group $\mathbf{G}$ over an algebraically closed field of characteristic zero is called *reductive* if all its finite-dimensional representations are completely reducible (that is, we do not require $\mathbf{G}$ to be connected).

Let X be a scheme and ℓ a prime number invertible on X.

By a $\overline{\mathbf{Q}}_\ell$-sheaf on X, we always mean a constructible étale $\overline{\mathbf{Q}}_\ell$-sheaf. Perverse sheaves (when X is an algebraic variety defined over a field k) are always considered with respect to the middle perversity. We include a short survey of the most important properties of perverse sheaves in Appendix A, but recall here some of the definitions. An ℓ-adic complex is said to be *semiperverse* if the inequality

$$\dim \operatorname{supp} \mathscr{H}^i(\mathrm{M}) \leqslant -i$$

holds for any integer i. This is equivalent to asking that the perverse cohomology sheaves ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M})$ are zero for $i \geqslant 1$ (see [8, Prop. 1.3.7]).

We say that a complex M in $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X}, \overline{\mathbf{Q}}_\ell)$ has *perverse amplitude* $[a, b]$ if its perverse cohomology sheaves ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M})$ are zero for $i \notin [a, b]$.

A *stratification* $\mathscr{X}$ of X is a finite set-theoretic partition of the associated reduced scheme $\mathrm{X}^{\mathrm{red}}$ into non-empty reduced locally-closed subschemes of X, called the *strata* of $\mathscr{X}$.

Let $\mathscr{X}$ be a stratification of X, and let $\mathscr{F}$ be an ℓ-adic sheaf on X. The sheaf $\mathscr{F}$ is said to be *tame and constructible along* $\mathscr{X}$ if it is tamely ramified, as in [109, §1.3.1], and if its restriction to any strat of $\mathscr{X}$ is a lisse sheaf. More generally, a complex $\mathrm{M} \in \mathrm{D}^{\mathrm{b}}_c(\mathrm{X}, \overline{\mathbf{Q}}_\ell)$ is said to be tame and constructible along $\mathscr{X}$ if all its cohomology sheaves are tame and constructible along $\mathscr{X}$.

Let $f\colon \mathrm{X} \to \mathrm{Y}$ be a morphism of schemes. For an object M of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X}, \overline{\mathbf{Q}}_\ell)$, we write $\mathrm{R}f_!\mathrm{M} = \mathrm{R}f_*\mathrm{M}$ to indicate that the canonical "forget supports" morphism $\mathrm{R}f_!\mathrm{M} \to \mathrm{R}f_*\mathrm{M}$ is an isomorphism (and similarly for equality of cohomology groups with and without compact support).

Let $q \geqslant 1$ and $w \in \mathbf{Z}$ be integers. A complex number α is called a q-*Weil number* of weight w if α is algebraic over $\mathbf{Q}$ and all its Galois conjugates have modulus $q^{w/2}$. If k is a finite field, then a k-Weil number is a $|k|$-Weil number.

Throughout, for any prime ℓ, we consider a fixed isomorphism $\iota_0\colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$. Trace functions of ℓ-adic perverse sheaves are thus always identified with complex-valued functions through ι_0, and similarly ℓ-adic characters are identified with complex characters. On the other hand, purity of perverse sheaves (or lisse sheaves or ℓ-adic complexes) refers to purity in the sense of Deligne, i.e., pointwise purity means that the eigenvalues of Frobenius are Weil numbers of some weight; see the survey in Section A.3.

The following notation is used consistently in all the book, although frequently with reminders (some objects, such as character sheaves, will be defined later).

- $\mathrm{X} - \mathrm{Y}$: difference set (elements of X that are not in Y); also used in scheme-theoretic settings.
- $\mathrm{M}|\mathrm{X}$ or $\mathrm{M}_{|\mathrm{X}}$: restriction of an object M (or a section of a sheaf) to a subset or subscheme X.
- $|\mathrm{X}|$: cardinality of a set X.
- $\tau(\chi, \psi)$: (unnormalized) Gauss sum attached to a multiplicative character $\chi\colon k^\times \to \overline{\mathbf{Q}}_\ell^{\times}$ and an additive character $\psi\colon k \to \overline{\mathbf{Q}}_\ell^{\times}$ for a finite field k, i.e.

$$\tau(\chi, \xi) = \sum_{x \in k^\times} \chi(x)\psi(x). \tag{3}$$

- $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X}) = \mathrm{D}^{\mathrm{b}}_c(\mathrm{X}, \overline{\mathbf{Q}}_\ell)$: category of bounded constructible complexes of $\overline{\mathbf{Q}}_\ell$-sheaves on a scheme X such that the prime ℓ is invertible in X.

- $K(X) = K(X, \overline{\mathbf{Q}}_\ell)$: the Grothendieck group (or ring) of $D_c^b(X)$; it has a basis consisting of classes of simple perverse sheaves (see [98, § 0.8]).
- $\alpha^{\deg}$: for k a finite field and α an ℓ-adic unit, the ℓ-adic sheaf of rank 1 on $\mathrm{Spec}(k)$ on which the geometric Frobenius acts by multiplication by α; more generally, for $f\colon X \to \mathrm{Spec}(k)$ a scheme over k, the pullback of $\alpha^{\deg}$ to X.
- $M \otimes N$: derived tensor product of objects of $D_c^b(X)$.
- $M \boxtimes N$: for M an object of $D_c^b(X)$ and N an object of $D_c^b(Y)$, the object $p_1^*M \otimes p_2^*N$ on $X \times Y$, where p_1 and p_2 are the two projections.
- $\mathbf{Perv}(X) = \mathbf{Perv}(X, \overline{\mathbf{Q}}_\ell)$: the category of ℓ-adic perverse sheaves on X. A simple perverse sheaf will also sometimes be called an irreducible perverse sheaf.
- $D(M)$: the Verdier dual of a complex M.
- $\mathscr{H}^i(M)$: for $M \in D_c^b(X)$, the i-th cohomology sheaf of M.
- ${}^{\mathfrak{p}}\mathscr{H}^i(M)$: for $M \in D_c^b(X)$, the i-th perverse cohomology sheaf of M.
- $H^i(M) = H^i(X_{\bar{k}}, M)$: the étale cohomology groups of the pull-back of M to $X \times_k \bar{k}$.
- $H_c^i(M) = H_c^i(X_{\bar{k}}, M)$: the étale cohomology groups with compact support of M.
- $h^i(X_{\bar{k}}, M) = \dim H^i(X_{\bar{k}}, M)$.
- $h_c^i(X_{\bar{k}}, M) = \dim H_c^i(X_{\bar{k}}, M)$.
- $H^*(X_{\bar{k}}, M)$ or $H_c^*(X_{\bar{k}}, M)$: the graded vector space which is the direct sum of all cohomology spaces $H^i(X_{\bar{k}}, M)$ or $H_c^i(X_{\bar{k}}, M)$.
- $\chi(X_{\bar{k}}, M)$ or $\chi_c(X_{\bar{k}}, M)$: Euler–Poincaré characteristic for cohomology or cohomology with compact support.
- $t_M(x; k_n)$: Frobenius trace function of an object M of $D_c^b(X)$ for $x \in X(k_n)$. For $x \in X(k)$, we usually abbreviate it by $t_M(x) = t_M(x; k)$.
- $\langle M \rangle$: tannakian category generated by M.
- $\mathbf{G}_M^{\mathrm{ari}}$ (resp. $\mathbf{G}_M^{\mathrm{geo}}$): arithmetic (resp. geometric) tannakian group associated with a perverse sheaf M.
- $\widehat{G}(k_n)$: group of $\overline{\mathbf{Q}}_\ell$-characters of the finite group $G(k_n)$.
- $\widehat{G}$: disjoint union of the sets $\widehat{G}(k_n)$ for $n \geqslant 1$.
- $\Pi(G)$: for a semiabelian variety G, the $\overline{\mathbf{Q}}_\ell$-scheme of ℓ-adic characters of G.
- $\mathscr{L}_\chi$: character sheaf on G_{k_n} associated to a character $\chi \in \widehat{G}(k_n)$.
- $\mathscr{L}_{\chi(f)}$: for $f\colon X \to G$ and $\chi \in \widehat{G}(k_n)$, the sheaf $f^*\mathscr{L}_\chi$ on X.
- M_χ: for an object M of $D_c^b(G)$ and a character χ, the object $M \otimes \mathscr{L}_\chi$.

Moreover, the following notational conventions will be used (often with reminders).

- k: a finite field of characteristic p.
- ℓ: a prime different from p.
- $\bar{k}$: an algebraic closure of k.
- k_n: the extension of degree n of k inside $\bar{k}$.

– G: a connected commutative algebraic group (in particular of finite type) defined over k.
– T: a torus;
– U: a unipotent group;
– A: an abelian variety.
– $\mathscr{F}$: a $\overline{\mathbf{Q}}_\ell$-sheaf;
– $\mathscr{L}$: a $\overline{\mathbf{Q}}_\ell$-lisse sheaf of rank one.
– M, N: objects of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ or $\mathbf{Perv}(\mathrm{X})$.

Part 1

Theoretical foundations

CHAPTER 1

Preliminaries

In this chapter, we summarize some tools we use throughout the book, especially the properties of Sawin's quantitative sheaf theory [115] with an emphasis on commutative algebraic groups.

1.1. Specializations of perverse sheaves

We will frequently use the following result concerning specializations of perverse sheaves.

Proposition 1.1. *Let k be a field, $f\colon \mathrm{Y} \to \mathrm{X}$ a surjective affine morphism of varieties over k, and M a perverse sheaf on Y. For all closed points x outside of a closed strict subvariety of X, the object $\mathrm{M}|_{f^{-1}(x)}[-1]$ is perverse on $f^{-1}(x)$.*

This follows directly from [82, Ch. III, Lemma 6.3].

1.2. Review of quantitative sheaf theory

Let k be a field, $\bar{k}$ an algebraic closure of k, and ℓ a prime different from the characteristic of k.

Definition 1.2 (Complexity). Let $\mathrm{M}_{n+1,m+1}$ be the variety of $(n+1)\times(m+1)$ matrices of maximal rank, viewed as an affine scheme over k. For each $0 \leqslant m \leqslant n$, consider a geometric generic point a_m of $\mathrm{M}_{n+1,m+1}$ defined over an algebraically closed extension K of k, and let $l_{a_m}\colon \mathbf{P}^m_{\mathrm{K}} \to \mathbf{P}^n_{\mathrm{K}}$ denote the associated linear map.

(a) The *complexity* of an object M of $\mathrm{D}^{\mathrm{b}}_c(\mathbf{P}^n_k)$ is defined as

$$c(\mathrm{M}) = \max_{0\leqslant m\leqslant n} \sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^n_{\mathrm{K}}, \mathrm{M}\otimes l_{a_m*}\overline{\mathbf{Q}}_\ell) = \max_{0\leqslant m\leqslant n} \sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^m_{\mathrm{K}}, l_{a_m}^*\mathrm{M}),$$

where the last equality follows from the projection formula.

(b) Let (X, u) be a quasi-projective variety over k. For any object M of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$, the *complexity* of M with respect to u is defined as $c_u(\mathrm{M}) = c(u_!\mathrm{M})$.

The invariance of étale cohomology under base change between algebraically closed fields implies that the complexity is well-defined (i.e., it does not depend on the choice of fields of definition of the generic points a_m).

Lemma 1.3. *Let (X, u) be a quasi-projective variety over k and let M be an object of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$. The following inequality holds:*

$$\sum_{i\in\mathbf{Z}} h^i_c(\mathrm{X}_{\bar{k}}, \mathrm{M}) \leqslant c_u(\mathrm{M}). \tag{1.1}$$

Proof. This follows from the equality $h^i_c(\mathrm{X}_{\bar{k}}, \mathrm{M}) = h^i(\mathbf{P}^n_{\bar{k}}, u_!\mathrm{M})$ and the invariance of étale cohomology under extension of scalars between algebraically closed fields, combined with the fact that $l_{a_n}\colon \mathbf{P}^n_{\mathrm{K}} \to \mathbf{P}^n_{\mathrm{K}}$ is an isomorphism. □

DEFINITION 1.4. Let $f\colon (\mathrm{X},u)\to(\mathrm{Y},v)$ be a morphism of quasi-projective varieties over k with embedding dimensions n_{X} and n_{Y} respectively. For all integers $0\leqslant m_{\mathrm{X}}\leqslant n_{\mathrm{X}}$ and $0\leqslant m_{\mathrm{Y}}\leqslant n_{\mathrm{Y}}$, consider geometric generic points $a_{m_{\mathrm{X}}}$ of $\mathrm{M}_{n_{\mathrm{X}}+1,m_{\mathrm{X}}+1}$ and $b_{m_{\mathrm{Y}}}$ of $\mathrm{M}_{n_{\mathrm{Y}}+1,m_{\mathrm{Y}}+1}$ defined over an algebraically closed extension K of k, and let $l_{a_{m_{\mathrm{X}}}}\colon \mathbf{P}^{m_{\mathrm{X}}}_{\mathrm{K}}\to\mathbf{P}^{n_{\mathrm{X}}}_{\mathrm{K}}$ and $l_{b_{m_{\mathrm{Y}}}}\colon \mathbf{P}^{m_{\mathrm{Y}}}_{\mathrm{K}}\to\mathbf{P}^{n_{\mathrm{Y}}}_{\mathrm{K}}$ denote the associated linear maps. The *complexity* of f is defined as

$$c_{u,v}(f)=\max_{0\leqslant m_{\mathrm{X}}\leqslant n_{\mathrm{X}}}\max_{0\leqslant m_{\mathrm{Y}}\leqslant n_{\mathrm{Y}}}\sum_{i\in\mathbf{Z}}h^i_c(\mathrm{X}_{\mathrm{K}},u^*l_{a_{m_{\mathrm{X}}}*}\overline{\mathbf{Q}}_\ell\otimes f^*v^*l_{b_{m_{\mathrm{X}}}*}\overline{\mathbf{Q}}_\ell).$$

The main result of [115] establishes, among other things, the "continuity" of the six operations on the derived category with respect to the complexity. In this result and the remainder of this section, the implicit constants only depend on the embedding dimensions of the quasi-projective varieties, unless otherwise specified.

THEOREM 1.5 ([115, Th. 6.8, Prop. 6.14, Prop. 6.12]). *Let $f\colon (\mathrm{X},u)\to(\mathrm{Y},v)$ be a morphism of quasi-projective varieties over k. Let* M, N, P *be objects of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ and let* Q *be an object of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{Y})$. The following inequalities hold:*

(1) $c_u(\mathrm{M}\oplus\mathrm{N})\leqslant c_u(\mathrm{M})+c_u(\mathrm{N})$.
(2) $c_u(\mathrm{M}\otimes\mathrm{N})\ll c_u(\mathrm{M})c_u(\mathrm{N})$.
(3) *If* $\mathrm{M}\to\mathrm{N}\to\mathrm{P}$ *is a distinguished triangle, then* $c_u(\mathrm{N})\leqslant c_u(\mathrm{M})+c_u(\mathrm{P})$.
(4) $c_u(\mathrm{M}[k])=c_u(\mathrm{M})$ *for any* $k\in\mathbf{Z}$.
(5) $c_u(\mathrm{RHom}(\mathrm{M},\mathrm{N}))\leqslant c_{u,\mathrm{id}}(u)c_u(\mathrm{M})c_u(\mathrm{N})$.
(6) $c_v(\mathrm{R}f_!\mathrm{M})\ll c_{u,v}(f)c_u(\mathrm{M})$ *and* $c_v(\mathrm{R}f_*\mathrm{M})\ll c_{u,\mathrm{id}}(u)c_{v,\mathrm{id}}(v)c_{u,v}(f)c_u(\mathrm{M})$.
(7) $c_u(f^*\mathrm{Q})\ll c_{u,v}(f)c_v(\mathrm{Q})$ *and* $c_u(f^!\mathrm{Q})\ll c_{u,\mathrm{id}}(u)c_{v,\mathrm{id}}(v)c_{u,v}(f)c_v(\mathrm{Q})$.
(8) $c_{u\boxtimes v}(\mathrm{M}\boxtimes\mathrm{Q})\ll c_{u\boxtimes v,u}(p_1)c_{u\boxtimes v,v}(p_2)c_u(\mathrm{M})c_v(\mathrm{Q})$.

In the last of these inequalities, $u\boxtimes v$ is the composition of $u\times v$ with the Segre embedding and p_1, p_2 are the projections $\mathrm{X}\times\mathrm{Y}\to\mathrm{X}$ and $\mathrm{X}\times\mathrm{Y}\to\mathrm{Y}$, respectively.

REMARK 1.6. Although the notion of complexity on a quasi-projective scheme (X,u) depends on the quasi-projective immersion u, note that if v is another quasi-projective immersion of X, then applying the property (7) to the identity morphism between (X,u) and (X,v), we get

$$c_u(\mathrm{M})\asymp c_v(\mathrm{M})$$

for all objects M of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$, where the implied constants are essentially $c_{u,v}(\mathrm{Id})$ and $c_{v,u}(\mathrm{Id})$, up to constants depending on the embedding dimensions of u and v. Thus, as long as we only consider on X an absolutely bounded number of different quasi-projective immersions, we can think of the complexity as being essentially independent of them. (This is reminiscent of similar properties of height functions in diophantine geometry.)

The complexity can also be used to control the degree of the locus where a complex of sheaves is lisse, and of the locus where the generic base change theorem holds.

THEOREM 1.7 ([115, Th. 6.23]). *Let (X,u) be an irreducible quasi-projective variety over k. Let* M *be an object of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$. Let* Z *be the complement of the maximal open subset where* X *is smooth and* M *is lisse. Then the estimate*

$$\deg(u(\mathrm{Z}))\ll(3+s)c(u)c_u(\mathrm{M})$$

holds, where the degrees are computed in the projective space target of u, and s is the degree of the codimension 1 *part of the singular locus of* X.

Theorem 1.8 ([115, Th. 6.27]). *Let* (X, u), (Y, v) *and* (S, w) *be quasi-projective algebraic varieties over* k. *Let* $f\colon \mathrm{X} \to \mathrm{Y}$ *and* $g\colon \mathrm{Y} \to \mathrm{S}$ *be morphisms.*

For any object M *of* $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$, *there exists an integer* $\mathrm{C} \geqslant 0$, *depending only on* $c_u(\mathrm{M})$ *and the morphisms* (f, g, u, v, w), *and a dense open set* $\mathrm{U} \subset \mathrm{S}$ *such that:*

(i) *The image of the complement of* U *has degree* $\leqslant \mathrm{C}$.

(ii) *The object* $f_*\mathrm{M}$ *is of formation compatible with any base change* $\mathrm{S}' \to \mathrm{U} \subset \mathrm{S}$.

Proposition 1.9 ([115, Th. 6.15]). *Let* (X, u) *be a quasi-projective variety over* k. *Let* M *be an object of* $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$. *For each integer* i, *let* $\mathrm{M}_{i,1}, \ldots, \mathrm{M}_{i,n_i}$ *denote the Jordan–Hölder factors of the perverse cohomology sheaf* ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M})$. *Then the following estimate holds:*

$$\sum_{i\in\mathbf{Z}} \sum_{1\leqslant j\leqslant n_i} c_u(\mathrm{M}_{i,j}) \ll c_{u,\mathrm{id}}(u) c_u(\mathrm{M}).$$

We also recall the quantitative statement of the Riemann hypothesis over finite fields when interpreted as a quasi-orthogonality statement.

Theorem 1.10 ([115, Th. 7.13 (2)]). *Let* k *be a finite field and* ℓ *a prime different from the characteristic of* k. *Let* (X, u) *be a geometrically irreducible quasi-projective algebraic variety over* k. *Let* M *and* N *be geometrically simple* ℓ*-adic perverse sheaves on* X *that are pure of weight zero, with complex trace functions* t_{M} *and* t_{N} *respectively. Then the estimate*

$$\sum_{x\in\mathrm{X}(k)} t_{\mathrm{M}}(x)\overline{t_{\mathrm{N}}(x)} \ll c(u) c_u(\mathrm{M}) c_u(\mathrm{N}) |k|^{-1/2}$$

holds if M *and* N *are not geometrically isomorphic, whereas*

$$\sum_{x\in\mathrm{X}(k)} |t_{\mathrm{M}}(x)|^2 = 1 + \mathrm{O}(c(u) c_u(\mathrm{M})^2 |k|^{-1/2}).$$

In both estimates, the implied constants only depend on the embedding dimension and are effective.

Finally, we have pointwise bounds for the trace functions.

Proposition 1.11 ([115, Prop. 7.11 (2)]). *Let* k *be a finite field and* ℓ *a prime different from the characteristic of* k. *Let* (X, u) *be a quasi-projective algebraic variety over* k, *and let* M *be a non-punctual simple perverse sheaf on* X *which is pure of weight zero. For any* $n \geqslant 1$ *and* $x \in \mathrm{X}(k_n)$, *the following estimate holds:*

$$t_{\mathrm{M}}(x; k_n) \ll \frac{1}{|k_n|^{1/2}}.$$

1.3. Existence of rational points

The following lemma is standard.

Lemma 1.12. *Let* (X, u) *be a non-empty quasi-projective variety over a finite field* k *with embedding dimension* n. *There exists a finite extension* k' *of* k *with degree bounded in terms of* $(\dim(\mathrm{X}), \deg(\overline{u(\mathrm{X})}), n)$ *such that* $\mathrm{X}(k')$ *is non-empty.*

Proof. Write the variety $u(\mathrm{X})$ as $\mathrm{Z}-\mathrm{Y}$ for some closed subvarieties Z and Y. Then the degrees of Z and $\mathrm{Y} \cap \mathrm{Z}$ are bounded in terms of $c_u(\mathrm{X})$ by [115, Lemma 6.26], and the result then follows from the Lang–Weil bound (see [94, Th. 1]) applied to Z and to $\mathrm{Y} \cap \mathrm{Z}$. ∎

1.4. Structure of commutative algebraic groups

Let k be a field and let G be a commutative algebraic group over k. The algebraic variety G is quasi-projective (see, e.g., [**22**, Prop. A.3.5] or [**122**, Lemma 0BF7]. We will always assume that a quasi-projective immersion u of G is given, and the complexity of ℓ-adic complexes will be understood with respect to u (so that we sometimes write just $c(\mathrm{M})$ instead of $c_u(\mathrm{M})$). If G is either a power of $\mathbf{G}_a$ or of $\mathbf{G}_m$, we assume that u is simply the obvious embedding in the projective space of the same dimension. We will on occasion use auxiliary quasi-projective immersions and rely on Remark 1.6 to compare complexities.

Smooth connected commutative algebraic groups can be built as successive extensions of abelian varieties, tori, unipotent[1] and finite commutative group schemes. The most convenient formulation of this fact for us is the following statement, which follows from results of Barsotti–Chevalley and Rosenlicht (see for instance the account in the book of Brion, Samuel and Umae, combining [**17**, Cor. 5.5.2] with the structure theorem for connected affine commutative algebraic groups over perfect fields as a product of a unipotent group and a torus; see, e.g., [**16**, Th. 5.3.1 (2)]).

Proposition 1.13. *Let k be a finite field and let G be a connected commutative algebraic group over k. There exist an abelian variety A, a torus T, a unipotent group U and a finite commutative subgroup scheme N of $\mathrm{A}\times\mathrm{U}\times\mathrm{T}$, all defined over k, such that G is isomorphic to $(\mathrm{A}\times\mathrm{U}\times\mathrm{T})/\mathrm{N}$.*

We further recall that a finite commutative group scheme N over a perfect field has a unique direct product decomposition $\mathrm{N}=\mathrm{N}_r\times\mathrm{N}_l$ where N_r is reduced and N_l is local (i.e., equal to its connected component of the identity; see, e.g., [**16**, Prop. 2.5.4]).

1.5. Convolution

Let G be a commutative algebraic group over a field k. We denote by

$$m\colon \mathrm{G}\times\mathrm{G}\to\mathrm{G},\quad \mathrm{inv}\colon\mathrm{G}\to\mathrm{G},\quad e\in\mathrm{G}(k)$$

the group law, the inversion morphism, and the neutral element respectively.

Definition 1.14 (Convolution). The *convolution product* and the *convolution product with compact support* on G are the functors from $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})\times\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$ to $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$ defined as

$$\mathrm{M}*_*\mathrm{N}=\mathrm{R}m_*(\mathrm{M}\boxtimes\mathrm{N}),\qquad \mathrm{M}*_!\mathrm{N}=\mathrm{R}m_!(\mathrm{M}\boxtimes\mathrm{N})$$

for objects M and N of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$.

If G is projective, then so is the morphism m, and hence the two convolutions agree. In general, there is a canonical "forget supports" morphism

$$\mathrm{M}*_!\mathrm{N}\longrightarrow\mathrm{M}*_*\mathrm{N}.$$

We will write $\mathrm{M}*_!\mathrm{N}=\mathrm{M}*_*\mathrm{N}$ when this morphism is an isomorphism.

If u is a quasi-projective immersion of G, then we deduce from Theorem 1.5 (8) that for any objects M and N, the following estimates hold:

$$c_u(\mathrm{M}*_*\mathrm{N})\ll c_u(\mathrm{M})c_u(\mathrm{N}),\qquad c_u(\mathrm{M}*_!\mathrm{N})\ll c_u(\mathrm{M})c_u(\mathrm{N}),$$

where the implied constant depends on G, and is uniform in families (see [**115**, § 6.5]).

For an object M of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$, we define

$$\mathrm{M}^\vee=\mathrm{inv}^*\,\mathrm{D}(\mathrm{M}),$$

[1] In this book, "unipotent" only applies to commutative groups

where $\mathrm{D(M)}$ is the Verdier dual. Since $\mathrm{inv}^* = \mathrm{inv}^!$ commutes with D, the functor $\mathrm{M} \mapsto \mathrm{M}^\vee$ is an involution, in the sense that the functor $\mathrm{M} \mapsto (\mathrm{M}^\vee)^\vee$ is canonically isomorphic to the identity.

We denote by $\mathbf{1}$ the skyscraper sheaf supported at the neutral element e of G.

The basic formal properties of the convolution products are given by the following lemma:

LEMMA 1.15. *Let* M *and* N *be objects of* $\mathbf{Perv}(\mathrm{G})$*. There exist canonical isomorphisms*

$$\mathrm{Hom}(\mathbf{1}, \mathrm{M}^\vee *_* \mathrm{N}) \simeq \mathrm{Hom}(\mathrm{M}, \mathrm{N}) \simeq \mathrm{Hom}(\mathrm{M} *_! \mathrm{N}^\vee, \mathbf{1}), \tag{1.2}$$

$$\mathrm{D}(\mathrm{M} *_* \mathrm{N}) \simeq \mathrm{D}(\mathrm{M}) *_! \mathrm{D}(\mathrm{N}), \quad \mathrm{D}(\mathrm{M} *_! \mathrm{N}) \simeq \mathrm{D}(\mathrm{M}) *_* \mathrm{D}(\mathrm{N}), \tag{1.3}$$

$$\mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}) \otimes_{\overline{\mathbf{Q}}_\ell} \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{N}) \simeq \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M} *_! \mathrm{N}), \tag{1.4}$$

$$\mathrm{H}^*(\mathrm{G}_{\bar{k}}, \mathrm{M}) \otimes_{\overline{\mathbf{Q}}_\ell} \mathrm{H}^*(\mathrm{G}_{\bar{k}}, \mathrm{N}) \simeq \mathrm{H}^*(\mathrm{G}_{\bar{k}}, \mathrm{M} *_* \mathrm{N}). \tag{1.5}$$

In the first isomorphisms, the Hom spaces are taken in the category $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$*.*

PROOF. All these are consequences of the formal properties of the six operations on $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$. More precisely, all can be found in [69, 8.1.8, 8.1.9], except for the first statement. This is proved for tori in [50, p. 533]; however, the argument applies verbatim to any G, since it only uses formal properties of the six operations on $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$. ☐

From the adjunctions in (1.2) of Lemma 1.15, we see that for all $\mathrm{M} \in \mathbf{Perv}(\mathrm{G})$, the identity morphism $\mathrm{id}_\mathrm{M} \colon \mathrm{M} \to \mathrm{M}$ defines evaluation and coevaluation morphisms

$$\mathrm{ev} \colon \mathrm{M} *_! \mathrm{M}^\vee \longrightarrow \mathbf{1} \qquad \text{and} \qquad \mathrm{coev} \colon \mathbf{1} \longrightarrow \mathrm{M}^\vee *_* \mathrm{M}.$$

As a consequence of these properties, we note that $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$ is a symmetric monoidal $\overline{\mathbf{Q}}_\ell$-linear category with respect to either the convolution $(\mathrm{A}, \mathrm{B}) \mapsto \mathrm{A} *_! \mathrm{B}$ or $(\mathrm{A}, \mathrm{B}) \mapsto \mathrm{A} *_* \mathrm{B}$,

1.6. Character groups

In this section, we denote by k a finite field, by $\bar{k}$ an algebraic closure of k, and by k_n the extension of degree n of k in $\bar{k}$. Let ℓ be a prime number distinct from the characteristic of k.

Let G be a connected commutative algebraic group defined over k. For each $n \geqslant 1$, the *norm map* is the group homomorphism $\mathrm{N}_{k_n/k} \colon \mathrm{G}(k_n) \to \mathrm{G}(k)$ defined as

$$\mathrm{N}_{k_n/k}(x) = \prod_{i=0}^{n-1} \mathrm{Fr}^i_{k_n}(x).$$

For any $n \geqslant 1$, let $\widehat{\mathrm{G}}(k_n)$ be the group of characters $\chi \colon \mathrm{G}(k_n) \to \overline{\mathbf{Q}}_\ell^\times$. We denote by $\widehat{\mathrm{G}}$ the *disjoint union*

$$\widehat{\mathrm{G}} = \bigsqcup_{n \geqslant 1} \widehat{\mathrm{G}}(k_n)$$

(note that this is not a group; we also omit the dependency on ℓ from this notation).

Given any set $\mathrm{S} \subset \widehat{\mathrm{G}}$, we also define $\mathrm{S}(k_n) = \mathrm{S} \cap \widehat{\mathrm{G}}(k_n)$, so that S is the disjoint union of the subsets $\mathrm{S}(k_n)$.

Since G is geometrically irreducible (see, e.g. , [106, Cor. 1.35]), the estimate

$$|\widehat{\mathrm{G}}(k_n)| = |\mathrm{G}(k_n)| = |k|^{n \dim(\mathrm{G})} + \mathrm{O}(|k|^{(n-1/2)\dim(\mathrm{G})})$$

holds for $n \geqslant 1$ by the Lang–Weil estimates. If G is an abelian variety, we have more precisely

$$(|k|^{1/2} - 1)^{2n \dim(\mathrm{G})} \leqslant |\widehat{\mathrm{G}}(k_n)| \leqslant (|k|^{1/2} + 1)^{2n \dim(\mathrm{G})}$$

and if G is a torus, then

$$(|k|-1)^{n\dim(\mathrm{G})} \leqslant |\widehat{\mathrm{G}}(k_n)| \leqslant (|k|+1)^{n\dim(\mathrm{G})}.$$

These can be derived from the computation of the étale cohomology of abelian varieties combined with the trace formula, or from Steinberg's formula for tori; see, for instance, [105, Th. 15.1, Th. 19.1] for the case of abelian varieties and [19, Prop. 3.3.5] for the case of tori.

We now recall from [27, Sommes trig., 1.4] the *Lang torsor* construction and the basic properties of the associated character sheaves. There is an exact sequence of commutative algebraic groups[2]

$$1 \longrightarrow \mathrm{G}(k) \longrightarrow \mathrm{G} \xrightarrow{\mathfrak{L}} \mathrm{G} \longrightarrow 1,$$

where $\mathfrak{L}$ is the Lang isogeny $x \mapsto \mathrm{Fr}_k(x)\cdot x^{-1}$. The Lang isogeny is a Galois étale covering with Galois group $\mathrm{G}(k)$, and hence induces a surjective map $\pi_1^{\text{ét}}(\mathrm{G}, e) \to \mathrm{G}(k)$. Given a character $\chi \in \widehat{\mathrm{G}}(k)$, we denote by $\mathscr{L}_\chi$ the ℓ-adic lisse sheaf of rank one on G obtained by composing this map with χ^{-1} and we say that $\mathscr{L}_\chi$ is the *character sheaf* on G associated to χ.

For $x \in \mathrm{G}(k)$, the geometric Frobenius automorphism at x acts on the stalk of $\mathscr{L}_\chi$ at x by multiplication by $\chi(x)$. In particular, the lisse sheaf $\mathscr{L}_\chi$ is pure of weight zero.

If χ is the trivial character, then $\mathscr{L}_\chi$ is the constant sheaf $\overline{\mathbf{Q}}_\ell^\times$.

The dual $\mathrm{D}(\mathscr{L}_\chi)$ of a character sheaf is isomorphic to $\mathscr{L}_{\chi^{-1}}$, and for any two characters χ_1 and χ_2 there are canonical isomorphisms

$$\mathscr{L}_{\chi_1} \otimes \mathscr{L}_{\chi_2} \simeq \mathscr{L}_{\chi_1\chi_2}.$$

If $n \geqslant 1$ and $\chi \in \widehat{\mathrm{G}}(k_n)$ is non-trivial, then for all $i \in \mathbf{Z}$, the cohomology space $\mathrm{H}^i_c(\mathrm{G}_{\bar{k}}, \mathscr{L}_\chi)$ vanishes (see [27, Sommes trig., Th. 2.7*]). More generally, we have the following relative version.

Lemma 1.16. *Let $f\colon \mathrm{G} \to \mathrm{H}$ be a surjective morphism of commutative algebraic groups over k. Let $\chi \in \widehat{\mathrm{G}}(k)$. The complex $\mathrm{R}f_!\mathscr{L}_\chi$ vanishes unless $\mathscr{L}_\chi|\ker(f)^\circ$ is the constant sheaf, i.e., unless χ is trivial on $\ker(f)^\circ$.*

Proof. Let $\mathrm{M} = \mathrm{R}f_!\mathscr{L}_\chi$. Let $y \in \mathrm{H}$ and let $z \in \mathrm{G}$ be such that $f(z) = y$. By the proper base change theorem, the stalk of M at y is given by

$$\mathrm{M}_y = \mathrm{H}^*_c(f^{-1}(y)_{\bar{k}}, \mathscr{L}_\chi) = \mathrm{H}^*_c((z\ker(f))_{\bar{k}}, \mathscr{L}_\chi) = \mathrm{H}^*_c(\ker(f)_{\bar{k}}, [x \mapsto xz]^*\mathscr{L}_\chi|\ker(f)).$$

We write $\ker(f)$ as the disjoint union of cosets $u\ker(f)^\circ$ where u runs over a set of representatives of the group of connected components of $\ker(f)$. Thus,

$$\mathrm{H}^*_c(\ker(f)_{\bar{k}}, [x \mapsto xz]^*\mathscr{L}_\chi|\ker(f)) = \bigoplus_u \mathrm{H}^*_c(\ker(f)^\circ_{\bar{k}}, [x \mapsto xuz]^*\mathscr{L}_\chi|\ker(f)^\circ).$$

Since $\mathscr{L}_\chi$ is a character sheaf, the sheaf $[x \mapsto xuz]^*\mathscr{L}_\chi$ is geometrically isomorphic to $\mathscr{L}_\chi$, so that we have an isomorphism

$$\mathrm{M}_y \simeq \bigoplus_u \mathrm{H}^*_c(\ker(f)^\circ_{\bar{k}}, \mathscr{L}_\chi|\ker(f)^\circ),$$

and the result now follows from [27, Sommes trig., Th. 2.7*] as recalled above. ☐

[2] Note that it is here that the assumption that G is connected plays a role, since in general the image of the morphism $\mathfrak{L}$ is equal to the connected component of the neutral element.

Let $n \geqslant 1$ and $\chi \in \widehat{\mathrm{G}}(k)$. The base change of $\mathscr{L}_\chi$ to G_{k_n} is the character sheaf on G_{k_n} associated to the character $\chi \circ \mathrm{N}_{k_n/k}$ of $\mathrm{G}(k_n)$. In particular, the trace function of $\mathscr{L}_\chi$ on k_n is given by

$$t_{\mathscr{L}_\chi}(x; k_n) = \chi(\mathrm{N}_{k_n/k}(x)).$$

When there is no risk of confusion, we will still denote by $\mathscr{L}_\chi$ the pullback of the character sheaf associated to χ to $\bar{k}$. The previous remark shows that χ and $\chi \circ \mathrm{N}_{k_n/k}$ give rise to the same base change to $\bar{k}$.

Let $f\colon \mathrm{G} \to \mathrm{H}$ be a homomorphism of commutative algebraic groups defined over k. For any integer $n \geqslant 1$, let us denote by f_n the induced morphism $\mathrm{G}(k_n) \to \mathrm{H}(k_n)$; then we have dual homomorphisms $\widehat{f}_n\colon \widehat{\mathrm{H}}(k_n) \to \widehat{\mathrm{G}}(k_n)$ defined by $\chi \mapsto \chi \circ f_n$. The combination of all these maps gives a map $\widehat{f}\colon \widehat{\mathrm{H}} \to \widehat{\mathrm{G}}$, which we will often denote simply by $\chi \mapsto \chi \circ f$. We will sometimes say that a character $\chi \in \widehat{\mathrm{G}}$ *arises* from H if χ belongs to the image of $\widehat{f}$.

For $\chi \in \widehat{\mathrm{H}}(k_n)$, there is a canonical isomorphism $\mathscr{L}_{\widehat{f}(\chi)} \simeq f^*\mathscr{L}_\chi$.

For any object M of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$ and any character χ of $\mathrm{G}(k)$, we denote by

$$\mathrm{M}_\chi = \mathrm{M} \otimes \mathscr{L}_\chi$$

the "twist" of M by the character sheaf $\mathscr{L}_\chi$.

For all $\chi \in \widehat{\mathrm{G}}$, and all objects M and N of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$ (or $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G}_{\bar{k}})$), there are canonical isomorphisms

$$\mathrm{D}(\mathrm{M}_\chi) \simeq \mathrm{D}(\mathrm{M})_{\chi^{-1}}, \tag{1.6}$$

$$(\mathrm{M}_\chi)^\vee \simeq (\mathrm{M}^\vee)_\chi, \tag{1.7}$$

$$(\mathrm{M} *_* \mathrm{N})_\chi \simeq (\mathrm{M}_\chi *_* \mathrm{N}_\chi), \quad (\mathrm{M} *_! \mathrm{N})_\chi \simeq (\mathrm{M}_\chi *_! \mathrm{N}_\chi). \tag{1.8}$$

The first two properties follow from duality from $\mathrm{D}(\mathscr{L}_\chi) = \mathscr{L}_{\chi^{-1}}$, and the third from the projection formula combined with the canonical isomorphism $m^*\mathscr{L}_\chi \simeq \mathscr{L}_\chi \boxtimes \mathscr{L}_\chi$, where p_1 and p_2 are the projections $\mathrm{G} \times \mathrm{G} \to \mathrm{G}$ (see [69, 8.1.10 (4)]).

More generally, for any algebraic variety X over k, any morphism $f\colon \mathrm{X} \to \mathrm{G}$, and any object M of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$, we set

$$\mathrm{M}_\chi = \mathrm{M} \otimes f^*\mathscr{L}_\chi,$$

and we use the same notation for objects in $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G}_{\bar{k}})$ and $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X}_{\bar{k}})$, or in $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G}_{k_n})$ and $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X}_{k_n})$.

We will extensively (and often without comment) use the following standard lemma.

Lemma 1.17. *Let $f\colon \mathrm{X} \to \mathrm{G}$ be a morphism from an algebraic variety* X *to a connected commutative algebraic group* G, *both defined over k. Let $\chi \in \widehat{\mathrm{G}}$ be a character. Then the functor $\mathrm{M} \mapsto \mathrm{M}_\chi$ on $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ or $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X}_{\bar{k}})$ is t-exact for the standard and perverse t-structures. In particular, if* M *is perverse* (*resp. semiperverse*) *then so is M_χ.*

Proof. Let $i \in \mathbf{Z}$. Since $\mathscr{L}_\chi$ is a lisse sheaf on G, the pullback $f^*\mathscr{L}_\chi$ is lisse on X, and hence tensoring with $f^*\mathscr{L}_\chi$ is exact for the standard t-structure on $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ or $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X}_{\bar{k}})$ (i.e., the t-structure whose heart is the category of constructible sheaves concentrated in degree 0). There are thus canonical isomorphisms $\mathscr{H}^i(\mathrm{M} \otimes f^*\mathscr{L}_\chi) \simeq \mathscr{H}^i(\mathrm{M}) \otimes f^*\mathscr{L}_\chi$ for all i. Hence, looking at the support, we see that the functor $\mathrm{M} \mapsto \mathrm{M}_\chi$ is right t-exact for the perverse t-structure. It is also left t-exact since $\mathrm{D}(\mathrm{M}_\chi)$ is isomorphic to $\mathrm{D}(\mathrm{M})_{\chi^{-1}}$, hence the result. ∎

1.7. Complexity estimates for character sheaves

We keep the notation of the previous section. The first essential new ingredient for our work is the fact that the complexity of character sheaves on G is uniformly bounded.

Proposition 1.18. *Let* G *be a connected commutative algebraic group over* k *together with a quasi-projective immersion* u. *There exists a real number* $\mathrm{C} \geqslant 0$ *such that, for every* $n \geqslant 1$ *and for every character* $\chi \in \widehat{\mathrm{G}}(k_n)$, *the inequality* $c_u(\mathscr{L}_\chi) \leqslant \mathrm{C}$ *holds.*

Proof. We will proceed in several steps, first noting that we may assume that $n = 1$.

(1) If the result is true for the groups G_1 and G_2, then it is true for their product $\mathrm{G} = \mathrm{G}_1 \times \mathrm{G}_2$. Indeed, let $p_i\colon \mathrm{G} \to \mathrm{G}_i$ denote the two projections. Since any character χ of $\mathrm{G}(k)$ takes the form $(x_1, x_2) \mapsto \chi_1(x_1)\chi_2(x_2)$ for some characters χ_i of $\mathrm{G}_i(k)$, the corresponding character sheaf is the external product $\mathscr{L}_\chi = \mathscr{L}_{\chi_1} \boxtimes \mathscr{L}_{\chi_2}$, which has complexity bounded in terms of the complexity of $\mathscr{L}_{\chi_1}$ and that of $\mathscr{L}_{\chi_2}$, and hence bounded uniformly by assumption.

(More precisely, this is one case where we use Remark 1.6, since we most easily bound the complexity of $\mathscr{L}_{\chi_1} \boxtimes \mathscr{L}_{\chi_2}$ with respect to the composition v of the given quasi-projective immersions u_1 and u_2 of G_1 and G_2 and the Segre embedding using Theorem 1.5, as in [115, Prop. 6.12].)

(2) If the result holds for a group G, then for any finite subgroup scheme H (defined over k), the results holds for the quotient G/H (if this quotient is an algebraic group). To see this, we can further decompose $\mathrm{H} = \mathrm{H}_r \times \mathrm{H}_l$ where H_r is reduced and H_l is local, so that we may assume that H is either reduced or local. Let v be a quasi-projective embedding of G/H and let $\pi\colon \mathrm{G} \to \mathrm{G}/\mathrm{H}$ be the quotient morphism.

If H is reduced, then π is a finite étale covering, so for any lisse sheaf $\mathscr{L}$ on G/H, the sheaf $\mathscr{L}$ is a direct factor of $\pi_*\pi^*\mathscr{L}$, and we deduce

$$c_v(\mathscr{L}) \leqslant c_v(\pi_*\pi^*\mathscr{L}) \ll c_u(\pi^*\mathscr{L}).$$

This implies the result since $\pi^*\mathscr{L}$ is a character sheaf on G if $\mathscr{L}$ is a character sheaf on G/H.

If H is local, then the quotient morphism π is finite and radicial, and hence the adjunction map $\mathscr{L} \to \pi_*\pi^*\mathscr{L} = \pi_!\pi^*\mathscr{L}$ is an isomorphism (see, e.g. , [47, Cor. 5.3.10]). By Theorem 1.5 (6), the complexity $c_v(\mathscr{L}) = c_v(\pi_!\pi^*\mathscr{L})$ is hence $\ll c_u(\pi^*\mathscr{L})$, and the result again follows.

(3) The result is valid for tori and unipotent groups. For the former, since complexity is a geometric invariant, we may assume that we have a split torus, and the result then follows from (1) and the case of $\mathrm{G} = \mathbf{G}_m$, which is established in [115, Prop. 7.5].

Assume then that G is a unipotent group. Let $\mathrm{G}^\vee$ be its Serre dual (or more precisely, an algebraic group model of it; see Section 2.2 for details). There exists a lisse ℓ-adic sheaf $\mathscr{L}$ of rank one on $\mathrm{G}^\vee \times \mathrm{G}$ such that the character sheaves associated to characters of $\mathrm{G}(k)$ are in bijection with the points $a \in \mathrm{G}^\vee(k)$ by mapping $a \in \mathrm{G}^\vee(k)$ to the restriction of the sheaf $\mathscr{L}$ to $\{a\} \times \mathrm{G}$. Hence, by Theorem 1.5, the complexity of any character sheaf of G is bounded in terms of the complexity of the single sheaf $\mathscr{L}$.

(4) The result holds for abelian varieties by [115, Prop. 7.9], since abelian varieties are projective and any character sheaf is lisse on G.

(5) The general case now follows using the previous results and the dévissage of Proposition 1.13. This completes the proof of the proposition. $\square$

Remark 1.19. A potential alternative (more conceptual) approach to this result would be the following. For a character sheaf $\mathscr{L}$ on G, there is an isomorphism

$$m^*\mathscr{L} \simeq p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}$$

(recall that m is the multiplication map $\mathrm{G} \times \mathrm{G} \to \mathrm{G}$). If one could prove directly the estimate

$$c(\mathscr{L})^2 \ll c(p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}), \tag{1.9}$$

then we would deduce from Theorem 1.5 that

$$c(\mathscr{L})^2 \ll c(m^*\mathscr{L}) \ll c(\mathscr{L}),$$

and hence $c(\mathscr{L}) \ll 1$. Note that Proposition 1.18 shows that (1.9) is indeed true, and it is maybe not out of the question that one could provide a direct proof.

1.8. Arithmetic Fourier transforms

We continue with the notation of the previous section. Given an ℓ-adic complex M in $\mathrm{D}_c^{\mathrm{b}}(\mathrm{G})$, we can consider for any fixed $n \geqslant 1$ the discrete Fourier transform of the trace function $x \mapsto t_{\mathrm{M}}(x; k_n)$ on $\mathrm{G}(k_n)$, which we normalize to be the function from $\widehat{\mathrm{G}}(k_n)$ to $\overline{\mathbf{Q}}_\ell$, or $\mathbf{C}$, defined by

$$\chi \mapsto \mathrm{S}(\mathrm{M}, \chi) = \sum_{x \in \mathrm{G}(k_n)} \chi(x) t_{\mathrm{M}}(x; k_n).$$

This Fourier transform satisfies the usual formalism of commutative harmonic analysis (see, e.g., [14]). For instance the Fourier inversion formula

$$t_{\mathrm{M}}(x; k_n) = \frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi \in \widehat{\mathrm{G}}(k_n)} \mathrm{S}(\mathrm{M}, \chi) \bar{\chi}(x) \tag{1.10}$$

holds for any $x \in \mathrm{G}(k_n)$, and there is also a Plancherel formula

$$\sum_{x \in \mathrm{G}(k_n)} |t_{\mathrm{M}}(x; k_n)|^2 = \frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi \in \widehat{\mathrm{G}}(k_n)} |\mathrm{S}(\mathrm{M}, \chi)|^2.$$

Putting together the data of these discrete Fourier transforms on $\mathrm{G}(k_n)$ for all $n \geqslant 1$, we obtain what we call the *arithmetic Fourier transform of the complex* M, an element of the product set

$$\mathscr{C}(\widehat{\mathrm{G}}, \overline{\mathbf{Q}}_\ell) = \prod_{n \geqslant 1} \mathscr{C}(\widehat{\mathrm{G}}(k_n), \overline{\mathbf{Q}}_\ell),$$

where, for any set X and ring A, we denote by $\mathscr{C}(\mathrm{X}, \mathrm{A})$ the A-module of functions $f \colon \mathrm{X} \to \mathrm{A}$.

Combining the Fourier inversion formula (1.10) with the known injectivity theorem for trace functions (see Proposition A.22), we deduce a corresponding injectivity property of the discrete Fourier transform of complexes:

Proposition 1.20. *Let* M_1 *and* M_2 *be complexes in* $\mathrm{D}_c^{\mathrm{b}}(\mathrm{G})$ *such that for all* $n \geqslant 1$ *and all characters* $\chi \in \widehat{\mathrm{G}}(k_n)$*, the equality*

$$\sum_{x \in \mathrm{G}(k_n)} \chi(x) t_{\mathrm{M}_1}(x; k_n) = \sum_{x \in \mathrm{G}(k_n)} \chi(x) t_{\mathrm{M}_2}(x; k_n)$$

holds. Then the classes of M_1 *and* M_2 *in the Grothendieck group* $\mathrm{K}(\mathrm{G}) = \mathrm{K}(\mathrm{G}, \overline{\mathbf{Q}}_\ell)$ *are equal.*

Remark 1.21. In Chapter 6, we will establish a more refined statement where the equality of discrete Fourier transforms is only assumed to hold for characters in a "generic" set, as described in the next section.

1.9. Generic sets of characters

For an arbitrary connected commutative algebraic group, there is no obvious topology (or measure) on the set $\widehat{\mathrm{G}}$ of characters which would lead to a natural notion of sets containing "almost all" characters. We will use instead the following definition of a generic set of characters.

DEFINITION 1.22. Let k be a finite field and let G be a connected commutative algebraic group of dimension d over k. Let S be a subset of $\widehat{\mathrm{G}}$.

Let $i \geqslant 0$ be an integer. We say that S has *character codimension at least* i, which we denote sometimes by $\mathrm{ccodim}(\mathrm{S}) \geqslant i$, if the estimate

$$|\mathrm{S}(k_n)| \ll |k|^{n(d-i)} \tag{1.11}$$

holds for all integers $n \geqslant 1$.

We say that S is *generic* if $\widehat{\mathrm{G}} - \mathrm{S}$ has character codimension at least 1, i.e., if the estimate

$$\left|\widehat{\mathrm{G}}(k_n) - \mathrm{S}(k_n)\right| \ll |k|^{n(d-1)} \tag{1.12}$$

holds for all integers $n \geqslant 1$.

We now discuss the relation between the definition of generic sets and other notions that appear in the literature, in the case of unipotent and semiabelian varieties.

If G is unipotent, then the set of characters can be identified with the $\bar{k}$-points of a k-scheme $\mathrm{G}^{\vee}$; see again Section 2.2. If $\mathrm{S} \subset \widehat{\mathrm{G}}$ is algebraic (i.e., the disjoint union of the sets $\widetilde{\mathrm{S}}(k_n)$ for some subvariety $\widetilde{\mathrm{S}}$ of G), then the condition $\mathrm{ccodim}(\mathrm{S}) \geqslant i$ implies that the codimension of $\widetilde{\mathrm{S}}$ in $\mathrm{G}^{\vee}$ is at least i. Conversely, if $\widetilde{\mathrm{S}}$ is a closed subvariety of $\mathrm{G}^{\vee}$ over k, then $\mathrm{ccodim}(\widetilde{\mathrm{S}}(\bar{k})) \geqslant \mathrm{codim}_{\mathrm{G}^{\vee}}(\widetilde{\mathrm{S}})$.

Let G be a semiabelian variety over k. Let ℓ be a prime different from the characteristic of k. The set of ℓ-adic characters of G can be naturally identified with the set of $\overline{\mathbf{Q}}_\ell$-points of a $\overline{\mathbf{Q}}_\ell$-scheme, as we now recall. Let $\pi_1^t(\mathrm{G}_{\bar{k}})$ be the geometric tame étale fundamental group of G (see, for instance, the paper [81] of Kerz and Schmidt for various equivalent definitions; note that it is well-known that semiabelian varieties have good compactifications), and let $\Pi(\mathrm{G}, \overline{\mathbf{Q}}_\ell)$ be the group of continuous characters $\chi\colon \pi_1^t(\mathrm{G}_{\bar{k}}) \to \overline{\mathbf{Q}}_\ell^{\times}$. For any $n \geqslant 1$ and $\chi \in \widehat{\mathrm{G}}(k_n)$, the character sheaf $\mathscr{L}_\chi$ is tamely ramified (indeed, only the case of tori requires proof; since the question is geometric, we may assume that $\mathrm{G} = \mathbf{G}_m^d$ for some integer $d \geqslant 0$, and the result follows by induction from the well-known case of $\mathbf{G}_m$ and the multiplicativity of the tame fundamental group [108, Th. 5.1]), and hence corresponds to a point in $\Pi(\mathrm{G}, \overline{\mathbf{Q}}_\ell)$. For each $n \geqslant 1$, this leads to a natural injective map

$$\widehat{\mathrm{G}}(k_n) \hookrightarrow \Pi(\mathrm{G}, \overline{\mathbf{Q}}_\ell),$$

and we will identify $\widehat{\mathrm{G}}(k_n)$ this way with a subset of $\Pi(\mathrm{G}, \overline{\mathbf{Q}}_\ell)$.

There is a decomposition

$$\Pi(\mathrm{G}, \overline{\mathbf{Q}}_\ell) = \Pi(\mathrm{G}, \overline{\mathbf{Q}}_\ell)_{\ell'} \times \Pi(\mathrm{G}, \overline{\mathbf{Q}}_\ell)_{\ell},$$

where $\Pi(\mathrm{G}, \overline{\mathbf{Q}}_\ell)_{\ell'}$ is the group of torsion characters of order prime to ℓ and $\Pi(\mathrm{G}, \overline{\mathbf{Q}}_\ell)_\ell$ is the group of characters that factor through the maximal pro-ℓ quotient $\pi_1^t(\mathrm{G}_{\bar{k}})_\ell$ of $\pi_1^t(\mathrm{G}_{\bar{k}})$. Since $\pi_1^t(\mathrm{G}_{\bar{k}})_\ell$ is a free $\mathbf{Z}_\ell$-module of finite rank, by a result of Brion and Szamuely [18], we can identify $\Pi(\mathrm{G}, \overline{\mathbf{Q}}_\ell)_\ell$ with the $\overline{\mathbf{Q}}_\ell$-points of a scheme $\Pi(\mathrm{G})_\ell$, following the arguments of Gabber and Loeser [50, Section 3.3].

Letting $\Pi(\mathrm{G})$ be the disjoint union of the schemes $\Pi(\mathrm{G})_\ell$ indexed by $\chi \in \Pi(\mathrm{G}, \overline{\mathbf{Q}}_\ell)_{\ell'}$, we get

$$\Pi(\mathrm{G}, \overline{\mathbf{Q}}_\ell) = \Pi(\mathrm{G})(\overline{\mathbf{Q}}_\ell),$$

and as above we will identify $\widehat{\mathrm{G}}$ with a subset of $\Pi(\mathrm{G})(\overline{\mathbf{Q}}_\ell)$.

Let G' be a semiabelian variety over k and $f\colon \mathrm{G} \to \mathrm{G}'$ a homomorphism. There is a dual morphism $\Pi(\mathrm{G}') \to \Pi(\mathrm{G})$, denoted by $\chi \mapsto \chi \circ f$; if f is an inclusion, we also write simply $\chi \circ f = \chi_{|\mathrm{G}}$. The restriction of this map to the subset $\widehat{\mathrm{G}}'$ is the map $\widehat{f}\colon \widehat{\mathrm{G}}' \to \widehat{\mathrm{G}}$ previously defined.

Definition 1.23. Let G be a semiabelian variety over a finite field k, and let ℓ be a prime different from the characteristic of k.

(1) A subset $\mathrm{S} \subset \Pi(\mathrm{G})(\overline{\mathbf{Q}}_\ell)$ is a *translate of an algebraic cotorus* (abbreviated *tac*) if there exists a surjective morphism $\pi\colon \mathrm{G}_{\bar{k}} \to \mathrm{G}'$ of semiabelian varieties over $\bar{k}$, with non-trivial *connected* kernel, and a character $\chi_0 \in \Pi(\mathrm{G})(\overline{\mathbf{Q}}_\ell)$ such that

$$\mathrm{S} = \{\chi_0 \cdot (\chi' \circ \pi) \in \Pi(\mathrm{G})(\overline{\mathbf{Q}}_\ell) \mid \chi' \in \Pi(\mathrm{G}')(\overline{\mathbf{Q}}_\ell)\}.$$

We then say that S is *defined by the quotient* $\mathrm{G}_{\bar{k}} \to \mathrm{G}'$ *and the character* χ_0, and that S *has dimension* $\dim(\mathrm{G}'_{\bar{k}})$. The kernel of π is also called the *kernel of the tac*. If G' and π are defined over a finite extension k' of k, then we say that S is a tac of $\mathrm{G}_{k'}$.

(2) We say that a subset $\mathrm{S} \subset \Pi(\mathrm{G})(\overline{\mathbf{Q}}_\ell)$ contains *most* characters if the complement of S is contained in a finite union of tacs.

(3) We say that a subset $\mathrm{S} \subset \Pi(\mathrm{G})(\overline{\mathbf{Q}}_\ell)$ is *weakly generic* if it is a generic set in the sense of the Zariski topology in $\Pi(\mathrm{G})$, *i.e* it contains a dense open subset of $\Pi(\mathrm{G})$.

By extension, we shall say that a subset $\mathrm{S} \subset \widehat{\mathrm{G}}$ *contains most characters*, or is *weakly generic*, if its image in $\Pi(\mathrm{G})(\overline{\mathbf{Q}}_\ell)$ satisfies this property.

Remark 1.24. (1) The terminology "most" is used by Krämer and Weissauer [93]; Esnault and Kerz [36] speak of "quasi-linear" subsets. What we call "weakly generic" is usually called "generic" (see, for example, the papers [93], [90] and [50]).

(2) Let $\mathrm{S} \subset \Pi(\mathrm{G})(\overline{\mathbf{Q}}_\ell)$ be a subset that contains most characters. The Lang–Weil estimates imply that $\mathrm{S} \cap \widehat{\mathrm{G}}$ is generic in the sense of (1.12). Also, if $\mathrm{S} \subset \widehat{\mathrm{G}}$ is a generic set and $\Pi(\mathrm{G})(\overline{\mathbf{Q}}_\ell) - \mathrm{S}$ is not Zariski-dense, then S is weakly generic.

(3) The tac defined by π and χ_0 can also be interpreted as the set of characters χ such that the restriction of χ to $\ker(\pi)$ is equal to that of χ_0.

(4) If a tac S of G has dimension i, then $\mathrm{S} \cap \widehat{\mathrm{G}}$ has character codimension $\geqslant \dim(\mathrm{G}) - i$ since

$$|(\mathrm{S} \cap \widehat{\mathrm{G}})(k_n)| \leqslant |\mathrm{G}'(k_n)| \ll |k|^{ni}$$

if S is defined by the quotient $\mathrm{G} \to \mathrm{G}'$ and the character χ_0.

Lemma 1.25. *Let* G *be a semiabelian variety over a finite field* k. *Let* ℓ *be a prime different from the characteristic of* k. *Let* I *be a non-empty finite set and let* $(\mathrm{S}_i)_{i \in \mathrm{I}}$ *be a family of tacs in* G, *defined by quotient morphisms* $\pi_i\colon \mathrm{G}_{\bar{k}} \to \mathrm{G}_{i,\bar{k}}$ *and characters* $\chi_i \in \Pi(\mathrm{G})(\overline{\mathbf{Q}}_\ell)$.

Let K *be the subgroup of* $\mathrm{G}_{\bar{k}}$ *generated by the subgroups* $\ker(\pi_i)$. *The intersection* $\mathrm{S} = \bigcap \mathrm{S}_i$ *is non-empty if and only if the restriction of* χ_i *to* K *is independent of* i.

If this is the case, then S *is a tac, which is defined by the quotient morphism* $\pi\colon \mathrm{G}_{\bar{k}} \to \mathrm{G}_{\bar{k}}/\mathrm{K}$ *and any of the characters* χ_i.

Proof. We write $\mathrm{K}_i = \ker(\pi_i)$ for $i \in \mathrm{I}$. Since each K_i is connected by definition, the subgroup K generated by the K_i is also connected.

A character $\chi \in \Pi(\mathrm{G})(\overline{\mathbf{Q}}_\ell)$ belongs to S_i if and only if $\chi|_{\mathrm{K}_i} = \chi_i|_{\mathrm{K}_i}$. If $\chi \in \mathrm{S}$, then the restriction of χ_i to K must coincide with the restriction of χ to K, and is therefore independent of i.

Conversely, if this condition is satisfied, then pick any $i_0 \in \mathrm{I}$. The tac defined by $\mathrm{G}_{\bar{k}} \to \mathrm{G}_{\bar{k}}/\mathrm{K}$ and the character χ_{i_0} consists of characters χ such that $\chi|_{\mathrm{K}} = \chi_{i_0}|_{\mathrm{K}}$. This condition is equivalent to $\chi|_{\mathrm{K}_i} = \chi_{i_0}|_{\mathrm{K}_i}$ for all $i \in \mathrm{I}$. Since $\chi_i|_{\mathrm{K}_i} = \chi_{i_0}|_{\mathrm{K}_i}$, this tac is exactly the intersection of the S_i. □

1.10. Fourier–Mellin transforms on semiabelian varieties

Let k be a finite field and G a semiabelian variety over k. Let ℓ be a prime different from the characteristic of k. We use the notation of the previous section.

We recall here some results of Gabber and Loeser for tori [50], generalized by Krämer [90] to semiabelian varieties.

Let R be the ring of integers of a finite extension of $\mathbf{Q}_\ell$ and $\Omega_{\mathrm{G}} = \mathrm{R}[\![\pi_1^t(\mathrm{G}_{\bar{k}})_\ell]\!]$. We have

$$\Pi(\mathrm{G})_\ell = \mathrm{Spec}(\overline{\mathbf{Q}}_\ell \otimes_{\mathrm{R}} \Omega_{\mathrm{G}}).$$

Let $p\colon \mathrm{G}_{\bar{k}} \to \mathrm{Spec}(\bar{k})$ be the structural morphism. We denote by $\mathrm{can}_{\mathrm{G}}$ the tautological character

$$\mathrm{can}_{\mathrm{G}}\colon \pi_1^t(\mathrm{G}_{\bar{k}})_\ell \to \Omega_{\mathrm{G}}^\times,$$

which defines a lisse Ω_{G}-sheaf of rank one $\mathscr{L}_{\mathrm{G}}$ on $\mathrm{G}_{\bar{k}}$. Given an object N of $\mathrm{D}_c^{\mathrm{b}}(\mathrm{G}_{\bar{k}}, \mathrm{R})$, one defines the *Fourier–Mellin transforms of* N, with and without compact support, as the objects

$$\mathrm{FM}_!(\mathrm{N}) = \mathrm{R}p_!(\mathrm{N} \otimes_{\mathrm{R}} \mathscr{L}_{\mathrm{G}})$$
$$\mathrm{FM}_*(\mathrm{N}) = \mathrm{R}p_*(\mathrm{N} \otimes_{\mathrm{R}} \mathscr{L}_{\mathrm{G}})$$

of the category $\mathrm{D}_c^{\mathrm{b}}(\bar{k}, \Omega_{\mathrm{G}}) = \mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\Omega_{\mathrm{G}})$. Inverting ℓ and passing to the direct limit over all $\mathrm{R} \subset \overline{\mathbf{Q}}_\ell$ and all $\chi \in \Pi(\mathrm{G}, \overline{\mathbf{Q}}_\ell)_{\ell'}$, we then get two functors

$$\mathrm{FM}_!, \mathrm{FM}_*\colon \mathrm{D}_c^{\mathrm{b}}(\mathrm{G}_{\bar{k}}) \to \mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\Pi(\mathrm{G})),$$

where $\mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\Pi(\mathrm{G}))$ is the derived category of the category of coherent sheaves on $\Pi(\mathrm{G})$.

By (the generalization of) [50, Cor. 3.3.2], for an object N of $\mathrm{D}_c^{\mathrm{b}}(\mathrm{G}_{\bar{k}})$ and every $\chi \in \Pi(\mathrm{G})(\overline{\mathbf{Q}}_\ell)$, viewed as a closed immersion $i_\chi\colon \{\chi\} \to \Pi(\mathrm{G})$, there are canonical isomorphisms

$$\mathrm{L}i_\chi^*\mathrm{FM}_!(\mathrm{N}) \simeq \mathrm{R}p_!(\mathrm{N}_\chi) \quad \text{and} \quad \mathrm{L}i_\chi^*\mathrm{FM}_*(\mathrm{N}) \simeq \mathrm{R}p_*(\mathrm{N}_\chi),$$

where $\mathrm{L}i_\chi$ indicates *left-derived functors.*

1.11. A geometric lemma

A connected commutative algebraic group G is said to be *almost simple* if it has no proper connected closed subgroup. Examples of such groups are $\mathbf{G}_a$, $\mathbf{G}_m$ and simple abelian varieties.

We will use the following lemma in the proof of the general higher vanishing theorem.

Lemma 1.26. *Let k be a field. Let $s \geqslant 0$ be an integer. We denote $[s] = \{1, \ldots, s\}$. Let*

$$\mathrm{G} = \prod_{i=1}^{s} \mathrm{G}_i$$

be a product of almost simple connected commutative algebraic groups over k. Let $d = \dim(\mathrm{G})$.

For any subset $\mathrm{I} \subset [s]$, let

$$\mathrm{G}_{\mathrm{I}} = \prod_{i \in \mathrm{I}} \mathrm{G}_i,$$

which we identify with a subgroup of G *in the obvious way.*

Let $1 \leqslant i \leqslant d$. Let $\mathscr{E}_i$ be the set of subsets I *such that* $\dim(\mathrm{G_I}) > d - i$. *For each* $\mathrm{I} \in \mathscr{E}_i$, *let* $\mathrm{H_I}$ *be a non-trivial connected subgroup of* $\mathrm{G_I}$. *Then the algebraic subgroup* H *generated by all* $\mathrm{H_I}$ *has dimension at least* i.

Proof. We denote $d_i = \dim(\mathrm{G}_i)$ for $1 \leqslant i \leqslant s$.

We work by induction on s, and for each s, by induction on i. The case $s = 1$ is elementary, since $\{1\} \in \mathscr{E}_i$ then, hence $\mathrm{H} = \mathrm{H}_{[s]} = \mathrm{G}$ in that case. For any s, the result is also elementary for $i = 1$, since for $\mathrm{I} = [s] \in \mathscr{E}_1$, we have $\dim(\mathrm{H}) \geqslant \dim(\mathrm{H_I}) \geqslant 1$. Assume now that $2 \leqslant i \leqslant g$ and that the result is known for (s, i') for $i' < i$ as well as for (s', i) for any $s' < s$.

The subgroup $\mathrm{H}_{[s]} \subset \mathrm{G}$ is non-trivial, and hence there exists some integer $j \leqslant s$ such that the image of $\mathrm{H}_{[s]}$ under the projection $\mathrm{G} \to \mathrm{G}_j$ is non-trivial; this means that this image must be equal to G_j since all G_i are almost simple. Up to reordering the factors, we may assume that the projection of $\mathrm{H}_{[s]}$ on G_s is surjective.

If $d_s \geqslant i$, then we are done since we then have $\dim(\mathrm{H}_{[s]}) \geqslant \dim(\mathrm{G}_s) = d_s \geqslant i$. We therefore assume now that $d_s < i$.

Let $\mathrm{G}' = \mathrm{G}_1 \times \cdots \times \mathrm{G}_{s-1}$ and $i' = i - d_s$. The dimension of G' is $d' = d - d_s$. We have $1 \leqslant i' \leqslant d'$ and $d - i = d' - i'$. Each $\mathrm{J} \subset [s-1]$ with $\dim(\mathrm{G}'_\mathrm{J}) > d' - i' = d - i$ is an element of $\mathscr{E}_i$. By induction, applied to the subgroups $\mathrm{H_J}$ for $\mathrm{J} \in \mathscr{E}_{i'}$, the subgroup H' of G' generated by all $\mathrm{H_J}$ has dimension $\geqslant i' = i - d_s$.

To conclude, we observe that since H' is a subgroup of G' with dimension $\geqslant i - d_s$ and $\mathrm{H}_{[s]}$ is a subgroup of $\mathrm{G} = \mathrm{G}' \times \mathrm{G}_s$ such that the projection of $\mathrm{H}_{[s]}$ to G_s is surjective, the subgroup H that they generate together satisfies

$$\begin{aligned}\dim(\mathrm{H}) &= \dim(\mathrm{H}') + \dim(\mathrm{H}_{[s]}) - \dim(\mathrm{H}' \cap \mathrm{H}_{[s]})\\ &\geqslant \dim(\mathrm{H}') + \dim(\mathrm{H}_{[s]}) - \dim(\mathrm{G}' \cap \mathrm{H}_{[s]}) \geqslant i - d_s + d_s = i\end{aligned}$$

since $\dim(\mathrm{G}' \cap \mathrm{H}_{[s]}) + \dim(\mathrm{G}_s) = \dim(\mathrm{H}_{[s]})$. ☐

1.12. Geometric and arithmetic semisimplicity

Let k be a finite field, and $\bar{k}$ an algebraic closure of k. Let ℓ be a prime different from the characteristic of k.

For an algebraic variety X over k and a complex M in $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X}, \overline{\mathbf{Q}}_\ell)$, we will sometimes refer to properties of M (e.g., M being a simple or semisimple perverse sheaf) as *arithmetic*, and to the analogue for the base change of M to $\mathrm{M}_{\bar{k}}$ as being *geometric*. Thus we may speak of a geometrically simple perverse sheaf, or an arithmetically semisimple perverse sheaf.

We collect here some facts about certain relations between such properties.

Lemma 1.27. *Let* X *a geometrically irreducible algebraic variety over* k *and* $\mathscr{F}$ *a lisse* ℓ*-adic sheaf on* X. *If* $\mathscr{F}$ *is arithmetically semsimple, then it is geometrically semisimple.*

Proof. Using the correspondence between lisse sheaves and representations of the étale fundamental group, this follows, e.g., from [119, Lem. 5 (a)]. ☐

Lemma 1.28. *Let* (X, u) *be a quasi-projective variety over* k. *Let* M *be an arithmetically simple perverse sheaf on* X. *There exists a finite extension of* k *of degree bounded in terms of* $c_u(\mathrm{M})$ *such that the base change of* M *to* $\mathrm{X}_{k'}$ *is a direct sum of geometrically simple perverse sheaves on* k'.

In particular, M *is geometrically semisimple.*

Proof. By [8, Prop. 5.3.9 (ii)], there exists an integer $n \geqslant 1$ and a geometrically simple perverse sheaf N on X_{k_n} such that $M = f_{n*}N$, where $f_n \colon X_{k_n} \to X$ is the base change morphism. Since N is non-zero, we deduce that $n \ll c_u(M)$ by looking at the rank at a generic point of the support. The base change of M to k_n is then a direct sum of geometrically simple perverse sheaves. ☐

Lemma 1.29. *Let k be a finite field and $\bar{k}$ an algebraic closure of k. Let ℓ be a prime different from the characteristic of k. Let X be a smooth and geometrically connected quasi-projective variety over k. Two perverse sheaves M et N on X which are geometrically simple are geometrically isomorphic if and only if there exists $\alpha \in \overline{\mathbf{Q}}_\ell$ such that $M \simeq \alpha^{\deg} \otimes N$.*

This is a standard fact (see, e.g., [103, Lemme 4.4.4]).

1.13. A result from representation theory

The following basic fact from the representation theory of reductive groups will play a crucial role.

Proposition 1.30. *Let F be a field of characteristic zero and let G be a reductive algebraic group over F. Let V be a finite-dimensional faithful representation of G over F. Any finite-dimensional irreducible representation of G over F occurs in a tensor power $(V \oplus V^\vee)^{\otimes m}$ for some integer $m \geqslant 0$, where $V^\vee$ is the contragredient of V.*

See, for instance, [29, Prop. 3.1] for the proof.

CHAPTER 2

Generic vanishing theorems

Throughout this chapter, k denotes a finite field, $\bar{k}$ an algebraic closure of k, and k_n the extension of degree n of k inside $\bar{k}$ for each $n \geqslant 1$. We also fix once for all a prime number ℓ different from the characteristic of k. All complexes of sheaves and characters are tacitly understood to be ℓ-adic complexes and characters for this choice of ℓ.

2.1. Statement of the vanishing theorems

We now state our main vanishing theorems.

Theorem 2.1 (Generic vanishing). *Let* G *be a connected commutative algebraic group over* k *and let* M *be a perverse sheaf on* G*. The set of characters* $\chi \in \widehat{\mathrm{G}}$ *satisfying*

$$\begin{aligned} \mathrm{H}^i(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = \mathrm{H}^i_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = 0 \quad &\textit{for all } i \neq 0, \\ \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = \mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)& \end{aligned} \tag{2.1}$$

is generic in the sense of Definition 1.22.

This gives the first part of Theorem 1 from the introduction.

Remark 2.2. Various versions of Theorem 2.1 have been proved by the following authors:

(1) Katz–Laumon [77, Th. 2.1.3, Scholie 2.3.1] in the case of powers of the additive group and Saibi [112, Th. 3.1] in the case of unipotent groups; in both cases, the generic set is a Zariski-dense open subset of the k-scheme parameterizing characters.

(2) Gabber–Loeser [50, Cor. 2.3.2] for tori, with "generic" replaced by a condition implying "weakly-generic" in the sense of Definition 1.23; see also [50, Th. 7.2.1], for "most" characters in codimension 1.

(3) Weissauer [127, Vanishing Th., p. 561] for abelian varieties, with "generic" replaced by "most", and Krämer [90, Th. 2.1] for semiabelian varieties, with "weakly generic" characters.

We will in fact prove the following stronger result, which also controls the "stratification" arising from the non-vanishing of other cohomology groups; this has a number of useful applications.

Theorem 2.3 (Stratified vanishing). *Let* G *be a connected commutative algebraic group of dimension* d *over* k*, and* M *a perverse sheaf on* G*. There exist subsets*

$$\mathscr{S}_d \subset \cdots \subset \mathscr{S}_0 = \widehat{\mathrm{G}}$$

such that the following holds:

(1) *For* $0 \leqslant i \leqslant d$*, the subset* $\mathscr{S}_i$ *has character codimension at least* i*.*

(2) *For $0 \leqslant i \leqslant d$, any $\chi \in \widehat{\mathrm{G}}$ such that at least one of the cohomology groups*

$$\mathrm{H}^i(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi), \qquad \mathrm{H}^{-i}(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi), \qquad \mathrm{H}^i_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi), \qquad \mathrm{H}^{-i}_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) \tag{2.2}$$

is non-zero belongs to $\mathscr{S}_i$.

(3) *For $\chi \in \mathscr{S}_0 - \mathscr{S}_1$, the equality $\mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = \mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ holds.*

(4) *If G is a torus or an abelian variety, then $\mathscr{S}_i$ is a finite union of tacs of G of dimension $\leqslant d - i$.*

(5) *If G is a unipotent group, then $\mathscr{S}_i$ is the set of closed points of a closed subvariety of dimension $\leqslant d - i$ of the Serre dual $\mathrm{G}^\vee$.*

Concretely, this implies that for $0 \leqslant i \leqslant d$, the estimate

$$|\{\chi \in \widehat{\mathrm{G}}(k_n) \mid \mathrm{H}^i_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) \neq 0 \text{ or } \mathrm{H}^{-i}_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) \neq 0 \\ \text{or } \mathrm{H}^i(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) \neq 0 \text{ or } \mathrm{H}^{-i}(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) \neq 0\}| \ll |k_n|^{d-i}$$

holds for all $n \geqslant 1$, and so this implies the second part of Theorem 1.

Note that Theorem 2.1 is a straightforward consequence of Theorem 2.3, since the set of characters satisfying (2.1) contains the generic set $\mathscr{S}_0 - \mathscr{S}_1$.

Remark 2.4. We expect that this result should be true with the stronger information that the implied constants in (1.11) for the subsets $\mathscr{S}_i$ depend only on the complexity of M. A result of this type would be especially useful for applications to "horizontal" equidistribution theorems.

However, we can only prove this at the current time in the following cases:

(1) if G is a unipotent group (use the equality of Fourier transforms of [112, Th. 3.1] combined with Theorem 1.7);

(2) if G is a geometrically simple abelian variety (see Corollary 2.19);

(3) and probably, although we have not checked this in full details, if $\mathrm{G} = \mathrm{U} \times \mathbf{G}_m$ where U is unipotent.

The issues that arise in attempting to handle the general case are:

– For tori, the use of de Jong's theorem on alterations, where we do not control the number of exceptional components that appear (thus, a suitably effective version of de Jong's theorem, or an effective form of embedded resolution of singularities, would probably imply the desired conclusion in this case).

– For abelian varieties, the need to find and control the complexity of an alteration that "moderates" certain perverse sheaves, to apply results of Orgogozo.

Corollary 2.5. *Let G be a connected commutative algebraic group over k. Let M be an object of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$. Then for generic $\chi \in \widehat{\mathrm{G}}$ and any $i \in \mathbf{Z}$, we have canonical isomorphisms*

$$\mathrm{H}^i_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) \simeq \mathrm{H}^i(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) \simeq \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, {}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M}_\chi)).$$

Proof. The proof is similar to that of [90, Cor. 2.3]; see also the proof of Corollary 2.18 below. ☐

We will prove Theorem 2.3 in Section 2.5. Before doing this, we need to establish some preliminaries concerning perverse sheaves on the basic building blocks of Proposition 1.13, namely (in rough order of difficulty) unipotent groups, tori and abelian varieties.

Note that proving either Theorem 1 or Theorem 2 for a given group G only involves the corresponding material for groups of the types which actually appear in Proposition 1.13 applied to G. In particular, for instance, the proof of Theorem 3 (and other similar statements) only depends on the case of tori, i.e., on Section 2.3.

To facilitate orientation, we list below the key statements about each type of groups; Section 2.5 only requires these statements from the next three sections.

(1) Unipotent groups: Proposition 2.7.

(2) Tori: Corollary 2.15.

(3) Abelian varieties: Corollary 2.26 and the auxiliary Theorem 2.23, due to Orgogozo [109].

2.2. The case of unipotent groups

We begin by summarizing the duality theory of commutative unipotent groups; a good account can also be found in [15, App. F].

Let U be a connected unipotent commutative algebraic group over a finite field k of characteristic p. The functor that sends a perfect k-scheme S (i.e., a scheme for which the absolute Frobenius is an automorphism) to the extension group

$$\mathrm{Ext}^1(\mathrm{U}\times_k \mathrm{S}, \mathbf{Q}_p/\mathbf{Z}_p) = \varinjlim_m \mathrm{Ext}^1(\mathrm{U}\times_k \mathrm{S}, p^{-m}\mathbf{Z}_p/\mathbf{Z}_p)$$

in the category of commutative group schemes over S (with $\mathbf{Q}_p/\mathbf{Z}_p$ viewed as a constant group scheme) is representable by a connected commutative group scheme U^* over k, called the *Serre dual* of U. This goes back to a remark by Serre [117, p. 55] and was developed by Bégueri in [6, Prop. 1.2.1] and Saibi [112]. Morever, if $m \geqslant 1$ is such that $p^m\mathrm{U} = 0$, then the natural morphism

$$\mathrm{Ext}^1(\mathrm{U}\times_k \mathrm{S}, p^{-m}\mathbf{Z}_p/\mathbf{Z}_p) \longrightarrow \varinjlim_m \mathrm{Ext}^1(\mathrm{U}\times_k \mathrm{S}, p^{-m}\mathbf{Z}_p/\mathbf{Z}_p)$$

is an isomorphism.

Let A be a finite abelian group. For each integer $n \geqslant 1$, the short exact sequence

$$1 \longrightarrow \mathrm{U}(k_n) \longrightarrow \mathrm{U}_{k_n} \xrightarrow{x\longmapsto \mathrm{Fr}_{k_n}(x)\cdot x^{-1}} \mathrm{U}_{k_n} \longrightarrow 1$$

induces an isomorphism

$$\mathrm{Hom}(\mathrm{U}(k_n), \mathrm{A}) \xrightarrow{\sim} \mathrm{Ext}^1(\mathrm{U}_{k_n}, \mathrm{A})$$

(see [15, Prop. F.2]).

Let $m \geqslant 1$ be such that $p^m\mathrm{U} = 0$. We take $\mathrm{A} = p^{-m}\mathbf{Z}_p/\mathbf{Z}_p \simeq \mathbf{Z}/p^m\mathbf{Z}$. For any integer $n \geqslant 1$, we obtain an isomorphism

$$\mathrm{Hom}(\mathrm{U}(k_n), \mathrm{A}) \xrightarrow{\sim} \mathrm{U}^*(k_n).$$

Fix now a faithful character $\psi\colon p^{-m}\mathbf{Z}_p/\mathbf{Z}_p \to \overline{\mathbf{Q}}_\ell^\times$. We then obtain, for any $n \geqslant 1$, an isomorphism

$$\mathrm{U}^*(k_n) \xrightarrow{\sim} \widehat{\mathrm{U}}(k_n).$$

Saibi [112, Lemma 1.5.4.1] (see also [15, Remark F.1 (ii)]) proved that there exists a connected commutative unipotent algebraic group $\mathrm{U}^\vee$ and a bi-extension $\mathscr{L}_{\mathrm{U},\mathrm{U}^\vee}$ of $\mathrm{U}^\vee \times \mathrm{U}$ by $\mathbf{Q}_p/\mathbf{Z}_p$ such that the bi-extension induces an isomorphism between the perfectization of $\mathrm{U}^\vee$ and U^*. Together with the above character ψ, this induces isomorphisms

$$\beta_n\colon \mathrm{U}^\vee(k_n) \to \widehat{\mathrm{U}}(k_n)$$

for all $n \geqslant 1$. (See also [15, Remark F.4 (ii)] for a different approach to the construction of the finite-type model $\mathrm{U}^\vee$.) We also write ψ_x for the character $\beta_n(x)$.

We denote by $\mathscr{L}_{\mathrm{U},\mathrm{U}^\vee,\psi}$ the associated lisse ℓ-adic sheaf of rank 1 on $\mathrm{U}^\vee \times \mathrm{U}$; its trace functions are given by

$$t_n(x, y; k_n) = \beta_n(x)(y)$$

for all $n \geqslant 1$ and $(x, y) \in \mathrm{U}(k_n) \times \mathrm{U}^\vee(k_n)$.

Example 2.6. Fix a non-trivial additive character $\psi\colon k \to \overline{\mathbf{Q}}_\ell$. Suppose that $\mathrm{U} = \mathbf{G}_a^d$ for some $d \geqslant 0$. We denote

$$x \cdot y = \sum_{i=1}^{d} x_i y_i$$

for $(x, y) \in \mathrm{U} \times \mathrm{U}$.

There exists a choice of bi-extension with $\mathrm{U}^\vee = \mathrm{U}$, and the isomorphisms

$$\beta_n\colon (\mathbf{G}_a^d)(k_n) \longrightarrow \widehat{\mathbf{G}}_a^d(k_n)$$

are given by $x \mapsto \psi_x$ where

$$\psi_x(y) = \psi(\mathrm{Tr}_{k_n/k}(x \cdot y)).$$

We come back to the general case. Let $p\colon \mathrm{U} \times_k \mathrm{U}^\vee \to \mathrm{U}$ and $p^\vee\colon \mathrm{U} \times_k \mathrm{U}^\vee \to \mathrm{U}^\vee$ denote the projections. The *Fourier transform* is the equivalence of categories $\mathrm{FT}_\psi\colon \mathrm{D}_c^{\mathrm{b}}(\mathrm{U}) \to \mathrm{D}_c^{\mathrm{b}}(\mathrm{U}^\vee)$ defined by

$$\mathrm{FT}_\psi(\mathrm{M}) = \mathrm{R}p_!^\vee(p^*(\mathrm{M}) \otimes \mathscr{L}_{\mathrm{U},\mathrm{U}^\vee,\psi}) = \mathrm{R}p_*^\vee(p^*(\mathrm{M}) \otimes \mathscr{L}_{\mathrm{U},\mathrm{U}^\vee,\psi}),$$

where the second equality (more precisely, the fact that the natural transformation "forget supports" from the left-hand side to the right-hand side is an isomorphism) is [112, Th. 3.1]. A corollary of this is that the Fourier transform is compatible with Verdier duality, in that there is a canonical functorial isomorphism

$$\mathrm{D}(\mathrm{FT}_\psi(\mathrm{M})) \simeq \mathrm{FT}_{\psi^{-1}}(\mathrm{D}(\mathrm{M}))(\dim \mathrm{U})$$

for each object M of $\mathrm{D}_c^{\mathrm{b}}(\mathrm{U})$, see [112, Cor. 3.2.1]. We refer the reader to Saibi's article [112] for the other main properties of the ℓ-adic Fourier transform on unipotent groups, such as the formula for the inverse Fourier transform.

By the proper base change theorem and the definition of Fourier transform using $p_!^\vee$, for all $a \in \mathrm{U}^\vee(\bar{k})$ and $i \in \mathbf{Z}$, there are natural isomorphisms

$$\mathrm{H}_c^i(\mathrm{U}_{\bar{k}}, \mathrm{M}_{\psi_a}) = \mathscr{H}^i(\mathrm{FT}_\psi(\mathrm{M}))_a. \tag{2.3}$$

Since unipotent groups are affine, it follows from Artin's vanishing theorem that the Fourier transform shifts the perverse degree by the dimension of U. In particular, if M is perverse, then so is $\mathrm{FT}_\psi(\mathrm{M})[\dim(\mathrm{U})]$.

Proposition 2.7. *Let* U *be a connected unipotent commutative algebraic group of dimension d over k. Fix a locally-closed immersion u (resp. $u^\vee$) of* U *(resp. $\mathrm{U}^\vee$) into some projective space to compute the complexity. Let* M *be an object of $\mathrm{D}_c^{\mathrm{b}}(\mathrm{U})$ of perverse amplitude $[a, b]$.*

There exists an integer $\mathrm{C} \geqslant 0$, depending only on $c_u(\mathrm{M})$, and a stratification (S_i) of $\mathrm{U}^\vee$ such that every strat S_i is either empty or has dimension $d - i$, with the following properties:

1. *The sum of the degrees of the irreducible components of $u(\mathrm{S}_i)$ is at most* C.
2. *For each $\xi \in \mathrm{S}_i(\bar{k})$, the vanishing $\mathrm{H}_c^j(\mathrm{U}_{\bar{k}}, \mathrm{M}_{\psi_\xi}) = 0$ holds for all $j \notin [a, b + i]$.*

In particular, the estimate

$$|\mathrm{S}_i(k_n)| \ll |k_n|^{d-i} \tag{2.4}$$

holds for all $n \geqslant 1$, with an implicit constant that only depends on $c_u(\mathrm{M})$.

Moreover, for any $\xi \in \mathrm{S}_0(\bar{k})$ and any $j \in \mathbf{Z}$, we have

$$\mathrm{H}^j_c(\mathrm{U}_{\bar{k}}, \mathrm{M}_{\psi_\xi}) = \mathrm{H}^j(\mathrm{U}_{\bar{k}}, \mathrm{M}_{\psi_\xi}).$$

Proof. Since the Fourier transform shifts the perverse degree by d, the complex $\mathrm{FT}_\psi(\mathrm{M})$ has perverse amplitude $[a+d, b+d]$. By Theorem 1.5, the complexity $c_u(\mathrm{FT}_\psi(\mathrm{M}))$ is bounded in terms of $c_u(\mathrm{M})$.

By Theorems 1.7 and 1.8, there exists a smooth open subscheme $\mathrm{S}_0 \subset \mathrm{U}^\vee$, with closed complement Y_0 of degree bounded in terms of $c_u(\mathrm{FT}_\psi(\mathrm{M}))$, and hence in terms of $c_u(\mathrm{M})$, such that the restriction of $\mathrm{FT}_\psi(\mathrm{M})$ to S_0 has lisse cohomology sheaves and such that $\mathrm{FT}_\psi(\mathrm{M})$ is of formation compatible with any base change $\mathrm{S}' \to \mathrm{S}_0 \subset \mathrm{U}^\vee$ (this follows from the formula for the Fourier transform in terms of $p^\vee_*$). Up to replacing S_0 by a smaller open subset we may assume that S_0 is affine (and this does not increase the complexity of the complement).

In particular, using (2.3) and this compatibility, we obtain the following equality for $a \in \mathrm{S}_0(\bar{k})$:

$$\mathrm{H}^i_c(\mathrm{U}_{\bar{k}}, \mathrm{M}_{\psi_a}) = \mathscr{H}^i(\mathrm{FT}_\psi(\mathrm{M}))_a = \mathrm{H}^i(\mathrm{U}_{\bar{k}}, \mathrm{M}_{\psi_a}).$$

By a slight generalization of [8, Cor. 4.1.10. *ii*], the pullback by a closed immersion with affine complement of a complex of perverse amplitude $[a, b]$ has perverse amplitude $[a-1, b]$. Therefore, the restriction of $\mathrm{FT}_\psi(\mathrm{M})$ to Y_0 has perverse amplitude $[a+d-1, b+d]$. Proceeding by induction, we construct a stratification $(\mathrm{S}_i)_{0 \leqslant i \leqslant d}$ of $\mathrm{U}^\vee$ into strats S_i such that

(1) each S_i is smooth, empty or equidimensional of dimension $d - i$;
(2) the closure of each S_i has degree bounded in terms of $c_u(\mathrm{M})$;
(3) the restriction of $\mathrm{FT}_\psi(\mathrm{M})$ to each S_i has lisse cohomology sheaves and is of perverse amplitude $[a+d-i, b+d]$.

Let $0 \leqslant i \leqslant d$. On each connected component of S_i, the support of the cohomology sheaves of $\mathrm{FT}_\psi(\mathrm{M})$ is either empty or equal to S_i (since these sheaves are lisse). However, the definition of perversity implies the inequality

$$\dim \operatorname{supp} \mathscr{H}^j(\mathrm{FT}_\psi(\mathrm{M})_{|\mathrm{S}_i}) \leqslant -j + b + d$$

for all integers j. Since S_i has dimension $d - i$, the non-vanishing of $\mathscr{H}^j(\mathrm{FT}_\psi(\mathrm{M})_{|\mathrm{S}_i})$ implies therefore the inequality

$$d - i \leqslant -j + b + d, \quad \text{i.e. } j \leqslant b + i.$$

Since S_i is smooth of dimension $d-i$ (so the dualizing complex on S_i is $\overline{\mathbf{Q}}_\ell[d-i](d-i)$ and the Verdier dual of a lisse sheaf is the naive dual) and the cohomology sheaves on S_i are lisse, duality implies that $\mathrm{D}(\mathrm{FT}_\psi(\mathrm{M})_{|\mathrm{S}_i})$ also has lisse cohomology sheaves, given by the formula

$$\mathscr{H}^j(\mathrm{D}(\mathrm{FT}_\psi(\mathrm{M}))_{|\mathrm{S}_i}) = (\mathscr{H}^{-j-2d+2i}(\mathrm{FT}_\psi(\mathrm{M})_{|\mathrm{S}_i}))^\vee(d-i)$$

for all j.

Thus, arguing as above, the perversity condition shows that the condition $\mathscr{H}^j(\mathrm{FT}_\psi(\mathrm{M})_{|\mathrm{S}_i}) \neq 0$ implies

$$d - i \leqslant j + 2d - 2i - a - d + i, \quad \text{i.e. } j \geqslant a.$$

We conclude that the cohomology sheaves of the complex $\mathrm{FT}_\psi(\mathrm{M})_{|\mathrm{S}_i}$ are concentrated in degrees $[a, b+i]$. By proper base change, this implies assertion (2) of the proposition and concludes the proof. ∎

Remark 2.8. This result is a generalization to all unipotent groups, and a quantification by means of the complexity, of some of the Fouvry–Katz–Laumon stratification results for additive exponential sums [77, 39]. It may have interesting applications to analytic number theory, since the quantitative form means that it may be used over varying finite fields, e.g. $\mathbf{F}_p$ as $p \to +\infty$ as in Chapter 6 (although a referee pointed out that the complexity of the Fourier transform of a perverse sheaf on a non-additive unipotent group, such as Witt vectors of length 2, will usually have to depend on p).

2.3. Perverse sheaves on tori

In this section, we generalize some of the results of Gabber and Loeser [50] about perverse sheaves on tori. We begin with a generalization of [50, Th. 4.1.1'], which is proved in loc. cit. under the assumption that resolution of singularities and simplification of ideals hold for varieties of dimension at most the dimension of the torus in question. The structure of our proof is the same, but we are able to replace the appeal to resolution of singularities with de Jong's theorem on alterations [24].

Theorem 2.9. *Let* T *be a torus over* $\bar{k}$ *and let* M *be an object of* $\mathrm{D}_c^{\mathrm{b}}(\mathrm{T})$. *For all characters* $\chi \in \Pi(\mathrm{T})(\overline{\mathbf{Q}}_\ell)$ *outside of a finite union of tacs, the equality* $\mathrm{H}^i(\mathrm{T}, \mathrm{M}_\chi) = \mathrm{H}^i_c(\mathrm{T}, \mathrm{M}_\chi)$ *holds for all* $i \in \mathbf{Z}$.

As in [50], the proof of Theorem 2.9 relies on the auxiliary proposition stated below. We pick a smooth compactification of T by a simple normal crossing divisor $j\colon \mathrm{T} \to \bar{\mathrm{T}}$ (for example, the projective space), and denote by $i\colon \bar{\mathrm{T}} - \mathrm{T} \to \bar{\mathrm{T}}$ the complementary closed immersion. Given any morphism $\varphi\colon \mathrm{W} \to \bar{\mathrm{T}}$ of varieties over k, denote by $j_{\mathrm{W}}\colon \varphi^{-1}(\mathrm{T}) \to \mathrm{W}$ and $i_{\mathrm{W}}\colon \varphi^{-1}(\bar{\mathrm{T}} - \mathrm{T}) \to \mathrm{W}$ the corresponding open and closed immersions. Recall the Ω_{T}-sheaf of rank one $\mathscr{L}_{\mathrm{T}}$ on T from Section 1.10. In particular, for $\mathrm{N} \in \mathrm{D}_c^{\mathrm{b}}(\varphi^{-1}(\mathrm{T}))$ and $\xi \in \varphi^{-1}(\bar{\mathrm{T}} - \mathrm{T})$, the stalk $j_{\mathrm{W}*}(\mathrm{N} \otimes \varphi^*(\mathscr{L}_{\mathrm{T}})))_\xi$ is a complex of coherent sheaves on $\Pi(\mathrm{T})$.

Proposition 2.10. *With notation as above, let* N *be an object of* $\mathrm{D}_c^{\mathrm{b}}(\varphi^{-1}(\mathrm{T}))$. *There exists a finite union* $\mathscr{S}$ *of tacs in* $\widehat{\mathrm{T}}$ *such that, for any* $r \geqslant 0$ *and any* $\xi \in \varphi^{-1}(\bar{\mathrm{T}} - \mathrm{T})$, *the support of the module* $(\mathrm{R}^r j_{\mathrm{W}*}(\mathrm{N} \otimes \varphi^*(\mathscr{L}_{\mathrm{T}})))_\xi$ *is contained in* $\mathscr{S}$.

Proof. The idea of the proof is to reduce to the situation of [50, Prop. 4.3.1'].

We use induction on the dimension of W. We can then readily assume that N is a lisse sheaf on a locally-closed irreducible subvariety U of $\varphi^{-1}(\mathrm{T})$, extended by zero to $\varphi^{-1}(\mathrm{T})$. We can assume further that U is dense in W. Now the monodromy of N can be assumed to be pro-ℓ. Indeed, consider the finite étale cover $f\colon \mathrm{U}' \to \mathrm{U}$ associated to the ℓ-Sylow subgroup of the monodromy group of N, and let W' be the normalization of W in the function field of U. The sheaf N is a direct factor of $f_*f^*\mathrm{N}$, and it suffices to prove the theorem for $f^*\mathrm{N}$ and W'. Hence, we assume that the monodromy of N is pro-ℓ.

By de Jong's theorem [24, Th. 4.1], there exists an alteration $f\colon \mathrm{W}' \to \mathrm{W}$ such that W' is smooth and the reduction of the complement of $f^{-1}(\mathrm{U})$ in W' is a strict normal crossing divisor. Since we are working over a perfect field, we can further assume that the alteration f is generically étale. Hence, there exists a dense open subset U_0 of U such that f is finite étale over $f^{-1}(\mathrm{U}_0)$. By induction, it is enough to prove the result for U_0 and $\mathrm{N}_{|\mathrm{U}_0}$, and hence by the same argument as above, it is enough to prove it for $f_*f^*\mathrm{N}_{|\mathrm{U}_0}$. By proper base change, it is then enough to prove the result for W' and $f^*\mathrm{N}_{|\mathrm{U}_0}$. By a last dévissage, it is finally enough to prove it for $f^*\mathrm{N}$.

We are now in a situation where we can suppose that W is smooth, that the complements of $\varphi^{-1}(\mathrm{T})$ and U in W are strict normal crossing divisors, and that the monodromy of N is pro-ℓ.

This is exactly the situation at the end of the proof of [50, Prop. 4.3.1', starting from p. 544, line -4] (with N replacing A there) and the remaining argument is identical to that of loc. cit. □

PROOF OF THEOREM 2.9. The fact that Proposition 2.10 implies Theorem 2.9 is completely similar to the fact that Proposition 4.3.1' implies Théorème 4.1.1' in [50]. We keep the notation introduced before the statement of Proposition 2.10, and apply Proposition 2.10 with $\mathrm{W} = \bar{\mathrm{T}}$, with φ the identity morphism and $\mathrm{N} = \mathrm{M}$, so that $j_{\mathrm{W}} = j$ and $i_{\mathrm{W}} = i$.

Let $\chi \in \widehat{\mathrm{T}}$ such that χ does not belong to the finite number of tacs of $\widehat{\mathrm{T}}$ given by Proposition 2.10. According to [50, Prop. 4.5.1 (2)], this implies that the object $i^*\mathrm{R}j_*(\mathrm{M}_\chi) \in \mathrm{D}^{\mathrm{b}}_c(\bar{\mathrm{T}} - \mathrm{T})$ is trivial, and hence its cohomology complex

$$\mathrm{R}\Gamma(\bar{\mathrm{T}} - \mathrm{T}, i^*\mathrm{R}j_*(\mathrm{M}_\chi))$$

is also trivial. But this last complex is isomorphic to the cone of the morphism

$$\mathrm{R}s_!(\mathrm{M}_\chi) \to \mathrm{R}s_*(\mathrm{M}_\chi),$$

where $s\colon \mathrm{T}_{\bar{k}} \to \mathrm{Spec}(\bar{k})$ is the structure morphism, hence the theorem. □

We now use Proposition 2.10 to deduce a relative version of Theorem 2.9.

THEOREM 2.11. *Let T be a torus over $\bar{k}$, let S be an arbitrary scheme over $\bar{k}$, and let $\mathrm{G} = \mathrm{S} \times \mathrm{T}$. Denote by $p\colon \mathrm{G} \to \mathrm{S}$ the projection. Let $\mathrm{N} \in \mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$.*

For $\chi \in \Pi(\mathrm{T})(\overline{\mathbf{Q}}_\ell)$ away from a finite union of tacs $\mathscr{S}$, we have $\mathrm{R}p_!(\mathrm{N}_\chi) = \mathrm{R}p_*(\mathrm{N}_\chi)$.

In particular, if N is a perverse sheaf, then for χ not in $\mathscr{S}$, the complex $\mathrm{R}p_!(\mathrm{N}_\chi) = \mathrm{R}p_*(\mathrm{N}_\chi)$ is a perverse sheaf on S.

PROOF. This is similar to Theorem 2.9. We apply Proposition 2.10 with $\mathrm{W} = \mathrm{S} \times_{\bar{k}} \bar{\mathrm{T}}$, and check that, for each character χ away from the finite union of tacs given by the proposition, the object $i^*_{\mathrm{W}}\mathrm{R}j_{\mathrm{W}*}(\mathrm{N} \otimes \mathscr{L}_\chi)$ is trivial, which follows from the immediate extension of [50, Prop. 4.7.2 (ii)] to an arbitrary base scheme S (instead of just tori). □

THEOREM 2.12. *Let T be a d-dimensional torus over k, let S be an arbitrary scheme over k, and define $\mathrm{X} = \mathrm{T} \times \mathrm{S}$. Let i be an integer such that $1 \leqslant i \leqslant d$.*

Let M be a perverse sheaf on X. There exist a finite extension k' of k and a family $(\mathrm{S}_f)_{f\in\mathscr{F}}$ of tacs of $\mathrm{T}_{k'}$ of dimension $\leqslant d - i$ with the property that for any $\chi \in \widehat{\mathrm{T}}_{k'}$ which does not belong to the union of the S_f there exists a quotient torus $\mathrm{T}_{k'} \to \mathrm{Z}$ of dimension $i - 1$ such that

$$\mathrm{R}q_{\mathrm{S}!}\mathrm{M}_\chi = \mathrm{R}q_{\mathrm{S}*}\mathrm{M}_\chi$$

and this complex is perverse on $\mathrm{B} \times_{k'} \mathrm{S}_{k'}$.

PROOF. Up to replacing k by a finite extension, we can assume that T_k is split, and thus reduce to $\mathrm{T} = \mathbf{G}_m^d$.

Now let $1 \leqslant i < d$. For each subset I of $[d] = \{1, \ldots, d\}$ of size $i - 1$, we apply Theorem 2.11 with $(\mathrm{T}, \mathrm{S}) = (\mathbf{G}_m^{[d]-\mathrm{I}}, \mathbf{G}_m^{\mathrm{I}} \times \mathrm{S})$ over $\bar{k}$, so that the projection p in the theorem is then the canonical projection

$$q_{\mathrm{I}}\colon \mathbf{G}_m^d \times \mathrm{S} = \mathbf{G}_m^{[d]} \times \mathrm{S} \to \mathbf{G}_m^{\mathrm{I}} \times \mathrm{S}.$$

We obtain a finite union of tacs of $\mathbf{G}_m^{[d]-\mathrm{I}}$ such that for characters χ of $\mathbf{G}_m^{[d]-\mathrm{I}}$ outside of this finite union, we have

$$\mathrm{R}q_{\mathrm{I}!}(\mathrm{M}_\chi) = \mathrm{R}q_{\mathrm{I}*}(\mathrm{M}_\chi)$$

and this complex is perverse.

Let

$$(\pi_{\mathrm{I},j}\colon \mathbf{G}_m^{[d]-\mathrm{I}} \to \mathrm{Y}_{\mathrm{I},j}, \chi_{\mathrm{I},j})_{j\in \mathrm{X}_\mathrm{I}}$$

be the quotient morphisms and characters defining this finite family of tacs. For $j \in \mathrm{X}_\mathrm{I}$, we define $\mathrm{K}_{\mathrm{I},j} = \ker(\pi_{\mathrm{I},j})$; this is a non-trivial subtorus of $\mathbf{G}_m^{[d]-\mathrm{I}}$, which we identify with a subtorus of $\mathbf{G}_m^d$ using the canonical embedding $\mathbf{G}_m^{[d]-\mathrm{I}} \to \mathbf{G}_m^d$. In addition, we define $\chi'_{\mathrm{I},j} \in \Pi(\mathbf{G}_m^d)(\overline{\mathbf{Q}}_\ell)$ to be the character that is trivial on $\mathbf{G}_m^\mathrm{I}$ and coincides with $\chi_{\mathrm{I},j}$ on $\mathbf{G}_m^{[d]-\mathrm{I}}$.

Let $\mathscr{F}$ be the set of all maps f from the subsets of $[d]$ of size $i-1$ to the disjoint union of the X_I that send a subset I to an element $j \in \mathrm{X}_\mathrm{I}$ for each I; this set is finite. For $f \in \mathscr{F}$, let S_f be the intersection of the tacs of $\mathbf{G}_m^d$ defined by

$$(\mathbf{G}_m^d \to \mathbf{G}_m^d/\mathrm{K}_{\mathrm{I},f(\mathrm{I})}, \chi'_{\mathrm{I},f(\mathrm{I})}).$$

We claim that the family $(\mathrm{S}_f)_{f\in\mathscr{F}}$ (to be precise, the subfamily where S_f is not empty) satisfies the assertions of the theorem.

Indeed, first of all Lemma 1.25 shows that S_f is either empty or is again a tac; moreover, in the second case, it is defined by the projection $\mathbf{G}_d^m \to \mathbf{G}_m^d/\mathrm{T}_f$ where T_f is the subtorue of $\mathbf{G}_m^d$ generated by the $\mathrm{K}_{\mathrm{I},f(\mathrm{I})}$ (as subtori of $\mathbf{G}_m^d$). By Lemma 1.26 applied to $\mathrm{G}_i = \mathbf{G}_m$ for all i and the subgroups $\mathrm{K}_{\mathrm{I},f(\mathrm{I})}$, we have $\dim(\mathrm{T}_f) \geqslant i$ for all such f, and hence the quotient

$$p_f\colon \mathbf{G}_m^d \to \mathrm{Y}_f = \mathbf{G}_m^d/\mathrm{T}_f$$

has image of dimension $\leqslant d-i$, as desired.

Finally, let $\chi \in \widehat{\mathbf{G}}_m^d$ be a character that does not belong to any of the tacs S_f. This implies that there exists some $f \in \mathscr{F}$, some subset $\mathrm{I} \subset [d]$ of size $i-1$ and some $j \in \mathrm{X}_\mathrm{I}$ such that the restriction χ_I of χ to $\mathbf{G}_m^{[d]-\mathrm{I}}$ is not equal to $\chi_{\mathrm{I},j}$.

We can write $\chi = \chi_\mathrm{I}\chi'$ where χ' is a character of $\mathbf{G}_m^\mathrm{I}$. Then, considering the quotient $q\colon \mathbf{G}_m^d \to \mathbf{G}_m^\mathrm{I}$, the base change q_S is the canonical projection q_I and from the application of Theorem 2.11 to q_I, we obtain

$$\mathrm{R}_{q\mathrm{S}*}(\mathrm{M}_\chi) = \mathrm{R}_{q\mathrm{S}*}(\mathrm{M}_{\chi_\mathrm{I}}) \otimes \mathscr{L}_{\chi'} = \mathrm{R}_{q\mathrm{S}!}(\mathrm{M}_{\chi_\mathrm{I}}) \otimes \mathscr{L}_{\chi'} = \mathrm{R}q_{\mathrm{S}!}(\mathrm{M}_\chi),$$

and the fact that this object is perverse. ☐

We deduce two corollaries that are sometimes more convenient for applications. The first one is Theorem 2.3 for tori.

Corollary 2.13. *Let* T *be a torus of dimension d over k. Let $\mathrm{M} \in \mathbf{Perv}(\mathrm{T})$. For $-d \leqslant i \leqslant d$, the sets*

$$\{\chi \in \widehat{\mathrm{T}} \mid \mathrm{H}^i(\mathrm{T}_{\bar{k}}, \mathrm{M}_\chi) \neq 0\}, \qquad \{\chi \in \widehat{\mathrm{T}} \mid \mathrm{H}^i_c(\mathrm{T}_{\bar{k}}, \mathrm{M}_\chi) \neq 0\}$$

are contained in a finite union of tacs of T *of dimension $\leqslant d-|i|$, and in particular they have character codimension at least $|i|$.*

Proof. We apply Theorem 2.12 to $|i|$ and claim that the characters in either of these sets belong to the union of the tacs S_f that arise. Indeed, if χ is not in any S_f, then there exists a quotient torus $\mathrm{T}_{k'} \to \mathrm{Z}$ of dimension $i-1$ such that $\mathrm{R}q_{\mathrm{S}!}\mathrm{M}_\chi = \mathrm{R}q_{\mathrm{S}*}\mathrm{M}_\chi$, and hence

$$\mathrm{H}^i(\mathrm{T}_{\bar{k}}, \mathrm{M}_\chi) = \mathrm{H}^i(\mathrm{B}_{\bar{k}}, \mathrm{R}q_{\mathrm{S}*}\mathrm{M}_\chi) = 0$$

since $\mathrm{R}q_{\mathrm{S}*}\mathrm{M}_\chi$ is a perverse sheaf and $\dim(\mathrm{B}) = i-1$. The argument is similar for cohomology with compact support. ☐

Remark 2.14. We recall that, concretely, this corollary implies that for $|i| \leqslant d$, the estimate

$$\Big|\{\, \chi \in \widehat{\mathrm{T}}(k_n) \mid \mathrm{H}^i(\mathrm{T}_{\bar{k}}, \mathrm{M}_\chi) \neq 0 \text{ or } \mathrm{H}^i_c(\mathrm{T}_{\bar{k}}, \mathrm{M}_\chi) \neq 0 \,\}\Big| \ll |k_n|^{d-|i|}$$

holds for all $n \geqslant 1$.

The following "stratified" statement is also a useful formulation of the result.

Corollary 2.15. *Let* T *be a torus of dimension* d *over* k *and* S *a variety over* k*. Set* $\mathrm{X} = \mathrm{T} \times \mathrm{S}$ *and let* q *denote the projection* $q\colon \mathrm{X} \to \mathrm{S}$*. Let* M *be a perverse sheaf on* X*. There exists a finite extension* k' *of* k *and a partition of* $\widehat{\mathrm{T}}_{k'}$ *into subsets* $(\mathrm{S}_i)_{0 \leqslant i \leqslant d}$ *of character codimension* $\geqslant i$ *such that, for any* i *and* $\chi \in \mathrm{S}_i$*, the object* $\mathrm{R}q_!(\mathrm{M}_\chi)$ *of* $\mathrm{D}^{\mathrm{b}}_c(\mathrm{S})$ *has perverse amplitude* $[0, i]$*.*

Proof. Using the notation of the proof of the theorem, for any integer i with $1 \leqslant i \leqslant d$, let k'_i be the finite extension arising from its application to i and let $\mathscr{F}_i$ be the corresponding family of tacs. Define $\widetilde{\mathrm{S}}_i$ to be the union of the S_f for $f \in \mathscr{F}_i$ for $1 \leqslant i \leqslant d$.

Let k' be the compositum of all k'_i. Define $\mathrm{S}_0 = \widehat{\mathrm{T}} - \widetilde{\mathrm{S}}_1$ and $\mathrm{S}_i = \widetilde{\mathrm{S}}_i - \widetilde{\mathrm{S}}_{i+1}$ for $1 \leqslant i \leqslant d$. These sets form a partition of $\widehat{\mathrm{T}}_{k'}$, and since $\mathrm{S}_i \subset \widetilde{\mathrm{S}}_i$ for $i \geqslant 1$, they have character codimension $\geqslant i$. This property is also clear for $i = 0$.

Let $0 \leqslant i \leqslant d$, and let $\chi \in \mathrm{S}_i$. Then $\chi \notin \widetilde{\mathrm{S}}_{i+1}$, and hence the theorem provides a projection $q_{\mathrm{S}}\colon \mathbf{G}_m^d \times \mathrm{S} \to \mathrm{Z} \times \mathrm{S}$ with $\dim(\mathrm{Z}) = i$ such that $\mathrm{R}q_{\mathrm{S}!}\mathrm{M}_\chi$ is perverse. Composing with the projection $r\colon \mathrm{Z} \times \mathrm{S} \to \mathrm{S}$, which is affine and hence such that $\mathrm{R}r_!$ preserves objects with perverse amplitude $[0, +\infty]$ (by Artin's vanishing theorem), it follows that $\mathrm{R}q_!\mathrm{M}_\chi$ has perverse amplitude $[0, i]$. ☐

2.4. Perverse sheaves on abelian varieties

In this section, we will review and extend some results of Krämer and Weissauer on perverse sheaves on abelian varieties.

2.4.1. Statement of the results and corollaries. Let k be a finite field, and $\bar{k}$ an algebraic closure of k.

Let X be an abelian variety over k. We fix a projective embedding u of X_k. For subvarieties of X, the degree means the degree of the image by u; for a tac of S defined by $\pi\colon \mathrm{X} \to \mathrm{A}$ and χ, we will say that the *degree* of S is the degree of the image $u(\ker(\pi))$.

For a perverse sheaf M on X, a combination of the main result of Weissauer [127] and of the machinery developped by Krämer and Weissauer [93] implies that for most characters $\chi \in \widehat{\mathrm{X}}$, we have $\mathrm{H}^i(\mathrm{X}_{\bar{k}}, \mathrm{M}_\chi) = 0$ for all $i \neq 0$; we will show here that this result can be made quantitative using the complexity of M, and will then establish a relative version (see Section 2.4.3).

Theorem 2.16. *Let* X *be an abelian variety over* k*. Let* M *be a perverse sheaf on* X*.*

There exist an integer $c \geqslant 0$ *depending only on* $c_u(\mathrm{M})$*, a finite extension* k' *of* k *of degree* $\leqslant c$*, and a finite family* $(\mathrm{S}_f)_{f \in \mathrm{F}}$ *of tacs of* $\mathrm{X}_{k'}$ *with* $|\mathrm{F}| \leqslant c$*, each of degree at most* c*, such that any* $\chi \in \widehat{\mathrm{X}}_{k'}$ *which does not belong to the union of the* S_f *satisfies*

$$\mathrm{H}^i(\mathrm{X}_{\bar{k}}, \mathrm{M}_\chi) = 0$$

for all $i \neq 0$*.*

We will prove this below, but first we establish some corollaries.

Corollary 2.17. *Let* $\mathrm{M} \in \mathrm{D}_c^{\mathrm{b}}(\mathrm{X})$ *be a complex on* X.

There exist an integer $c \geqslant 0$*, depending only on* $c_u(\mathrm{M})$*, a finite extension* k' *of* k *of degree* $\leqslant c$*, and a finite family* $(\mathrm{S}_f)_{f\in\mathrm{F}}$ *of tacs of* $\mathrm{X}_{k'}$*, each of degree at most* c*, with* $|\mathrm{F}| \leqslant c$*, such that for any* $\chi \in \widehat{\mathrm{X}}_{k'}$ *which does not belong to the union of the* S_f*, there is a canonical isomorphism*

$$\mathrm{H}^i(\mathrm{X}_{\bar{k}}, \mathrm{M}_\chi) \simeq \mathrm{H}^0(\mathrm{X}_{\bar{k}}, {}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M})_\chi)$$

for all $i \in \mathbf{Z}$.

Proof. This is the same argument as in the proof of Corollary 2.5; the dependency on $c_u(\mathrm{M})$ is obtained by means of Proposition 1.9 to control the perverse cohomology sheaves of M. ☐

Alternatively, the next corollary may be more convenient for applications.

Corollary 2.18. *Let* $\mathrm{M} \in \mathrm{D}_c^{\mathrm{b}}(\mathrm{X}_k)$ *be a complex on* X. *The set* $\mathscr{S}$ *of characters* $\chi \in \widehat{\mathrm{X}}$ *such that we have isomorphisms*

$$\mathrm{H}^i(\mathrm{X}_{\bar{k}}, \mathrm{M}_\chi) \simeq \mathrm{H}^0(\mathrm{X}_{\bar{k}}, {}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M})_\chi)$$

for all $i \in \mathbf{Z}$ *is generic, and the implicit constant in* (1.12) *depends only on* $c_u(\mathrm{M})$.

In particular, if M *is a perverse sheaf, then the set of* χ *such that* $\mathrm{H}^i(\mathrm{X}_{\bar{k}}, \mathrm{M}_\chi) = 0$ *for all* $i \neq 0$ *is generic and the implicit constant in* (1.12) *depends only on* $c_u(\mathrm{M})$.

Proof. Assume first that M is a perverse sheaf. We apply Theorem 2.16 to M, and use the notation there. For $n \geqslant 1$, let $k'_n = k'k_n$. For any $\chi \in \widehat{\mathrm{X}}(k_n) - \mathscr{S}(k_n)$, the corresponding character in $\widehat{\mathrm{X}}(k'_n)$ belongs to $\mathrm{S}_f(k'_n)$ for some $f \in \mathrm{F}$. Let A_f be the abelian variety such that S_f is defined by $\pi_f \colon \mathrm{X}_{k'} \to \mathrm{A}_f$; we have

$$|\widehat{\mathrm{X}}(k_n) - \mathscr{S}(k_n)| \leqslant \sum_{f\in\mathrm{F}} |\mathrm{A}_f(k'_n)| \leqslant |\mathrm{F}|\ (|k'k_n|^{1/2} + 1)^{2\dim(\mathrm{A}_f)} \ll |k_n|^{\dim(\mathrm{X})-1},$$

where the implied constant depends only on $c_u(\mathrm{M})$ by the theorem.

Now in the general case, recalling that ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M}_\chi)$ is canonically isomorphic to ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M})_\chi$ for all i and all χ, we have the convergent perverse spectral sequences

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(\mathrm{X}_{\bar{k}}, {}^{\mathfrak{p}}\mathscr{H}^j(\mathrm{M})_\chi) \Rightarrow \mathrm{H}^{i+j}(\mathrm{X}_{\bar{k}}, \mathrm{M}_\chi).$$

By the previous case applied to each of the finitely many perverse cohomology sheaves, the set of χ such that $\mathrm{H}^i(\mathrm{X}_{\bar{k}}, {}^{\mathfrak{p}}\mathscr{H}^j(\mathrm{M})_\chi) = 0$ for all $i \neq 0$ and all j is generic; for any such character, the spectral sequence degenerates and we obtain isomorphisms

$$\mathrm{H}^i(\mathrm{X}_{\bar{k}}, \mathrm{M}_\chi) \simeq \mathrm{H}^0(\mathrm{X}_{\bar{k}}, {}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M})_\chi).$$

Applying Proposition 1.9, we see that the last statement concerning the implicit constant in (1.12) holds. ☐

Corollary 2.19. *Let* X *be a geometrically simple abelian variety over* k. *Let* M *be a perverse sheaf on* X. *Then there exists a constant* c *depending only on* $c_u(\mathrm{M})$ *and a finite set* $\mathscr{S} \subset \Pi(\mathrm{X})(\overline{\mathbf{Q}}_\ell)$ *of cardinality at most* c *such that for* $\chi \in \Pi(\mathrm{X})(\overline{\mathbf{Q}}_\ell) - \mathscr{S}$,

$$\mathrm{H}^i(\mathrm{X}_{\bar{k}}, \mathrm{M}_\chi) = 0 \text{ for } i \neq 0.$$

Proof. Since X is a geometrically simple abelian variety, then a tac of X contains a single character. Hence, the result follows from Theorem 2.16. ☐

2.4.2. Proof of the results. We now proceed with the proof of Theorem 2.16. As we indicated, the first ingredient is a quantitative version of a result of Weissauer [127].

Proposition 2.20. *Let* X *be an abelian variety over* k *with a projective embedding* u*, and let* M *be a geometrically simple perverse sheaf on* X *such that* $\chi(\mathrm{X}_{\bar{k}}, \mathrm{M}) = 0$*.*

There exists a tac S *on* X *with kernel an abelian subvariety* A *of degree bounded in terms of* $c_u(\mathrm{M})$*, such that*

$$\bigoplus_i \mathrm{H}^i(\mathrm{X}_{\bar{k}}, \mathrm{M}_\chi) \neq 0$$

if and only if χ *is in* S*.*

Moreover, M *is invariant by translation by* A*.*

Proof. For any perverse sheaf N on X, we denote by $\mathscr{S}_1(\mathrm{N})$ the set of characters χ such that for some $i \neq 0$ the cohomology group $\mathrm{H}^i(\mathrm{X}_{\bar{k}}, \mathrm{N})$ is non-zero.

By [127, Th. 3 and Lem. 6], there exists a maximal abelian variety $\mathrm{A}_{\bar{k}}$ of $\mathrm{X}_{\bar{k}}$ such that M is invariant by translation by $\mathrm{A}_{\bar{k}}$, and this abelian variety is non-trivial.

Denoting by $q\colon \mathrm{X}_{\bar{k}} \to \mathrm{X}_{\bar{k}}/\mathrm{A}_{\bar{k}}$ the quotient morphism, this is equivalent to the fact that M is isomorphic over $\bar{k}$ to a perverse sheaf of the form $\mathscr{L}_{\chi_0} \otimes q^*(\widetilde{\mathrm{M}})[\dim(\mathrm{A})]$ for some character $\chi_0\colon \pi_1(\mathrm{X}_{\bar{k}}) \to \overline{\mathbf{Q}_\ell}^\times$ and some simple perverse sheaf $\widetilde{\mathrm{M}}$ on $\mathrm{X}_{\bar{k}}/\mathrm{A}_{\bar{k}}$.

We claim first that $\mathrm{A}_{\bar{k}}$ is defined over k and that the degree of $\mathrm{A}_{\bar{k}}$ in the image of u is bounded in terms of $c_u(\mathrm{M})$.

The fact that $\mathrm{A}_{\bar{k}}$ is defined over k is implicit in the proof of the existence of A by Weissauer. We recall his argument. First, a perverse sheaf $\mathscr{P}_{\mathrm{M}}$ is defined as follows ([127, p. 563]): the evaluation morphism $\mathrm{ev}\colon \mathrm{M}^\vee * \mathrm{M} \to \mathbf{1}$ (see Section 1.5) induces morphisms of perverse sheaves ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M}^\vee * \mathrm{M})[-i] \to \mathbf{1}$ for all i, and each ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M}^\vee * \mathrm{M})[-i]$ is a direct sum of shifted irreducible perverse sheaves. Since by Lemma 1.15 $\dim \mathrm{Hom}(\mathrm{M}^\vee * \mathrm{M}, \mathbf{1}) = \dim \mathrm{Hom}(\mathrm{M}, \mathrm{M}) = 1$, there is a unique integer $\nu \geqslant 0$ and a unique perverse irreducible summand $\mathscr{P}_{\mathrm{M}}[-\nu]$ of ${}^{\mathfrak{p}}\mathscr{H}^\nu(\mathrm{M}^\vee * \mathrm{M})[-\nu]$ on which the restriction of ev is not zero (see [127, p. 563 and Remark (2), p. 569] for details).

Since $\chi(\mathrm{X}_{\bar{k}}, \mathrm{M}) = 0$, we have $\nu \geqslant 1$, and by [127, Lemma 2], it follows that $\mathscr{S}_1(\mathrm{M}) = \mathscr{S}_1(\mathscr{P}_{\mathrm{M}})$. Weissauer shows (see [127, Prop. 2]) that there exists an abelian subvariety A of X of dimension $\nu \geqslant 1$, with closed immersion $i\colon \mathrm{A} \to \mathrm{X}$, and a character χ_0 such that there is a geometric isomorphism $\mathscr{P}_{\mathrm{M}} \simeq \mathscr{L}_{\chi_0} \otimes i_* \overline{\mathbf{Q}}_\ell[\nu]$. Since A can therefore be recovered as the support of $\mathscr{P}_{\mathrm{M}}$, it is defined over k.

The perverse sheaf $\mathscr{P}_{\mathrm{M}}$ is invariant by translation by $\mathrm{A}_{\bar{k}}$, and then so is M by [127, Remark (2)].

Moreover, we have $c_u(\mathscr{P}_{\mathrm{M}}) \ll c_u(\mathrm{M})$ by the definition of $\mathscr{P}_{\mathrm{M}}$ and Proposition 1.9, and $\deg(u \circ i(\mathrm{A})) \ll c_u(\mathscr{P}_{\mathrm{M}}) \ll c_u(\mathrm{M})$ by Theorem 1.7.

Let $q\colon \mathrm{X} \to \mathrm{X}/\mathrm{A}$ be the quotient morphism and χ be a character not in the tac S of X_k defined by (q, χ_0^{-1}). We now compute for every $i \in \mathbf{Z}$ that

$$\mathrm{H}^i(\mathrm{X}_{\bar{k}}, \mathscr{P}_{\mathrm{M}\chi}) = \mathrm{H}^i((\mathrm{X}/\mathrm{A})_{\bar{k}}, \mathrm{R}q_*(\mathscr{P}_{\mathrm{M}\chi}) = \mathrm{H}^i((\mathrm{X}/\mathrm{A})_{\bar{k}}, \mathrm{R}q_*(\mathscr{L}_{\chi\cdot\chi_0}) \otimes \mathbf{1}[\dim(\mathrm{A})]).$$

Since χ is not in the tac S, the restriction of $\mathscr{L}_{\chi\cdot\chi_0}$ to $\mathrm{A}_{\bar{k}}$ is non-trivial, and hence we have $\mathrm{R}q_*(\mathscr{L}_{\chi\cdot\chi_0}) = 0$ by Lemma 1.16, and therefore $\mathrm{H}^i(\mathrm{X}_{\bar{k}}, \mathscr{P}_{\mathrm{M}\chi}) = 0$ for all i.

Conversely, if $\chi = \chi_0^{-1} \cdot (\widetilde{\chi} \circ q)$, then we have

$$\mathrm{H}^*(\mathrm{X}_{\bar{k}}, \mathscr{P}_{\mathrm{M}\chi}) = \mathrm{H}^*(\mathrm{A}_{\bar{k}}, \overline{\mathbf{Q}}_\ell) \otimes \mathrm{H}^*((\mathrm{X}/\mathrm{A})_{\bar{k}}, \mathscr{L}_{\widetilde{\chi}} \mathbf{1}[\dim(\mathrm{A})]),$$

by the Künneth formula, and this is non-zero. ∎

Proof of Theorem 2.16. We follow the method used by Krämer and Weissauer to prove [93, Th. 1.1], keeping track of the complexity.

Since X is an abelian variety, the two convolution products of Section 1.5 coincide; for an object M of $\mathrm{D}^{\mathrm{b}}_{c}(\mathrm{X})$ and an integer $n \geqslant 1$, we denote by M^{*n} the n-th iterated convolution product of M.

We recall the axiomatic framework of [93, Section 5], specialized to our situation as in [93, Example 5.1]. Let **D** be the full subcategory of $\mathrm{D}^{\mathrm{b}}_{c}(\mathrm{X}_{\bar{k}})$ whose objects are direct sums of shifts of geometrically semisimple perverse sheaves which are obtained by pullback from X_{k_n} for some $n \geqslant 1$. Let $\mathbf{P} \subset \mathbf{Perv}(\mathrm{X}_{\bar{k}})$ be the corresponding subcategory of perverse sheaves, namely that with objects the geometrically semisimple perverse sheaves arising by pullback from X_{k_n} for some $n \geqslant 1$. Then the categories **P** and **D** satisfy the axioms (D1), (D2), and (D3) of [93, Section 5], namely:

(D1) The category **D** is stable under degree shift, convolution and perverse truncation functors; the category **P** is the heart of this t-structure, and is a semisimple abelian category.

(D2) Any object M of **D** can be written (non-canonically) as a direct sum

$$\bigoplus_{n\in\mathbf{Z}} {}^{\mathfrak{p}}\mathscr{H}^{m}(\mathrm{M})[-m].$$

(D3) The Hard Lefschetz Theorem holds for objects of **D**.

Let **N** be the full subcategory of **D** whose objects are the complexes N such that all geometrically simple constituents of all perverse cohomology sheaves ${}^{\mathfrak{p}}\mathscr{H}^{i}(\mathrm{N})$ for $i \in \mathbf{Z}$ have Euler–Poincaré characteristic equal to 0. By [93, Cor. 6.4], the category **N** satisfies the axioms (N1), (N2), (N3) and (N4) of [93, Section 5], namely:

(N1) We have $\mathbf{N} * \mathbf{D} \subset \mathbf{N}$ and the category **N** is stable under direct sums, retracts, degree shifts, perverse truncation and duality;

(N2) If N is an object of **N**, then for most characters χ, we have $\mathrm{H}^{i}(\mathrm{X}_{\bar{k}}, \mathrm{N}_{\chi}) = 0$ for all i;

(N3) The category **N** contains all objects M of **D** such that $\mathrm{H}^{i}(\mathrm{X}_{\bar{k}}, \mathrm{N}) = 0$ for all $i \in \mathbf{Z}$;

(N4) The category **N** contains all simple objects of **P** with zero Euler–Poincaré characteristic.

(Note that we will not make use of this version of (N2).)

By [93, Theorem 9.1], every $\mathrm{M} \in \mathbf{P}$ is an **N**-multiplier, meaning that for all integers $i \neq 0$ and any integer $r \geqslant 1$, every subquotient of ${}^{\mathfrak{p}}\mathscr{H}^{i}((\mathrm{M} \oplus \mathrm{M}^{\vee})^{*r})$ lies in **N**.

We now argue as in the proof of [93, Lemma 8.2] to prove Theorem 2.16 for a perverse sheaf M on X.

Step 1. We assume that M is arithmetically simple. By Lemma 1.28, the base change of M to $\bar{k}$ is an object of **P**. We denote $g = \dim(\mathrm{X})$; by (D2), we have

$$\mathrm{M}_{\bar{k}}^{*(g+1)} \simeq \bigoplus_{m\in\mathbf{Z}} \mathrm{M}_m[m],$$

for some objects M_m of **P**, which are in fact objects of **N** for $m \neq 0$ since M is an **N**-multiplier.

By Proposition 1.9, the number of integers m such that M_m is non-zero is bounded in terms of $c_u(\mathrm{M})$, and similarly $c_u(\mathrm{M}_m)$ is bounded in terms of $c_u(\mathrm{M})$. By the semisimplicity property in (D1), each M_m is a direct sum of simple perverse sheaves in **N**, and by Proposition 1.9, the number and the complexity of these constituents are bounded in terms of $c_u(\mathrm{M})$. We denote by $\mathscr{C}$ the finite set of all these simple perverse sheaves. By Lemma 1.28, there exists a finite extension k' of k, of degree bounded in terms of $c_u(\mathrm{M})$, such that any element C of $\mathscr{C}$ is defined over k'.

We apply Proposition 2.20 to each $\mathrm{C} \in \mathscr{C}$. Let $\mathscr{S}_{\mathrm{C}}$ denote the corresponding tac; it is of degree bounded in terms of $c_u(\mathrm{M})$.

We claim that if $\chi \in \widehat{\mathrm{X}}$ does not belong to the union of the tacs $\mathscr{S}_{\mathrm{C}}$, then we have

$$\mathrm{H}^i(\mathrm{X}_{\bar{k}}, \mathrm{M}_\chi) = 0$$

for all $i \neq 0$. This statement will conclude the proof of Theorem 2.16 for M.

Let χ be a character that is not in any of the tacs $\mathscr{S}_{\mathrm{C}}$. Since $\mathrm{M}_\chi^{*(g+1)}$ is isomorphic to $(\mathrm{M}^{*(g+1)})_\chi$ and $\mathrm{H}^*(\mathrm{X}_{\bar{k}}, \mathrm{C}_\chi) = 0$ for $\chi \notin \mathscr{S}_{\mathrm{C}}$, we have

$$\mathrm{H}^i(\mathrm{X}_{\bar{k}}, \mathrm{M}_\chi^{*(g+1)}) = \mathrm{H}^i(\mathrm{X}_{\bar{k}}, \mathrm{M}_{0\chi}),$$

for any $i \in \mathbf{Z}$. The right-hand side vanishes if $|n| > g$ since M_0 is perverse. Finally, by the compatibility between convolution and the Künneth formula (see Lemma 1.15 below) we also have an isomorphism

$$\mathrm{H}^*(\mathrm{X}_{\bar{k}}, \mathrm{M}_\chi^{*(g+1)}) \simeq \mathrm{H}^*(\mathrm{X}_{\bar{k}}, \mathrm{M}_\chi)^{\otimes(g+1)},$$

and by comparing we see that only the space $\mathrm{H}^0(\mathrm{X}_{\bar{k}}, \mathrm{M}_\chi)$ may be non-zero, which establishes the claim.

Step 2. Now let M be an arbitrary perverse sheaf on X. By Proposition 1.9, the number of geometric Jordan–Hölder factors of M is bounded in terms of $c_u(\mathrm{M})$, and hence also the number of arithmetic Jordan-Hölder factors; we then apply the first step to each of the terms of a composition series for M, and deduce the corresponding result for M. ☐

2.4.3. The relative version. Our next goal is to establish a relative version of Theorem 2.16. The arguments over the complex numbers of Krämer and Weissauer in [93, Section 2] do not apply to our situation over finite fields, since they rely on Verdier stratifications. We instead use a constructibility result of Orgogozo [109], which is a stratification result, locally for the alteration topology.

Theorem 2.21. *Let* S *be a quasi-projective scheme over* k*, and let* A *be an abelian variety over* k*. Let* $\mathrm{X} = \mathrm{A} \times \mathrm{S}$*, and denote by* $f\colon \mathrm{X} \to \mathrm{S}$ *the canonical morphism. Fix a projective embedding* u *of* X.

Let $\alpha\colon \mathrm{X}' \to \mathrm{X}$ *be an alteration defined over* k*, and* $\mathscr{X}'$ *a stratification of* X'.

Let $a \leqslant b$ *be integers. Let* M *be an object of* $\mathrm{D}_c^{\mathrm{b}}(\mathrm{X})$ *with perverse amplitude* $[a, b]$ *such that* $\alpha^*\mathrm{M}$ *is tame and constructible along* $\mathscr{X}'$.

There exist an integer $d \geqslant 1$*, a finite extension* k' *of* k *and a finite family* $(\mathrm{S}_f)_{f \in \mathscr{F}}$ *of tacs of* $\mathrm{A}_{k'}$*, such that*

(1) *The integer* d *and the size of* $\mathscr{F}$ *are bounded in terms of* $c_u(\mathrm{M})$ *and the data* $(\mathrm{X}, \mathrm{X}', \alpha, \mathscr{X}')$*,*
(2) *Each tac* S_f *has degree at most* d*,*
(3) *The degree of* k' *is at most* d*,*

with the property that for any $\chi \in \widehat{\mathrm{A}}_{k'}$ *which does not belong to the union of the* S_f*, the object* $\mathrm{R}f_*(\mathrm{M}_\chi)$ *has perverse amplitude* $[a, b]$.

By [109, Prop. 1.6.7], for any object M of $\mathrm{D}_c^{\mathrm{b}}(\mathrm{X})$, there does exist an alteration $\alpha\colon \mathrm{X}' \to \mathrm{X}$ (in fact, a finite surjective morphism) and a stratification $\mathscr{X}'$ of X' such that $\alpha^*\mathrm{M}$ is tame and constructible along $\mathscr{X}'$. In particular, the following corollary follows.

Corollary 2.22. *Let* S *be a quasi-projective scheme over* k *and let* A *be an abelian variety over* k*. Define* $\mathrm{X} = \mathrm{A} \times \mathrm{S}$ *and denote* $f\colon \mathrm{A} \times \mathrm{S} \to \mathrm{S}$ *the projection.*

Let $a \leqslant b$ be integers and let M *be an object of* $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ *with perverse amplitude* $[a,b]$*. There exist a finite extension* k' *of* k *and a finite family* $(\mathrm{S}_f)_{f\in\mathscr{F}}$ *of tacs of* $\mathrm{A}_{k'}$ *such that for any character* $\chi \in \widehat{\mathrm{A}}_{k'}$ *that does not belong to the union of the* S_f*, the object* $\mathrm{R}f_*(\mathrm{M}_\chi)$ *has perverse amplitude* $[a,b]$*.*

For the proof of Theorem 2.21, we use the following special case of [109, Th. 3.1.1].

THEOREM 2.23 (Orgogozo). *Let* $f\colon \mathrm{X}\to\mathrm{Y}$ *be a proper morphism defined over* k*. Let* $\alpha\colon \mathrm{X}'\to\mathrm{X}$ *be an alteration and* $\mathscr{X}'$ *a stratification of* X'*. Then there exist an alteration* $\beta\colon \mathrm{Y}'\to\mathrm{Y}$ *and a stratification* $\mathscr{Y}'$ *of* Y' *such that for any object* M *of* $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$*, the condition that* $\alpha^*(\mathrm{M})$ *is tame and constructible along* $\mathscr{X}'$ *implies that* $\beta^*\mathrm{R}f_*\mathrm{M}$ *is tame and constructible along* $\mathscr{Y}'$*.*

PROOF OF THEOREM 2.21. By shifting and Verdier duality, it is enough to prove the weaker statement where "M is of perverse amplitude $[a,b]$" is replaced by "M is semiperverse".

Apply Theorem 2.23 to the proper morphism $f\colon \mathrm{A}\times\mathrm{S}\to\mathrm{S}$ and to the alteration α. We obtain an alteration $\beta\colon \mathrm{S}'\to\mathrm{S}$ and a stratification $\mathscr{S}'$ of S' such that $\beta^*\mathrm{R}f_*\mathrm{M}$ is tame and constructible along $\mathscr{S}'$. Note that since any $\mathscr{L}_\chi$ is lisse and tame, $\alpha^*\mathrm{M}_\chi$ is tame and constructible along $\mathscr{X}'$ (see [109, 5.2.5] for details), and hence the complex $\beta^*\mathrm{R}f_*\mathrm{M}_\chi$ is also tame and constructible along $\mathscr{S}'$ for any $\chi\in\widehat{\mathrm{A}}$.

Consider the image of the stratification $\mathscr{S}'$ by β. By Chevalley's theorem, it is a covering of S by constructible sets, but not necessarily a partition. Refine this covering and remove redundant strats in order to obtain a stratification $\mathscr{S}$ of S where all strats are equidimensional. Then refine the stratification $\mathscr{S}'$ in such a way that preimages by β of strats of $\mathscr{S}$ are union of strats of $\mathscr{S}'$ and that β induces surjective morphisms from each strat of $\mathscr{S}'$ to a strat of $\mathscr{S}$.

Let $\chi\in\widehat{\mathrm{A}}$. Even if the complex $\mathrm{R}f_*\mathrm{M}_\chi$ is not necessarily constructible along $\mathscr{S}$, it has the property that for any strat S_i of $\mathscr{S}$, the support of the restriction of each cohomology sheaf of $\mathrm{R}f_*\mathrm{M}_\chi$ to S_i is either S_i or empty, since the analogue property holds for $\beta^*\mathrm{R}f_*\mathrm{M}_\chi$ and the stratification $\mathscr{S}'$, and β is surjective from a strat of $\mathscr{S}'$ to one of $\mathscr{S}$.

Consider now the preimage of the stratification $\mathscr{S}$ by f, and also the image of the stratification $\mathscr{X}'$ of X' by α. Choose a stratification $\mathscr{X}$ of X that refines both these coverings of X, with the property that for any strats X_i and S_j of $\mathscr{X}$ and $\mathscr{S}$ such that $f(\mathrm{X}_i)\subset\mathrm{S}_j$, the restriction of f to X_i is smooth (in particular, that X_i is equidimensional above S_j). Now refine $\mathscr{X}'$ similarly to $\mathscr{S}'$, in such a way that preimages by α of strats of $\mathscr{X}$ are union of strats of $\mathscr{X}'$ and α induces surjective morphisms from any strat of $\mathscr{X}'$ to a strat of $\mathscr{X}$.

By Lemma 1.12, up to replacing k with a finite extension of degree bounded in terms of $c_u(\mathrm{M})$ (and the fixed data $(\mathrm{X},\mathrm{X}',\alpha,\mathscr{X}')$), we can assume that each strat S_i of $\mathscr{S}$ has a k-rational point s_i. We now apply Corollary 2.17 for each i to the restriction M_{s_i} of M to $f^{-1}(s_i)\simeq\mathrm{A}$ for each i, obtaining extensions k_i of k and families $(\mathrm{S}_{f,i})_{f\in\mathscr{F}_i}$ of tacs of A_{k_i} satisfying the properties of this corollary.

Let k' be the compositum of all k_i, which has degree bounded in terms of $c_u(\mathrm{M})$ and the fixed data. We claim that for any character $\chi\in\widehat{\mathrm{A}}_{k'}$ that belongs to none of the tacs $\mathrm{S}_{f,i}$ for any i, the object $\mathrm{R}f_*\mathrm{M}_\chi$ is semiperverse. This will conclude the proof.

Suppose that the claim fails for some χ. Then there exists an integer $k\in\mathbf{Z}$ such that

$$\dim\operatorname{Supp}(\mathscr{H}^k(\mathrm{R}f_*(\mathrm{M}_\chi)))>-k.$$

Since $\operatorname{Supp}(\mathscr{H}^k(\mathrm{R}f_*(\mathrm{M}_\chi)))$ is a union of strats of $\mathscr{S}$, there is a strat $\mathrm{S}_i\subset\operatorname{Supp}(\mathscr{H}^k(\mathrm{R}f_*(\mathrm{M}_\chi)))$ of $\mathscr{S}$ of dimension $>-k$. In particular, we have $\mathscr{H}^k(\mathrm{R}f_*(\mathrm{M}_\chi))_{s_i}\neq 0$. By proper base change, we have $\mathscr{H}^k(\mathrm{R}f_*(\mathrm{M}_\chi))_{s_i}=\mathrm{H}^k(\mathrm{A}_{\bar k}\times\{\,s_i\,\},\mathrm{M}_{s_i\chi})$, and hence the latter is also non-zero. From the assumption on χ and Corollary 2.17, we have

$$\mathrm{H}^k(\mathrm{A}_{\bar{k}} \times \{ s_i \}, \mathrm{M}_{s_i\chi}) \simeq \mathrm{H}^0(\mathrm{A}_{\bar{k}} \times \{ s_i \}, {}^{\mathfrak{p}}\mathscr{H}^k(\mathrm{M}_{s_i})_\chi),$$

and hence ${}^{\mathfrak{p}}\mathscr{H}^k(\mathrm{M}_{s_i}) = {}^{\mathfrak{p}}\mathscr{H}^0(\mathrm{M}_{s_i}[k]) \neq 0$. By definition of the perverse t-structure, this implies that there exists some $r \in \mathbf{Z}$ such that

$$\dim \mathrm{Supp}(\mathscr{H}^r(\mathrm{M}_{s_i})) \geqslant -r + k.$$

The support of $\mathscr{H}^r(\mathrm{M})$ is a union of strats of $\mathscr{X}$, so there exists a strat $\mathrm{X}_j \subset \mathrm{Supp}(\mathscr{H}^r(\mathrm{M}))$ of $\mathscr{X}$ with $\dim(\mathrm{X}_j \cap \mathrm{A} \times \{ s_i \}) = \dim \mathrm{Supp}(\mathscr{H}^r(\mathrm{M}_{s_i}))$. Since X_j is equidimensional over S_i and $\dim(\mathrm{S}_i) > -k$, we conclude that

$$\dim \mathrm{Supp}(\mathscr{H}^r(\mathrm{M})) \geqslant \dim(\mathrm{X}_j) \geqslant -r + k + \dim(\mathrm{S}_i) > -r,$$

contradicting the semiperversity of M. ☐

We now prove a vanishing theorem for higher cohomology groups of perverse sheaves on abelian varieties. We begin with an analogue of Theorem 2.12.

Proposition 2.24. *Let* A *be a g-dimensional algebraic variety over k, let* S *be a quasi-projective scheme over k, and define* $\mathrm{X} = \mathrm{A} \times \mathrm{S}$. *Fix a projective embedding u of* X.

Let $\alpha \colon \mathrm{X}' \to \mathrm{X}$ *be an alteration and* $\mathscr{X}'$ *a stratification of* X'.

Let i be an integer with $1 \leqslant i \leqslant g$. *Let* $a \leqslant b$ *be integers.*

Let M *be an object of* $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ *with perverse amplitude* $[a, b]$ *such that* $\alpha^*\mathrm{M}$ *is tame and constructible along* $\mathscr{X}'$. *There exist a finite extension k' of k and a family* $(\mathrm{S}_f)_{f \in \mathscr{F}}$ *of tacs of* $\mathrm{A}_{k'}$ *of dimension* $\leqslant d - i$ *with the property that for any* $\chi \in \widehat{\mathrm{A}}_{k'}$ *which does not belong to the union of the* S_f *there exists a quotient abelian variety* $q \colon \mathrm{A}_{k'} \to \mathrm{B}$ *of dimension at most* $i - 1$ *such that* $\mathrm{R}q_{\mathrm{S}*}\mathrm{M}_\chi$ *has perverse amplitude* $[a, b]$.

Moreover, the degree of k' over k and the size of $\mathscr{F}$ *depend only on* $c_u(\mathrm{M})$ *and* $(\mathrm{X}, \mathrm{X}', \alpha, \mathscr{X}')$.

Proof. As in the proof of Theorem 2.23, we can work with each perverse cohomology sheaf, and it is therefore enough to prove the proposition for $a = b = 0$, which means that M is perverse.

By Poincaré's complete reducibility theorem, up to replacing k with a finite extension, there exists an isogeny $f \colon \mathrm{A} \to \mathrm{B}$ over k where B is a product of geometrically simple abelian varieties. We first claim that it is enough to prove the proposition for B.

To see this, we assume that the statement holds for B. Consider the base change $f_{\mathrm{B}} \colon \mathrm{X} \to \mathrm{B} \times \mathrm{S}$. Since f is finite, $f_{\mathrm{B}*}(\mathrm{M})$ is perverse for every perverse sheaf M on X. By Theorem 2.23, we find an alteration $\beta \colon \mathrm{B}' \to \mathrm{B} \times \mathrm{S}$ and a stratification of B' such that $\beta^* f_{\mathrm{B}*}(\mathrm{M}_\chi)$ is tame and adapted for every M such that $\alpha^*\mathrm{M}$ is tame and adapted to $\mathscr{X}'$. Then the proposition can be applied to $f_{\mathrm{B}*}(\mathrm{M}_\chi)$. Let N be the kernel of the isogeny f. Choose up to $|\mathrm{N}|$ characters of A whose restrictions to N run over the character group of N. Then the proposition for A follows by applying the result for B to the objects $f_{\mathrm{B}*}(\mathrm{M}_\chi)$, where χ varies among this finite set of characters. This proves the claim.

So we assume that $\mathrm{A} = \mathrm{A}_1 \times \cdots \times \mathrm{A}_s$ is a product of geometrically simple abelian varieties. Set $g_j = \dim(\mathrm{A}_j)$ for all j. For any subset $\mathrm{I} \subset [s]$, let

$$\mathrm{A}_{\mathrm{I}} = \prod_{i \in \mathrm{I}} \mathrm{A}_i,$$

viewed as a subvariety of A, and let $\mathrm{A}_{\mathrm{I}}^{\perp} = \mathrm{A}_{[s]-\mathrm{I}}$ be the kernel of the canonical projection $\mathrm{A} \to \mathrm{A}_{\mathrm{I}}$.

Fix an integer $1 \leqslant i \leqslant g = \dim(\mathrm{A})$. Let $\mathscr{E}$ be the set of subsets $\mathrm{I} \subset [s]$ such that $\dim(\mathrm{A}_\mathrm{I}) < i$; for $\mathrm{I} \in \mathscr{E}$, we have $\dim(\mathrm{A}_\mathrm{I}^\perp) > g - i$.

Fix $\mathrm{I} \in \mathscr{E}$. Let $p\colon \mathrm{A} \times \mathrm{S} \to \mathrm{A}_\mathrm{I} \times \mathrm{S}$ be the projection. We apply Theorem 2.21 to p and M, i.e., with (A, S) there equal to $(\mathrm{A}_\mathrm{I}^\perp, \mathrm{A} \times \mathrm{S})$. Up to replacing k by a finite extension k', we obtain a finite family $(\mathrm{S}_{\mathrm{I},j})_{j \in \mathrm{X}_\mathrm{I}}$ of tacs of $\mathrm{A}_{\mathrm{I},k'}^\perp$ such that the object $\mathrm{R}p_*(\mathrm{M}_\chi)$ is perverse on $\mathrm{A}_\mathrm{I} \times \mathrm{S}$ for any $\chi \in \widehat{\mathrm{A}}_{\mathrm{I},k'}^\perp$ not in the union of these tacs. Let

$$(\pi_{\mathrm{I},j}, \chi_{\mathrm{I},j})_{j \in \mathrm{X}_\mathrm{I}}$$

be the projection and characters defining these tacs, and let $\mathrm{K}_{\mathrm{I},j} = \ker(\pi_{\mathrm{I},j})$, viewed as a subgroup of $\mathrm{A}_{k'}$.

Let $\mathscr{F}$ be the set of all maps f from $\mathscr{E}$ to the disjoint union of the $\mathscr{S}_\mathrm{I}$ that send a subset I to an element $j \in \mathrm{X}_\mathrm{I}$ for each I; this set is finite. For $f \in \mathscr{F}$, let S_f be the intersection of the tacs of $\mathrm{A}_{k'}$ defined by

$$(\mathrm{A}_{k'} \to \mathrm{A}_{k'}/\mathrm{K}_{\mathrm{I},f(\mathrm{I})}, \chi'_{\mathrm{I},f(\mathrm{I})})$$

for $\mathrm{I} \in \mathscr{E}$.

We claim that the family $(\mathrm{S}_f)_{f \in \mathscr{F}}$ (to be precise, the subfamily where S_f is not empty) satisfies the assertions of the theorem.

Indeed, first of all Lemma 1.25 shows that S_f is either empty or is again a tac; moreover, in the second case, it is defined by the projection $\mathrm{A}_{k'} \to \mathrm{A}_{k'}/\mathrm{B}_f$ where B_f is the abelian subvariety in $\mathrm{A}_{k'}$ generated by the $\mathrm{K}_{\mathrm{I},f(\mathrm{I})}$, viewed as subvarieties of $\mathrm{A}_{k'}$. For such f, by Lemma 1.26 applied to A and the subgroups $\mathrm{K}_{\mathrm{I},f(\mathrm{I})}$, we have $\dim(\mathrm{B}_f) \geqslant i$, and hence the quotient

$$p_f\colon \mathrm{A}_{k'} \to \mathrm{A}_{k'}/\mathrm{B}_f$$

has image of dimension $\leqslant d - i$.

Finally, let $\chi \in \widehat{\mathrm{A}}_{k'}$ be a character that does not belong to any of the tacs S_f. This implies that there exists some $f \in \mathscr{F}$, some subset $\mathrm{I} \subset \mathscr{E}$ and some $j \in \mathrm{X}_\mathrm{I}$ such that the restriction χ_I of χ to $\mathrm{A}_{\mathrm{I},k'}^\perp$ is not equal to $\chi_{\mathrm{I},j}$.

We can write $\chi = \chi_\mathrm{I}\chi'$ where χ' is a character of $\mathrm{A}_{\mathrm{I},k'}$. Then, considering the particular quotient $q\colon \mathrm{A}_{k'} \to \mathrm{A}_{\mathrm{I},k'}$, the base change q_S is the canonical projection q_I and hence

$$\mathrm{R}_{q_\mathrm{S}*}\mathrm{M}_\chi = \mathrm{R}_{q\mathrm{S}*}(\mathrm{M}_{\chi_\mathrm{I}}) \otimes \mathscr{L}_{\chi'}$$

is perverse. ☐

As in the case of tori, we state two further consequences that are useful in applications.

Corollary 2.25. *Let* A *be an abelian variety defined over* k *of dimension* g. *Let* M *be a perverse sheaf on* A. *For* $-g \leqslant i \leqslant g$, *the sets*

$$\{\chi \in \widehat{\mathrm{A}} \mid \mathrm{H}^i(\mathrm{A}_{\bar{k}}, \mathrm{M}_\chi) \neq 0\}$$

are contained in a finite union of tacs of A *of dimension* $\leqslant g - |i|$, *and in particular they have character codimension at least* $|i|$.

Proof. We argue as in the proof of Corollary 2.13 using the previous theorem (with $a = b = 0$), as we may since we have recalled that one can find an alteration α of A such that the pull-back $\alpha^*\mathrm{M}$ is tame. ☐

Corollary 2.26. *Let* A *be a* g*-dimensional algebraic variety over* k, *let* S *be a quasi-projective scheme over* k, *and define* $\mathrm{X} = \mathrm{A} \times \mathrm{S}$. *Fix a projective embedding* u *of* X *and denote by* q *the projection* $\mathrm{X} \to \mathrm{S}$.

Let $\alpha\colon \mathrm{X}' \to \mathrm{X}$ be an alteration and $\mathscr{X}'$ a stratification of X'.

Let M be a perverse sheaf on X such that $\alpha^*\mathrm{M}$ is tame and constructible along $\mathscr{X}'$. There exists a finite extension k'/k and a partition of $\widehat{\mathrm{A}}_{k'}$ into subsets $(\mathrm{S}_i)_{0\leqslant i\leqslant g}$ of character codimension $\geqslant i$ such that for any i and $\chi \in \mathrm{S}_i$, the object $\mathrm{R}q_!\mathrm{M}_\chi$ has perverse amplitude $[-i, i]$.

Moreover, for any integer $n \geqslant 1$, we have

$$|\mathrm{S}_i(k_n)| \ll |k|^{n(g-i)}, \tag{2.5}$$

where the implied constant depends only on $(c_u(\mathrm{M}), \mathrm{X}, \mathrm{X}', \alpha, \mathscr{X}')$.

Proof. We argue as in the proof of Corollary 2.15 for the first part; to deduce (2.5), we simply note for each $i \leqslant g$, the number of tacs in Proposition 2.24 is bounded in terms of the indicated data, and for each tac S of dimension i, the number of characters in $\mathrm{S}(k_n)$ is $\leqslant (|k_n|^{1/2}+1)^{2i} \ll |k|^{ni}$. ☐

2.5. Proof of the general vanishing theorem

We can now prove Theorem 2.3.

We consider the dévissage of Proposition 1.13. Namely, let A be an abelian variety, T a torus, U a unipotent group and N a finite commutative subgroup scheme of $\mathrm{A} \times \mathrm{U} \times \mathrm{T}$ such that G is isomorphic to $(\mathrm{A} \times \mathrm{U} \times \mathrm{T})/\mathrm{N}$. Further, we write $\mathrm{N} = \mathrm{N}_r \times \mathrm{N}_l$ where N_r is reduced and N_l is local.

Let M be a perverse sheaf on G.

Step 1. We claim that it is enough to prove the theorem for the group $\widetilde{\mathrm{G}} = \mathrm{A} \times \mathrm{U} \times \mathrm{T}$.

Indeed, since $\mathrm{N} = \mathrm{N}_r \times \mathrm{N}_l$, the quotient morphism $p\colon \widetilde{\mathrm{G}} \to \mathrm{G}$ can be factored as the composition of an étale isogeny and a purely inseparable one. The latter is a universal homeomorphism, and since universal homeomorphisms preserve the étale site, and since pull-back by a finite étale map preserves perversity, it follows that the pull-back $p^*(\mathrm{M})$ is perverse.

Assume that the result of Theorem 2.3 holds for $p^*(\mathrm{M})$ on $\widetilde{\mathrm{G}}$. Then we obtain the vanishing theorem for M as follows. Let $\mathscr{S}'_i$ be the subsets of loc. cit. for $p^*(\mathrm{M})$ on $\widetilde{\mathrm{G}}$, and define $\mathscr{S}_i$ to be the set of $\chi \in \widehat{\mathrm{G}}$ such that $\chi \circ p \in \mathscr{S}'_i$. Since G has the same dimension as $\widetilde{\mathrm{G}}$ and $\mathscr{S}'_i$ has character codimension i, do does $\mathscr{S}_i$.

If $\chi \in \widehat{\mathrm{G}}$, then the projection formula gives isomorphisms

$$\mathrm{H}^i(\widetilde{\mathrm{G}}_{\bar{k}}, p^*(\mathrm{M}_\chi)) = \mathrm{H}^i(\widetilde{\mathrm{G}}_{\bar{k}}, p^*(\mathrm{M})_{\chi\circ p})$$

for all $i \in \mathbf{Z}$.

The vanishing of $\mathrm{H}^i(\widetilde{\mathrm{G}}, p^*(\mathrm{M}_\chi)_{\chi\circ p})$ implies that of $\mathrm{H}^i(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$, since the latter space is a direct summand of the former. A similar argument applies for compactly-supported cohomology, which shows that the characters $\chi \in \widehat{\mathrm{G}}$ such that any of the groups (2.2) is non-zero belong to $\mathscr{S}_i$.

Finally, suppose that $\chi \in \mathscr{S}_0 - \mathscr{S}_1$, so that $\chi \circ p \in \mathscr{S}'_0 - \mathscr{S}'_1$. Since the forget support map is functorial, the forget support morphism

$$\mathrm{H}^0_c(\widetilde{\mathrm{G}}_{\bar{k}}, p^*(\mathrm{M}_\chi)) \to \mathrm{H}^0(\widetilde{\mathrm{G}}_{\bar{k}}, p^*(\mathrm{M}_\chi))$$

induces by restriction the forget support morphism

$$\mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) \to \mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi),$$

and since the former is an isomorphism (from our assumption that Theorem 2.3 holds for $\widetilde{\mathrm{G}}$), so is the latter. This concludes the proof of the claim of Step 1.

Step 2. We now assume that $G = A \times U \times T$. We fix a quasi-projective immersion u of G. Let $d_A = \dim(A)$, $d_U = \dim(U)$, $d_T = \dim(T)$, and $d = d_A + d_U + d_T = \dim(G)$. We denote by $p_T\colon A \times U \times T \to A \times U$ the canonical projection.

Up to replacing k by a finite extension, we can assume that T is split. By applying Theorem 2.12 and Corollary 2.15 with $S = A \times U$, we can partition $\widehat{T}$ into subsets $(S_i)_{0\leqslant i\leqslant d_T}$ of character codimension $\geqslant i$ such that

(1) if $\chi \in S_i$, then the complex $Rp_{T!}(M_\chi) \in D^b_c(A \times U)$ is of perverse amplitude $[-i, i]$.

(2) if $\chi \in \widehat{T} - S_1$, then $Rp_{T!}(M_\chi) = Rp_{T*}(M_\chi)$.

We now wish to apply Proposition 2.24 to $A \times U$, but we first need to find an alteration that moderates all complexes $Rp_{T!}(M_\chi)$.

Let $j\colon T \to \bar{T} = (\mathbf{P}^1)^{d_T}$ be the obvious compactification of T. By [109, Prop. 1.6.7], there exists an alteration $\alpha\colon X \to A \times U \times \bar{T}$ and a stratification $\mathscr{X}$ of X such that $\alpha^*(j_!M)$ is tame and constructible along $\mathscr{X}$. For each character $\chi \in \widehat{T}$, the sheaf $j_!(\mathscr{L}_\chi)$ is tame, and hence $\alpha^*(j_!M_\chi)$ is also constructible and tame along $\mathscr{X}$ (see [109, 5.2.5] for details).

We apply Theorem 2.23 to the proper projection $A \times U \times \bar{T} \to A \times U$. This provides us with an alteration $\beta\colon X' \to A \times U$ and a stratification $\mathscr{X}'$ of X′ such that the complex $\beta^* Rp_{T!}(M_\chi)$ is tame and constructible along $\mathscr{X}'$ for every $\chi \in \widehat{T}$. Moreover, by Proposition 1.18 and Theorem 1.5, the complexity of $Rp_{T!}(M_\chi)$ is bounded independently of $\chi \in \widehat{T}$.

We can now apply Corollary 2.26 to $S = U$ and the complexes $Rp_{T!}(M_\chi)$. For each character $\chi \in \widehat{T}$, we obtain a partition $(S_{\chi,j})_{0\leqslant j\leqslant d_A}$ of $\widehat{A}$ into subsets such that $S_{\chi,j}$ has character codimension at least j, with the property that for $(\chi, \xi) \in S_i \times S_{\chi,j}$, the complex $Rp_{A!}(Rp_{T!}(M_\chi))_\xi)$ has perverse amplitude $[-i - j, i + j]$.

By Proposition 1.18 and Theorem 1.5, the complexity of the object $Rp_{A!}(Rp_{T!}(M_\chi))_\xi)$ is bounded independently of $(\chi, \xi) \in S_i \times S_{\chi,j}$. Hence, by applying Proposition 2.7 to these objects we find for each (χ, ξ) a partition $(S_{\chi,\xi,m})_{0\leqslant m\leqslant d_U}$ of $\widehat{U}$ such that the set $S_{\chi,\xi,m}$ has character codimension at least m and, moreover, we have

$$H^n_c(G_{\bar{k}}, M_{\chi\xi\psi}) = 0$$

for each $\psi \in S_{\chi,\xi,m}$ unless $n \in [-i - j, i + j + m]$.

For $0 \leqslant r \leqslant d$, we now define $\widetilde{\mathscr{S}_r}$ to be the set of characters $(\chi, \xi, \psi) \in \widehat{G}$ such that

$$\psi \in S_{\chi,\xi,m}, \qquad \xi \in S_{\chi,j}, \qquad \chi \in S_i$$

for some i, j, m such that $i + j + m \geqslant r$.

For any integer $n \geqslant 1$, we have

$$|\widetilde{\mathscr{S}_r}(k_n)| = \sum_{i+j+m\geqslant r} \sum_{\chi\in S_i(k_n)} \sum_{\xi\in S_{\chi,j}(k_n)} |S_{\chi,\xi,m}(k_n)| \ll |k|^{n(d-(i+j+m))} \ll |k|^{n(d-r)}$$

by (2.4) and (2.5) (note that the uniformity with respect to the perverse sheaf in these estimates, and the uniform bound on the complexity, are crucial to control the sums over χ and ξ). Thus the set $\widetilde{\mathscr{S}_r}$ has character codimension at least r.

By construction of the sets S_i, $S_{\chi,j}$ and $S_{\chi,\xi,m}$, the condition $H^i_c(G_{\bar{k}}, M_{\chi\xi\psi}) \neq 0$, for $(\chi, \xi, \psi) \in \widehat{G}$, implies that $(\chi, \xi, \psi) \in \mathscr{S}_{|i|}$. We apply a similar argument with D(M) to obtain the analogue conclusion for ordinary cohomology and set $\mathscr{S}_i$ to be the intersection of the set $\widetilde{\mathscr{S}_i}$ for M and of the analogue for D(M). By construction, the sets $\mathscr{S}_i$ satisfy the first two claims of Theorem 2.3.

We now establish the last claims of Theorem 2.3.

First, let $(\chi, \xi, \psi) \in \widehat{\mathrm{G}} - \mathscr{S}_1$. By construction of S_1 through Theorem 2.12 (see point (2) above), we have $\mathrm{R}p_{\mathrm{T}!}(\mathrm{M}_\chi) = \mathrm{R}p_{\mathrm{T}*}(\mathrm{M}_\chi)$. Moreover $p_{\mathrm{A}!} = p_{\mathrm{A}*}$ since p_A is proper, and by the last claim of Proposition 2.7, we obtain

$$\mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_{\chi\xi\psi}) = \mathrm{H}^0(\mathrm{U}_{\bar{k}}, \mathrm{R}p_{\mathrm{A}*}\mathrm{R}p_{\mathrm{T}*}\mathrm{M}_{\chi\xi\psi}) = \mathrm{H}^0_c(\mathrm{U}_{\bar{k}}, \mathrm{R}p_{\mathrm{A}!}\mathrm{R}p_{\mathrm{T}!}\mathrm{M}_{\chi\xi\psi}) = \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_{\chi\xi\psi}).$$

Finally, if G is a torus (resp. an abelian variety) then we use Corollary 2.13 (resp. Corollary 2.25) to prove that the sets $\mathscr{S}_i$ are contained in a finite union of tacs of G of dimension $\leqslant d - i$.

This finally concludes the proof. ∎

Remark 2.27. In view of the relative generic vanishing theorems for tori and abelian varieties (Theorem 2.11 and Corollary 2.22), one could hope that the generic vanishing theorem for a perverse sheaf M on a product $\mathrm{A} \times \mathrm{T}$ of an abelian variety A with a torus T could be proved by applying the relative generic vanishing theorems to the image of M under both projections (with compact support), and looking at products of characters χ and η which satisfy the desired properties for the image objects. However, this does not work, as explained in Example 2.28 below. (A similar remark can be made by looking at the diagonal inside $\mathbf{G}_m \times \mathbf{G}_a$; see Chapter 9 for the study of the arithmetic Fourier transform on this group.)

Nevertheless, we still expect, in the case $\mathrm{G} = \mathrm{A} \times \mathrm{T}$, that the exceptional set for the generic vanishing theorem is contained in a finite union of tacs, see Section 12, Problem (1). A step in this direction is that only finitely many abelian varieties can appear in the tacs we construct in the proof. Indeed they need to be of degree bounded in terms of the complexity of M, and there are only finitely many abelian subvarieties of a given degree inside a fixed abelian variety (up to translation), as follows from example from [95].

Example 2.28. Let E be an elliptic curve and $p\colon \mathrm{E} \times \mathbf{G}_m \to \mathrm{E}$ and $q\colon \mathrm{E} \times \mathbf{G}_m \to \mathbf{G}_m$ be the two projections. Let $i\colon \mathrm{C} \to \mathrm{E} \times \mathbf{G}_m$ be a closed one-dimensional irreducible subvariety of $\mathrm{E} \times \mathbf{G}_m$, with dominant projections to both E and $\mathbf{G}_m$. Let $\mathrm{M} = i_{!*}\overline{\mathbf{Q}}_\ell[1]$ be the perverse sheaf on $\mathrm{E} \times \mathbf{G}_m$ which is the intermediate extension of the shifted constant sheaf on C.

We claim that both $p_!\mathrm{M}$ and $q_!\mathrm{M}$ are perverse. Indeed, we have $q_!\mathrm{M} = q_{|\mathrm{C}!}i_{!*}\overline{\mathbf{Q}}_\ell[1]$, which is perverse since $q_{|\mathrm{C}}\colon \mathrm{C} \to \mathbf{G}_m$ is a finite morphism, hence is t-exact by [8, Cor. 2.2.6]. Moreover $p_!\mathrm{M} = p_{|\mathrm{C}!}i_{!*}\overline{\mathbf{Q}}_\ell[1]$ and $p_{|\mathrm{C}}$ can be factored as a finite morphism followed by an affine open immersion, both of which are t-exact by [8, Cor. 4.1.3]. It follows that the trivial characters on $\mathbf{G}_m$ and E are in the generic sets for the relative vanishing Theorem 2.11 applied to p and for Corollary 2.22 applied to q. However, $\mathrm{H}^1_c(\mathrm{E} \times \mathbf{G}_m, \mathrm{M}) \simeq \mathrm{H}^2_c(\mathrm{C}, \overline{\mathbf{Q}}_\ell) \neq 0$, hence the trivial character is *not* in the generic set given by Theorem 2.1 applied to $\mathrm{E} \times \mathbf{G}_m$ and M.

Remark 2.29. Once we have reduced the proof of Theorem 2.3 to a product, the order in which we handle the toric, unipotent and abelian variety parts of G in the proof is essentially dictated by the fact that the current versions of Theorem 2.11 (the relative vanishing theorem for tori) and its corollaries are not uniform in terms of the complexity of the input object M.

However, if the toric part has dimension 1, it is not difficult to obtain such a statement, and thus to vary the proof. This is not entirely anecdotal, because the choice of order has implications on the structure of the sets $\mathscr{S}_i$ in Theorem 2.3.

We describe the special case of $\mathrm{G} = \mathbf{G}_m \times \mathbf{G}_a$, which will be used in Chapter 9. We note first that if M is a perverse sheaf on $\mathbf{G}_m$ over k, then the vanishing

$$\mathrm{H}^i(\mathbf{G}_{m,\bar{k}}, \mathrm{M}_\chi) = \mathrm{H}^i_c(\mathbf{G}_{m,\bar{k}}, \mathrm{M}_\chi) = 0$$

holds for $i \neq 0$ and χ outside of a set $\mathscr{S} \subset \widehat{\mathbf{G}}_m$ such that $|\mathscr{S}(k_n)| \ll 1$ for all $n \geqslant 1$, where the implied constant depends only on the complexity of M (one reduces to the case of a simple perverse sheaf, and then one can apply Lemma A.15, for instance). In particular, for $\chi \notin \mathscr{S}$, the complex M_χ is a perverse sheaf.

Theorem 2.30 (Stratified vanishing for $\mathbf{G}_m \times \mathbf{G}_a$). *Let* M *a perverse sheaf on* $\mathrm{G} = \mathbf{G}_m \times \mathbf{G}_a$. *There exist subsets*

$$\mathscr{S}_2 \subset \mathscr{S}_1 \subset \mathscr{S}_0 = \widehat{\mathrm{G}} \quad \textit{and} \quad \mathscr{T} \subset \widehat{\mathbf{G}}_a$$

such that the following holds:

(1) *For* $n \geqslant 1$, *we have* $|\mathscr{S}_2(k_n)| \ll 1$, $|\mathscr{T}(k_n)| \ll 1$ *and* $|\mathscr{S}_1(k_n)| \ll |k_n|$.
(2) *For* $0 \leqslant i \leqslant 2$, *any* $\chi \in \widehat{\mathrm{G}}$ *such that at least one of the cohomology groups*

$$\mathrm{H}^i(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi), \qquad \mathrm{H}^{-i}(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi), \qquad \mathrm{H}^i_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi), \qquad \mathrm{H}^{-i}_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$$

is non-zero belongs to $\mathscr{S}_i$.
(3) *For* $\chi \in \mathscr{S}_0 - \mathscr{S}_1$, *the equality* $\mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = \mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ *holds.*
(4) *For* $\psi \in \widehat{\mathbf{G}}_a - \mathscr{T}$, *the set* $\mathscr{T}_\psi$ *of all* $\chi \in \widehat{\mathbf{G}}_m$ *such that* $\psi \boxtimes \chi \in \mathscr{S}_1$ *satisfies the bound* $|\mathscr{T}_\psi(k_n)| \ll 1$ *for all* $n \geqslant 1$, *with an implied constant that only depends on* $c(\mathrm{M})$.

Proof. Let $p \colon \mathrm{G} \to \mathbf{G}_m$ be the projection. For ψ varying in $\widehat{\mathbf{G}}_a$, the complexes $\mathrm{R}p_!(\mathrm{M}_\psi)$ on $\mathbf{G}_m$ have bounded complexity. By Proposition 2.7, we can partition $\widehat{\mathbf{G}}_a$ in subsets S_0 and S_1, with S_1 of character codimension $\geqslant 1$, such that $\mathrm{R}p_!(\mathrm{M}_\psi)$ is perverse if $\psi \in \mathrm{S}_0$.

Let $\psi \in \mathrm{S}_0$. Then by the elementary remark before the statement, the set $\mathrm{S}_{0,\psi}$ of $\chi \in \widehat{\mathbf{G}}_m$ such that $\mathrm{R}p_!(\mathrm{M}_\psi)_\chi$ is not perverse has the property that $|\mathrm{S}_{0,\psi}(k_n)| \ll 1$ for all $n \geqslant 1$, where the implied constant depends only on $c(\mathrm{M})$. The result now follows with

$$\begin{aligned} \mathscr{S}_0 &= \widehat{\mathrm{G}}, \\ \mathscr{S}_1 &= (\widehat{\mathbf{G}}_m \times \mathrm{S}_0) \cup \{\chi \boxtimes \psi \mid \psi \in \mathrm{S}_0 \text{ and } \chi \in \mathrm{S}_{\psi,1}\}, \\ \mathscr{T} &= \mathrm{S}_0, \end{aligned}$$

and $\mathscr{S}_2$ the set of characters such that one of

$$\mathrm{H}^2(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi), \qquad \mathrm{H}^{-2}(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi), \qquad \mathrm{H}^2_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi), \qquad \mathrm{H}^{-2}_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$$

is non-zero (which satisfies $|\mathscr{S}_2(k_n)| \ll 1$ for all n by the observation before the theorem; note that $\mathscr{S}_2 \subset \mathscr{S}_1$ because the existence for given $(\chi, \psi) \in \widehat{\mathrm{G}}$ of non-zero H^2 or H^2_c implies that $\mathrm{R}p_!(\mathrm{M}_\psi)$ is not perverse). ∎

CHAPTER 3

Tannakian categories of perverse sheaves

3.1. Introduction

Throughout this chapter, k denotes a finite field and $\bar{k}$ an algebraic closure of k. We denote by ℓ a prime number different from the characteristic of k. All complexes we consider are ℓ-adic complexes.

Let G be a connected commutative algebraic group over k.

Let M be a perverse sheaf on G. We wish to define a "symmetry group" that governs the statistical behavior of the arithmetic Fourier transform

$$\mathrm{S}(\mathrm{M},\chi) = \sum_{x\in \mathrm{G}(k_n)} \chi(x) t_{\mathrm{M}}(x;k_n)$$

for $\chi \in \widehat{\mathrm{G}}(k_n)$. The fundamental mechanism for this is that the symmetry group $\mathbf{G}$ should come with a faithful linear representation $\mathbf{G} \subset \mathbf{GL}_r$ for some $r \geqslant 0$, and to almost all characters χ there should be assigned an element (or conjugacy class) $\mathrm{Fr}_\chi \in \mathbf{G}$ such that $\mathrm{S}(\mathrm{M},\chi)$ is the trace of Fr_χ.

The idea behind the construction of the group $\mathbf{G}$ (following Katz [74]) is based on the fact that we have a "geometric" control on the algebra structure on the space of arithmetic Fourier transforms through the link with convolution: for two objects M_1 and M_2 on G, we have

$$\mathrm{S}(\mathrm{M}_1,\chi)\mathrm{S}(\mathrm{M}_2,\chi) = \sum_{x\in \widehat{\mathrm{G}}(k_n)} \chi(x)(t_{\mathrm{M}_1} * t_{\mathrm{M}_2})(x;k_n),$$

where

$$(t_{\mathrm{M}_1} * t_{\mathrm{M}_2})(x;k_n) = \sum_{y\in \mathrm{G}(k_n)} t_{\mathrm{M}_1}(y;k_n) t_{\mathrm{M}_2}(y^{-1}x;k_n),$$

for $x \in \mathrm{G}(k_n)$, is the convolution product in the classical sense of Fourier analysis on $\mathrm{G}(k_n)$.

It is fundamental that by the proper base change theorem and the trace formula, we can view this function as a trace function, namely

$$(t_{\mathrm{M}_1} * t_{\mathrm{M}_2})(x;k_n) = t_{\mathrm{M}_1 *_! \mathrm{M}_2}(x;k_n),$$

where $\mathrm{M}_1 *_! \mathrm{M}_2$ is the convolution with compact support (Section 1.5).

This geometric interpretation suggests to use the convolution as "tensor operation" to define a *tannakian category*, which would be equivalent to the category of representations of the desired symmetry group.

In essence, this is what we will do. However, there are some significant issues to handle:

– The first one, already present in the work of Katz for $\mathbf{G}_m$, has to do with the fact that convolution with compact support does not always preserve perverse sheaves (for instance, if G has dimension d, then the convolution $\mathscr{L}_{\chi_1}[d] *_! \mathscr{L}_{\chi_2}[d]$ is not perverse) or duality (because duality transforms the convolution $*_!$ into the convolution $*_*$, which is different in general).

We can solve this first problem using a suitable quotient category where the two geometric convolution products turn out to coincide (this idea goes back to Gabber and Loeser and was also used by Katz).

– A related issue is that weights do not always behave well under convolution, in the case of affine groups at least. Since weights dictate the size of the sums $\mathrm{S}(\mathrm{M},\chi)$, this is a crucial issue for our intended applications. This is again related to the difference between the two geometric convolutions, each of which leads in practice to inequalities in one direction for the weights.

– Finally there is a major new difficulty in comparison with the work of Katz. The link between the abstract tannakian ideas and the arithmetic Fourier transform is that for a character $\chi \in \widehat{\mathrm{G}}(k_n)$, the formula

$$\mathrm{S}(\mathrm{M},\chi) = \mathrm{Tr}(\mathrm{Fr}_{k_n} \mid \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi))$$

should hold. This is in fact (by the generic vanishing theorem) only true in general for a generic set of χ where the contributions of H^i_c in the trace formula vanish for $i \neq 0$. But we also want "higher-order" versions of this formula to hold, namely for instance

$$\mathrm{S}(\mathrm{M},\chi)^2 = \mathrm{Tr}(\mathrm{Fr}_{k_n} \mid \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi *_! \mathrm{M}_\chi)),$$

and so on for further powers (intuitively, this is because understanding the limits of averages of such expressions is necessary to apply the Weyl equidistribution criterion, as we will do in the next chapter). This amounts roughly to requesting that $\mathrm{M} \mapsto \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ should be compatible with convolution and so should (roughly) the generic vanishing theorem.

Thus we need to distinguish various types of characters depending on their behavior with respect to operations of this type.

3.2. Categories of objects defined over finite fields

We denote by $\mathbf{D}(\mathrm{G})$ and $\mathbf{P}(\mathrm{G})$ the full subcategories of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G}_{\bar{k}})$ and $\mathbf{Perv}(\mathrm{G}_{\bar{k}})$ respectively whose objects are defined over some finite extension of the base field k. These categories are stable by direct sum, shifts and duality. Moreover, the perverse cohomology sheaves of an object of $\mathbf{D}(\mathrm{G})$ belong to $\mathbf{P}(\mathrm{G})$.

We recall from Section 1.5 the definition and properties of the two convolution bifunctors $(\mathrm{M},\mathrm{N}) \mapsto \mathrm{M} *_* \mathrm{N}$ and $(\mathrm{M},\mathrm{N}) \mapsto \mathrm{M} *_! \mathrm{N}$ for objects M and N of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$ or $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G}_{\bar{k}})$. These are compatible with base change, so that the convolutions on $\mathrm{G}_{\bar{k}}$ preserve the category $\mathbf{D}(\mathrm{G})$. In addition, the functor $\mathrm{M} \mapsto \mathrm{M}^\vee$ also induces a functor on $\mathbf{D}(\mathrm{G})$ and $\mathbf{P}(\mathrm{G})$.

3.3. Weakly unramified characters

Definition 3.1 (Weakly unramified characters). Let M be an object of $\mathbf{P}(\mathrm{G})$. A character $\chi \in \widehat{\mathrm{G}}$ is said to be *weakly unramified* for M if the following holds:

$$\mathrm{H}^i(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = \mathrm{H}^i_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = 0 \quad \text{for all } i \neq 0,$$
$$\mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = \mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi).$$

We denote by $\mathscr{X}_w(\mathrm{M})$ the set of weakly unramified characters for M.

Remark 3.2. (1) The terminology is suggested by analogy with the case of the additive group, in which the characters for which generic vanishing holds correspond to points at which the Fourier transform is lisse. However, we will see that the generic vanishing condition is not in general strong enough to obtain the properties we seek (namely, that the assignment $\mathrm{M} \mapsto \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ defines a fiber functor on a suitable tannakian category of perverse sheaves on $G_{\bar{k}}$), see Remark 3.25 and Example 9.15. We will introduce unramified characters in Definition 3.24, as well as the variant of Frobenius-unramified characters in Definition 3.35.

(2) Since G is not proper in general, the condition on the cohomology groups with support is not implied by the one on the cohomology groups without support.

With this definition, we can reformulate the Stratified Generic Vanishing Theorem 2.1 as follows:

THEOREM 3.3. *The subset $\mathscr{X}_w(\mathrm{M}) \subset \widehat{\mathrm{G}}$ of weakly unramified characters for an object* M *of* $\mathbf{P}(\mathrm{G})$ *is generic.*

3.4. Negligible objects

In general, none of the two convolution bifunctors on the derived category preserves the subcategory of perverse sheaves. As first observed in the case of tori by Gabber and Loeser [50], there is however a suitable quotient of the category $\mathbf{P}(\mathrm{G})$ on which both convolution functors induce the same bifunctor.

DEFINITION 3.4. An object M of $\mathbf{P}(\mathrm{G})$ is said to be *negligible* if the set of characters $\chi \in \widehat{\mathrm{G}}$ satisfying $\mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = 0$ is generic. An object N of $\mathbf{D}(\mathrm{G})$ is said to be *negligible* if all its perverse cohomology objects ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{N})$ are negligible.

We denote by $\mathrm{Neg}_{\mathbf{P}}(\mathrm{G})$ and $\mathrm{Neg}_{\mathbf{D}}(\mathrm{G})$ the full subcategories of $\mathbf{P}(\mathrm{G})$ and $\mathbf{D}(\mathrm{G})$ respectively consisting of negligible objects.

We denote by $\mathrm{K}_{\mathrm{neg}}(\mathrm{G})$ the subgroup of the Grothendieck group $\mathrm{K}(\mathrm{G})$ generated by classes of negligible perverse sheaves, or equivalently by classes of negligible objects.

Given an object M of $\mathbf{P}(\mathrm{G})$, set

$$\mathscr{N}(\mathrm{M}) = \{\chi \in \widehat{\mathrm{G}} \mid \mathrm{H}^i(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = \mathrm{H}^i_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = 0 \text{ for all } i\}.$$

Using Theorem 2.1, we see that M is negligible if and only if $\mathscr{N}(\mathrm{M})$ is a generic subset of $\widehat{\mathrm{G}}$. For $\mathrm{M} \in \mathrm{Neg}_{\mathbf{D}}(\mathrm{G})$, we set

$$\mathscr{N}(\mathrm{M}) = \bigcup_i \mathscr{N}({}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M})).$$

It follows from the definition that, for each negligible perverse sheaf M (resp. object of $\mathrm{Neg}_{\mathbf{D}}(\mathrm{G})$), the perverse sheaf $\mathrm{M}^\vee$ is also negligible (resp. the complex $\mathrm{M}^\vee$ is negligible).

EXAMPLE 3.5. Any character sheaf $\mathscr{L}_\chi$ on G is negligible. More generally, let $f\colon \mathrm{G} \to \mathrm{H}$ be a surjective morphism of algebraic groups such that the dimension d of the kernel $\ker(f)$ is positive. Let $\eta \in \widehat{\mathrm{G}}$ and let N be any object of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{H})$. We claim that the object $\mathrm{M} = (f^*\mathrm{N})_\eta$ is negligible.

Indeed, let $i \in \mathbf{Z}$. We can factor $f = f_1 \circ f_2$, where f_2 is smooth of relative dimension d and f_1 is radicial. Then $f_2^*[d]$ is t-exact (see [8, § 4.2.4]), and so is tensoring by $\mathscr{L}_\eta$ (Lemma 1.17), so there is a canonical isomorphism

$${}^{\mathfrak{p}}\mathscr{H}^i((f^*\mathrm{N})_\eta) \simeq f_2^*({}^{\mathfrak{p}}\mathscr{H}^{i-d}(f_1^*(\mathrm{N})))_\eta.$$

For $\chi \in \widehat{\mathrm{G}}$, the projection formula leads to canonical isomorphisms

$$\mathrm{H}^*(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) \simeq \mathrm{H}^*(\mathrm{G}_{\bar{k}}, f_2^*({}^{\mathfrak{p}}\mathscr{H}^{i-d}(f_1^*(\mathrm{N}))) \otimes \mathscr{L}_{\eta\chi}) \simeq \mathrm{H}^*(\mathrm{H}_{\bar{k}}, {}^{\mathfrak{p}}\mathscr{H}^{i-d}(f_1^*(\mathrm{N})) \otimes \mathrm{R}f_{2!}\mathscr{L}_{\eta\chi}).$$

The complex $\mathrm{R}f_{2!}\mathscr{L}_{\eta\chi}$ is zero if the restriction of $\eta\chi$ to the subgroup $\ker(f_2)^\circ$ is not the trivial character (see Lemma 1.16). Since this condition defines a generic set of characters χ, we deduce that ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M})$ is negligible, and the result follows.

Remark 3.6. Intuitively, to say that M is negligible means that the arithmetic Fourier transform of M (see Section 1.8) satisfies $\mathrm{S}(\mathrm{M},\chi)=0$ for χ in a generic subset of $\widehat{\mathrm{G}}$. To illustrate this concrete aspect, we show how it explains the previous example. Thus consider $\mathrm{M}=(f^*\mathrm{N})_\eta$, with notation as above for some $\eta\in\widehat{\mathrm{G}}(k)$. Let $\chi\in\widehat{\mathrm{G}}(k_n)$; the corresponding value of the Fourier transform is

$$\begin{aligned}\mathrm{S}(\mathrm{M},\chi)=\sum_{x\in\mathrm{G}(k_n)}\chi(x)t_{\mathrm{M}}(x;k_n)&=\sum_{x\in\mathrm{G}(k_n)}\chi(x)(\eta\circ\mathrm{N}_{k_n/k})(x)t_{\mathrm{N}}(f(x);k_n)\\&=\sum_{y\in\mathrm{H}(k_n)}t_{\mathrm{N}}(y;k_n)\sum_{\substack{x\in\mathrm{G}(k_n)\\f(x)=y}}\chi(\eta\circ\mathrm{N}_{k_n/k})(x),\end{aligned}$$

and the inner sum is either empty or a sum of a character over the k_n-points of a coset of the kernel of f, which vanishes unless $\chi=(\eta\circ\mathrm{N}_{k_n/k})^{-1}$ on the kernel of f.

In some cases, one can show that, conversely, all simple negligible perverse sheaves are of the form $(f^*\mathrm{N})_\eta$ for some quotient morphism f with kernel of dimension at least 1. This is for instance the case for abelian varieties, by a result of Weissauer [127, Lemma 6, Th. 3] (see also Remark 5.15) and we will prove later that this is also the case for $\mathbf{G}_a\times\mathbf{G}_m$ (see Section 9.4).

This structural property is however not always true. For instance, if G is a unipotent group of dimension at least 2 (e.g., $\mathrm{G}=\mathbf{G}_a^d$ with $d\geqslant 2$), with Serre dual $\mathrm{G}^\vee$, then we can take any object $\mathrm{N}\in\mathrm{D}_c^{\mathrm{b}}(\mathrm{G}^\vee)$ whose support S has codimension at least 1, and the inverse Fourier transform M of N will be a negligible object on G. If S is not a translate of a subgroup of G, then the object M is not obtained by pullback from any quotient of G. (In the terminology of [39, §4], in the case of $\mathbf{G}_a^d$, such objects are said to have A-number equal to 0, and they play a delicate role in certain analytic applications.)

We recall that a full subcategory S of an abelian category C is said to be a *Serre subcategory* if it is not empty, stable by extension and by subobject and quotient. A strictly full triangulated subcategory S of a triangulated category C is said to be *thick* if, for any morphism $f\colon \mathrm{X}\to\mathrm{Y}$ in C which factors through an object of S, and which appears in a distinguished triangle

$$\mathrm{X}\xrightarrow{f}\mathrm{Y}\to\mathrm{Z}$$

with Z object of S, the objects X and Y are in S.

Lemma 3.7. *The category* $\mathrm{Neg}_{\mathbf{P}}(\mathrm{G})$ *is a Serre subcategory of* $\mathbf{P}(\mathrm{G})$*, and* $\mathrm{Neg}_{\mathbf{D}}(\mathrm{G})$ *is a thick triangulated subcategory of* $\mathbf{D}(\mathrm{G})$*.*

Proof. Fix an exact sequence $\mathrm{X}\to\mathrm{Y}\to\mathrm{Z}$ in $\mathbf{P}(\mathrm{G})$ such that X and Z are objects of $\mathrm{Neg}_{\mathbf{P}}(\mathrm{G})$. By Theorem 3.3, there is a generic set of characters $\chi\in\widehat{\mathrm{G}}$ that are weakly unramified for X, Y, and Z. From the long exact sequence in cohomology, we find that for any such χ, the vanishing $\mathrm{H}^i(\mathrm{G}_{\bar{k}},\mathrm{Y}_\chi)=\mathrm{H}^i_c(\mathrm{G}_{\bar{k}},\mathrm{Y}_\chi)=0$ holds for all i, and hence Y is negligible. The first statement follows easily. An argument of Gabber–Loeser (see [50, Prop. 3.6.1(i)]) then implies that $\mathrm{Neg}_{\mathbf{D}}(\mathrm{G})$ is a thick triangulated subcategory of $\mathbf{D}(\mathrm{G})$. ☐

Lemma 3.8. *For all objects* M *and* N *of* $\mathbf{D}(\mathrm{G})$*, the following properties hold:*

(1) *The cone of the canonical morphism* $\mathrm{M}*_!\mathrm{N}\to\mathrm{M}*_*\mathrm{N}$ *lies in* $\mathrm{Neg}_{\mathbf{D}}(\mathrm{G})$*.*
(2) *If* M *belongs to* $\mathrm{Neg}_{\mathbf{D}}(\mathrm{G})$*, then so do* $\mathrm{M}*_!\mathrm{N}$ *and* $\mathrm{M}*_*\mathrm{N}$ *for each object* N*.*
(3) *If* M *and* N *belong to* $\mathbf{P}(\mathrm{G})$*, then* ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M}*_!\mathrm{N})$ *and* ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M}*_*\mathrm{N})$ *lie in* $\mathrm{Neg}_{\mathbf{P}}(\mathrm{G})$ *for each non-zero integer* i*.*

We omit the proof, which is the same as that of [90, Lem. 4.3].

3.5. Tannakian categories

By results of Gabriel [51] for abelian categories and Verdier (see the treatment in the book [107] of Neeman) for triangulated categories, we can define the quotient of an abelian or triangulated category by a Serre or thick subcategory. This allows us to make the following definition.

Definition 3.9 (Convolution categories). The *convolution category* of G, denoted $\overline{\mathbf{D}}(\mathrm{G})$, is the quotient category of $\mathbf{D}(\mathrm{G})$ by $\mathrm{Neg}_{\mathbf{D}}(\mathrm{G})$; it is a triangulated category.

The *perverse convolution category* of G, denoted $\overline{\mathbf{P}}(\mathrm{G})$, is the quotient abelian category of $\mathbf{P}(\mathrm{G})$ by $\mathrm{Neg}_{\mathbf{P}}(\mathrm{G})$.

Those two constructions are compatible, in the sense that the t-structure on $\mathbf{D}(\mathrm{G})$ induces a t-structure on $\overline{\mathbf{D}}(\mathrm{G})$ whose heart is the category $\overline{\mathbf{P}}(\mathrm{G})$ (see [50, Prop. 3.6.1]).

Since the functor $\mathrm{N} \mapsto \mathrm{N}^\vee$ preserves negligible objects, it induces a functor on $\overline{\mathbf{P}}(\mathrm{G})$ (resp. on $\overline{\mathbf{D}}(\mathrm{G})$), which is still an involution.

Proposition 3.10. *With notation as above, the following properties hold:*

1. *The convolution products $*_!$ and $*_*$ induce bifunctors on $\overline{\mathbf{D}}(\mathrm{G}) \times \overline{\mathbf{D}}(\mathrm{G})$.*
2. *The canonical forget support morphisms $\mathrm{M} *_! \mathrm{N} \to \mathrm{M} *_* \mathrm{N}$ induce isomorphisms in $\overline{\mathbf{D}}(\mathrm{G})$, and define by passing to the quotient a convolution bifunctor denoted*
$$* : \overline{\mathbf{D}}(\mathrm{G}) \times \overline{\mathbf{D}}(\mathrm{G}) \to \overline{\mathbf{D}}(\mathrm{G}).$$
3. *The subcategory $\overline{\mathbf{P}}(\mathrm{G})$ of $\overline{\mathbf{D}}(\mathrm{G})$ is stable under the convolution $*$.*
4. *The categories $\overline{\mathbf{D}}(\mathrm{G})$ and $\overline{\mathbf{P}}(\mathrm{G})$, endowed with the bifunctor $*$, are symmetric $\overline{\mathbf{Q}}_\ell$-linear monoidal categories with unit object $\mathbf{1}$ the image of the skyscraper sheaf at the neutral element of* G.

Proof. The fact that $*_!$ and $*_*$ induce functors on $\overline{\mathbf{D}}(\mathrm{G}) \times \overline{\mathbf{D}}(\mathrm{G})$ follows from Lemma 3.8 (2). That they agree is Lemma 3.8 (1). The stability of $\overline{\mathbf{P}}(\mathrm{G})$ under $*$ is Lemma 3.8 (3). The fact that we obtain symmetric $\overline{\mathbf{Q}}_\ell$-linear monoidal categories is now clear. The last assertion follows from the canonical isomorphisms $\mathbf{1} *_! \mathrm{M} \simeq \mathbf{1} *_* \mathrm{M} \simeq \mathrm{M}$ which exist for any complex M. ☐

It is also very useful that there exists a natural subcategory of $\mathbf{P}(\mathrm{G})$ that is equivalent to the perverse convolution category.

Definition 3.11. The *internal convolution category* of G is the full subcategory $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$ of the category $\mathbf{P}(\mathrm{G})$ whose objects are perverse sheaves that have no subobject or quotient in $\mathrm{Neg}_{\mathbf{P}}(\mathrm{G})$.

Proposition 3.12. *The localization functor $\mathbf{P}(\mathrm{G}) \to \overline{\mathbf{P}}(\mathrm{G})$ restricts to an equivalence of categories*
$$\mathbf{P}_{\mathrm{int}}(\mathrm{G}) \longrightarrow \overline{\mathbf{P}}(\mathrm{G}),$$
hence the convolution product bifunctor $*$ on $\overline{\mathbf{P}}(\mathrm{G})$ induces a convolution bifunctor $*_{\mathrm{int}}$ on $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$.

Proof. The argument is the same as that of Gabber and Loeser [50, Déf.-Prop. 3.7.2]. ☐

The convolution product on $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$ will sometimes be called the *internal* or *middle* convolution.

Remark 3.13. One can give a more explicit form of the equivalence of categories above, and of the internal convolution.

First, Gabber and Loeser (loc. cit.) give an explicit quasi-inverse functor $\mathrm{M} \mapsto \mathrm{M}_{\mathrm{int}}$ to the equivalence of categories $\mathbf{P}_{\mathrm{int}}(\mathrm{G}) \to \overline{\mathbf{P}}(\mathrm{G})$. Namely, let M be an object of $\mathbf{P}(\mathrm{G})$. Let M_t be the largest subobject of M that belongs to $\mathrm{Neg}_{\mathbf{P}}(\mathrm{G})$ and let M^t be the smallest subobject of M such that M/M^t belongs to $\mathrm{Neg}_{\mathbf{P}}(\mathrm{G})$. Define $\mathrm{M}_{\mathrm{int}} = \mathrm{M}^t/(\mathrm{M}^t \cap \mathrm{M}_t)$. Then we have canonical isomorphisms

$$\mathrm{M}_{\mathrm{int}} \simeq (\mathrm{M}^t + \mathrm{M}_t)/\mathrm{M}_t,$$

and the assignment $\mathrm{M} \mapsto \mathrm{M}_{\mathrm{int}}$ is a functor which factors through $\overline{\mathbf{P}}(\mathrm{G})$ and induces a quasi-inverse of the localization functor.

In particular, this implies that if M is a semisimple object of $\overline{\mathbf{P}}(\mathrm{G})$, then $\mathrm{M}_{\mathrm{int}}$ is the sum of all the simple constituents of M that are not in $\mathrm{Neg}_{\mathbf{P}}(\mathrm{G})$.

Second, it follows from the argument in [50, Déf.-Prop. 3.7.3] that for M and N in $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$, there are canonical isomorphisms

$$\mathrm{M} *_{\mathrm{int}} \mathrm{N} \to {}^{\mathfrak{p}}\mathscr{H}^0(\mathrm{M} *_! \mathrm{N})_{\mathrm{int}} \to {}^{\mathfrak{p}}\mathscr{H}^0(\mathrm{M} *_* \mathrm{N})_{\mathrm{int}}.$$

Recall from Section 1.5 that for $\mathrm{M} \in \mathbf{P}(\mathrm{G})$, the identity morphism $\mathrm{id}_{\mathrm{M}} \colon \mathrm{M} \to \mathrm{M}$ defines evaluation and coevaluation maps

$$\mathrm{ev} \colon \mathrm{M} *_! \mathrm{M}^\vee \to \mathbf{1} \qquad \text{and} \qquad \mathrm{coev} \colon \mathbf{1} \to \mathrm{M}^\vee *_* \mathrm{M}.$$

They correspond to maps in $\overline{\mathbf{P}}(\mathrm{G})$ which we denote in the same way.

Proposition 3.14. *The monoidal category $\overline{\mathbf{P}}(\mathrm{G})$ is rigid. That is, for each object M of $\overline{\mathbf{P}}(\mathrm{G})$, the morphisms*

$$\mathrm{M} \simeq \mathrm{M} * \mathbf{1} \xrightarrow{\mathrm{id}_{\mathrm{M}} * \mathrm{coev}} \mathrm{M} * \mathrm{M}^\vee * \mathrm{M} \xrightarrow{\mathrm{ev} * \mathrm{id}_{\mathrm{M}}} \mathbf{1} * \mathrm{M} \simeq \mathrm{M}$$

$$\mathrm{M}^\vee \simeq \mathbf{1} * \mathrm{M}^\vee \xrightarrow{\mathrm{coev} * \mathrm{id}_{\mathrm{M}}} \mathrm{M}^\vee * \mathrm{M} * \mathrm{M}^\vee \xrightarrow{\mathrm{id}_{\mathrm{M}} * \mathrm{ev}} \mathrm{M}^\vee * \mathbf{1} \simeq \mathrm{M}^\vee$$

are the identity on M and on $\mathrm{M}^\vee$ respectively.

Proof. The argument is the same as that of Krämer in [90, Th. 5.2]. ☐

For any object M of $\overline{\mathbf{P}}(\mathrm{G})$ (resp. of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$), we denote by $\langle \mathrm{M} \rangle$ the subcategory of $\overline{\mathbf{P}}(\mathrm{G})$ (resp. of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$) which is tensor-generated by M, i.e., the full subcategory whose objects are the subquotients of all convolution powers of $\mathrm{M} \oplus \mathrm{M}^\vee$.

Our next goal is to prove the following crucial result:

Theorem 3.15. *The categories $\overline{\mathbf{P}}(\mathrm{G})$ and $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$ are neutral tannakian categories.*

In particular, for any object M of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$ or of $\mathbf{P}(\mathrm{G})$, the category $\langle \mathrm{M} \rangle$ is a neutral tannakian category over $\overline{\mathbf{Q}}_\ell$.

Recall that this means that there exists a fiber functor, namely a faithful exact tensor functor from $\overline{\mathbf{P}}(\mathrm{G})$ to the category $\mathrm{Vect}_{\overline{\mathbf{Q}}_\ell}$ of finite dimensional $\overline{\mathbf{Q}}_\ell$-vector spaces.

We begin the proof with an auxiliary result. Recall that the trace $\mathrm{Tr}(f) \in \overline{\mathbf{Q}}_\ell = \mathrm{End}(\mathbf{1})$ of an endomorphism f of $\mathrm{M} \in \mathbf{P}_{\mathrm{int}}(\mathrm{G})$ is defined as the composition

$$\mathbf{1} \xrightarrow{\mathrm{coev}} \mathrm{M} *_{\mathrm{int}} \mathrm{M}^\vee \xrightarrow{f *_{\mathrm{int}} \mathrm{id}_{\mathrm{M}^\vee}} \mathrm{M} *_{\mathrm{int}} \mathrm{M}^\vee \xrightarrow{\mathrm{ev}} \mathbf{1}.$$

The dimension of $\mathrm{M} \in \mathbf{P}_{\mathrm{int}}(\mathrm{G})$ is then intrinsically defined as $\dim(\mathrm{M}) = \mathrm{Tr}(\mathrm{id}_{\mathrm{M}})$. It is, a priori, an element of $\overline{\mathbf{Q}}_\ell$.

Proposition 3.16. *Let M be an object of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$ and let C be the cone of the canonical morphism*

$$\mathrm{M} *_! \mathrm{M}^\vee \to \mathrm{M} *_* \mathrm{M}^\vee.$$

For any character $\chi \in \widehat{\mathrm{G}}$ *in the generic set*

$$\mathscr{X}_w(\mathrm{M}) \cap \mathscr{N}(\mathrm{C}),$$

the following equality holds:

$$\dim \mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = \dim(\mathrm{M}). \tag{3.1}$$

In particular, $\dim(\mathrm{M})$ *is a non-negative integer, and there exists a generic set of characters* χ *such that the dimension of* $\mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ *is independent of* χ*.*

Proof. We need to determine the morphism

$$\mathbf{1} \xrightarrow{\mathrm{coev}} \mathrm{M} *_{\mathrm{int}} \mathrm{M}^\vee \xrightarrow{\mathrm{ev}} \mathbf{1}.$$

Twisting by χ and taking cohomology, the sequence above induces a sequence

$$\overline{\mathbf{Q}}_\ell \to \mathrm{H}^*(\mathrm{G}_{\bar{k}}, (\mathrm{M} *_{\mathrm{int}} \mathrm{M}^\vee)_\chi) \to \overline{\mathbf{Q}}_\ell.$$

By Lemma 3.8, the object C is in $\mathrm{Neg}_{\mathbf{D}}(\mathrm{G})$ so that for $\chi \in \mathscr{N}(\mathrm{C})$, we have a canonical isomorphism

$$\mathrm{H}^*(\mathrm{G}_{\bar{k}}, (\mathrm{M} *_{\mathrm{int}} \mathrm{M}^\vee)_\chi) \simeq \mathrm{H}^*(\mathrm{G}_{\bar{k}}, (\mathrm{M} *_* \mathrm{M}^\vee)_\chi).$$

By Lemma 1.15, there is also a canonical isomorphism

$$\mathrm{H}^*(\mathrm{G}_{\bar{k}}, (\mathrm{M} *_* \mathrm{M}^\vee)_\chi) \simeq \mathrm{H}^*(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) \otimes \mathrm{H}^*(\mathrm{G}_{\bar{k}}, (\mathrm{M}_\chi)^\vee),$$

If $\chi \in \mathscr{X}_w(\mathrm{M})$, then we also get $\mathrm{H}^*(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = \mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ and $\mathrm{H}^*(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi^\vee) = \mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)^\vee$, and therefore the sequence above becomes

$$\overline{\mathbf{Q}}_\ell \to \mathrm{End}(\mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)) \to \overline{\mathbf{Q}}_\ell.$$

Since the evaluation and coevaluation maps are sent to evaluation and covevaluation maps in vector spaces (see the proof of [90, Th. 5.2]), this composition is the multiplication by the dimension of $\mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$, which is therefore equal to the dimension of M in $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$. ☐

Proof of Theorem 3.15. By Proposition 3.14, the equivalent categories $\overline{\mathbf{P}}(\mathrm{G})$ and $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$ are $\overline{\mathbf{Q}}_\ell$-linear rigid tensor symmetric categories. Since the unit $\mathbf{1}$ is (the image of) a skyscraper sheaf, we have $\mathrm{End}(\mathbf{1}) \simeq \overline{\mathbf{Q}}_\ell$.

Proposition 3.16 and Theorem 3.3 imply that the dimension $\dim(\mathrm{M})$ of every object M of $\overline{\mathbf{P}}(\mathrm{G})$ is a non-negative integer. By a theorem of Deligne [30, Th. 7.1], it follows that the category $\overline{\mathbf{P}}(\mathrm{G})$ is a tannakian category. A further theorem of Deligne (see the proof by Coulembier in [23, Th.6.4.1]) implies that it is indeed neutral (i.e., there exists a fiber functor defined over $\overline{\mathbf{Q}}_\ell$). ☐

Remark 3.17. (1) In Example 9.15, we will give examples to show that there may exist weakly unramified characters for which formula (3.1) does not hold.

(2) In this book, we will exclusively consider from now on the categories $\langle \mathrm{M} \rangle$ generated by a single object. A simpler proof that these are neutral tannakian categories is then provided by combining [30, Th. 7.1] with [30, Cor. 6.20].

Corollary 3.18. *Let* M *be an object of* $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$*. There exists an affine algebraic group* $\mathbf{G}$ *over* $\overline{\mathbf{Q}}_\ell$ *such that the category* $\langle \mathrm{M} \rangle$ *is equivalent to the category* $\mathrm{Rep}_{\overline{\mathbf{Q}}_\ell}(\mathbf{G})$ *of finite-dimensional* $\overline{\mathbf{Q}}_\ell$*-representations of* $\mathbf{G}$*. If* M *is semisimple, then the group* $\mathbf{G}$ *is reductive and the category* $\langle \mathrm{M} \rangle$ *is semisimple.*

PROOF. The first part follows from the tannakian reconstruction theorem [32, Th. 2.11]. If M is semisimple then since the category of representations of $\mathbf{G}$ is equivalent to the category $\langle \mathrm{M} \rangle$ generated by the semisimple object M, it follows, e.g., from [106, Th. 22.42] that the group $\mathbf{G}$ is reductive, and that every object $\mathrm{N} \in \langle \mathrm{M} \rangle$ is semisimple. □

DEFINITION 3.19. For any object M of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$ or of $\overline{\mathbf{P}}(\mathrm{G})$, we denote by $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ the affine algebraic group over $\overline{\mathbf{Q}}_\ell$ given by the corollary, and we say that it is the *geometric tannakian group* of the object M.

EXAMPLE 3.20. (1) Let $\mathrm{G} = \mathbf{G}_m$. A perverse sheaf N on $\mathbf{G}_m$ is negligible if and only if it is a successive extension of shifted Kummer sheaves $\mathscr{L}_\chi[1]$ for some characters χ, and it follows that the category $\mathbf{P}_{\mathrm{int}}(\mathbf{G}_m)$ is the same as the category $\mathcal{P}$ of Katz [74, Ch. 2] (see also Section B.1).

(2) Let $\mathrm{G} = \mathbf{G}_a$. Fix an additive character ψ of k. By the proper base change theorem, a perverse sheaf N on $\mathbf{G}_a$ is negligible if and only if its Fourier transform $\mathrm{FT}_\psi(\mathrm{N})$ is punctual, which means that N is a finite direct sum of Artin–Schreier sheaves $\mathscr{L}_{\psi(yx)}[1]$ for some $y \in \mathbf{G}_a$. This implies that the category $\mathbf{P}_{\mathrm{int}}(\mathbf{G}_a)$ coincides with the category of perverse sheaves on $\mathbf{G}_a$ with "property $\mathcal{P}$", as defined by Katz [70, (2.6.2)] (this follows by combining Cor. 2.6.14, Cor. 2.6.15 and Lemma 2.6.13 of [70]; see Remark 2.10.4 in loc. cit.).

3.6. Euler–Poincaré characteristic and Grothendieck groups

Proposition 3.16 has some other useful corollaries which we state now.

PROPOSITION 3.21. *Let* M *be an object of* $\mathrm{D}_c^{\mathrm{b}}(\mathrm{G})$.

(1) *There exists a generic set* $\mathscr{X} \subset \widehat{\mathrm{G}}$ *such that the Euler–Poincaré characteristic* $\chi(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ *is independent of* $\chi \in \mathscr{X}$.
(2) *If* M *is negligible, then* $\chi(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = 0$ *for all* χ *in a generic set of characters. The converse holds if* M *is a perverse sheaf.*
(3) *If* G *is a semiabelian variety, then the Euler–Poincaré characteristic* $\chi(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ *is independent of* $\chi \in \widehat{\mathrm{G}}$ *and it is non-negative if* M *is a perverse sheaf.*

PROOF. The decomposition

$$\mathrm{M} = \sum_{i \in \mathbf{Z}} (-1)^i \ {}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M})$$

in the Grothendieck group K(G), together with Lemma 1.17, implies that

$$\chi(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = \sum_{i \in \mathbf{Z}} (-1)^i \ \chi(\mathrm{G}_{\bar{k}}, {}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M})_\chi)$$

for all $\chi \in \widehat{\mathrm{G}}$. Thus the first statement is an immediate consequence of Proposition 3.16, combined wih the generic vanishing theorem, applied to each perverse cohomology sheaf.

If N is a negligible perverse sheaf, then by definition we get $\mathrm{H}^*(\mathrm{G}_{\bar{k}}, \mathrm{N}_\chi) = 0$ for a generic set of characters, hence also $\chi(\mathrm{G}_{\bar{k}}, \mathrm{N}_\chi) = 0$ for a generic set of characters. The previous formula shows that this is also true for any complex M.

Conversely, assume that M is a perverse sheaf and $\chi(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = 0$ for all χ in a generic set. Combined with the generic vanishing theorem, this implies that $\mathrm{H}^*(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = 0$ for χ generic, hence M is negligible.

If G is a semiabelian variety, then the Euler–Poincaré characteristic $\chi(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ is independent of χ by a result of Deligne (see [63]), because all the $\chi \in \widehat{\mathrm{G}}$ are tame. In this case, the tannakian dimension of a perverse sheaf on G is therefore the same as its Euler–Poincaré characteristic. □

Corollary 3.22. *A perverse sheaf* M *in* $\mathbf{P}(\mathrm{G})$ *is negligible if and only if its class in the Grothendieck group* $\mathrm{K}(\mathrm{G})$ *belongs to the subgroup* $\mathrm{K}_{\mathrm{neg}}(\mathrm{G})$ *generated by classes of negligible perverse sheaves.*

Proof. It suffices to prove that a perverse sheaf M is negligible if the class of M in K(G) can be expressed as a finite sum

$$\mathrm{M} = \sum_{i\in\mathrm{I}} \varepsilon_i \mathrm{M}_i$$

in K(G), where M_i is a negligible perverse sheaf for all $i \in \mathrm{I}$ and $\varepsilon_i \in \{-1, 1\}$. Such a formula implies the equality

$$\chi(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = \sum_{i\in\mathrm{I}} \varepsilon_i \chi(\mathrm{G}_{\bar{k}}, \mathrm{M}_{i,\chi})$$

for all $\chi \in \widehat{\mathrm{G}}$. For a generic set of characters we have $\chi(\mathrm{G}_{\bar{k}}, \mathrm{M}_{i,\chi}) = 0$ for all $i \in \mathrm{I}$, since M_i is negligible by assumption, hence $\chi(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = 0$ for a generic set of characters; thus M is negligible by Proposition 3.21, (2). ☐

Corollary 3.23. *Suppose that* G *is a semiabelian variety. Let* M *be a negligible perverse sheaf on* G*. The Euler–Poincaré characteristic of* M *is* 0 *and the set of characters* $\chi \in \widehat{\mathrm{G}}$ *such that the space* $\mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ *is non-zero is contained in a finite union of tacs.*

Proof. The fact that $\chi(\mathrm{M}) = 0$ has been stated in Proposition 3.21. By Theorem 2.16, there exists a finite family (S_f) of tacs of G such that $\mathrm{H}^i(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = 0$ for all $i \neq 0$ and χ not belonging to the union $\mathscr{S}$ of these tacs. For any χ not in $\mathscr{S}$, we then deduce by loc. cit. that

$$\dim \mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = \chi(\mathrm{M}_\chi) = \chi(\mathrm{M}) = 0.$$

☐

3.7. Arithmetic fiber functors

We now address the question of constructing arithmetic fiber functors that will be used to define conjugacy classes of elements in the tannakian groups.

Definition 3.24 (Unramified characters). Let M be an object of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$. A weakly unramified character $\chi \in \widehat{\mathrm{G}}$ for M is said to be *unramified for* M if the functor

$$\mathrm{N} \longmapsto \omega_\chi(\mathrm{N}) = \mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{N}_\chi)$$

is a fiber functor on the category $\langle \mathrm{M} \rangle \subset \mathbf{P}_{\mathrm{int}}(\mathrm{G})$. We denote by

$$\mathscr{X}(\mathrm{M}) \subset \mathscr{X}_w(\mathrm{M}) \subset \widehat{\mathrm{G}}$$

the set of unramified characters for M. We say that the perverse sheaf M is *generically unramified* if the subset $\mathscr{X}(\mathrm{M}) \subset \widehat{\mathrm{G}}$ is generic.

Remark 3.25. In Example 9.15, we will give examples to show that there may exist weakly unramified characters which are not unramified. An example is given by the sheaf $\mathrm{M} = \mathscr{L}_{\eta(f)}(1/2)[1]$ on $\mathbf{G}_m \times \mathbf{G}_a$, where f is a polynomial of degree d such that $f(0) \neq 0$ and η is a multiplicative character such that η^d is non-trivial.

We shall prove that every character (χ, a) is weakly unramified for M, that $\dim(\mathrm{H}^0(\mathbf{G}_m \times \mathbf{G}_a, \mathrm{M}_{(\chi,a)})$ is $d + 1$ if $a \neq 0$ but d if $a = 0$, implying that $(\chi, 0)$ is not unramified for M.

We expect that all semisimple objects of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$ are generically unramified. We can currently only prove this property for the three fundamental types of algebraic groups.

Theorem 3.26. *If* G *is a torus, an abelian variety or a unipotent group, then any semisimple object of* $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$ *is generically unramified.*

For tori or abelian varieties, we need a general technical criterion ensuring that an object M is generically unramified.

Lemma 3.27. *Let* M *be a semisimple object of* $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$. *Set* $\mathrm{L} = \mathrm{M} \oplus \mathrm{M}^\vee$. *For each* $m \geqslant 2$, *let* C_m *be the cone of the canonical morphism* $\mathrm{L}^{*_!^m} \to \mathrm{L}^{*_*^m}$. *All characters* χ *in*

$$\mathscr{X}_w(\mathrm{M}) \cap \bigcap_{m \geqslant 2} \mathscr{N}(\mathrm{C}_m) \tag{3.2}$$

are unramified for M.

Proof. Let χ be a character in the set (3.2). By Proposition 1.30, every object N of $\langle \mathrm{M} \rangle$ is a direct sum of direct factors of m-fold convolution products $\mathrm{L}^{*_{\mathrm{int}}^m}$ for some integers m. By the definition of (3.2) and Lemma 1.15, we have canonical isomorphisms

$$\mathrm{H}^*(\mathrm{G}_{\bar{k}}, \mathrm{L}_\chi^{*_{\mathrm{int}}^m}) \simeq \mathrm{H}^*(\mathrm{G}_{\bar{k}}, \mathrm{L}_\chi^{*_*^m}) \simeq \mathrm{H}^*(\mathrm{G}_{\bar{k}}, \mathrm{L}_\chi)^{\otimes m}$$

for any m.

By (3.2) again, we have $\mathrm{H}^*(\mathrm{G}_{\bar{k}}, \mathrm{L}_\chi) = \mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{L}_\chi)$, and hence $\omega_\chi(\mathrm{L}^{*_{\mathrm{int}}^m}) = \omega_\chi(\mathrm{L})^{\otimes m}$. This proves that the functor ω_χ is compatible with the tensor product; other compatibilities are elementary, and the functor ω_χ is exact on $\langle \mathrm{M} \rangle$, hence the result (see [32, Prop. 1.19]). ☐

Proof of Theorem 3.26 for abelian varieties. If G is an abelian variety, then both convolution functors are canonically isomorphic; hence, all objects C_m vanish and the set (3.2) is the same as $\mathscr{X}_w(\mathrm{M}) = \mathscr{X}(\mathrm{M})$, which is generic. ☐

Remark 3.28. There is a more precise result if G is an abelian variety. Indeed, we have recalled that $\mathscr{X}_w(\mathrm{M}) = \mathscr{X}(\mathrm{M})$ for any semisimple object of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$, and by the strong form of the Stratified Generic Vanishing Theorem (Theorem 2.3), it follows that the set of ramified characters is contained in a finite union of tacs of G.

Proof of Theorem 3.26 for tori. We use the notation of the previous lemma. For a torus G, a result of Gabber and Loeser [50, Prop. 3.9.3 (iv)] implies that there is an inclusion $\mathscr{N}(\mathrm{C}_2) \subset \mathscr{N}(\mathrm{C}_m)$ for all integers $m \geqslant 2$. So the set

$$\mathscr{X}_w(\mathrm{M}) \cap \bigcap_{m \geqslant 2} \mathscr{N}(\mathrm{C}_m) = \mathscr{X}_w(\mathrm{M}) \cap \mathscr{N}(\mathrm{C}_2)$$

is generic, by the generic vanishing theorem and the definition of negligible objects. ☐

Finally we consider unipotent groups.

Proof of Theorem 3.26 for G unipotent. We denote by $\mathrm{G}^\vee$ a form of the Serre dual of G, and we fix an additive character ψ to compute the Fourier transform FT_ψ on G (see Section 2.2).

Let M be a semisimple object of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$. We claim that there exists a dense open set $\mathrm{V} \subset \mathrm{G}^\vee$ such that for all objects N and N′ of $\langle \mathrm{M} \rangle$, the restriction of $\mathrm{FT}_\psi(\mathrm{N})$ to V is lisse and there exists a canonical isomorphism

$$\mathrm{FT}_\psi(\mathrm{N} *_{\mathrm{int}} \mathrm{N}')|\mathrm{V} \to (\mathrm{FT}_\psi(\mathrm{N}) \otimes \mathrm{FT}_\psi(\mathrm{N}'))|\mathrm{V}. \tag{3.3}$$

Indeed, if this claim holds, then it is elementary that for any $a \in \mathrm{V}(\bar{k})$, the corresponding character $\psi_a \in \widehat{\mathrm{G}}$ is unramified for M.

The claim above follows in turn from a more general statement: for all objects M_1 and M_2 of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$, and for any open dense subset $\mathrm{W} \subset \mathrm{G}^\vee$ such that the Fourier transforms $\mathrm{FT}_\psi(\mathrm{M}_1)$ and $\mathrm{FT}_\psi(\mathrm{M}_2)$ are lisse on W, there exists a canonical isomorphism

$$\mathrm{FT}_\psi(\mathrm{M}_1 *_{\mathrm{int}} \mathrm{M}_2)|\mathrm{W} \to (\mathrm{FT}_\psi(\mathrm{M}_1) \otimes \mathrm{FT}_\psi(\mathrm{M}_2))|\mathrm{W}.$$

Indeed, the isomorphism shows in particular that the Fourier transform of $\mathrm{M}_1 *_{\mathrm{int}} \mathrm{M}_2$ is also lisse on W; since the same is true of the dual $\mathrm{D}(\mathrm{M}_1)$, it follows that the Fourier transform of any object of $\langle \mathrm{M}_1 \rangle$ is lisse on W, leading to the previous claim (with $\mathrm{V} = \mathrm{W}$).

We now prove the general statement above. Let $\mathrm{M} = {}^{\mathfrak{p}}\mathscr{H}^0(\mathrm{M}_1 *_! \mathrm{M}_2)$. By definition of $\mathrm{M}_1 *_{\mathrm{int}} \mathrm{M}_2$, we have $\mathrm{M}_1 *_{\mathrm{int}} \mathrm{M}_2 = \mathrm{M}_{\mathrm{int}}$ (see Remark 3.13).

Let ${}^{\mathfrak{p}}\tau_{\leqslant 0}$ and ${}^{\mathfrak{p}}\tau_{\geqslant 0}$ be the perverse truncation functors. We have canonical morphisms

$$ {}^{\mathfrak{p}}\tau_{\leqslant 0}(\mathrm{M}_1 *_! \mathrm{M}_2) \to \mathrm{M}_1 *_! \mathrm{M}_2 \tag{3.4}$$

and

$$ {}^{\mathfrak{p}}\tau_{\leqslant 0}(\mathrm{M}_1 *_! \mathrm{M}_2) \to {}^{\mathfrak{p}}\tau_{\geqslant 0}({}^{\mathfrak{p}}\tau_{\leqslant 0}(\mathrm{M}_1 *_! \mathrm{M}_2)) = {}^{\mathfrak{p}}\mathscr{H}^0(\mathrm{M}_1 *_! \mathrm{M}_2) = \mathrm{M}. \tag{3.5}$$

By Lemma 3.8, the mapping cones of both morphisms are negligible. By the vanishing theorem for unipotent groups (Proposition 2.7), there is a dense open subset W' of W such that the induced morphisms

$$\mathrm{FT}_\psi({}^{\mathfrak{p}}\tau_{\leqslant 0}(\mathrm{M}_1 *_! \mathrm{M}_2))|\mathrm{W}' \to \mathrm{FT}_\psi(\mathrm{M}_1 *_! \mathrm{M}_2)|\mathrm{W}' \tag{3.6}$$

and

$$\mathrm{FT}_\psi({}^{\mathfrak{p}}\tau_{\leqslant 0}(\mathrm{M}_1 *_! \mathrm{M}_2))|\mathrm{W}' \to \mathrm{FT}_\psi(\mathrm{M})|\mathrm{W}' \tag{3.7}$$

are isomorphisms. Inverting (3.6) and composing with (3.7), we obtain a canonical isomorphism

$$\mathrm{FT}_\psi(\mathrm{M}_1 *_! \mathrm{M}_2)|\mathrm{W}' \to \mathrm{FT}_\psi(\mathrm{M})|\mathrm{W}'. \tag{3.8}$$

Let M^t be the smallest subobject of M such that M/M^t is negligible. We then have a canonical injection $\mathrm{M}^t \to \mathrm{M}$ with negligible cokernel and a canonical surjection $\mathrm{M}^t \to \mathrm{M}_{\mathrm{int}}$ with negligible kernel, by Remark 3.13. By the vanishing theorem for unipotent groups (Proposition 2.7), up to replacing W' by a smaller dense open subset, we can assume that the canonical morphisms

$$\mathrm{FT}_\psi(\mathrm{M}^t)|\mathrm{W}' \to \mathrm{FT}_\psi(\mathrm{M})|\mathrm{W}' \tag{3.9}$$

and

$$\mathrm{FT}_\psi(\mathrm{M}^t)|\mathrm{W}' \to \mathrm{FT}_\psi(\mathrm{M}_{\mathrm{int}})|\mathrm{W}' \tag{3.10}$$

are isomorphisms. Inverting (3.9) and composing with (3.10), we get a canonical isomorphism

$$\mathrm{FT}_\psi(\mathrm{M})|\mathrm{W}' \to \mathrm{FT}_\psi(\mathrm{M}_{\mathrm{int}})|\mathrm{W}' = \mathrm{FT}_\psi(\mathrm{M}_1 *_{\mathrm{int}} \mathrm{M}_2)|\mathrm{W}'. \tag{3.11}$$

Composing (3.8) and (3.11), we get a canonical isomorphism

$$\mathrm{FT}_\psi(\mathrm{M}_1 *_! \mathrm{M}_2)|\mathrm{W}' \simeq \mathrm{FT}_\psi(\mathrm{M}_1 *_{\mathrm{int}} \mathrm{M}_2)|\mathrm{W}'. \tag{3.12}$$

Denote by $j\colon \mathrm{W}' \to \mathrm{W}$ the open immersion. By the definition of the category $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$, the Fourier transform $\mathrm{FT}_\psi(\mathrm{M}_1 *_{\mathrm{int}} \mathrm{M}_2)$ (which is a perverse sheaf up to shift) has no shifted perverse component supported in $\mathrm{G}^\vee - \mathrm{W}'$ (such a component would be negligible), and therefore we have a canonical isomorphism

$$j_{!*}j^*(\mathrm{FT}_\psi(\mathrm{M}_1 *_{\mathrm{int}} \mathrm{M}_2)|\mathrm{W}) \simeq \mathrm{FT}_\psi(\mathrm{M}_1 *_{\mathrm{int}} \mathrm{M}_2)|\mathrm{W} \tag{3.13}$$

by the properties of the intermediate extension functor $j_{!*}$ (see Proposition A.9).

By Lemma 1.15, there is a canonical isomorphism $\mathrm{FT}_\psi(\mathrm{M}_1 *_! \mathrm{M}_2) \simeq \mathrm{FT}_\psi(\mathrm{M}_1) \otimes \mathrm{FT}_\psi(\mathrm{M}_2)$. Since $\mathrm{FT}_\psi(\mathrm{M}_1)$ and $\mathrm{FT}_\psi(\mathrm{M}_2)$ are lisse on W, we have also a canonical isomorphim

$$j_{!*}j^*((\mathrm{FT}_\psi(\mathrm{M}_1) \otimes \mathrm{FT}_\psi(\mathrm{M}_2))|\mathrm{W}) \simeq (\mathrm{FT}_\psi(\mathrm{M}_1) \otimes \mathrm{FT}_\psi(\mathrm{M}_2))|\mathrm{W},$$

hence a canonical isomorphism

$$j_{!*}j^*(\mathrm{FT}_\psi(\mathrm{M}_1 *_! \mathrm{M}_2)|\mathrm{W}) \simeq (\mathrm{FT}_\psi(\mathrm{M}_1) \otimes \mathrm{FT}_\psi(\mathrm{M}_2))|\mathrm{W}. \tag{3.14}$$

We now apply the functor $j_{!*}$ to the isomorphism (3.12), and use (3.13) and (3.14) to obtain the desired canonical isomorphism (3.3); this concludes the proof of the claim. ☐

3.8. The arithmetic tannakian group

In this section, we consider the situation over the finite field k. Base change $\mathrm{M} \mapsto \mathrm{M}_{\bar{k}}$ gives a functor $\mathbf{Perv}(\mathrm{G}) \to \mathbf{P}(\mathrm{G})$. For a perverse sheaf M on G, we define the set of unramified characters for M as $\mathscr{X}(\mathrm{M}) = \mathscr{X}(\mathrm{M}_{\bar{k}})$.

We denote by $\mathrm{Neg}^{\mathrm{ari}}_{\mathbf{P}}(\mathrm{G})$ (resp. $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{G})$) the full subcategory of $\mathbf{Perv}(\mathrm{G})$ whose objects are the perverse sheaves M such that $\mathrm{M}_{\bar{k}}$ is an object of $\mathrm{Neg}_{\mathbf{P}}(\mathrm{G})$ (resp. of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$). As in the geometric case, we find that $\mathrm{Neg}^{\mathrm{ari}}_{\mathbf{P}}(\mathrm{G})$ is a Serre subcategory of $\mathbf{Perv}(\mathrm{G})$ and that the localization functor induces an equivalence from $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{G})$ to the quotient abelian category $\overline{\mathbf{P}}^{\mathrm{ari}}(\mathrm{G}) = \mathbf{Perv}(\mathrm{G})/\mathrm{Neg}^{\mathrm{ari}}_{\mathbf{P}}(\mathrm{G})$.

Also similarly to the geometric case, the two convolution bifunctors on $\mathbf{Perv}(\mathrm{G})$ induce equivalent bifunctors on $\overline{\mathbf{P}}^{\mathrm{ari}}(\mathrm{G})$ (compare with Proposition 3.10). The categories $\overline{\mathbf{P}}^{\mathrm{ari}}(\mathrm{G})$ and $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{G})$ are then rigid symmetric $\overline{\mathbf{Q}}_\ell$-linear tensor categories, with unit object $\mathbf{1}$ still the skyscraper sheaf at the unit of G, which again satisfies $\mathrm{End}(\mathbf{1}) \simeq \overline{\mathbf{Q}}_\ell$.

Let M be a perverse sheaf on G. To distinguish between the arithmetic and geometric situations, we denote from now on by $\langle \mathrm{M}\rangle^{\mathrm{ari}}$ (resp. $\langle \mathrm{M}\rangle^{\mathrm{geo}}$) the subcategory of $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{G}) \simeq \overline{\mathbf{P}}^{\mathrm{ari}}(\mathrm{G})$ (resp. of $\mathbf{P}_{\mathrm{int}}(\mathrm{G}) \simeq \overline{\mathbf{P}}(\mathrm{G})$) that is tensor-generated by (the image of) M (resp. by $\mathrm{M}_{\bar{k}}$). Base change $\mathrm{N} \mapsto \mathrm{N}_{\bar{k}}$ gives a functor from $\langle \mathrm{M}\rangle^{\mathrm{ari}}$ to $\langle \mathrm{M}\rangle^{\mathrm{geo}}$.

THEOREM 3.29. *Let* M *be an object of* $\mathbf{Perv}(\mathrm{G})$*. The categories* $\langle \mathrm{M}\rangle^{\mathrm{ari}}$ *and* $\langle \mathrm{M}\rangle^{\mathrm{geo}}$ *are neutral* $\overline{\mathbf{Q}}_\ell$*-linear tannakian categories. There exist algebraic groups* $\mathbf{G}^{\mathrm{geo}}_{\mathrm{M}}$ *and* $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$ *over* $\overline{\mathbf{Q}}_\ell$ *such that* $\langle \mathrm{M}\rangle^{\mathrm{ari}}$ *is equivalent to the category* $\mathrm{Rep}_{\overline{\mathbf{Q}}_\ell}(\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}})$ *and* $\langle \mathrm{M}\rangle^{\mathrm{geo}}$ *is equivalent to the category* $\mathrm{Rep}_{\overline{\mathbf{Q}}_\ell}(\mathbf{G}^{\mathrm{geo}}_{\mathrm{M}})$*.*

Moreover, if r *is the tannakian dimension of* M*, then the objects* M *and* $\mathrm{M}_{\bar{k}}$ *of* $\langle \mathrm{M}\rangle^{\mathrm{ari}}$ *and* $\langle \mathrm{M}\rangle^{\mathrm{geo}}$*, respectively, correspond to faithful representations of* $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$ *and* $\mathbf{G}^{\mathrm{geo}}_{\mathrm{M}}$ *in* $\mathbf{GL}_r(\overline{\mathbf{Q}}_\ell)$*.*

PROOF. The case of $\langle \mathrm{M}\rangle^{\mathrm{geo}}$ is dealt with by Theorem 3.15 and Corollary 3.18. The case of $\langle \mathrm{M}\rangle^{\mathrm{ari}}$ follows by the same argument because Proposition 3.16 also applies to $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{G})$.

The last assertion is a tautological consequence of the formalism. ☐

DEFINITION 3.30. In the context of Theorem 3.29, we call $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$ the *arithmetic tannakian group* of M, and $\mathbf{G}^{\mathrm{geo}}_{\mathrm{M}}$ its *geometric tannakian group.*

PROPOSITION 3.31. *Let* M *be an object of* $\mathbf{Perv}(\mathrm{G})$*. The functor of base change to* $\bar{k}$ *is a tensor functor from* $\langle \mathrm{M}\rangle^{\mathrm{ari}}$ *to* $\langle \mathrm{M}\rangle^{\mathrm{geo}}$ *that induces a morphism* $\varphi\colon \mathbf{G}^{\mathrm{geo}}_{\mathrm{M}} \to \mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$*. This morphism is a closed immersion.*

PROOF. The first assertion is immediate, and it implies by the tannakian formalism the existence of the homomorphism φ. According to [32, Prop. 2.21 (b)], this morphism φ is a closed immersion if and only if every object of $\langle \mathrm{M}\rangle^{\mathrm{geo}}$ is isomorphic to a subquotient of an object in the essential image of the base-change functor.

Let N be such an object of $\langle \mathrm{M} \rangle^{\mathrm{geo}}$, viewed as an object of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$. By definition of the category $\mathbf{P}(\mathrm{G})$, there exists a finite extension k_n of k in $\bar{k}$ such that N is the base change to $\bar{k}$ of a perverse sheaf N_1 on G_{k_n}. Then N is a subquotient of the base change of the perverse sheaf $f_{n*}\mathrm{N}_1$ to $\mathrm{G}_{\bar{k}}$, where $f_n \colon \mathrm{Spec}(k_n) \to \mathrm{Spec}(k)$ is the canonical morphism, hence the result. □

From now on, we will identify the geometric tannakian group of a perverse sheaf M on G with its image in the arithmetic tannakian group.

We recall the convention from Section 1.12 concerning properties over k and $\bar{k}$. Let M be a perverse sheaf on G. We view $\langle \mathrm{M} \rangle^{\mathrm{ari}}$ as a subcategory of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$, so that the weights of an object $\mathrm{N} \in \langle \mathrm{M} \rangle^{\mathrm{ari}}$ are well-defined.

THEOREM 3.32. *Let* M *be a perverse sheaf on* G*. Assume that* M *is arithmetically semisimple and pure of weight zero. Let r be the tannakian dimension of* M*.*

(1) *The groups $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$ and $\mathbf{G}^{\mathrm{geo}}_{\mathrm{M}}$ are reductive subgroups of $\mathbf{GL}_r$.*

(2) *Every object* N *of $\langle \mathrm{M} \rangle^{\mathrm{ari}}$ is arithmetically semisimple and pure of weight zero, and every object* N *of $\langle \mathrm{M} \rangle^{\mathrm{geo}}$ is semisimple.*

PROOF. Since any pure perverse sheaf on G is geometrically semisimple by [8, Th. 5.3.8], the assertions for $\langle \mathrm{M} \rangle^{\mathrm{geo}}$ follow. The same proof is also valid for $\langle \mathrm{M} \rangle^{\mathrm{ari}}$, since M is arithmetically semisimple, so that the group $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$ is also reductive, and all objects of $\langle \mathrm{M} \rangle^{\mathrm{ari}}$ are arithmetically semisimple.

We now prove the purity statement. Since M is pure of weight zero, it follows from the description of $\mathrm{M}_{\mathrm{int}}$ in Remark 3.13 that the corresponding object of $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{G})$ is also pure of weight zero, and similarly for its dual

For any perverse sheaves N_1 and N_2 on G that are pure of weight zero, the convolution $\mathrm{N}_1 *_{\mathrm{int}} \mathrm{N}_2$ is also pure of weight zero. Indeed, by Deligne's Riemann Hypothesis [28, 3.3.1], the object $\mathrm{N}_1 *_! \mathrm{N}_2$ is mixed of weights $\leqslant 0$. Hence, the quotient $\mathrm{N}_1 *_{\mathrm{int}} \mathrm{N}_2$ of $\mathrm{N}_1 *_! \mathrm{N}_2$ is also mixed of weights $\leqslant 0$ by [8, Prop. 5.3.1]. Thanks to Lemma 1.15, the same applies to the Verdier dual $\mathrm{D}(\mathrm{N}_1 *_{\mathrm{int}} \mathrm{N}_2)$, which implies the claim.

Hence, the property of being pure of weight zero is preserved by convolution, duality and taking subobjects. Thus we conclude that every object N of $\langle \mathrm{M} \rangle^{\mathrm{ari}}$ is pure of weight zero. □

We now show that the tannakian groups coincide with those of Katz for the multiplicative group using the category $\mathcal{P}$ (see [74, Ch. 2] and Section B.1), and with monodromy groups of the Fourier transform for unipotent groups.

PROPOSITION 3.33. *Let* M *be a perverse sheaf on* G*. Assume that* M *is arithmetically semisimple and pure of weight zero.*

1. *If $\mathrm{G} = \mathbf{G}_m$, then the arithmetic and geometric tannakian groups of* G *coincide with those defined by Katz using the category $\mathcal{P}$.*
2. *If* G *is unipotent of dimension d, and ψ is a fixed additive character used to define its Fourier transform, then there exists a dense open subset* U *of the Serre dual $\mathrm{G}^\vee$ such that $(\mathrm{FT}_\psi\, \mathrm{M}_{\mathrm{int}})|\mathrm{U}$ is isomorphic to a lisse sheaf $\mathscr{F}$ on* U*, pure of weight d, placed in degree 0. The arithmetic and geometric tannakian groups of* M *coincide with the arithmetic and geometric monodromy groups of the lisse sheaf $\mathscr{F}$.*

PROOF. In the case of $\mathbf{G}_m$, the statement follows directly from Example 3.20 (1) (see also Section B.1 for the definition of $\mathcal{P}$).

Suppose then that G is unipotent. To prove the first assertion of (2), we may assume that M is simple and non-negligible. Its Fourier transform is then a simple d-shifted perverse sheaf on the Serre dual $\mathrm{G}^\vee$, pure of weight d, and with support equal to $\mathrm{G}^\vee$ (since the object M would be negligible if the support were smaller). Thus it is a single lisse sheaf, pure of weight d, on an open dense subset of $\mathrm{G}^\vee$.

For the second part of (2), we note that by (the proof of) Theorem 3.26 for unipotent groups, the convolution product on $\langle \mathrm{M}\rangle^{\mathrm{ari}}$ can be identified with the tensor product on the subcategory generated by $\mathscr{F}$ of the category of lisse sheaves on U. The result then follows. ☐

3.9. Frobenius conjugacy classes

We keep working over the finite field k and use the same notation as in the previous subsection. For any finite extension k_n of k, we denote by Fr_{k_n} the geometric Frobenius automorphism of k_n.

For an object M of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$, an integer $n \geqslant 1$ and a character $\chi \in \widehat{\mathrm{G}}(k_n)$, we denote by $\mathrm{Fr}_{\mathrm{M},k_n}(\chi)$ the automorphism of the $\overline{\mathbf{Q}}_\ell$-vector space $\mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ induced by the action of Fr_{k_n}. Recall from A.3 the notions of weights and purity.

Let r be the dimension of this space. If the automorphism $\mathrm{Fr}_{\mathrm{M},k_n}(\chi)$ is pure of weight zero, for instance if M is pure of weight 0 and χ is weakly unramified for M, then there is a unique conjugacy class $\Theta_{\mathrm{M},k_n}(\chi)$ in the complex unitary group $\mathbf{U}_r(\mathbf{C})$ containing the semisimple part of $\iota_0(\mathrm{Fr}_{\mathrm{M},k_n}(\chi))$.

We call $\mathrm{Fr}_{\mathrm{M},k_n}(\chi)$ the *Frobenius automorphism of* M *associated to* χ *over* k_n and $\Theta_{\mathrm{M},k_n}(\chi)$ the *unitary Frobenius conjugacy class of* M *associated to* χ *over* k_n.

Suppose now that M is an arithmetically semisimple perverse sheaf on G.

Let $n \geqslant 1$ and let $\chi \in \widehat{\mathrm{G}}(k_n)$ be an *unramified* character for M, so that the functor $\omega_\chi \colon \mathrm{N} \mapsto \mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{N}_\chi)$ is a fiber functor on the tannakian category $\langle \mathrm{M}\rangle^{\mathrm{ari}}$. For any object N of $\langle \mathrm{M}\rangle^{\mathrm{ari}}$, the Frobenius automorphism Fr_{k_n} now induces an automorphism of $\omega_\chi(\mathrm{N})$, and thus defines an automorphism of the fiber functor ω_χ. By the tannakian formalism, this corresponds to a unique conjugacy class in $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}(\overline{\mathbf{Q}}_\ell)$. We denote by $\mathbf{Fr}_{\mathrm{M},k_n}(\chi)$ the corresponding conjugacy class of $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}(\mathbf{C})$, and call it the *Frobenius conjugacy class of* M *associated to* χ *over* k_n.

Suppose furthermore that M is pure of weight zero. Let K_{M} be a maximal compact subgroup of the reductive group $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}(\mathbf{C})$. Since all objects of $\langle \mathrm{M}\rangle^{\mathrm{ari}}$ are pure of weight zero (by Theorem 3.32), the eigenvalues of any element of the conjugacy class $\mathbf{Fr}_{\mathrm{M},k_n}(\chi)$ are complex numbers with modulus 1, so that the semisimple part of this conjugacy class is a unitary matrix. One can then deduce from the Peter–Weyl Theorem that the $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}(\mathbf{C})$-conjugacy class of the semisimple part of $\mathbf{Fr}_{\mathrm{M},k_n}(\chi)$ intersects K_{M} in a unique conjugacy class, which is denoted $\boldsymbol{\Theta}_{\mathrm{M},k_n}(\chi)$, and is called the *unitary Frobenius conjugacy class* of M associated to χ. (See, e.g., [78, 9.2.4] for this argument.)

For an unramified character χ, the space $\omega_\chi(\mathrm{M})$ has dimension r, the tannakian dimension of M, and the conjugacy class of $\mathbf{Fr}_{\mathrm{M},k_n}(\chi)$ in the automorphism group of $\mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ coincides with that of $\mathrm{Fr}_{\mathrm{M},k_n}(\chi)$, and similarly for $\boldsymbol{\Theta}_{\mathrm{M},k_n}(\chi)$.

When $k_n = k$, we will sometimes use simply the notation $\mathrm{Fr}_{\mathrm{M}}(\chi)$, $\Theta_{\mathrm{M}}(\chi)$, $\mathbf{Fr}_{\mathrm{M}}(\chi)$, $\boldsymbol{\Theta}_{\mathrm{M}}(\chi)$.

We have the following important consequences of the formalism.

Lemma 3.34. *Let* M *be an arithmetically semisimple perverse sheaf on* G *that is pure of weight zero and of tannakian dimension* $r \geqslant 0$.

(1) *Let* $\chi \in \mathscr{X}_w(\mathrm{M})(k)$ *be a weakly unramified character for* M. *For any integer* $n \geqslant 1$, *we have*

$$\mathrm{Tr}(\Theta_{\mathrm{M},k_n}(\chi)) = \mathrm{Tr}(\Theta_{\mathrm{M}}(\chi)^n) = \sum_{x \in \mathrm{G}(k_n)} \chi(\mathrm{N}_{k_n/k}(x)) t_{\mathrm{M}}(x; k_n),$$

where t_{M} *is the trace function of* M *and the trace on the left is that on* $\mathbf{GL}_r$.

(2) *Let* $\chi \in \mathscr{X}(\mathrm{M})(k)$ *be an unramified character. Let* ϱ *be an algebraic* $\overline{\mathbf{Q}}_\ell$*-representation of* $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ *and denote by* $\varrho(\mathrm{M})$ *the corresponding object of* $\langle \mathrm{M} \rangle^{\mathrm{ari}}$. *The character* χ *is unramified for* $\varrho(\mathrm{M})$ *and*

$$\mathrm{Tr}(\varrho(\mathbf{Fr}_{\mathrm{M}}(\chi))) = \mathrm{Tr}(\mathrm{Fr}_k \mid \mathrm{H}_c^0(\mathrm{G}_{\bar{k}}, \varrho(\mathrm{M})_\chi)).$$

Proof. (1) By definition, we have

$$\mathrm{Tr}(\Theta_{\mathrm{M}}(\chi)^n) = \mathrm{Tr}(\mathrm{Fr}_{\mathrm{M}}(\chi)^n) = \mathrm{Tr}(\mathrm{Fr}_k^n \mid \mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)).$$

Since χ is weakly unramified, we have $\mathrm{H}_c^i(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = 0$ for all $i \neq 0$ and $\mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = \mathrm{H}_c^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$, so that we can write

$$\mathrm{Tr}(\Theta_{\mathrm{M}}(\chi)^n) = \sum_{i \in \mathbf{Z}} (-1)^i \, \mathrm{Tr}(\mathrm{Fr}_k^n \mid \mathrm{H}_c^i(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)) = \sum_{x \in \mathrm{G}(k)} \chi(\mathrm{N}_{k_n/k}(x)) t_{\mathrm{M}}(x; k_n),$$

by the trace formula.

(2) The fact that χ is unramified for $\varrho(\mathrm{M})$ follows from the definition and Proposition 1.30, and the formula follows then from the definition of the Frobenius conjugacy class of χ for $\varrho(\mathrm{M})$. ☐

3.10. Frobenius-unramified characters

Because weakly unramified characters do not always give rise to fiber functors, and moreover we do not always know if there exist sufficiently many (if any) unramified characters, we introduce an intermediate notion.

Definition 3.35 (Frobenius-unramified characters). Let M be an object of $\mathbf{Perv}(\mathrm{G})$ which is arithmetically semisimple and pure of weight zero, of tannakian dimension r. Let ϱ be a representation of $\mathbf{GL}_r$ and let N be the object of $\langle \mathrm{M} \rangle^{\mathrm{ari}}$ corresponding to the restriction of ϱ to $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$. Let $n \geqslant 1$ and let $\chi \in \mathscr{X}_w(\mathrm{M})(k_n)$ be a weakly unramified character for M. We say that χ is *Frobenius-unramified for* ϱ if χ is weakly unramified for N and if the formula

$$\mathrm{Tr}(\varrho(\Theta_{\mathrm{M},k_n}(\chi))^v) = \mathrm{Tr}(\mathrm{Fr}_{k_n}^v \mid \mathrm{H}_c^0(\mathrm{G}_{\bar{k}}, \mathrm{N}_\chi))$$

holds for all integers $v \geqslant 1$, or equivalently if

$$\det(1 - \varrho(\Theta_{\mathrm{M},k_n}(\chi))\mathrm{T}) = \det(1 - \mathrm{T}\,\mathrm{Fr}_{k_n} \mid \mathrm{H}_c^0(\mathrm{G}_{\bar{k}}, \mathrm{N}_\chi)).$$

The disjoint union over n of the set of Frobenius-unramified characters is denoted $\mathscr{X}_{\mathrm{F}}(\varrho)$.

Remark 3.36. (1) The key point is that since ϱ is a representation of $\mathbf{GL}_r$, we can consider the conjugacy class of $\varrho(\mathrm{Fr}_{\mathrm{M},k_n}(\chi))$ (in $\mathbf{GL}(\mathrm{V})$, where ϱ is a representation on V); a priori, this is not meaningful for a representation of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$, unless we know that elements of the conjugacy class of the Frobenius automorphism of $\mathrm{H}_c^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ are conjugate to some element of the arithmetic tannakian group, which is unique up to conjugacy in $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$.

(2) We will also sometimes write $\mathscr{X}_{\mathrm{F}}(\varrho) = \mathscr{X}_{\mathrm{F}}(\mathrm{N})$, although this set depends on M, since we view N as an object of $\langle \mathrm{M} \rangle^{\mathrm{ari}}$. When confusion might arise, we may also write $\mathscr{X}_{\mathrm{F}}(\mathrm{N})_{\mathrm{M}}$.

Any unramified character for M is Frobenius-unramified for all objects of $\langle \mathrm{M}\rangle^{\mathrm{ari}}$, by Lemma 3.34, (2). But in contrast to unramified characters, we can prove in all cases that the set of Frobenius-unramified characters is generic.

Proposition 3.37. *Let* M *be an object of* $\mathbf{Perv}(\mathrm{G})$ *which is arithmetically semisimple and pure of weight zero and of tannakian dimension* $r \geqslant 0$*. For any representation* ϱ *of* $\mathbf{GL}_r$*, the set* $\mathscr{X}_{\mathrm{F}}(\varrho)$ *is generic.*

Proof. We first observe that it is straightforward that if two representations ϱ_1 and ϱ_2 of $\mathbf{GL}_r$ have the property that $\mathscr{X}_{\mathrm{F}}(\varrho_1)$ and $\mathscr{X}_{\mathrm{F}}(\varrho_2)$ are generic, then the sets $\mathscr{X}_{\mathrm{F}}(\varrho_1 \oplus \varrho_2)$, $\mathscr{X}_{\mathrm{F}}(\varrho_1 \otimes \varrho_2)$ and $\mathscr{X}_{\mathrm{F}}(\varrho_1^{\vee})$ are also generic. Indeed, consider the case of the tensor product, the others being similar (and in fact simpler). Let N_i be the object corresponding to ϱ_i. For χ generic, we have

$$\mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, (\mathrm{N}_1 *_{\mathrm{int}} \mathrm{N}_2)_\chi) = \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, (\mathrm{N}_1 *_! \mathrm{N}_2)_\chi) \simeq \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{N}_{1,\chi}) \otimes \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{N}_{2,\chi})$$

as well as

$$\mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{N}_{i,\chi}) = \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{N}_{i,\chi})$$

for $i = 1$ and $i = 2$, all these isomorphisms being compatible with Frobenius. Thus

$$\det(1 - \mathrm{T}\,\mathrm{Fr}_{k_n} \mid \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, (\mathrm{N}_1 *_{\mathrm{int}} \mathrm{N}_2)_\chi)) = \det(1 - \mathrm{T}\,\mathrm{Fr}_{k_n} \mid \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{N}_{1,\chi}))\det(1 - \mathrm{T}\,\mathrm{Fr}_{k_n} \mid \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{N}_{2,\chi}))$$

for χ generic, which then establishes the claim concerning $\mathscr{X}_{\mathrm{F}}(\varrho_1 \otimes \varrho_2)$ using the definition of $\mathscr{X}_{\mathrm{F}}(\varrho_i)$ and the assumption that these are generic sets.

A first consequence of this observation is that we may assume that ϱ is irreducible to prove the proposition. Recall next that every irreducible representation ϱ of $\mathbf{GL}_r$ is isomorphic to one of the form $\varrho = \varrho_0 \otimes \det(\cdot)^k$ for some representation ϱ_0 given by a Schur functor $\mathbf{S}_\lambda$ and some integer $k \in \mathbf{Z}$ (see, e.g., [49, Prop. 15.47]). Since the determinant is itself a Schur functor, and $\det(\cdot)^{-k}$ is the contragredient of $\det(\cdot)^k$, the previous observation reduces the proof to the case where $\varrho = \mathbf{S}_\lambda$ for some λ.

In this case, ϱ is given by the image of an explicit projector (see, e.g., [49, § 6.1, th. 6.3]), and hence makes sense for any symmetric monoidal category where idempotents split. In particular, this applies to $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$ with either of the two convolutions, since $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$ is known to have this property (e.g., by combining the fact that $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$ is equivalent to the bounded derived category of the category of perverse sheaves, by a theorem of Beilinson [7, Th. 1.3], and the fact that the bounded derived category of an abelian category is idempotent complete, by a result of Balmer and Schlichting [3, Cor. 2.10]). We will denote by $\varrho_!(\mathrm{M})$ (resp. $\varrho_*(\mathrm{M})$) the action of these functors on M for the symmetric monoidal structures given by the convolution $(\mathrm{A}, \mathrm{B}) \mapsto \mathrm{A} *_! \mathrm{B}$ (resp. by $(\mathrm{A}, \mathrm{B}) \mapsto \mathrm{A} *_* \mathrm{B}$).

Since taking cohomologgy with compact support (resp. cohomology) is an additive monoidal functor for the convolution $\mathrm{A} *_! \mathrm{B}$ (resp. for $\mathrm{A} *_* \mathrm{B}$), by the Künneth formula, the explicit description of the idempotent defining ϱ provides isomorphisms

$$\mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \varrho_!(\mathrm{M})_\chi) \simeq \varrho(\mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)) \tag{3.15}$$

$$\mathrm{H}^*(\mathrm{G}_{\bar{k}}, \varrho_*(\mathrm{M})_\chi) \simeq \varrho(\mathrm{H}^*(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)) \tag{3.16}$$

for every χ, which are also compatible with Frobenius, where the Schur functor acts on the right-hand sides in the category of bounded complexes of $\overline{\mathbf{Q}}_\ell$-vector spaces.

Let $\mathrm{N} = \varrho(\mathrm{M})$ be the object of $\langle \mathrm{M}\rangle^{\mathrm{ari}}$ corresponding to ϱ. Since ϱ is assumed to be a Schur functor, there exists an integer $l \geqslant 0$ and an embedding $\mathrm{N} \to \mathrm{M}_l = \mathrm{M}^{*_{\mathrm{int}} l}$ (see, e.g., [49, § 6.1]).

We obtain a commutative square

$$\begin{array}{ccc} \varrho_!(\mathrm{M}) & \longrightarrow & \varrho_*(\mathrm{M}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{M}_{l,!} & \longrightarrow & \mathrm{M}_{l,*} \end{array}$$

where $\mathrm{M}_{l,!} = \mathrm{M}^{*_! l}$ and $\mathrm{M}_{l,*} = \mathrm{M}^{*_* l}$. This implies, in particular, that the cone C of the morphism

$$\varrho_!(\mathrm{M}) \to \varrho_*(\mathrm{M})$$

is negligible, since this is the case for the cone of the bottom morphism by Lemma 3.8. Applying Remark 3.13, there exists a generic set $\mathscr{X}$ of characters such that for $\chi \in \mathscr{X}$, we have isomorphisms

$$\mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{N}_\chi) \simeq \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \varrho_!(\mathrm{M})_\chi) \tag{3.17}$$

which are compatible with Frobenius.

Let finally $\chi \in \mathscr{X}$ be a character which is weakly unramified for both M and N. Then we have isomorphisms

$$\mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{N}_\chi) \simeq \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{N}_\chi) \simeq \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \varrho_!(\mathrm{M})_\chi) \simeq \varrho(\mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)) \simeq \varrho(\mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi))$$

compatible with Frobenius (the first and fourth of these are given by the theorem, the second is (3.17) and the third is (3.15)), and hence

$$\det(1 - \mathrm{T}\,\mathrm{Fr}_{k_n} \mid \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{N}_\chi)) = \det(1 - \varrho(\Theta_{\mathrm{M},k_n}(\chi))\mathrm{T}).$$

Since this holds for a generic set of characters (by Theorem 3.3), we obtain the desired result. ☐

Corollary 3.38. *Let* M *be an object of* $\mathbf{Perv}(\mathrm{G})$ *which is arithmetically semisimple and pure of weight zero and of tannakian dimension* $r \geqslant 0$*. If the group* $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$ *is finite, then* M *is generically unramified.*

Proof. The fact that the tannakian group is finite implies that any object of $\langle \mathrm{M} \rangle^{\mathrm{ari}}$ is a subobject of a *direct sum* of copies of a single object $\mathrm{N} = \mathrm{M}^{*_{\mathrm{int}} m} *_{\mathrm{int}} (\mathrm{M}^\vee)^{*_{\mathrm{int}} l}$ for some (fixed) integers m and l (see [32, Prop. 2.20 (a)]). Any Frobenius-unramified character for M is then an unramified character for M. ☐

3.11. Group-theoretic properties

We continue with the notation of the previous sections.

The following basic proposition establishes the relation between the geometric and arithmetic tannakian groups.

Proposition 3.39. *Let* M *be a geometrically semisimple object of* $\mathbf{Perv}(\mathrm{G})$*. The geometric tannakian group* $\mathbf{G}^{\mathrm{geo}}_{\mathrm{M}}$ *is a normal subgroup of the arithmetic tannakian group* $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$*.*

Proof. The proof is identical with that of [74, Lemma 6.1]. ☐

Proposition 3.40. *Let* M *be an arithmetically semisimple object of* $\mathbf{Perv}(\mathrm{G})$*. Assume that* M *is pure of weight zero.*

1. *The quotient* $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}/\mathbf{G}^{\mathrm{geo}}_{\mathrm{M}}$ *is of multiplicative type.*
2. *Let* V *be a geometrically trivial object of* $\langle \mathrm{M} \rangle^{\mathrm{ari}}$ *which corresponds to a faithful representation of the group* $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}/\mathbf{G}^{\mathrm{geo}}_{\mathrm{M}}$*. Any character* $\chi \in \widehat{\mathrm{G}}$ *is unramified for* V*, and the class* ξ *of the Frobenius conjugacy class of any such character is independent of* χ *and generates a Zariski-dense subgroup of* $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}/\mathbf{G}^{\mathrm{geo}}_{\mathrm{M}}$*.*

(3) *For any $n \geqslant 1$ and any character $\chi \in \widehat{\mathrm{G}}(k_n)$ unramified for* M, *the image in* $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}/\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ *of the Frobenius conjugacy class* $\mathbf{Fr}_{\mathrm{M},k}(\chi)$ *is equal to* ξ^n.

Proof. This follows by the same arguments as in [74, Lemma 7.1] (checking first that, using the structure of geometrically trivial objects as direct sums of $\alpha^{\deg} \otimes \delta_1$ for suitable α, it is indeed straightforward that all characters are unramified for such objects). ☐

We will also use the following result in Chapter 9.

Proposition 3.41. *Let* G_1 *and* G_2 *be connected commutative algebraic groups over* k *and let* $p\colon \mathrm{G}_1 \to \mathrm{G}_2$ *be a morphism of algebraic groups. Let* M *be a perverse sheaf on* G_1 *which is arithmetically semisimple and pure of weight zero.*

Let $\chi_1 \in \widehat{\mathrm{G}}_1(k)$ *be a character such that we have* $\mathrm{R}p_!(\mathrm{M}_{\chi_1}) = \mathrm{R}p_*(\mathrm{M}_{\chi_1})$. *Assume further that* $\mathrm{N} = \mathrm{R}p_!(\mathrm{M}_{\chi_1})$ *is perverse and arithmetically semisimple.*

(1) *The object* N *is pure of weight zero.*
(2) *Let* $n \geqslant 1$ *and let* $\chi \in \mathscr{X}_w(\mathrm{N})(k_n)$ *be a character such that* $\chi_1 \cdot (\chi \circ p)$ *is weakly unramified for* M. *Then the conjugacy classes* $\Theta_{\mathrm{M},k_n}(\chi_1 \cdot (\chi \circ p))$ *and* $\Theta_{\mathrm{N},k_n}(\chi)$ *satisfy*
$$\det(1 - \mathrm{T}\Theta_{\mathrm{M},k_n}(\chi_1 \cdot (\chi \circ p))) = \det(1 - \mathrm{T}\Theta_{\mathrm{N},k_n}(\chi)) \in \mathbf{C}[\mathrm{T}]$$
and in particular
$$\det(\Theta_{\mathrm{M},k_n}(\chi_1 \cdot (\chi \circ p)))) = \det(\Theta_{\mathrm{N},k_n}(\chi)).$$

Proof. It suffices to consider the case where $\chi \in \widehat{\mathrm{G}}(k)$. For any $n \geqslant 1$, the exponential sums
$$\mathrm{S}_n = \sum_{x \in \mathrm{G}_1(k_n)} t_{\mathrm{M}}(x; k_n)(\chi_1 \cdot (\chi \circ p))(\mathrm{N}_{k_n/k}(x))$$
$$\mathrm{S}'_n = \sum_{y \in \mathrm{G}_2(k_n)} t_{\mathrm{N}}(y; k_n)\chi(\mathrm{N}_{k_n/k}(y))$$
are equal by the trace formula. Hence, the corresponding L-functions
$$\exp\Bigl(\sum_{n \geqslant 1} \mathrm{S}_n \frac{\mathrm{T}^n}{n}\Bigr), \qquad \exp\Bigl(\sum_{n \geqslant 1} \mathrm{S}'_n \frac{\mathrm{T}^n}{n}\Bigr)$$
are also equal. But these L-functions coincide with the (reversed) characteristic polynomials of the conjugacy classes $\Theta_{\mathrm{M},k}(\chi_1 \cdot (\chi \circ p))$ and $\Theta_{\mathrm{N},k}(\chi)$, by Lemma 3.34 (1), hence the result. ☐

Remark 3.42. If the morphism $p\colon \mathrm{G}_1 \to \mathrm{G}_2$ is affine, then the condition $\mathrm{R}p_!(\mathrm{M}_{\chi_1}) = \mathrm{R}p_*(\mathrm{M}_{\chi_1})$ implies that N is perverse.

We will give an application when the group G_2 is the multiplicative group. For this we need a lemma.

Lemma 3.43. *Let* N *be a simple perverse sheaf on* $\mathbf{G}_m$ *over* k *which is an object of the category* $\mathbf{P}_{\mathrm{int}}^{\mathrm{ari}}(\mathbf{G}_m)$. *Assume that* N *is pure of weight* 0 *and of tannakian dimension* 1. *Suppose that there exists an integer* $d \geqslant 1$ *and a finite set* $\mathscr{Y} \subset \widehat{\mathbf{G}}_m$ *such that for all* $n \geqslant 1$ *and for* $\chi \in \widehat{\mathbf{G}}_m(k_n) - \mathscr{Y}(k_n)$, *the determinant* $\det(\mathbf{\Theta}_{\mathrm{N},k_n})^d$ *depends only on* n. *Then* N *is geometrically of finite order.*

Proof. If N is not geometrically of finite order, then the perverse sheaf N is a hypergeometric sheaf of generic rank at least 1 (see Section B.4 and Theorem B.4 for reminders of the definition of hypergeometric sheaves and for this result, due to Katz). But these hypergeometric sheaves do not have the indicated property, e.g. because the $\mathbf{\Theta}_{\mathrm{N},k_n}(\chi)$ become equidistributed in $\mathbf{S}^1$ as χ varies among unramified characters in $\widehat{\mathbf{G}}_m(k_n)$ (see Theorem B.4, (3) and [74, Th. 7.2] or Theorem 4.11). $\square$

PROPOSITION 3.44. *Let* G *be a connected commutative algebraic group over* k *and let* $p\colon \mathrm{G} \to \mathbf{G}_m$ *be a non-trivial morphism of algebraic groups. Let* M *be a perverse sheaf on* G *which is arithmetically semisimple and pure of weight zero.*

Let $\chi_1 \in \widehat{\mathrm{G}}(k)$ *be a character such that the equality* $\mathrm{R}p_!(\mathrm{M}_{\chi_1}) = \mathrm{R}p_*(\mathrm{M}_{\chi_1})$ *holds. Assume further that the complex* $\mathrm{N} = \mathrm{R}p_!(\mathrm{M}_{\chi_1})$ *is a perverse sheaf on* $\mathbf{G}_m$ *and is arithmetically semisimple. It is then pure of weight zero.*

Suppose that the set of $\chi \in \widehat{\mathbf{G}}_m$ *such that* $\chi_1(\chi \circ p)$ *is unramified for the object* $\det(\mathrm{M})$ *is generic, and that the tannakian determinant of* N *is arithmetically* (*resp. geometrically*) *of infinite order. Then the tannakian determinant of* M *is arithmetically* (*resp. geometrically*) *of infinite order.*

PROOF. We begin by proving that the determinant is arithmetically of infinite order in both cases. Let $n \geqslant 1$ and let $\chi \in \widehat{\mathbf{G}}_m(k_n)$ be a character such that $\chi_1(\chi \circ p)$ is unramified for the object $\det(\mathrm{M})$. We then have

$$\boldsymbol{\Theta}_{\det(\mathrm{M}),k_n}(\chi) = \det\big(\Theta_{\mathrm{M},k_n}(\chi_1(\chi \circ p))\big) = \det\big(\Theta_{\mathrm{N},k_n}(\chi)\big) \tag{3.18}$$

by Proposition 3.41. By assumption this is valid for all but finitely many $\chi \in \widehat{\mathbf{G}}_m$, and moreover N has determinant which is arithmetically of infinite order, so that the arithmetic tannakian group of $\det(\mathrm{M})$ must be infinite.

It remains to deduce that the geometric tannakian determinant of M has infinite order if the same property holds for N. If not, then $\det(\mathrm{M})^d$ would be geometrically trivial for some integer $d \geqslant 1$. In this case, for any $n \geqslant 1$ and any character $\chi \in \widehat{\mathrm{G}}(k_n)$ which is Frobenius-unramified for det, the determinant $\det(\Theta_{\mathrm{M},k_n}(\chi))^d$ only depends on n (see Proposition 3.40, (2)). By (3.18) and Lemma 3.43, the tannakian determinant of N (which is an object of tannakian dimension 1 on $\mathbf{G}_m$) is geometrically of finite order, which contradicts the assumption. $\square$

REMARK 3.45. If $\mathrm{G} = \mathrm{T} \times \mathbf{G}_m$ for some torus T and p is the projection on $\mathbf{G}_m$ then, according to Theorem 2.11 applied to p and M, the assumption that $\mathrm{R}p_!\mathrm{M}_{\chi_1} = \mathrm{R}p_*\mathrm{M}_{\chi_1}$ and that this complex is a perverse sheaf is true for all χ_1 outside of a finite union of tacs of T. Moreover, by varying χ_1, we can always find such a character for which $\chi_1(\chi \circ p)$ is unramified for generic χ, since M is generically unramified by Theorem 3.26.

Using further work of Katz, we can give a sufficient criterion to apply this proposition.

COROLLARY 3.46. *Let* G *be a connected commutative algebraic group over* k *and let* $p\colon \mathrm{G} \to \mathbf{G}_m$ *be a non-trivial morphism of algebraic groups. Let* M *be a perverse sheaf on* G *which is arithmetically semisimple and pure of weight zero.*

Let $\chi_1 \in \widehat{\mathrm{G}}(k)$ *be a character satisfying* $\mathrm{R}p_!(\mathrm{M}_{\chi_1}) = \mathrm{R}p_*(\mathrm{M}_{\chi_1})$. *Assume that* $\mathrm{N} = \mathrm{R}p_!(\mathrm{M}_{\chi_1})$ *is a perverse sheaf on* $\mathbf{G}_m$*, which is arithmetically semisimple and of the form* $\mathscr{F}[1]$ *for some middle extension sheaf* $\mathscr{F}$ *(see Example A.12 for the definition of middle extension sheaves). Let*

$$(e_1, \ldots, e_l), \qquad (f_1, \ldots, f_m)$$

be the sizes of the unipotent Jordan blocks in the tame monodromy representation of $\mathscr{F}$ *at* 0 *and* ∞ *respectively.*

Suppose that the set of $\chi \in \widehat{\mathbf{G}}_m$ *such that* $\chi_1(\chi \circ p)$ *is unramified for the object* $\det(\mathrm{M})$ *is generic. If we have*

$$\sum_i e_i - \sum_j f_j \neq 0,$$

then the tannakian determinant of M *is geometrically of infinite order.*

Proof. According to the previous proposition, it suffices to show that the tannakian determinant of N is geometrically of infinite order. By [74, Th. 16.1], the condition implies that the determinant of the Frobenius action on Deligne's fiber functor $\omega_{\mathrm{Del}}(\mathrm{N})$ is not unitary (see Section B.2 for the definition of this functor), and the result follows from Katz's classification of objects of tannakian dimension 1 on $\mathbf{G}_m$ (Theorem B.4). ☐

3.12. External products

The following proposition concerns objects on a product $\mathrm{G} = \mathrm{G}_1 \times \mathrm{G}_2$, and is useful for constructing various examples (see for instance Section 9.4).

Proposition 3.47. *Assume that* $\mathrm{G} = \mathrm{G}_1 \times \mathrm{G}_2$ *for connected commutative algebraic groups* G_1 *and* G_2*. For any objects* $\mathrm{M}_i \in \mathrm{D}^{\mathrm{b}}_c(\mathrm{G}_i)$*, there exist natural isomorphisms*

$$\mathrm{M}_1 \boxtimes \mathrm{M}_2 \simeq (\mathrm{M}_1 \boxtimes \mathbf{1}_{\mathrm{G}_2}) *_! (\mathbf{1}_{\mathrm{G}_1} \boxtimes \mathrm{M}_2) \simeq (\mathrm{M}_1 \boxtimes \mathbf{1}_{\mathrm{G}_2}) *_* (\mathbf{1}_{\mathrm{G}_1} \boxtimes \mathrm{M}_2).$$

Moreover, if M_1 *and* M_2 *are perverse sheaves on* G_1 *and* G_2 *with tannakian rank* r_1 *and* r_2*, respectively, then the object* $\mathrm{M}_1 \boxtimes \mathrm{M}_2 \in \mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$ *is perverse and has tannakian rank* $r_1 r_2$*.*

Proof. We use coordinates (x_1, x_1, y_1, y_2) on $\mathrm{G} \times \mathrm{G}$ with x_i and y_i coordinates on G_i. Let $m_{12} \colon \mathrm{G} \times \mathrm{G} \to \mathrm{G}$ be the multiplication map for G, and $m_i \colon \mathrm{G}_i \times \mathrm{G}_i \to \mathrm{G}_i$ those for G_i.

Let N be the object on the right-hand side of the first isomorphism to be established. By definition, we have

$$\mathrm{N} = (\mathrm{M}_1 \boxtimes \mathbf{1}_{\mathrm{G}_2}) *_! (\mathbf{1}_{\mathrm{G}_1} \boxtimes \mathrm{M}_2) = m_{12,!}((\mathrm{M}_1 \boxtimes \mathbf{1}_{\mathrm{G}_2}) \boxtimes (\mathbf{1}_{\mathrm{G}_1} \boxtimes \mathrm{M}_2)).$$

Let $s \colon \mathrm{G} \to \mathrm{G}$ be the involution given by $(x_1, x_2, y_1, y_2) \mapsto (x_1, y_2, y_1, x_2)$. We have $m_{12} = m_{12} \circ s$, and hence

$$\mathrm{N} = m_{12,!} s_!((\mathrm{M}_1 \boxtimes \mathbf{1}_{\mathrm{G}_2}) \boxtimes (\mathbf{1}_{\mathrm{G}_1} \boxtimes \mathrm{M}_2)).$$

Since s is an involution, we have $s_! = s_* = s^*$, and therefore

$$s_!((\mathrm{M}_1 \boxtimes \mathbf{1}_{\mathrm{G}_2}) \boxtimes (\mathbf{1}_{\mathrm{G}_1} \boxtimes \mathrm{M}_2)) = p_1^*(\mathrm{M}_1 \boxtimes \mathrm{M}_2) \otimes p_2^*(\mathbf{1}_{\mathrm{G}}),$$

where p_1, $p_2 \colon \mathrm{G} \times \mathrm{G} \to \mathrm{G}$ are the two projections. Thus, using the definition again, we obtain an isomorphism

$$\mathrm{N} \simeq m_{12!}(p_1^*(\mathrm{M}_1 \boxtimes \mathrm{M}_2) \otimes p_2^*(\mathbf{1}_{\mathrm{G}})) = (\mathrm{M}_1 \boxtimes \mathrm{M}_2) *_! \mathbf{1}_{\mathrm{G}},$$

which is isomorphic to $\mathrm{M}_1 \boxtimes \mathrm{M}_2$ since $\mathbf{1}_{\mathrm{G}}$ is the unit for convolution.

We obtain similarly the second isomorphism

$$\mathrm{M}_1 \boxtimes \mathrm{M}_2 \simeq (\mathrm{M}_1 \boxtimes \mathbf{1}_{\mathrm{G}_2}) *_* (\mathbf{1}_{\mathrm{G}_1} \boxtimes \mathrm{M}_2) \simeq (\mathrm{M}_1 \boxtimes \mathbf{1}_{\mathrm{G}_2}) *_* (\mathbf{1}_{\mathrm{G}_1} \boxtimes \mathrm{M}_2).$$

It is classical that $\mathrm{M}_1 \boxtimes \mathrm{M}_2$ is perverse if M_1 and M_2 are, and the final assertion then results from the fact that

$$\mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_1 \boxtimes \mathrm{M}_2) \simeq \mathrm{H}^*_c(\mathrm{G}_{1,\bar{k}}, \mathrm{M}_1) \otimes \mathrm{H}^*_c(\mathrm{G}_{2,\bar{k}}, \mathrm{M}_2),$$

and the generic vanishing theorem. ☐

Remark 3.48. Concretely, this proposition reflects the convolution formula

$$f_1(x_1) f_2(x_2) = \sum_{(y_1, y_2) \in \mathrm{G}(k_n)} f_1(y_1) \delta_2(y_2) \delta_1(x_1 y_1^{-1}) f_2(x_2 y_2^{-1})$$

for any functions $f_i \colon \mathrm{G}_i(k_n) \to \mathbf{C}$, where the δ_i are Dirac masses at the unit element of G_i.

3.13. The rank 1 tannakian group

Given the group G over k, we can form the subcategory $\overline{\mathbf{P}}^1(\mathrm{G})$ of $\overline{\mathbf{P}}(\mathrm{G})$ (resp. $\mathbf{P}^1_{\mathrm{int}}(\mathrm{G})$ of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$) additively generated by objects of tannakian rank 1. This is again a tannakian category, since the convolution (resp. dual) of objects of rank 1 is of rank 1.

Proposition 3.49. *Let $\mathbf{L}(\mathrm{G})$ be the group of isomorphism classes of objects of $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$ of rank 1. The tannakian group of $\overline{\mathbf{P}}^1(\mathrm{G})$ and $\mathbf{P}^1_{\mathrm{int}}(\mathrm{G})$ is the pro-algebraic subgroup of*

$$\mathbf{GL}_1^{\mathbf{L}(\mathrm{G})}$$

defined by the equations

$$\prod_{1\leqslant i\leqslant m} x_{\mathrm{L}_i}^{n_i} = 1$$

for all integers $m \geqslant 0$, all families $(\mathrm{L}_i)_{1\leqslant i\leqslant m}$ of elements of $\mathbf{L}(\mathrm{G})$ and all families $(n_i)_{1\leqslant i\leqslant m}$ of integers such that the object

$$\mathrm{L}_1^{*_{\mathrm{int}} n_1} *_{\mathrm{int}} \cdots *_{\mathrm{int}} \mathrm{L}_m^{*_{\mathrm{int}} n_m}$$

is isomorphic to $\mathbf{1}_{\mathrm{G}}$, or equivalently that the relation

$$\mathrm{L}_1^{n_1}\cdots \mathrm{L}_m^{n_m} = 1$$

holds in the group $\mathbf{L}(\mathrm{G})$.

Proof. This amounts to proving that, for the object

$$\mathrm{M} = \mathrm{L}_1 \oplus \cdots \oplus \mathrm{L}_m,$$

the tannakian group $\mathbf{G}$ is the subgroup of $\mathbf{GL}_1^m$ determined by the equations

$$\prod_{1\leqslant i\leqslant m} x_i^{n_i} = 1 \tag{3.19}$$

for all $(n_1, \ldots, n_m) \in \mathbf{Z}^m$ such that

$$\mathrm{L}_1^{n_1}\cdots \mathrm{L}_m^{n_m} = 1. \tag{3.20}$$

Being a group of multiplicative type, $\mathbf{G}$ is characterized by its character group, and the character group of $\mathbf{G} \subset \mathbf{GL}_1^m$ is $\mathbf{Z}^m/\mathrm{H}$, where

$$\mathrm{H} = \{\chi\colon\ \mathbf{GL}_1^m \to \mathbf{G}_m \mid \mathbf{G} \subset \ker(\chi)\}.$$

A character χ of $\mathbf{GL}_1^m$ restricts to a character of $\mathbf{G}$, so we can form the object $\chi(\mathrm{M})$ in $\mathbf{P}^1_{\mathrm{int}}(\mathrm{G})$. Since the image $\chi(\mathbf{G})$ of $\mathbf{G}$ by χ is the tannakian group of $\chi(\mathrm{M})$, we then have

$$\mathbf{G} \subset \ker(\chi) \text{ if and only if } \chi(\mathrm{M}) \simeq \mathbf{1}_{\mathrm{G}}.$$

On the other hand, there is a natural isomorphism

$$\chi(\mathrm{M}) \simeq \mathrm{L}_1^{*_{\mathrm{int}} n_1} *_{\mathrm{int}} \cdots *_{\mathrm{int}} \mathrm{L}_m^{*_{\mathrm{int}} n_m}$$

(indeed, this holds when χ is the character $(x_1, \ldots, x_m) \mapsto x_i$, and then the general case follows by the compatibility with convolution and tensor product, which for characters is just the ordinary product), and therefore H is the subgroup of $\mathbf{Z}^m$ formed by the tuples $(n_1, \ldots, n_m)$ which satisfy (3.20). This means that $\mathbf{G}$ coincides with the subgroup of $\mathbf{GL}_1^m$ determined by the equations (3.19), and concludes the proof. □

Remark 3.50. This result is in fact valid, with the same proof, in any tannakian category. In particular, the analogue holds for the categories generated by objects of rank 1 in $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{G})$. Of course, as usual, the arithmetic tannakian group may be bigger than the geometric one.

EXAMPLE 3.51. If G is a torus, then the group $\mathbf{L}(\mathrm{T})$ has been determined by Gabber and Loeser [50, Th 8.6.1], who denote it $\mathbf{H}_{\mathrm{int}}(\mathrm{G})$.

Precisely, let $r \geqslant 0$ be such that G is isomorphic to $\mathbf{G}_m^r$. Let $\mathscr{S}$ be the set of one-dimensional subtori of $\mathbf{G}^r_{m,\bar{k}}$. For each torus $\mathrm{T} \in \mathscr{S}$, denote by i_{T} the closed immersion $\mathrm{T} \to \mathbf{G}^r_{m,\bar{k}}$ and choose an isomorphism $\varphi_{\mathrm{T}} \colon \mathbf{G}_{m,\bar{k}} \to \mathrm{T}$. Recall that $\Pi(\mathbf{G}_{m,\bar{k}}, \overline{\mathbf{Q}}_\ell)$ is the set of continuous tame characters of $\mathbf{G}_{m,\bar{k}}$ (see Section 1.10), and write the basis vectors of the free abelian group $\mathbf{Z}^{(\mathscr{S} \times \Pi(\mathbf{G}_{m,\bar{k}}, \overline{\mathbf{Q}}_\ell))}$ as (T, χ). Then Gabber and Loeser prove that there is an isomorphism

$$(\bar{k}^\times)^r \times \mathbf{Z}^{(\mathscr{S} \times \Pi(\mathbf{G}_{m,\bar{k}}, \overline{\mathbf{Q}}_\ell))} \to \mathbf{L}(\mathbf{G}^r_{m,\bar{k}})$$

that maps $(\lambda, (\mathrm{T}, \chi))$ to the object

$$\delta_\lambda *_{\mathrm{int}} \mathrm{R}(i_{\mathrm{T}} \circ \varphi_{\mathrm{T}})_*(j^* \mathscr{L}_\psi \otimes \mathscr{L}_\chi)[1],$$

where $j \colon \mathbf{G}_{m,\bar{k}} \to \mathbf{A}^1$ is the open immersion.

CHAPTER 4

Equidistribution theorems

4.1. Equidistribution on average

Along with the classical form of equidistribution that goes back in principle to Weyl and appears in Deligne's equidistribution theorem, we will apply a useful variant that allows us to avoid the assumption that the geometric and the arithmetic tannakian groups are equal, at the cost of getting slightly weaker statements.

DEFINITION 4.1. Let X be a locally compact topological space and let μ be a Borel probability measure on X. Let $(\mathrm{Y}_n, \Theta_n)_{n\geqslant 1}$ be a sequence of pairs of finite sets Y_n and maps $\Theta_n\colon \mathrm{Y}_n \to \mathrm{X}$.

(1) We say that (Y_n, Θ_n), or simply (Y_n) when the maps Θ_n are clear from the context, *becomes μ-equidistributed on average as $n \to \infty$* if the sets Y_n are non-empty for all large enough n and if the sequence of probability measures

$$\mu_{\mathrm{N}} = \frac{1}{\mathrm{N}'} \sum_{\substack{1\leqslant n\leqslant \mathrm{N}\\ \mathrm{Y}_n\neq\varnothing}} \frac{1}{|\mathrm{Y}_n|} \sum_{y\in\mathrm{Y}_n} \delta_{\Theta_n(y)}, \qquad \mathrm{N}' = |\{n \leqslant \mathrm{N} \mid \mathrm{Y}_n \neq \varnothing\}|,$$

defined on X for large enough N, converges weakly to μ as N goes to infinity, i.e., for any bounded continuous function $f\colon \mathrm{X} \to \mathbf{C}$, the following holds:

$$\lim_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}'} \sum_{\substack{1\leqslant n\leqslant \mathrm{N}\\ \mathrm{Y}_n\neq\varnothing}} \frac{1}{|\mathrm{Y}_n|} \sum_{y\in\mathrm{Y}_n} f(\Theta_n(y)) = \int_{\mathrm{X}} f d\mu. \tag{4.1}$$

(2) The sequence (Y_n, Θ_n), or simply (Y_n), *becomes μ-equidistributed as $n \to \infty$* if the sets Y_n are non-empty for all large enough n and if the sequence of probability measures

$$\widetilde{\mu}_n = \frac{1}{|\mathrm{Y}_n|} \sum_{y\in\mathrm{Y}_n} \delta_{\Theta_n(y)},$$

defined on X for large enough n, converges weakly to μ as n goes to infinity, i.e., for any bounded continuous function $f\colon \mathrm{X} \to \mathbf{C}$, the following holds:

$$\lim_{n\to+\infty} \frac{1}{|\mathrm{Y}_n|} \sum_{y\in\mathrm{Y}_n} f(\Theta_n(y)) = \int_{\mathrm{X}} f d\mu. \tag{4.2}$$

REMARK 4.2. (1) In practice, since $\mathrm{N}' \sim \mathrm{N}$ as $\mathrm{N} \to +\infty$, we will sometimes not distinguish between N and N', and use the convention that those terms for which Y_n is empty are omitted from the sum over n when discussing equidistribution on average.

(2) Since convergence of a sequence (x_n) of complex numbers implies that of its Cesàro means $(\mathrm{N}^{-1} \sum_{1\leqslant n\leqslant \mathrm{N}} x_n)$, with the same limit, equidistribution implies equidistribution on average.

4.2. The basic estimate

We state here a preliminary estimate that will be the key analytic step in the proof of our equidistribution results, including Theorem 2 from the introduction.

We denote as usual by k a finite field with algebraic closure $\bar{k}$, and by k_n the extension of k of degree n in $\bar{k}$. We fix a prime ℓ distinct from the characteristic of k.

PROPOSITION 4.3. *Let* G *be a commutative connected algebraic group over* k*. Let* M *be an* ℓ*-adic perverse sheaf on* G *that is arithmetically semisimple and pure of weight zero, and of tannakian dimension* r*. Let* N *be an object of* $\langle \mathrm{M}\rangle^{\mathrm{ari}}$*.*

For all $n \geqslant 1$ *such that* $\mathscr{X}_{\mathrm{F}}(\mathrm{N})(k_n)$ *is not empty, the following estimate holds:*

$$\frac{1}{|\mathscr{X}_{\mathrm{F}}(\mathrm{N})(k_n)|}\sum_{\chi\in\mathscr{X}_{\mathrm{F}}(\mathrm{N})(k_n)}\mathrm{Tr}(\mathrm{Fr}_{k_n}\mid \mathrm{H}^0_c(\mathrm{G}_{\bar{k}},\mathrm{N}_\chi)) = t_{\mathrm{N}}(e;k_n)+\mathrm{O}(|k_n|^{-1/2}). \tag{4.3}$$

PROOF. We fix a quasi-projective embedding u of G. Let d denote the dimension of G, and put $\mathscr{X}=\mathscr{X}_{\mathrm{F}}(\mathrm{N})$. For each non-zero integer i, consider the subset

$$\mathscr{A}_i=\{\chi\in\widehat{\mathrm{G}}\mid \mathrm{H}^i_c(\mathrm{G}_{\bar{k}},\mathrm{N}_\chi)\neq 0\}$$

consisting of those characters χ such that N_χ has non-trivial cohomology with compact support in degree i. Then the left-hand side of (4.3) is equal to

$$\begin{aligned}\frac{1}{|\mathscr{X}(k_n)|}\sum_{\chi\in\mathscr{X}(k_n)}\mathrm{Tr}(\mathrm{Fr}_{k_n}\mid \mathrm{H}^0_c(\mathrm{G}_{\bar{k}},\mathrm{N}_\chi)) = {}&\\ &\frac{1}{|\mathscr{X}(k_n)|}\sum_{\chi\in\widehat{\mathrm{G}}(k_n)}\sum_{|i|\leqslant d}(-1)^i\,\mathrm{Tr}(\mathrm{Fr}_{k_n}\mid \mathrm{H}^i_c(\mathrm{G}_{\bar{k}},\mathrm{N}_\chi))\\ &-\frac{1}{|\mathscr{X}(k_n)|}\sum_{0<|i|\leqslant d}(-1)^i\sum_{\chi\in\mathscr{A}_i(k_n)}\mathrm{Tr}(\mathrm{Fr}_{k_n}\mid \mathrm{H}^i_c(\mathrm{G}_{\bar{k}},\mathrm{N}_\chi))\\ &-\frac{1}{|\mathscr{X}(k_n)|}\sum_{\chi\in\widehat{\mathrm{G}}(k_n)-\mathscr{X}(k_n)}\mathrm{Tr}(\mathrm{Fr}_{k_n}\mid \mathrm{H}^0_c(\mathrm{G}_{\bar{k}},\mathrm{N}_\chi)).\end{aligned} \tag{4.4}$$

By the Grothendieck–Lefschetz trace formula (see (A.5)), the equalities

$$\sum_{|i|\leqslant d}(-1)^i\,\mathrm{Tr}(\mathrm{Fr}_{k_n}\mid \mathrm{H}^i_c(\mathrm{G}_{\bar{k}},\mathrm{N}_\chi)) = \sum_{x\in\mathrm{G}(k_n)}t_{\mathrm{N}_\chi}(x;k_n)=\sum_{x\in\mathrm{G}(k_n)}\chi(x)t_{\mathrm{N}}(x;k_n)$$

hold for any character χ. Combined with the orthogonality of characters of $\mathrm{G}(k_n)$, this shows that the first summand in (4.4) is equal to

$$\frac{|\widehat{\mathrm{G}}(k_n)|}{|\mathscr{X}(k_n)|}t_{\mathrm{N}}(e;k_n)=t_{\mathrm{N}}(e;k_n)+\mathrm{O}(|k_n|^{-1})$$

since the set $\mathscr{X}$ is generic, so that the estimate $\frac{|\widehat{\mathrm{G}}(k_n)|}{|\mathscr{X}(k_n)|}=1+\mathrm{O}(|k_n|^{-1})$ holds.

We now turn to bounding the second and the third summands in the right-hand side of (4.4).

Since M is pure of weight zero, the same holds for N and N_χ by Theorem 3.32. It then follows from Deligne's Riemann Hypothesis (see Theorem A.19) that $\mathrm{H}^i_c(\mathrm{G}_{\bar{k}},\mathrm{N}_\chi)$ is mixed of weights $\leqslant i$ for any i, in particular the eigenvalues of Fr_{k_n} acting on this space have modulus at most $|k_n|^{i/2}$. Moreover, using (1.1) and Theorem 1.5 (2), we get

$$h^i_c(\mathrm{G}_{\bar{k}},\mathrm{N}_\chi)\leqslant c_u(\mathrm{N}_\chi)\ll c_u(\mathrm{N})c_u(\mathscr{L}_\chi)\ll c_u(\mathrm{N})$$

since the complexity $c_u(\mathscr{L}_\chi)$ is bounded independently of χ by Proposition 1.18. So the second term in (4.4) can be bounded by

$$\frac{1}{|\mathscr{X}(k_n)|}\sum_{0<|i|\leqslant d}\ \sum_{\chi\in\mathscr{A}_i(k_n)} h^i_c(\mathrm{G}_{\bar{k}},\mathrm{N}_\chi)|k_n|^{i/2}\leqslant\frac{1}{|\mathscr{X}(k_n)|}\sum_{0<|i|\leqslant d}\ \sum_{\chi\in\mathscr{A}_i(k_n)}|k_n|^{i/2}.$$

The Stratified Vanishing Theorem 2.3 applied to N gives the estimate

$$|\mathscr{A}_i(k_n)|\ll|k_n|^{d-|i|}\tag{4.5}$$

for i such that $0<|i|\leqslant d$. We split the sum over i into that over $1\leqslant i\leqslant d$ and that over $-d\leqslant i\leqslant -1$, and obtain

$$\begin{aligned}\frac{1}{|\mathscr{X}(k_n)|}\sum_{0<|i|\leqslant d}\ \sum_{\chi\in\mathscr{A}_i(k_n)}|k_n|^{i/2}&\ll\frac{1}{|\mathscr{X}(k_n)|}\sum_{1\leqslant i\leqslant d}|k_n|^{d-i/2}+\frac{1}{|\mathscr{X}(k_n)|}\sum_{-d\leqslant i\leqslant -1}|k_n|^{d+3i/2}\\ &\ll\frac{|k_n|^{d-1/2}}{|\mathscr{X}(k_n)|}.\end{aligned}\tag{4.6}$$

Thanks to the estimate $|\mathscr{X}(k_n)|=|k_n|^d+\mathrm{O}(|k_n|^{d-1})$, the last term is $\ll|k_n|^{-1/2}$ and tends to 0 as $n\to+\infty$.

Finally, the third term in (4.4) satisfies

$$\frac{1}{|\mathscr{X}(k_n)|}\sum_{\chi\in\widehat{\mathrm{G}}(k_n)-\mathscr{X}(k_n)}\mathrm{Tr}(\mathrm{Fr}_{k_n}\mid\mathrm{H}^0_c(\mathrm{G}_{\bar{k}},\mathrm{N}_\chi))\ll\frac{|\widehat{\mathrm{G}}(k_n)-\mathscr{X}(k_n)|}{|\mathscr{X}(k_n)|}\ll\frac{1}{|k_n|}\tag{4.7}$$

since $\mathrm{H}^0_c(\mathrm{G}_{\bar{k}},\mathrm{N}_\chi)$ is mixed of weights $\leqslant 0$ and has dimension bounded for all χ, and the set $\mathscr{X}$ is generic. This finishes the proof. □

4.3. Equidistribution for characteristic polynomials

Let k be a finite field, with an algebraic closure $\bar{k}$, and let G be a connected commutative algebraic group over k. Let ℓ be a prime number distinct from the characteristic of k.

Our most general equidistribution result concerns the characteristic polynomials of the unitary Frobenius conjugacy classes for weakly unramified characters. Equivalently, this is about the conjugacy classes in the ambient unitary group.

THEOREM 4.4. *Let* M *be an ℓ-adic perverse sheaf on* G *that is arithmetically semisimple and pure of weight zero. Let $r\geqslant 0$ be the tannakian dimension of* M. *Let $\mathrm{K}\subset\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}(\mathbf{C})\subset\mathbf{GL}_r(\mathbf{C})$ be a maximal compact subgroup of the arithmetic tannakian group of* M, *and denote by ν_{cp} the measure on the space $\mathbf{U}_r(\mathbf{C})^\sharp$ of conjugacy classes in the unitary group which is the direct image of the Haar probability measure μ on* K *by the quotient map $\mathrm{K}\to\mathbf{U}_r(\mathbf{C})^\sharp$. Then the families of unitary conjugacy classes $(\Theta_{\mathrm{M},k_n}(\chi))_{\chi\in\mathscr{X}_w(\mathrm{M})(k_n)}$ become ν_{cp}-equidistributed on average in $\mathbf{U}_r(\mathbf{C})^\sharp$ as $n\to+\infty$.*

REMARK 4.5. (1) To be precise, in terms of Definition 4.1, we consider the equidistribution on average of pairs $(\mathscr{X}_w(\mathrm{M})(k_n),\Theta_n)$ with $\Theta_n(\chi)=\Theta_{\mathrm{M},k_n}(\chi)$.

(2) The set $\mathbf{U}_r(\mathbf{C})^\sharp$ can be identified with the set of characteristic polynomials of unitary matrices of size r, or equivalently with the quotient topological space $(\mathbf{S}_1)^r/\mathfrak{S}_r$ (by mapping a matrix to the set of eigenvalues, with multiplicity) so the statement means that the characteristic polynomials of the Frobenius automorphisms for weakly unramified characters tend to be distributed like the characteristic polynomials of random elements of K (hence the notation ν_{cp}).

PROOF. Let $\mathscr{X} = \mathscr{X}_w(\mathrm{M})$. It suffices to check the equality (4.1) for f taken in a set of continuous functions on $\mathbf{U}_r(\mathbf{C})^\sharp$ that span a dense subset of the Banach space $\mathscr{C}(\mathbf{U}_r(\mathbf{C})^\sharp)$ of all continuous complex-valued functions on $\mathbf{U}_r(\mathbf{C})^\sharp$ (since probability measures on $\mathbf{U}_r(\mathbf{C})^\sharp$ are continuous functionals on $\mathscr{C}(\mathbf{U}_r(\mathbf{C})^\sharp)$ by the Riesz representation theorem). Thanks to the Peter–Weyl Theorem, it suffices to prove the equality

$$\lim_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}} \sum_{1\leqslant n\leqslant \mathrm{N}} \frac{1}{|\mathscr{X}(k_n)|} \sum_{\chi\in\mathscr{X}(k_n)} \mathrm{Tr}(\varrho(\Theta_{\mathrm{M},k_n}(\chi))) = \int_{\mathrm{K}} \mathrm{Tr}(\varrho(g))d\mu(g)$$

for any irreducible unitary representation ϱ of $\mathbf{U}_r(\mathbf{C})$. In fact, we will prove this for any unitary representation ϱ, not necessarily irreducible.

By the Peter–Weyl Theorem again, the right-hand side is the multiplicity of the trivial representation in the representation of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ that corresponds to the restriction of ϱ to K. We denote by $\mathrm{N} = \varrho(\mathrm{M})$ the object of $\langle \mathrm{M}\rangle^{\mathrm{ari}}$ that corresponds to this restriction of ϱ.

Let $\mathscr{X}_{\mathrm{N}} = \mathscr{X}_{\mathrm{F}}(\mathrm{N})_{\mathrm{M}}$ be the set of Frobenius-unramified characters for N. We have

$$\lim_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}} \sum_{1\leqslant n\leqslant \mathrm{N}} \frac{1}{|\mathscr{X}(k_n)|} \sum_{\chi\in(\mathscr{X}-\mathscr{X}_{\mathrm{N}})(k_n)} \mathrm{Tr}(\varrho(\Theta_{\mathrm{M},k_n}(\chi))) = 0,$$

since $\mathscr{X}_{\mathrm{N}}$ is generic (by Proposition 3.37) and the upper-bound

$$|\,\mathrm{Tr}(\varrho(\Theta_{\mathrm{M},k_n}(\chi)))| \leqslant \dim(\varrho)$$

holds for all $\chi \in \mathscr{X}(k_n)$.

By the definition of Frobenius-unramified characters, we have

$$\frac{1}{|\mathscr{X}(k_n)|} \sum_{\chi\in\mathscr{X}_{\mathrm{N}}(k_n)} \mathrm{Tr}(\varrho(\Theta_{\mathrm{M},k_n}(\chi))) = \frac{1}{|\mathscr{X}(k_n)|} \sum_{\chi\in\mathscr{X}_{\mathrm{N}}(k_n)} \mathrm{Tr}(\mathrm{Fr}_{k_n} \mid \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{N}_\chi))$$

for $n \geqslant 1$. Since $\mathscr{X}$ and $\mathscr{X}_{\mathrm{N}}$ are both generic, we have $\frac{|\mathscr{X}_{\mathrm{N}}(k_n)|}{|\mathscr{X}(k_n)|} = 1 + \mathrm{O}(\frac{1}{|k_n|})$. By Proposition 4.3, we deduce that

$$\frac{1}{|\mathscr{X}(k_n)|} \sum_{\chi\in\mathscr{X}_{\mathrm{N}}(k_n)} \mathrm{Tr}(\varrho(\Theta_{\mathrm{M},k_n}(\chi))) = t_{\mathrm{N}}(e; k_n) + \mathrm{O}(|k_n|^{-1/2}),$$

where e is the identity of G.

We decompose the semisimple perverse sheaf N as a direct sum

$$\mathrm{N} = \bigoplus_{r\geqslant 0} \bigoplus_{i\in \mathrm{I}(r)} \mathrm{N}_{r,i}$$

of pairwise non-isomorphic arithmetically simple perverse sheaves $\mathrm{N}_{r,i}$ of support of dimension r. For $r \geqslant 1$, we get the pointwise bound

$$t_{\mathrm{N}_{r,i}}(e; k_n) \ll \frac{1}{\sqrt{|k_n|}}.$$

using Proposition 1.11.

The punctual objects $\mathrm{N}_{0,i}$ are of the form $\alpha_i^{\deg} \otimes \delta_{x_i}$ for some unitary scalars α_i and some points x_i. If $x_i \neq e$, then

$$t_{\mathrm{N}_{0,i}}(e; k_n) = 0.$$

Thus, if we denote by $\mathrm{J} \subset \mathrm{I}(0)$ the subset where $x_i = e$ (which has cardinality equal to the multiplicity of the trivial representation in the restriction of ϱ to $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$), then the formula

$$\frac{1}{|\mathscr{X}(k_n)|} \sum_{\chi \in \mathscr{X}_{\mathrm{N}}(k_n)} \mathrm{Tr}(\varrho(\Theta_{\mathrm{M},k_n}(\chi))) = \sum_{i \in \mathrm{J}} \alpha_i^n + \mathrm{O}(|k_n|^{-1/2}) \tag{4.8}$$

holds. The subset $\mathrm{J}^0 \subset \mathrm{J}$ where $\alpha_i = 1$ has cardinality equal to the multiplicity of the trivial representation in the restriction of ϱ to $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$. Averaging over n and using

$$\lim_{\mathrm{N} \to +\infty} \frac{1}{\mathrm{N}} \sum_{1 \leqslant n \leqslant \mathrm{N}} \alpha_i^n = 0$$

for $i \in \mathrm{J} - \mathrm{J}^0$, we conclude that

$$\lim_{\mathrm{N} \to +\infty} \frac{1}{\mathrm{N}} \sum_{1 \leqslant n \leqslant \mathrm{N}} \frac{1}{|\mathscr{X}(k_n)|} \sum_{\chi \in \mathscr{X}_{\mathrm{N}}(k_n)} \mathrm{Tr}(\varrho(\Theta_{\mathrm{M},k_n}(\chi))) = |\mathrm{J}^0| + \mathrm{O}(|k_n|^{-1/2}),$$

which gives the desired result. □

It is useful to state the following corollary of the proof, which is a diophantine version of Schur's Lemma in our context.

Corollary 4.6 (Schur's Lemma). *Let* M *and* N *be geometrically simple ℓ-adic perverse sheaves on* G *which are pure of weight zero and are objects of* $\mathbf{P}_{\mathrm{int}}^{\mathrm{ari}}(\mathrm{G})$. *Let* $\mathscr{X}$ *be the set of characters which are weakly unramified for* $\mathrm{M} \oplus \mathrm{N}^\vee$. *We have*

$$\lim_{\mathrm{N} \to +\infty} \frac{1}{\mathrm{N}} \sum_{n \leqslant \mathrm{N}} \frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi \in \mathscr{X}(k_n)} \mathrm{S}(\mathrm{M} *_{\mathrm{int}} \mathrm{N}^\vee, \chi) = \begin{cases} 1 & \textit{if } \mathrm{M} \textit{ is arithmetically isomorphic to } \mathrm{N}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Proposition 4.3 applied to the perverse sheaf $\mathrm{M} \oplus \mathrm{N}^\vee$ and the object $\mathrm{Q} = \mathrm{Hom}(\mathrm{N}, \mathrm{M})$ of the category $\langle \mathrm{M} \oplus \mathrm{N}^\vee \rangle^{\mathrm{ari}}$ (the homomorphisms are in the category $\mathbf{P}_{\mathrm{int}}^{\mathrm{ari}}(\mathrm{G})$) implies that

$$\frac{1}{|\mathscr{X}_{\mathrm{F}}(\mathrm{Q})(k_n)|} \sum_{\chi \in \mathscr{X}_{\mathrm{F}}(\mathrm{Q})(k_n)} \mathrm{S}(\mathrm{Q}, \chi) = t_{\mathrm{Q}}(e; k_n) + \mathrm{O}(|k_n|^{-1/2})$$

for any $n \geqslant 1$, where

$$\mathrm{S}(\mathrm{Q}, \chi) = \sum_{x \in \mathrm{G}(k_n)} \chi(x) t_{\mathrm{Q}}(x; k_n).$$

Since $\mathscr{X}_{\mathrm{F}}(\mathrm{Q})$ is generic, and since there is a canonical isomorphism $\mathrm{Q} \to \mathrm{M} *_{\mathrm{int}} \mathrm{N}^\vee$, we deduce that

$$\frac{1}{\mathrm{N}} \sum_{n \leqslant \mathrm{N}} \frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi \in \mathscr{X}(k_n)} \mathrm{S}(\mathrm{M} *_{\mathrm{int}} \mathrm{N}^\vee, \chi) = \frac{1}{\mathrm{N}} \sum_{n \leqslant \mathrm{N}} t_{\mathrm{Q}}(e; k_n) + \mathrm{O}(|k_n|^{-1/2})$$

for all $\mathrm{N} \geqslant 1$. Arguing as in the last part of the proof of Theorem 4.4, we see that the right-hand side converges to the multiplicity of the trivial representation in the representation corresponding to Q; by the classical form of Schur's Lemma, this is either 1 or 0, depending on whether M is isomorphic to N or not. □

Remark 4.7. The proof of Theorem 4.4 allows us to see clearly what is involved in the use of the Cesàro mean in the average equidistribution.

First, we can see that it is necessary in general, unless $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}} = \mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ (see Section 4.6 for statements under this assumption, in particular Proposition 4.18).

Second, we see that the use of the Cesàro average can be generalized to establish the convergence to the limit ν_{cp} of any sequence of average measures of the form

$$\sum_{n\geqslant 1}\frac{\varphi_{\mathrm{N}}(n)}{|\mathscr{X}(k_n)|}\sum_{\chi\in\mathscr{X}(k_n)}\delta_{\Theta_{\mathrm{M},k_n}(\chi)},$$

where $\varphi_{\mathrm{N}}(n)$ are non-negative coefficients that are bounded and satisfy the equality

$$\lim_{\mathrm{N}\to+\infty}\sum_{n\geqslant 1}\varphi_{\mathrm{N}}(n)\alpha^n=\begin{cases}0 & \text{if } \alpha\neq 1,\\ 1 & \text{if } \alpha=1\end{cases} \tag{4.9}$$

for any complex number α of modulus 1. The Cesàro case corresponds to $\varphi_{\mathrm{N}}(n)=1/\mathrm{N}$ for all $n\leqslant\mathrm{N}$ and $\varphi_{\mathrm{N}}(n)=0$ for $n\geqslant\mathrm{N}$, but there are many other possibilities. (In classical terms, as expounded for instance by Hardy [57], these φ_{N} define a "summation method", and it is elementary that the requirements amounts essentially[1] to asking that this summation method gives the "right" sum $1/(1-\alpha)$ to the geometric series for $|\alpha|=1$ and $\alpha\neq 1$.)

It is also instructive to view the average probabilistically, interpreting φ_{N} as the law of a random variable X_{N} with values in positive integers. The condition above is the requirement that the equality

$$\lim_{\mathrm{N}\to+\infty}\mathbf{E}(e^{i\theta\mathrm{X}_{\mathrm{N}}})=0$$

holds for all $\theta\in\mathbf{R}/2\pi\mathbf{Z}-\{0\}$.

Besides the Cesàro case, where X_{N} is a random variable uniform on $\{1,\ldots,\mathrm{N}\}$, consider a Poisson distribution X_{N} with parameter $\lambda_{\mathrm{N}}>0$, shifted to have support in the positive integers, i.e., let

$$\mathbf{P}(\mathrm{X}_{\mathrm{N}}=n)=\varphi_{\mathrm{N}}(n)=e^{-\lambda_{\mathrm{N}}}\frac{\lambda_{\mathrm{N}}^{n-1}}{(n-1)!}$$

for any positive integers N and n. The condition above becomes the limit

$$\mathbf{E}(e^{i\theta\mathrm{X}_{\mathrm{N}}})=\exp(i\theta+\lambda_{\mathrm{N}}(e^{i\theta}-1))\to 0$$

as $\mathrm{N}\to+\infty$ for $\theta\in\mathbf{R}/2\pi\mathbf{Z}-\{0\}$, which holds provided $\lambda_{\mathrm{N}}\to+\infty$, since the modulus of the left-hand side is $\exp(\lambda_{\mathrm{N}}(\cos(\theta)-1))$.

Intuitively, this means that if we pick a positive integer n according to a Poisson distribution with large parameter, then pick uniformly a random $\chi\in\mathscr{X}(k_n)$, then the Frobenius conjugacy class $\Theta_{\mathrm{M},k_n}(\chi)$ will be distributed like a random $\mathbf{U}_r(\mathbf{C})$-conjugacy class of an element of the maximal compact subgroup K. (A whimsical enough way to do this – according to the Rényi–Turan form of the Erdős–Kac Theorem, see e.g. [65, Prop. 4.14] – would be to pick a large integer $m\geqslant 1$ and to take n to be the number of prime factors of m, which corresponds roughly to having $\lambda_{\mathrm{N}}=\log\log\mathrm{N}$.)

Note however that are also many cases where the condition (4.9) is not true. The most obvious is when $\varphi_{\mathrm{N}}(\mathrm{N})=1$ and $\varphi_{\mathrm{N}}(n)=0$ for $n\neq\mathrm{N}$, corresponding to a limit without extra average at all. In addition, the condition implies that for any integers $q\geqslant 1$ and $a\in\mathbf{Z}$, we have

$$\mathbf{P}(\mathrm{X}_{\mathrm{N}}\equiv a\,(\mathrm{mod}\,q))=\frac{1}{q}\sum_{b\,(\mathrm{mod}\,q)}e^{-2i\pi ab/q}\mathbf{E}(e^{2i\pi b\mathrm{X}_{\mathrm{N}}/q})\to\frac{1}{q},$$

so there is a strong arithmetic restriction that $\mathrm{X}_{\mathrm{N}}\,(\mathrm{mod}\,q)$ converge to the uniform probability measure modulo q for all $q\geqslant 1$.

[1] Precisely, we need that the series $\sum a_n$ with $a_1=\alpha$ and $a_n=\alpha^n-\alpha^{n-1}$ for $n\geqslant 2$ has "sum" $\alpha+(\alpha-1)/(1-\alpha)=0$ for $|\alpha|=1$ and $\alpha\neq 1$.

Similar remarks apply in an obvious manner to our other equidistribution statements, e.g. to Theorem 2.

4.4. Equidistribution for arithmetic Fourier transforms

We now deduce from Theorem 4.4 the equidistribution of the exponential sums defined by

$$\mathrm{S}(\mathrm{M},\chi)=\sum_{x\in\mathrm{G}(k_n)}\chi(x)t_{\mathrm{M}}(x;k_n).$$

In fact, note that these sums make sense for *all* characters $\chi\in\widehat{\mathrm{G}}(k_n)$, and we can indeed prove equidistribution for all of them. This implies Theorem 2 from the introduction. As a final addition, we prove an equidistribution statement for the arithmetic Fourier transforms of all objects M of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$ which are mixed semiperverse sheaves of weights $\leqslant 0$. This is of interest especially in more analytic applications, since the condition of being semiperverse and that of being mixed of weights $\leqslant 0$ are much more flexible, and easier to check, than those of being perverse and pure.

THEOREM 4.8. *Let k be a finite field and let G be a connected commutative algebraic group over k. Let ℓ be a prime number distinct from the characteristic of k.*

Let M be an object of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$. Assume that M is semiperverse and mixed of weights $\leqslant 0$. Let N be the maximal perverse subsheaf of weight 0 of the arithmetic semisimplification of the perverse cohomology sheaf ${}^{\mathfrak{p}}\mathscr{H}^0(\mathrm{M})$.

Let $r\geqslant 0$ be the tannakian dimension of N. Let $\mathrm{K}\subset\mathbf{G}^{\mathrm{ari}}_{\mathrm{N}}(\mathbf{C})\subset\mathbf{GL}_r(\mathbf{C})$ be a maximal compact subgroup of the arithmetic tannakian group of N. Denote by μ the Haar probability measure on K and by ν its image by the trace.

The families of exponential sums $\mathrm{S}(\mathrm{M},\chi)$ for $\chi\in\widehat{\mathrm{G}}(k_n)$ become ν-equidistributed on average as $n\to+\infty$.

PROOF. We first assume that M is perverse and pure of weight 0, so that the object N coincides with M. We then observe that, by the generic vanishing theorem, it suffices to prove that the families of exponential sums associated to $\chi\in\mathscr{X}_w(\mathrm{M})$ become ν-equidistributed on average, since for any bounded continuous function $f\colon\mathbf{C}\to\mathbf{C}$, we have

$$\Big|\frac{1}{|\widehat{\mathrm{G}}(k_n)|}\sum_{\chi\in(\widehat{\mathrm{G}}-\mathscr{X}_w(\mathrm{M}))(k_n)}f(\mathrm{Tr}(\Theta_{\mathrm{M},k_n}(\chi)))\Big|\leqslant\|f\|_\infty\frac{|(\widehat{\mathrm{G}}-\mathscr{X}_w(\mathrm{M}))(k_n)|}{|\widehat{\mathrm{G}}(k_n)|}\to 0$$

because $\mathscr{X}_w(\mathrm{M})$ is generic. But since $\mathrm{Tr}_*(\nu_{cp})=\nu$, this equidistribution follows from Theorem 4.4 by considering the composition $\mathrm{K}\to\mathbf{U}_r(\mathbf{C})^\sharp\xrightarrow{\mathrm{Tr}}\mathbf{C}$.

We now consider the general case. We denote by M_0 the arithmetic semisimplification of the perverse sheaf ${}^{\mathfrak{p}}\mathscr{H}^0(\mathrm{M})$, and by N' the perverse sheaf such that $\mathrm{M}_0=\mathrm{N}\oplus\mathrm{N}'$, defined using the weight filtration on M_0; the perverse sheaf N' is mixed of weights $\leqslant -1$.

Since M is semiperverse of weights $\leqslant 0$, we have ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M})=0$ for $i\geqslant 1$, and ${}^{\mathfrak{p}}\mathscr{H}^{-i}(\mathrm{M})$ is of weights $\leqslant -i\leqslant -1$ for all $i\geqslant 1$ (see [8, Th. 5.4.1]).

For any $n\geqslant 1$ and $\chi\in\widehat{\mathrm{G}}(k_n)$, we have the equality

$$\mathrm{S}(\mathrm{M},\chi)=\mathrm{S}(\mathrm{N},\chi)+\mathrm{S}(\mathrm{N}',\chi)+\sum_{i\geqslant 1}(-1)^i\mathrm{S}({}^{\mathfrak{p}}\mathscr{H}^{-i}(\mathrm{M}),\chi)\tag{4.10}$$

by (A.4).

By generic vanishing and the trace formula (see Theorem 6.1 below, applied to $\mathrm{N}'(-1/2)$ and ${}^{\mathfrak{p}}\mathscr{H}^{-i}(\mathrm{M})(-1/2)$ for $i \geqslant 1$, which are mixed perverse sheaves of weights $\leqslant 0$), there exists a generic subset $\mathscr{X} \subset \widehat{\mathrm{G}}$ such that we have

$$\mathrm{S}(\mathrm{N}', \chi) + \sum_{i \geqslant 1} (-1)^i \mathrm{S}({}^{\mathfrak{p}}\mathscr{H}^{-i}(\mathrm{M}), \chi) \ll \frac{1}{|k_n|^{1/2}} \tag{4.11}$$

for all $n \geqslant 1$ and $\chi \in \mathscr{X}(k_n)$. This implies that the sequence (ϖ_n) of probability measures defined as averages of delta masses at the points

$$\mathrm{S}(\mathrm{N}', \chi) + \sum_{i \geqslant 1} (-1)^i \mathrm{S}({}^{\mathfrak{p}}\mathscr{H}^{-i}(\mathrm{M}), \chi)$$

for all $\chi \in \widehat{\mathrm{G}}(k_n)$ converges to zero in probability, i.e., that for any fixed real number $\varepsilon > 0$, the limit

$$\lim_{n \to +\infty} \varpi_n(\{|t| > \varepsilon\}) = 0$$

holds.

By the first case applied to the perverse sheaf N, the sums $\mathrm{S}(\mathrm{N}, \chi)$ become ν-equidistributed on average as $n \to +\infty$, and the formula (4.10) ensures then that the same holds for the $\mathrm{S}(\mathrm{M}, \chi)$ (see, e.g., [88, Cor. B.4.2] for the simple probabilistic argument that leads to this conclusion). ☐

Remark 4.9. (1) As we will see later, it is often of interest to attempt to apply equidistribution of exponential sums to the test function $z \mapsto z^m$ or $z \mapsto |z|^m$ for some integer $m \geqslant 1$. Such functions are continuous but not bounded on $\mathbf{C}$, so that Theorem 4.8 does not apply, and Theorem 4.4 only gives the equidistribution for weakly unramified characters. In these attempts, the contribution of the other characters may therefore need to be handled separately (see for instance the proof of Theorem 9.11).

(2) See Chapter 7 for an application of this theorem to a question of independence of ℓ of tannakian groups.

(3) The measure ν is also the image by the trace of measure ν_{cp} on characteristic polynomials appearing in Theorem 4.4. It is often called the *Sato–Tate measure* associated to M.

Example 4.10. Let $k = \mathbf{F}_p$, and let ψ be the additive character on k such that $\psi(x) = e(x/p)$ for $x \in k$. Let $\mathrm{X} \subset \mathrm{G}$ be a locally-closed subvariety of G of dimension $d \geqslant 1$, and let $f \colon \mathrm{X} \to \mathbf{A}^1$ be a non-zero function on X. Then there is a semiperverse sheaf M on G, mixed of weights 0, such that the trace function of M is given by the formula

$$t_{\mathrm{M}}(x; \mathbf{F}_{p^n}) = \begin{cases} (-1)^d p^{-nd/2} e(\mathrm{Tr}_{\mathbf{F}_{p^n}/\mathbf{F}_p}(f(x))/p) & \text{if } x \in \mathrm{X}(\mathbf{F}_{p^n}) \\ 0 & \text{otherwise.} \end{cases}$$

for $n \geqslant 1$ and $x \in \mathrm{G}(k_n)$, namely

$$\mathrm{M} = j_! f^* \mathscr{L}_\psi[d](d/2),$$

where $j \colon \mathrm{X} \to \mathrm{G}$ is the natural immersion.

Hence Theorem 4.8 implies that the exponential sums

$$\frac{1}{p^{nd/2}} \sum_{x \in \mathrm{X}(\mathbf{F}_{p^n})} \chi(x) e\Big(\frac{f(x)}{p}\Big)$$

for $\chi \in \widehat{\mathrm{G}}(\mathbf{F}_{p^n})$ *always satisfy some* equidistribution theorem on average.

A similar property holds if we fix a non-trivial multiplicative character η of $\mathbf{F}_p^\times$ and an invertible function $g\colon \mathrm{X} \to \mathbf{G}_m$, and consider the exponential sums

$$\frac{1}{p^{nd/2}} \sum_{x \in \mathrm{X}(\mathbf{F}_{p^n})} \chi(x)\eta(g(x))$$

(using the object $j_! g^* \mathscr{L}_\eta[d](d/2)$, which is also mixed and semiperverse of weights $\leqslant 0$).

4.5. Equidistribution for conjugacy classes

We keep the notation of the previous sections. If the object M that we consider is generically unramified, then we can prove equidistribution at the level of the Frobenius conjugacy classes in the maximal compact subgroup of the arithmetic tannakian group.

Theorem 4.11 (Equidistribution on average). *Let k be a finite field and let G be a connected commutative algebraic group over k. Let ℓ be a prime number distinct from the characteristic of k.*

Let M be an ℓ-adic perverse sheaf on G that is arithmetically semisimple, pure of weight zero and generically unramified. Let $\mathscr{X} = \mathscr{X}(\mathrm{M})$ be the set of unramified characters for M. Let K be a maximal compact subgroup of the arithmetic tannakian group $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}(\mathbf{C})$ of M, and denote by $\mu^\sharp$ the direct image of the Haar probability measure μ on K by the projection to the set $\mathrm{K}^\sharp$ of conjugacy classes of K.

The families of unitary Frobenius conjugacy classes $(\boldsymbol{\Theta}_{\mathrm{M},k_n}(\chi))_{\chi \in \mathscr{X}(k_n)}$ become μ-equidistributed on average in $\mathrm{K}^\sharp$ as $n \to +\infty$.

Precisely, we are considering here the equidistribution on average of the pairs $(\mathscr{X}(k_n), \Theta_n)$ where $\Theta_n(\chi) = \boldsymbol{\Theta}_{\mathrm{M},k_n}(\chi)$.

Proof. By Theorem 3.29 and the definition of generic sets, we know that $|\mathscr{X}(k_n)| \sim |\mathrm{G}(k_n)|$ as $n \to +\infty$, and hence the sets of unramified conjugacy classes are non-empty for n large enough.

By the Peter-Weyl theorem, any continuous central function $f\colon \mathrm{K} \to \mathbf{C}$ is a uniform limit of linear combinations of characters of finite-dimensional unitary irreducible representations of K, and hence it suffices to prove the formula (4.1) when f is such a character. For the trivial representation, both sides are equal to 1. If the representation is non-trivial, then the integral on the right-hand side vanishes, and we are reduced to showing that the limit on the left-hand side exists and is equal to 0. We thus consider a non-trivial irreducible representation ϱ of K, which we identify with a non-trivial irreducible algebraic $\overline{\mathbf{Q}}_\ell$-representation of the arithmetic tannakian group $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ by Weyl's unitarian trick (see, e.g., [68, 3.2] for this step); applying the next proposition then completes the proof. ☐

Proposition 4.12. *With notation as in Theorem 4.11, let ϱ be a non-trivial irreducible unitary representation of K, identified with a non-trivial irreducible representation of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$.*

(1) *If the restriction of ϱ to $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ is non-trivial, then*

$$\frac{1}{|\mathscr{X}(k_n)|} \sum_{\chi \in \mathscr{X}(k_n)} \mathrm{Tr}(\varrho(\boldsymbol{\Theta}_{\mathrm{M},k_n}(\chi))) \ll \frac{1}{\sqrt{|k_n|}} \tag{4.12}$$

for all n such that $\mathscr{X}(k_n)$ is not empty.

(2) *If the restriction of ϱ to $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ is trivial, then*

$$\lim_{\mathrm{N} \to +\infty} \frac{1}{\mathrm{N}} \sum_{\substack{1 \leqslant n \leqslant \mathrm{N} \\ \mathscr{X}(k_n) \neq \varnothing}} \frac{1}{|\mathscr{X}(k_n)|} \sum_{\chi \in \mathscr{X}(k_n)} \mathrm{Tr}(\varrho(\boldsymbol{\Theta}_{\mathrm{M},k_n}(\chi))) = 0. \tag{4.13}$$

PROOF. (1) We assume that the restriction of ϱ to the geometric tannakian group is non-trivial.

Let $\varrho(\mathrm{M})$ denote the object of the tannakian category $\langle \mathrm{M}\rangle^{\mathrm{ari}}$ corresponding to the representation ϱ of the arithmetic tannakian group $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$; this is a simple perverse sheaf on G.

We have $\mathscr{X} \subset \mathscr{X}_{\mathrm{F}}(\varrho(\mathrm{M}))$. Applying Proposition 4.3 to the object $\mathrm{N} = \varrho(\mathrm{M})$, we obtain

$$\frac{1}{|\mathscr{X}(k_n)|} \sum_{\chi \in \mathscr{X}(k_n)} \mathrm{Tr}(\varrho(\mathbf{\Theta}_{\mathrm{M},k_n}(\chi))) = t_{\varrho(\mathrm{M})}(e; k_n) + \mathrm{O}(|k_n|^{-1/2})$$

since the conjugacy class $\mathbf{\Theta}_{\mathrm{M},k_n}(\chi)$ coincides with $\Theta_{\mathrm{M},k_n}(\chi)$ when χ is unramified for M.

Since $\varrho(\mathrm{M})$ is a simple perverse sheaf on G, the classification of [8, Th. 4.3.1 (ii)] shows that there exist an irreducible closed subvariety $s\colon \mathrm{Y} \to \mathrm{G}$ of dimension r, an open dense smooth subvariety $j\colon \mathrm{U} \to \mathrm{Y}$, and an irreducible lisse $\overline{\mathbf{Q}}_\ell$-sheaf $\mathscr{F}$ on U such that $\varrho(\mathrm{M}) = s_* j_{!*}\mathscr{F}[r]$. Since the functors s_* and $j_{!*}$ are weight-preserving, the sheaf $\mathscr{F}$ is pure of weight $-r$.

If $r = 0$, then Y consists of a closed point of G, which must be different from the neutral element e, since otherwise $\varrho(\mathrm{M})$ would be geometrically trivial, contrary to the assumption in (1). In that case, we have therefore $t_{\varrho(\mathrm{M})}(e; k_n) = 0$. On the other hand, if $r \geqslant 1$ we get

$$t_{\varrho(\mathrm{M})}(e; k_n) \ll \frac{1}{\sqrt{|k_n|}} \tag{4.14}$$

(by Proposition 1.11), which concludes the proof of (1).

(2) We assume that the restriction of the representation ϱ to $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ is trivial. Then ϱ has dimension 1 since the quotient $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}/\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ is abelian (Proposition 3.40).

Let Q be the set of integers $n \geqslant 1$ such that $\mathscr{X}(k_n)$ is not empty; it contains all sufficiently large integers. It follows from Proposition 3.40 that there exists an element ξ of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}/\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$, generating a Zariski-dense subgroup of this group, such that $\varrho(\mathbf{\Theta}_{\mathrm{M},k_m}(\chi)) = \varrho(\xi)^n$ for any $n \geqslant 1$ and any $\chi \in \widehat{\mathrm{G}}(k_n)$ unramified for M. Moreover, we have $\varrho(\xi) \neq 1$, since otherwise the representation ϱ would be trivial. We conclude that

$$\frac{1}{\mathrm{N}} \sum_{1 \leqslant n \leqslant \mathrm{N}} \frac{1}{|\mathscr{X}(k_n)|} \sum_{\substack{\chi \in \mathscr{X}(k_n) \\ \mathscr{X}(k_n) \neq \varnothing}} \mathrm{Tr}(\varrho(\mathbf{\Theta}_{\mathrm{M},k_n}(\chi))) = \frac{1}{\mathrm{N}} \sum_{\substack{1 \leqslant n \leqslant \mathrm{N} \\ n \in \mathrm{Q}}} \varrho(\xi)^n$$

converges to 0 as $\mathrm{N} \to +\infty$ by summing a non-trivial geometric progression.

$\square$

REMARK 4.13. For certain reductive groups $\mathbf{G} \subset \mathbf{GL}_r(\mathbf{C})$, a conjugacy class in a maximal compact subgroup K of $\mathbf{G}$ is determined by its characteristic polynomial (equivalently, the exterior powers of the standard representation generate the representation ring of $\mathbf{G}$). If $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}(\mathbf{C})$ has this property, then Theorem 4.4 implies a version of Theorem 4.11, even if M is not generically unramified.

If $\mathbf{G}$ is semisimple, this property holds, for instance, for $\mathbf{SL}_r(\mathbf{C}) \subset \mathbf{GL}_r(\mathbf{C})$, for $\mathbf{Sp}_{2r}(\mathbf{C}) \subset \mathbf{GL}_{2r}(\mathbf{C})$, and for $\mathbf{G}_2(\mathbf{C}) \subset \mathbf{GL}_7(\mathbf{C})$. Indeed, the first two cases are explained by Katz in [68, Lemma 13.1, Remark 13.2]; in the third case, we note that the second fundamental representation of $\mathbf{G}_2(\mathbf{C})$ is virtually $\bigwedge^2 \mathrm{Std} - \mathrm{Std}$ (see, e.g., [49, p. 353]) so that the exterior powers of the standard 7-dimensional representation generate the representation ring.

We deduce immediately from Theorem 4.11 a useful corollary, analogue to some classical consequences of the Chebotarev density theorem.

Corollary 4.14. *Let k be a finite field and let G be a connected commutative algebraic group over k. Let M be a perverse sheaf on G which is arithmetically semisimple, pure of weight zero and generically unramified.*

Let S be any finite subset of $\widehat{\mathrm{G}}$. The union of the unitary Frobenius conjugacy classes of M associated to unramified characters in $\widehat{\mathrm{G}} - \mathrm{S}$ is dense in a maximal compact subgroup of $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}(\mathbf{C})$.

4.6. Equidistribution without average

We continue again with the previous notation. If we make the extra assumption that the geometric and the arithmetic tannakian groups coincide, then the equidistribution of Frobenius conjugacy classes holds without averaging over n. We summarize the variants of the previous theorems in this situation.

Theorem 4.15 (Equidistribution without average). *Let M be an ℓ-adic perverse sheaf on G that is arithmetically semisimple, pure of weight zero. We assume that the inclusion $\mathbf{G}^{\mathrm{geo}}_{\mathrm{M}} \subset \mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$ is an equality.*

Let $r \geqslant 0$ be the tannakian dimension of M. Let $\mathrm{K} \subset \mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}(\mathbf{C}) \subset \mathbf{GL}_r(\mathbf{C})$ be a maximal compact subgroup of the arithmetic tannakian group of M. Denote by μ the Haar probability measure on K, by ν_{cp} its direct image by the map $\mathrm{K} \to \mathbf{U}_r(\mathbf{C})$, by ν its image by the trace, and by $\mu^\sharp$ its image by the map $\mathrm{K} \to \mathrm{K}^\sharp$.

1. *The families of unitary Frobenius conjugacy classes $(\Theta_{\mathrm{M},k_n}(\chi))_{\chi \in \mathscr{X}_w(\mathrm{M})(k_n)}$ become ν_{cp}-equidistributed as $n \to +\infty$.*
2. *The families of exponential sums $\mathrm{S}(\mathrm{M},\chi)$ for $\chi \in \widehat{\mathrm{G}}(k_n)$ become ν-equidistributed as $n \to +\infty$.*
3. *If M is generically unramified, then the family of conjugacy classes $(\boldsymbol{\Theta}_{\mathrm{M},k_n}(\chi))_{\chi \in \mathscr{X}(\mathrm{M})(k_n)}$ become $\mu^\sharp$-equidistributed as n goes to infinity.*

Proof. This follows from the Weyl Criterion as in the proof of Theorems 4.4, 4.8 and 4.11; in the case of the last statement, for instance, we use only the first part of Proposition 4.12 (as we may since a non-trivial irreducible representation of $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$ is a non-trivial irreducible representation of $\mathbf{G}^{\mathrm{geo}}_{\mathrm{M}}$ under the assumption). ☐

Remark 4.16. There is an obvious further variant of Theorems 4.15 and of the case of mixed semiperverse objects of weights $\leqslant 0$ of 4.8: if M is mixed semiperverse of weights $\leqslant 0$, with N as in Theorem 4.8 such that $\mathbf{G}^{\mathrm{ari}}_{\mathrm{N}} = \mathbf{G}^{\mathrm{geo}}_{\mathrm{N}}$, then the discrete Fourier transform becomes equidistributed towards the measure ν without average over n.

There is a converse to Theorem 4.15. In fact, there is a statement which is valid for an individual representation of the unitary group (this will be useful in Chapter 8).

Proposition 4.17. *Let M be an ℓ-adic perverse sheaf on G that is arithmetically semisimple and pure of weight zero. Let r be the tannakian dimension of M and let $\mathscr{X} = \mathscr{X}_w(\mathrm{M})$ be the set of weakly unramified characters for M. Let ϱ be a finite-dimensional unitary representation of $\mathbf{U}_r(\mathbf{C})$. Assume that the sequence*

$$\frac{1}{|\mathscr{X}(k_n)|} \sum_{\chi \in \mathscr{X}(k_n)} \mathrm{Tr}(\varrho(\Theta_{\mathrm{M},k_n}(\chi))),$$

defined for all integers $n \geqslant 1$ such that $\mathscr{X}(k_n)$ is not empty, has a limit. Then this limit is equal to the multiplicity of the trivial representation in the restriction of ϱ to $\mathbf{G}^{\mathrm{geo}}_{\mathrm{M}}$, and the latter equals the multiplicity of the trivial representation in ϱ.

Proof. We use the notation in the proof of Theorem 4.4. Taking the equality (4.8) into account, the assumption of the statement means that the limit

$$\lim_{n\to+\infty}\sum_{i\in \mathrm{J}}\alpha_i^n$$

exists, where the complex numbers α_i have modulus 1 and the set J has cardinality equal to the multiplicity of the trivial representation in the restriction of ϱ to $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$. We claim that the existence of this limit implies the equality $\alpha_i = 1$ for all $i \in \mathrm{J}$, so that the limit is equal to $|\mathrm{J}|$, as desired.

Indeed, let $\mathrm{L} \subset \mathrm{J}$ be the set of i where $\alpha_i \neq 1$. The sequence

$$\sum_{i\in \mathrm{L}}\alpha_i^n$$

converges as well, and its limit must be zero since it converges to 0 on average over $n \leqslant \mathrm{N}$. However, the lower bound

$$\limsup_{n\to+\infty}\Big|\sum_{i\in\mathrm{L}}\alpha_i^n\Big| \geqslant |\mathrm{L}|^{1/2}$$

holds (see, e.g., [64, Lemma 11.41]), so we deduce that L is empty, which proves the claim. ☐

A more global form of this converse, for generically unramified objects, is the following:

Proposition 4.18. *Let* M *be an* ℓ*-adic perverse sheaf on* G *that is arithmetically semisimple and pure of weight zero. Assume that* M *is generically unramified. Let* r *be the tannakian dimension of* M *and let* $\mathscr{X} = \mathscr{X}_w(\mathrm{M})$ *be the set of unramified characters for* M*. If the sequence of probability measures*

$$\frac{1}{|\mathscr{X}(k_n)|}\sum_{\chi\in\mathscr{X}(k_n)}\delta_{\mathbf{\Theta}_{\mathrm{M},k_n}(\chi)},$$

defined when $\mathscr{X}(k_n)$ *is not empty, converges weakly to some probability measure, then we have the equality* $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}} = \mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$.

Proof. Suppose that $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}} \neq \mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$. By Proposition 3.40, there exists an element $\xi \neq 1$ of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}/\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ which generates a Zariski-dense subgroup of this group, which is abelian. Thus there exists an irreducible representation ϱ of the quotient $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}/\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ such that $\varrho(\xi) \neq 1$; for any $n \geqslant 1$ and any $\chi \in \widehat{\mathrm{G}}(k_n)$ unramified for M, the equality $\varrho(\mathbf{\Theta}_{\mathrm{M},k_n}(\chi)) = \varrho(\xi)^n$ holds.

Let $\mathscr{X}$ be the set of characters unramified for M. Then

$$\frac{1}{|\mathscr{X}(k_n)|}\sum_{\chi\in\mathscr{X}(k_n)}\mathrm{Tr}(\varrho(\mathbf{\Theta}_{\mathrm{M},k_n}(\chi))) = \varrho(\xi)^n$$

for all $n \geqslant 1$ for which $\mathscr{X}(k_n)$ is not empty. Since $\varrho(\xi) \neq 1$, this quantity does not converge as $n \to +\infty$, which implies the proposition by contraposition. ☐

4.7. Horizontal equidistribution

The proof of Theorem 4.11 relies crucially on the estimates in the stratified vanishing theorem 2.3. We expect (see Remark 2.4) that the implied constants in these estimates depend only on the complexity of the perverse sheaf M (as is the case for unipotent groups).

Under the assumption that such a statement is valid, and in fact that this holds for the size of the set of unramified characters, one can obtain equidistribution statements for finite fields when their size tends to infinity (for instance, for $\mathbf{F}_p$ as $p \to +\infty$; compare with [74, Ch. 28–29]).

We include a conditional statement of this type, anticipating some progress soon concerning the underlying uniformity question. We leave to the interested reader the task of formulating variants similar to Theorems 4.8 and 4.4.

THEOREM 4.19 (Horizontal equidistribution). *Let ℓ be a prime number. Let $\mathrm{N} \geqslant 1$ be an integer and let (G, u) be a quasi-projective commutative group scheme over $\mathbf{Z}[1/\ell\mathrm{N}]$ such that, for all primes $p \nmid \ell\mathrm{N}$, the fiber G_p of G over $\mathbf{F}_p$ is a connected commutative algebraic group for which the estimate*

$$|\widehat{\mathrm{G}}_p(\mathbf{F}_{p^n}) - \mathscr{X}(\mathrm{M})(\mathbf{F}_{p^n})| \ll c_u(\mathrm{M}) p^{n(\dim(\mathrm{G}_p)-1)}$$

holds for all primes p and $n \geqslant 1$ and all arithmetically semisimple objects M in $\mathrm{Perv}_{\mathrm{int}}(\mathrm{G}_p)$ which are generically unramified.

Let $(\mathrm{M}_p)_{p\nmid \mathrm{N}\ell}$ be a sequence of arithmetically semisimple objects in $\mathrm{Perv}_{\mathrm{int}}(\mathrm{G}_p)$ which are pure of weight zero. Suppose that the tannakian dimension r of M_p is independent of p, and that for all p, we have $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_p} = \mathbf{G}^{\mathrm{geo}}_{\mathrm{M}_p}$, and that this common reductive group is conjugate to a fixed subgroup $\mathbf{G}$ of $\mathbf{GL}_r(\overline{\mathbf{Q}}_\ell)$.

Let K be a maximal compact subgroup of $\mathbf{G}(\mathbf{C})$ and let $\mu^\sharp$ be the direct image of the Haar probability measure on K to $\mathrm{K}^\sharp$.

Let $\mathscr{X}_p$ be the set of characters $\chi \in \widehat{\mathrm{G}}_p(\mathbf{F}_p)$ which are unramified for the object M_p.

If we have $c_u(\mathrm{M}_p) \ll 1$ for all $p \nmid \mathrm{N}\ell$, then the families of conjugacy classes $(\mathbf{\Theta}_{\mathrm{M}_p,\mathbf{F}_p}(\chi))_{\chi\in\mathscr{X}_p}$ become $\mu^\sharp$-equidistributed in $\mathrm{K}^\sharp$ as $p \to +\infty$.

PROOF. The argument follows that of Theorem 4.11; it suffices to prove the estimate

$$\frac{1}{|\mathscr{X}_p|} \sum_{\chi\in\mathscr{X}_p} \mathrm{Tr}\left(\varrho(\mathbf{\Theta}_{\mathrm{M}_p,\mathbf{F}_p}(\chi))\right) \ll \frac{1}{\sqrt{p}}$$

for all $p \nmid \mathrm{N}\ell$. The proof of this is similar to the first part of Proposition 4.12, noting that, under our assumptions, the implied constants in the key bounds (4.14), (4.5), (4.6) and (4.7) are independent of p, since the complexity of M_p is bounded independently of p, and hence also that of $\varrho(\mathrm{M}_p)$ by [115, Prop. 6.33]. ∎

REMARK 4.20. (1) For G unipotent, results of this form are unconditional by Proposition 2.7 (the case of $\mathbf{G}_a$ essentially goes back to Katz [68], whereas the case of an arbitrary power of $\mathbf{G}_a$ follows from [115, Th. 7.22]). For $\mathrm{G} = \mathbf{G}_m$, a similar statement is proved by Katz in [74, Th. 28.1].

(2) The result is also unconditional in the case of abelian varieties (see Remark 2.4). We expect that a careful look at the proof of the generic vanishing theorem will also show that it is unconditional for $\mathbf{G}_m \times \mathbf{G}_a$. The case of tori of dimension $\geqslant 2$ is however not yet known.

4.8. Objects of rank 1

In this section, we apply the general equdistribution results to objects in the tannakian subcategory of $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{G})$ additively generated by objects of tannakian rank 1. The corresponding arithmetic tannakian groups are computed (in principle) in Proposition 3.49.

PROPOSITION 4.21. *Let $r \geqslant 1$ be an integer and let*

$$\mathrm{M} = \mathrm{L}_1 \oplus \cdots \oplus \mathrm{L}_r$$

where L_i is an object of $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{G})$ of tannakian rank 1. Let

$$\mathrm{H} = \{(n_1, \ldots, n_r) \in \mathbf{Z}^r \mid \mathrm{L}_1^{*_{\mathrm{int}} n_1} *_{\mathrm{int}} \cdots *_{\mathrm{int}} \mathrm{L}_r^{*_{\mathrm{int}} n_r} \simeq \mathbf{1}_{\mathrm{G}}\},$$

and let

$$\mathrm{K} = \{(x_i) \in (\mathbf{S}^1)^r \mid \prod_{1 \leqslant i \leqslant r} x_i^{n_i} = 1 \text{ for all } (n_1, \dots, n_r) \in \mathrm{H}\}.$$

Then the unitary Frobenius conjugacy classes of M *are equidistributed on average in* K.

Proof. This follows from Theorem 4.4, on noting that the arithmetic tannakian $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ group of M is abelian (it can be viewed as a subgroup of the diagonal subgroup $\mathbf{GL}_1^r \subset \mathbf{GL}_r$); hence, the conjugacy classes of elements of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ are just singletons, and in particular are the same as the conjugacy classes in $\mathbf{GL}_r$. ☐

As an application, we explain how to recover a theorem of Rojas-León [110, Th. 1], which concerns the equidistribution properties of Gauss sums.

Theorem 4.22 (Rojas-León). *Let $r \geqslant 1$ be an integer. Let $(\alpha_i)_{1 \leqslant i \leqslant r}$ be a family of non-constant morphisms $\alpha_i \colon \mathbf{G}_m \to \mathbf{G}_m^r$ defined over k. Let $t = (t_i) \in (k^\times)^r$ and let $(\eta_i)_{1 \leqslant i \leqslant r}$ be characters of $k^\times$.*

The tuples

$$\Big(\frac{\chi(t)\tau(\psi, \eta_1 \cdot (\chi \circ \alpha_1))}{|k_n|^{1/2}}, \dots, \frac{\chi(t)\tau(\psi, \eta_r \cdot (\chi \circ \alpha_r))}{|k_n|^{1/2}}\Big)_{\chi \in \widehat{\mathbf{G}_m^r}(k_n)}$$

of Gauss sums are equidistributed on average in $\mathbf{C}^r$ according to the probability Haar measure on a closed subgroup $\mathrm{K} \subset (\mathbf{S}^1)^r \subset \mathbf{C}^r$.

Moreover, factor $\alpha_i = [x \mapsto x^{\mathrm{N}_i}] \circ \beta_i$ for some closed immersion β_i and some integer $\mathrm{N}_i \geqslant 1$. If, for each i with $1 \leqslant i \leqslant r$, the elements

$$\sum_{\omega^{\mathrm{N}_i} = \eta_i} \omega$$

of $\mathbf{Z}[\Pi(\mathbf{G}_{m,\bar{k}}, \overline{\mathbf{Q}}_\ell)]$ are linearly independent over $\mathbf{Z}$, then $\mathrm{K} = (\mathbf{S}^1)^r$ and equidistribution holds without average.

Proof. For simplicity of notation, we will assume that $t = 1$ and that each N_i is coprime to p. Let $j \colon \mathbf{G}_m \to \mathbf{A}^1$ be the open immersion. We recall that $\chi \mapsto \tau(\psi, \eta_i \cdot (\chi \circ \alpha_i))$ is the discrete Mellin transform on $\mathbf{G}_m$ of the trace function of the perverse sheaf

$$\mathrm{L}_i = \alpha_{i*}(j^* \mathscr{L}_\psi *_! \mathscr{L}_\chi[1])(1/2)$$

on $\mathbf{G}_m^r$, which is pure of weight 0. This is an object of tannakian rank 1.

Thus the first statement is a direct application of Proposition 4.21, with K a maximal compact subgroup for the arithmetic tannakian group of

$$\mathrm{M} = \mathrm{L}_1 \oplus \cdots \oplus \mathrm{L}_r$$

(note that we consider the Gauss sums in $\mathbf{C}$ to avoid excluding those boundedly many χ where some $\eta_i \cdot (\chi \circ \alpha_i)$ is trivial, for which the modulus is $1/\sqrt{|k_n|}$ instead of 1; these do not affect the equidistribution property).

Using Proposition 4.21 again, for the second statement we need to prove that under the stated assumptions, there is no convolution relation between the objects L_i, or equivalently no relation between their classes in the group $\mathrm{L}(\mathbf{G}_m^r)$; this implies that the arithmetic tannakian group of M is $\mathbf{G}_m^r$ (i.e., is as large as possible). In fact, we claim that the geometric tannakian group is already that large, which means that there are no geometric convolution relations between the objects L_i. This in particular also implies the equidistribution without average (see Theorem 4.15).

To prove the claim, we use the Gabber–Loeser isomorphism described in Example 3.51 to express the class of L_i in $\mathrm{L}(\mathbf{G}^r_{m,\bar{k}})$ as

$$\lambda_i = \Big(1, \sum_{\omega^{\mathrm{N}_i}=\eta_i} (\mathrm{T}_i, \omega)\Big),$$

where T_i is the image of β_i and (T_i, ω) is one of the basis vectors in the free abelian group generated by pairs of a one-dimensional subtorus and a tame character (see loc. cit.). (This fact is a form of the Hasse–Davenport relation; see [69, Th. 8.9.1].)

By definition of a free abelian group, a non-trivial linear relation can only exist if, for some one-dimensional subtorus $\mathrm{T} \subset \mathbf{G}^r_m$, the elements λ_i with $\mathrm{T}_i = \mathrm{T}$ are linearly dependent, and this in turn is equivalent with the elements

$$\sum_{\omega^{\mathrm{N}_i}=\eta_i} \omega$$

being linearly dependent, as claimed. $\square$

Part 2

Applications

Description of applications

The remainder of the book is devoted to applications of the theoretical results of the first part of this book. We split these applications in further chapters as follows:

(1) We define in Chapter 5 the analogue of L-functions for the Fourier–Mellin transforms. We establish with its help that the arithmetic tannakian group is infinite for many non-punctual objects on abelian varieties.

(2) We present in Chapter 6 the concrete analytic translation of the stratified vanishing theorem to stratification of estimates for exponential sums, in the spirit of Katz–Laumon [77] and Fouvry–Katz [39]. We also present a statement of "generic Fourier invertibility", which shows that two semsimple perverse sheaves are isomorphic in the category $\overline{\mathbf{P}}^{\mathrm{ari}}(\mathrm{G})$ if and only if the associated exponential sums coincide for a generic set of characters.

(3) In Chapter 7, we add a theoretical application of equidistribution in direction of *independence of* ℓ properties of the tannakian groups associated to a compatible system of ℓ-adic complexes.

(4) In applications of equidistribution to concrete perverse sheaves, the main issue is to determine the tannakian group. The main tool that we will use for this purpose is *Larsen's Alternative*, and its link with equidistribution. We present this result (and a new variant for the exceptional group $\mathbf{E}_6$) in Chapter 8.

(5) Then in the remaining chapters, we present examples of equidistribution for "concrete" groups, namely:
   - the product $\mathbf{G}_m \times \mathbf{G}_a$, which (apart from unipotent groups) is probably the simplest group of dimension $\geqslant 2$ (Chapter 9); this corresponds to rather natural families of exponential sums parameterized by both an additive character and a multiplicative character.
   - higher-dimensional tori, with applications to the study of the variance of arithmetic functions on $k[t]$ in arithmetic progressions modulo square-free polynomials (see Chapter 10).
   - the jacobian of a curve (Chapter 11); the application we present is a generalization of an unpublished result of Katz (which answered a question of Tsimerman).
   - in the same chapter, the intermediate jacobian of a smooth projective cubic hypersurface of dimension 3, which is an abelian variety of dimension 5 (see Chapter 11.2).

CHAPTER 5

Über eine neue Art von L-Reihen

5.1. $\widehat{\mathrm{L}}$-functions

Let k be a finite field, with algebraic closure $\bar{k}$ and intermediate extensions k_n. We fix as usual a prime ℓ different from the characteristic of k. Let G be a connected commutative algebraic group over k, and let d be its dimension. We denote by e the neutral element of G.

By analogy with algebraic varieties over k, we can define "L-functions" for objects of $\mathrm{D}_c^{\mathrm{b}}(\mathrm{G})$, where suitable characters $\chi \in \widehat{\mathrm{G}}$ play the role of primes in an "Euler product".

We denote by $\widehat{\mathrm{G}}^* \subset \widehat{\mathrm{G}}$ the set of characters such that $\chi \in \widehat{\mathrm{G}}^*(k_n)$ if and only if there is no $d \mid n$ with $d < n$ such that $\chi = \chi' \circ \mathrm{N}_{k_n/k_d}$. We say that elements of $\widehat{\mathrm{G}}^*$ are *primitive*, and for $\chi \in \widehat{\mathrm{G}}^*(k_n)$, we put $\deg(\chi) = n$. We then denote by $[\widehat{\mathrm{G}}]$ the quotient set of $\widehat{\mathrm{G}}^*$ by the equivalence relation defined by $\chi_1 \sim \chi_2$ if and only if $\deg(\chi_1) = \deg(\chi_2)$ and

$$\chi_2 = \chi_1 \circ \mathrm{Fr}^j_{k_{\deg(\chi_1)}}$$

for some integer $j \in \mathbf{Z}$. There are $\deg(\chi)$ primitive characters equivalent to a given $\chi \in \widehat{\mathrm{G}}^*$.

Definition 5.1 ($\widehat{\mathrm{L}}$-function). Let M be an object of $\mathrm{D}_c^{\mathrm{b}}(\mathrm{G})$. The *Fourier*-L-*function*, or $\widehat{\mathrm{L}}$-*function*, of M is the formal power series

$$\widehat{\mathrm{L}}(\mathrm{M}, \mathrm{T}) = \prod_{\chi \in [\widehat{\mathrm{G}}]} \det(1 - \mathrm{T}^{\deg(\chi)} \mathrm{Fr}_{k_{\deg(\chi)}} \mid \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi))^{-1} \in \overline{\mathbf{Q}}_\ell[[\mathrm{T}]].$$

This is similar to the definition

$$\mathrm{L}(\mathrm{M}, \mathrm{T}) = \prod_{x \in [\mathrm{X}]} \det(1 - \mathrm{T}^{\deg(x)} \mathrm{Fr}_{k_{\deg(x)}} \mid \mathrm{M}_x)^{-1} \in \overline{\mathbf{Q}}_\ell[[\mathrm{T}]]$$

of the L-function of M on an arbitrary algebraic variety X over k, with primitive characters replacing the set [X] of closed points of X.

Indeed, if G is unipotent of dimension d, and FT(M) denotes the Fourier transform of M on the (or "a") Serre dual $\mathrm{G}^\vee$ defined with respect to some additive character ψ, as in Section 2.2, then we obtain the identity

$$\widehat{\mathrm{L}}(\mathrm{M}, \mathrm{T}) = \mathrm{L}(\mathrm{FT}(\mathrm{M}), |k|^d \mathrm{T}),$$

(e.g. by the formula (5.2) below, since the stalk of FT(M) at the origin is canonically isomorphic to M by the proper base change theorem, and $|\mathrm{G}(k_n)| = |k|^{nd}$ in this case).

In general, however, we obtain "new" L-functions. Their fundamental properties, including rationaliy, are given by the next proposition.

Proposition 5.2. *Let* M *be an object of* $\mathrm{D}_c^{\mathrm{b}}(\mathrm{G})$. *We denote as usual*

$$\mathrm{S}(\mathrm{M}, \chi) = \sum_{x \in \mathrm{G}(k_n)} \chi(x) t_{\mathrm{M}}(x; k_n)$$

for $n \geqslant 1$ and $\chi \in \widehat{\mathrm{G}}(k_n)$.

(1) *The $\widehat{\mathrm{L}}$-function satisfies*

$$\widehat{\mathrm{L}}(\mathrm{M}, \mathrm{T}) = \exp\Big(\sum_{n \geqslant 1}\Big(\sum_{\chi \in \widehat{\mathrm{G}}(k_n)} \mathrm{S}(\mathrm{M}, \chi)\Big)\frac{\mathrm{T}^n}{n}\Big) \tag{5.1}$$

$$= \exp\Big(\sum_{n \geqslant 1} |\mathrm{G}(k_n)| t_{\mathrm{M}}(e; k_n)\frac{\mathrm{T}^n}{n}\Big). \tag{5.2}$$

(2) *The $\widehat{\mathrm{L}}$-function is a rational function; if M is a mixed complex, then the zeros and poles of $\widehat{\mathrm{L}}(\mathrm{M}, \mathrm{T})$ are $|k|$-Weil numbers of some weights.*

(3) *For any $\chi \in \widehat{\mathrm{G}}(k)$, the equality $\widehat{\mathrm{L}}(\mathrm{M}_\chi, \mathrm{T}) = \widehat{\mathrm{L}}(\mathrm{M}, \mathrm{T})$ holds.*

Proof. The proof of the formula (5.1), like in the classical case, is a simple consequence of the trace formula. Precisely, we apply the operator $f(\mathrm{T}) \mapsto \mathrm{T} d \log f(\mathrm{T})$ to both sides of this equality. On the left-hand side, after expressing the determinant as alternating product of the determinants on the various groups $\mathrm{H}^i_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$, we obtain

$$\mathrm{T} d \log \widehat{\mathrm{L}}(\mathrm{M}, \mathrm{T}) = \sum_{\chi \in [\widehat{\mathrm{G}}]} \deg(\chi) \sum_{m \geqslant 1} \mathrm{T}^{m \deg(\chi)} \operatorname{Tr}(\mathrm{Fr}^m_{k_{\deg(\chi)}} \mid \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)) =$$

$$\sum_{n \geqslant 1} \mathrm{T}^n \sum_{d \mid n} \sum_{\chi \in [\widehat{\mathrm{G}}](k_d)} d \operatorname{Tr}(\mathrm{Fr}^{n/d}_{k_d} \mid \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)).$$

On the right-hand side of (5.1), we obtain

$$\sum_{n \geqslant 1} \mathrm{T}^n \sum_{\chi \in \widehat{\mathrm{G}}(k_n)} \mathrm{S}(\mathrm{M}, \chi),$$

and hence the formula is equivalent with the fact that the identity

$$\sum_{d \mid n} \sum_{\chi \in [\widehat{\mathrm{G}}](k_d)} d \operatorname{Tr}(\mathrm{Fr}^{n/d}_{k_d} \mid \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)) = \sum_{\chi \in \widehat{\mathrm{G}}(k_n)} \mathrm{S}(\mathrm{M}, \chi) \tag{5.3}$$

holds for any integer $n \geqslant 1$.

Let $n \geqslant 1$. To establish (5.3) for n, we begin with the trace formula (A.5), which implies that

$$\mathrm{S}(\mathrm{M}, \chi) = \operatorname{Tr}(\mathrm{Fr}_{k_n} \mid \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)),$$

for any $\chi \in \widehat{\mathrm{G}}(k_n)$.

There exists a unique divisor d of n and a character $\chi_0 \in \widehat{\mathrm{G}}^*(k_d)$ such that $\chi = \chi_0 \circ \mathrm{N}_{k_n/k_d}$. The map sending χ to the equivalence class of χ_0 in $[\widehat{\mathrm{G}}]$ has image the subset of classes $[\eta]$ of primitive characters η with degree dividing n, and for any such class $[\eta]$, there are exactly $\deg([\eta])$ characters $\chi \in \widehat{\mathrm{G}}(k_n)$ mapping to $[\eta]$. Moreover, there are canonical isomorphisms

$$\mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) \simeq \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\eta),$$

with the actions of Fr_{k_n} corresponding to that of $\mathrm{Fr}^{n/d}_{k_d}$, so that

$$\mathrm{S}(\mathrm{M}, \chi) = \operatorname{Tr}(\mathrm{Fr}^{n/d}_{k_d} \mid \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\eta))$$

for all χ mapping to $[\eta]$. This implies the desired identity (5.3).

The second formula (5.2) for $\widehat{\mathrm{L}}(\mathrm{M},\mathrm{T})$ follows immediately from (5.1), since orthogonality of characters implies that the formula

$$\sum_{\chi\in\widehat{\mathrm{G}}(k_n)} \mathrm{S}(\mathrm{M},\chi) = |\mathrm{G}(k_n)| t_{\mathrm{M}}(e;k_n)$$

holds for all $n\geqslant 1$.

Using next the trace formula and the Riemann Hypothesis to compute $|\mathrm{G}(k_n)|$ as an alternating sum of $|k|$-Weil numbers, it follows that

$$|\mathrm{G}(k_n)| t_{\mathrm{M}}(e;k_n) = \sum_{i\in\mathrm{I}} \varepsilon_i\alpha_i^n$$

for some finite set I, some $\varepsilon_i\in\{-1,1\}$, and some $|k|$-Weil numbers α_i. The second assertion follows then from the usual power series expansion

$$\exp\Big(\sum_{n\geqslant 1}\frac{\alpha^n\mathrm{T}^n}{n}\Big) = \frac{1}{1-\alpha\mathrm{T}}.$$

The final assertion is clear either from the definition, or from the above, noting that $t_{\mathrm{M}_\chi}(e;k_n) = t_{\mathrm{M}}(e;k_n)$ for any $\chi\in\widehat{\mathrm{G}}(k)$ and $n\geqslant 1$. ☐

Remark 5.3. To illustrate the differences with L-functions, we note that if G is not unipotent, then the $\widehat{\mathrm{L}}$-function is very rarely a polynomial or the inverse of a polynomial, and does not satisfy in general any functional equation of the form

$$\widehat{\mathrm{L}}(\mathrm{M},\mathrm{T}) = (\text{simple quantities})\times\widehat{\mathrm{L}}(\mathrm{M}^\vee, q^\alpha\mathrm{T}^{-1}).$$

as is the case for the standard L-function of M (this is related to the remark of Boyarchenko and Drinfeld [**15**, §1.6, Example 1.8]).

To give a concrete example, take $\mathrm{G}=\mathbf{G}_m$. In this case, we deduce from (5.2) the formula

$$\widehat{\mathrm{L}}(\mathrm{M},\mathrm{T}) = \exp\Big(\sum_{n\geqslant 1}(|k|^n-1)\, t_{\mathrm{M}}(e;k_n)\frac{\mathrm{T}^n}{n}\Big) = \frac{\mathrm{L}(\mathrm{M}_e,|k|\mathrm{T})}{\mathrm{L}(\mathrm{M}_e,\mathrm{T})}$$

where M_e is the stalk of M at e (where $\mathrm{L}(\mathrm{M}_e,\mathrm{T})$ is the L-function of the stalk of M at e, viewed as a complex on $\{e\}$). If the L-function $\mathrm{L}(\mathrm{M}_e,\mathrm{T})$ is not constant, then there can never be cancellation in this quotient to obtain a polynomial or the inverse of a polynomial. If (say) we have

$$\mathrm{L}(\mathrm{M}_e,\mathrm{T}) = (1-\alpha\mathrm{T})(1-\alpha^{-1}\mathrm{T}),$$

then

$$\widehat{\mathrm{L}}(\mathrm{M},\mathrm{T}) = \frac{(1-|k|\alpha\mathrm{T})(1-|k|\alpha^{-1}\mathrm{T})}{(1-\alpha\mathrm{T})(1-\alpha^{-1}\mathrm{T})},$$

and this satisfies no simple functional relation.

We conclude with a result that will be useful in the next section when performing induction.

Proposition 5.4. *Let* G *be a semiabelian variety over* k. *Let* S *be a tac of* G_k *defined by a morphism* $\pi\colon \mathrm{G}\to\mathrm{G}'$ *over* k *and a character* $\chi_0\in\widehat{\mathrm{G}}(k)$, *and let* [S] *denote the classes in* $[\widehat{\mathrm{G}}]$ *of elements of* S. *Let* M *be an object of* $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$. *We then have*

$$\prod_{\chi\in[\mathrm{S}]}\det(1-\mathrm{T}^{\deg(\chi)}\,\mathrm{Fr}_{k_{\deg(\chi)}}\mid \mathrm{H}^*_c(\mathrm{G}_{\bar k},\mathrm{M}_\chi))^{-1} = \widehat{\mathrm{L}}(\mathrm{R}\pi_!\mathrm{M}_{\chi_0},\mathrm{T}).$$

Proof. We have $\chi \in [\mathrm{S}]$ if and only if $\chi = \chi_0 \cdot (\pi^*\eta)$ for some $\eta \in [\widehat{\mathrm{G}}']$, with $\deg(\chi) = \deg(\eta)$. By the projection formula, we have a canonical isomorphism

$$\mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) = \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_{\chi_0} \otimes \pi^*\mathscr{L}_\eta) \simeq \mathrm{H}^*_c(\mathrm{G}'_{\bar{k}}, \mathrm{R}\pi_!\mathrm{M}_{\chi_0} \otimes \mathscr{L}_\eta),$$

from which the identity

$$\det(1 - \mathrm{T}^{\deg(\chi)} \mathrm{Fr}_{k_{\deg(\chi)}} \mid \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi))^{-1} = \det(1 - \mathrm{T}^{\deg(\eta)} \mathrm{Fr}_{k_{\deg(\eta)}} \mid \mathrm{H}^*_c(\mathrm{G}'_{\bar{k}}, (\mathrm{R}\pi_!\mathrm{M}_{\chi_0})_\eta))^{-1}$$

follows for any $\chi \in [\mathrm{S}]$. ☐

5.2. Objects with finite arithmetic tannakian groups on abelian varieties

As a non-trivial application of $\widehat{\mathrm{L}}$-functions, we will show that they lead to a characterization of objects with finite arithmetic tannakian groups on abelian varieties. This is an analogue of a result of Katz (see [74, Th. 6.2], recalled in Theorem B.2, (1)) for $\mathbf{G}_m$, where in fact the $\widehat{\mathrm{L}}$-function appears implicitly (more precisely, where the logarithmic derivative $\mathrm{T}d\log\widehat{\mathrm{L}}(\mathrm{M},\mathrm{T})$ appears); similar results appear in a preprint of Weissauer [126].

More generally, inspired by the formulation used by Katz, we can prove a stronger statement.

Definition 5.5 (Quasi-unipotent object). Let G be a connected commutative algebraic group over k. An object M of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$ is said to be *quasi-unipotent* if it is generically unramified and if there exists an integer $m \geqslant 1$ such that for any unramified character $\chi \in \widehat{\mathrm{G}}$, the eigenvalues of Frobenius on $\mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ are roots of unity of order at most m.

Remark 5.6. (1) Any perverse sheaf M on G with $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$ finite is quasi-unipotent. Indeed, first M is generically unramified by Corollary 3.38. Let then m be the size of $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$. For any unramified character $\chi \in \widehat{\mathrm{G}}$, the Frobenius action on $\mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$ is "conjugate" to an element of $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$, so its eigenvalues are m-th roots of unity.

(2) If M is a quasi-unipotent perverse sheaf on G, then it follows from the definition that any object of $\langle \mathrm{M} \rangle$ is also quasi-unipotent.

(3) Let M be a quasi-unipotent object of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$. Let $g_0 \in \mathrm{G}(k)$. Then the translated object $\mathrm{M}' = [g \mapsto gg_0]^*\mathrm{M}$ is also quasi-unipotent. Indeed, since M' is canonically isomorphic to the convolution $\delta_{g_0} *_! \mathrm{M}$, we obtain for any $\chi \in \widehat{\mathrm{G}}$ a canonical isomorphism

$$\mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}'_\chi) \simeq \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, (\delta_{g_0^{-1}})_\chi) \otimes \mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi).$$

Noting that $\mathrm{H}^*_c(\mathrm{G}_{\bar{k}}, (\delta_{g_0^{-1}})_\chi) = \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, (\delta_{g_0^{-1}})_\chi)$, this shows already that χ is weakly-unramified for M if and only if it is for M'.

If χ is weakly-unramified for M', and belongs to $\widehat{\mathrm{G}}(k_n)$, then the Frobenius automorphism of k_n acts on $\mathrm{H}^0(\mathrm{G}_{\bar{k}}, (\delta_{g_0^{-1}})_\chi)$ by multiplication by $\chi(g_0^{-1})$, which is a root of unity of order bounded by the order of g_0 in $\mathrm{G}(k)$. Since M is quasi-unipotent, the eigenvalues of Frobenius on $\mathrm{H}^0(\mathrm{G}_{\bar{k}}, \mathrm{M}'_\chi)$ are roots of unity of order bounded independently of χ.

Theorem 5.7. *Let* A *be an abelian variety over* k*. Let* M *be an arithmetically semisimple perverse sheaf of weight zero in* $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{A})$ *which is non-zero. If* M *is quasi-unipotent, for instance if the group* $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$ *is finite, then* M *is punctual.*

Remark 5.8. As proved by Katz in the case of $\mathbf{G}_m$, one may expect that the conclusion of the theorem extends to objects with finite *geometric* tannakian group (see [74, Th. 6.4] or Theorem B.2, (2)). We do not know how to prove this in general (Katz's deduction of this fact from the analogue of Theorem 5.7 for $\mathbf{G}_m$ uses the classification of objects of tannakian rank 1, for instance, which we do not have in this setting). We will however prove a weaker statement in Section 8.7 which turns out to be sufficient for many applications, including those of Chapter 11.

Before giving the proof, we state two corollaries.

Corollary 5.9. *Let* M *be an arithmetically simple perverse sheaf of weight zero on an abelian variety* A *over* k *of dimension* $g \geqslant 1$*. Let* $\mathbf{G}$ *be the neutral component of* $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$*, and let* S *be the support of* M*. The restriction of the standard representation of* $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ *to* $\mathbf{G}$ *is irreducible unless there exists* $x \neq e$ *such that* $\mathrm{M} * \delta_x$ *is isomorphic to* M*. In particular, this holds unless there exists* $x \in \mathrm{A}$ *with* $x \neq e$ *such that* $x + \mathrm{S} = \mathrm{S}$*.*

Proof. Let P be an object of $\langle \mathrm{M} \rangle^{\mathrm{ari}}$ which is a faithful representation of the finite component group $\mathrm{C} = \mathbf{G}_{\mathrm{M}}^{\mathrm{ari}} / \mathbf{G}$. Its tannakian group is isomorphic to C, and hence the object P is punctual by Theorem 5.7. The points appearing in the decomposition of P generate a finite subgroup B of $\mathrm{A}(\bar{k})$, and each skyscraper sheaf for $x \in \mathrm{B}$ corresponds to a character χ_x of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ trivial on $\mathbf{G}$.

By a simple application of Frobenius reciprocity, a representation ϱ of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ restricts to an irreducible representation of $\mathbf{G}$ unless there exists $x \in \mathrm{C}$ such that $x \neq e$ and $\varrho \otimes \chi_x$ is isomorphic to ϱ. In terms of perverse sheaves on A, this condition (for the standard representation) means that $\mathrm{M} * \delta_x$ is isomorphic to M, which is the first assertion. Since it also implies that $\mathrm{S} + x = \mathrm{S}$, this concludes the proof. ☐

Corollary 5.10. *Let* A *be an abelian variety over* k*. Let* M *be a non-zero arithmetically semisimple perverse sheaf of weight zero in* $\mathbf{P}_{\mathrm{int}}^{\mathrm{ari}}(\mathrm{A})$*. If* M *is quasi-unipotent, for instance if the group* $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ *is finite, then* $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ *is a finite abelian group which is naturally isomorphic to the dual of the subgroup of* $\mathrm{A}(\bar{k})$ *generated by the support of* M*.*

Proof. By Theorem 5.7, the object M is punctual. If F denotes its support, we have an isomorphism

$$\mathrm{M} = \bigoplus_{x \in \mathrm{F}} \alpha_x^{\deg} \otimes \delta_x$$

for some unitary scalars α_x, and therefore a geometric isomorphism of M with the direct sum of the δ_x for $x \in \mathrm{F}$. Let H be the subgroup of $\mathrm{A}(\bar{k})$ generated by F, which is a finite abelian group and let $\widehat{\mathrm{H}} = \mathrm{Hom}(\mathrm{H}, \overline{\mathbf{Q}}_\ell^{\times})$ be the dual group of H. We obtain an additive functor from the finite-dimensional $\overline{\mathbf{Q}}_\ell$-representations of $\widehat{\mathrm{H}}$ to $\langle \mathrm{M} \rangle^{\mathrm{geo}}$ by associating to the character "evaluation at x" of $\widehat{\mathrm{H}}$ the object δ_x. Since $\delta_x * \delta_y \simeq \delta_{x+y}$, this is a tensor functor, and it gives an equivalence of categories. Hence $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ is isomorphic to $\widehat{\mathrm{H}}$. ☐

We will prove Theorem 5.7 in the next two sections. In fact, since this case is somewhat easier, we will begin by assuming that the abelian variety A is simple (which is in a reasonable sense the generic case) before handling the general situation. The reader may skip the first case to read directly the proof of the general result.

We first prove two lemmas that are used in both proofs.

Lemma 5.11. *Let* R *be a commutative ring with unit and* λ *a non-archimedean valuation on* R*. Assume that* R *is complete with the topology given by* λ*.*

Let $(\alpha_i)_{i \in \mathrm{I}}$ *be a family of elements of* R *such that* $|\alpha_i|_\lambda \leqslant 1$ *for all* $i \in \mathrm{I}$*, and let* $(d_i)_{i \in \mathrm{I}}$ *be a family of positive integers such that*

$$\lim_{\mathrm{I}} d_i = +\infty,$$

where the limit is along the filter of the complements of finite subsets of I*.*

The product

$$\prod_{i\in \mathrm{I}}(1-\alpha_i \mathrm{T}^{d_i})$$

converges and is non-zero for T *such that* $|\mathrm{T}|_\lambda < 1$.

PROOF. Let $\mathrm{J}\subset \mathrm{K}$ be finite subsets of I. Then for $|\mathrm{T}|_\lambda \leqslant 1$, we compute that

$$\begin{aligned}\Big|\prod_{i\in \mathrm{K}}(1-\alpha_i \mathrm{T}^{d_i})-\prod_{i\in \mathrm{J}}(1-\alpha_i \mathrm{T}^{d_i})\Big|_\lambda &= \Big|\prod_{i\in \mathrm{J}}(1-\alpha_i \mathrm{T}^{d_i})\Big(\prod_{i\in \mathrm{K}-\mathrm{J}}(1-\alpha_i \mathrm{T}^{d_i})-1\Big)\Big|_\lambda \\ &\leqslant \Big|\prod_{i\in \mathrm{K}-\mathrm{J}}(1-\alpha_i \mathrm{T}^{d_i})-1\Big|_\lambda = \Big|\sum_{\varnothing\neq \mathrm{L}\subset \mathrm{K}-\mathrm{J}}(-1)^{|\mathrm{L}|}\sigma_{\mathrm{L}}\mathrm{T}^{d_{\mathrm{L}}}\Big|_\lambda\end{aligned}$$

where

$$\sigma_{\mathrm{L}}=\prod_{i\in \mathrm{L}}\alpha_i, \qquad d_{\mathrm{L}}=\sum_{i\in \mathrm{L}} d_i.$$

We note that $|\sigma_{\mathrm{L}}|_\lambda\leqslant 1$ for all L. Moreover, since the lower-bound

$$d_{\mathrm{L}}\geqslant \min_{i\in \mathrm{I}-\mathrm{J}} d_i$$

holds, the assumption that $d_i\to +\infty$ implies that for any integer $\mathrm{N}\geqslant 1$, we can choose J so that

$$\Big|\sum_{\varnothing\neq \mathrm{L}\subset \mathrm{K}-\mathrm{J}}(-1)^{|\mathrm{L}|}\sigma_{\mathrm{L}}\mathrm{T}^{d_{\mathrm{L}}}\Big|_\lambda \leqslant |\mathrm{T}|_\lambda^{\mathrm{N}}$$

for any finite set K containing J. The absolute convergence of the product follows when $|\mathrm{T}|_\lambda<1$ using the Cauchy criterion. In particular, the product can only be zero if some term is zero, and this is not the case if $|\mathrm{T}|_\lambda<1$. ☐

The next lemma gives basic structural information on zeros and poles of $\widehat{\mathrm{L}}(\mathrm{M},\mathrm{T})$, refining the last part of Proposition 5.2 in the case of abelian varieties.

DEFINITION 5.12. Let $f\in \overline{\mathbf{Q}}_\ell(\mathrm{X})$ be a non-zero rational function, k a finite field and $r\in\mathbf{Z}$. We denote by $\mathrm{wt}_{k,r}(f)$ the rational function

$$\prod_{\alpha \text{ of } k\text{-weight } -r}(1-\alpha\mathrm{T})^{v_\alpha(f)}$$

where α runs over elements of $\overline{\mathbf{Q}}_\ell$ which are k-Weil numbers of weight $-r$, and v_α is the order of f at α.

In other words (note the minus sign), the rational function $\mathrm{wt}_{k,r}(f)$ is (up to leading terms) "the part of f with zeros and poles of weight r". Below, we will sometimes write wt_r when the finite field k is clear from context.

The definition implies that the identity

$$\mathrm{wt}_{k,r}(f_1f_2)=\mathrm{wt}_{k,r}(f_1)\,\mathrm{wt}_{k,r}(f_2)$$

holds for any rational functions f_1 and f_2.

PROPOSITION 5.13. *Let* M *be a complex on an abelian variety* A *over* k *of dimension* $g\geqslant 0$. *Assume that* M *is pure of weight zero and that* M_e *has weights in* $[a,b]$.

(1) *The poles (resp. zeros) of* $\widehat{\mathrm{L}}(\mathrm{M},\mathrm{T})$ *are* k*-Weil numbers. Their weights are of the form* $-w-i$ *for some even (resp. odd) integer* i *with* $0 \leqslant i \leqslant 2g$ *and some integer* w *with* $a \leqslant w \leqslant b$.

If there exists such a zero or pole then there exists an eigenvalue of weight w *on* M_e*, and the formula*

$$\mathrm{wt}_{k,-w}(\widehat{\mathrm{L}}(\mathrm{M},\mathrm{T})) = \mathrm{wt}_{k,-w}(\det(1-\mathrm{T}\,\mathrm{Fr}_k \mid \mathrm{M}_e))^{-1}$$

holds.

(2) *If* M *is an arithmetically simple perverse sheaf, and if* e *belongs to the open set of the support of* M *where* M *is lisse, then the poles (resp. zeros) of* $\widehat{\mathrm{L}}(\mathrm{M},\mathrm{T})$ *have* k*-weights equal to* $\dim \mathrm{Supp}(\mathrm{M}) - i$ *for some integers* i *with* $0 \leqslant i \leqslant 2g$ *such that*

$$\dim \mathrm{Supp}(\mathrm{M}) \equiv i \pmod 2,$$

and there are poles and zeros of all these possible weights.

Proof. (1) By Proposition 5.2, we have

$$\widehat{\mathrm{L}}(\mathrm{M},\mathrm{T}) = \exp\Big(\sum_{n\geqslant 1} |\mathrm{A}(k_n)| t_{\mathrm{M}}(e;k_n)\frac{\mathrm{T}^n}{n}\Big).$$

This expression, combined with the purity of M and the structure of the cohomology of A, shows that $\widehat{\mathrm{L}}(\mathrm{M},\mathrm{T})$ has:

(i) Poles of the form

$$\mathrm{T} = \frac{1}{\alpha\beta},$$

where α is an eigenvalue of Frobenius on the stalk of M at e, and β is an eigenvalue of Frobenius on $\mathrm{H}^i(\mathrm{A}_{\bar{k}}, \overline{\mathbf{Q}}_\ell)$ for some even integer i with $0 \leqslant i \leqslant 2g$. Since α is pure of some weight w where $a \leqslant w \leqslant b$, and β is of weight i, such a pole is a $|k|$-Weil number of weight $-w-i$.

(ii) Zeros of the form

$$\mathrm{T} = \frac{1}{\alpha\beta},$$

where α is an eigenvalue of Frobenius on the stalk of M at e, and β is an eigenvalue of Frobenius on $\mathrm{H}^i(\mathrm{A}_{\bar{k}}, \overline{\mathbf{Q}}_\ell)$ for some odd integer i with $1 \leqslant i \leqslant 2g-1$. As above, such a zero is a $|k|$-Weil number of weight $-w-i$ where $a \leqslant w \leqslant b$.

The precise formula for the parts of weight $-w$ follows from the above since $\beta = 1$ is the unique eigenvalue of weight 0 on $\mathrm{H}^*(\mathrm{A}_{\bar{k}}, \overline{\mathbf{Q}}_\ell)$.

(2) If M is an arithmetically simple perverse sheaf and e is a point where M is lisse, then the eigenvalues α above have weight $w = -\dim(\mathrm{supp}(\mathrm{M}))$, and there is at least one α since the stalk at e is non-zero. Thus the poles and zeros above have weight $\dim(\mathrm{supp}(\mathrm{M})) - i$. ☐

5.3. Perverse sheaves with finitely many ramified characters

In this section, we prove Theorem 5.7 in the case of an arithmetically semisimple perverse sheaf of weight zero in $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{A})$ which has the property that the set of ramified characters for M is finite. This applies in particular, for instance, if the abelian variety A is simple, since the set of ramified characters is a finite union of tacs of A (see Remark 3.28), and each tac is reduced to a single character if A is simple.

Let M be an arithmetically semisimple perverse sheaf of weight zero such that the set $\mathscr{S}$ of ramified characters for M is finite. We will prove the following:

Proposition 5.14. *Under the above assumptions, if* M *is quasi-unipotent and non-punctual, then it is negligible.*

In Theorem 5.7, we assume that M is quasi-unipotent and that $\mathrm{M}_{\mathrm{int}}$ is non-zero; comparing with the proposition, it implies that M must be punctual.

We now prove the proposition. After a finite extension of k, we may assume that each $\chi \in \mathscr{S}$ is in $\widehat{\mathrm{A}}(k)$.

One reduces using Lemma 1.28 to the case of M geometrically simple. We denote by S the support of M and by r its dimension; we have $r \geqslant 1$ since M is not punctual. We denote by U a smooth open dense subset of S such that M is lisse on U.

Let $n \geqslant 1$ and let $a \in \mathrm{A}(k_n)$. We denote $\mathrm{M}^{(a)} = [x \mapsto x + a]^*\mathrm{M}$, which is a simple perverse sheaf on A_{k_n}. The stalk of $\mathrm{M}^{(a)}$ at e is canonically isomorphic to the stalk M_a of M at a. We note that the set of ramified characters for $\mathrm{M}^{(a)}$ is also contained in $\mathscr{S}$, and that $\mathrm{M}^{(a)}$ is quasi-unipotent (see Remark 5.6, (3)).

We then write

$$\widehat{\mathrm{L}}(\mathrm{M}^{(a)}, \mathrm{T}) = \widehat{\mathrm{L}}_0(\mathrm{M}^{(a)}, \mathrm{T}) \prod_{\chi \in \mathscr{S}} \det(1 - \mathrm{T}\,\mathrm{Fr}_k \mid \mathrm{H}^*(\mathrm{A}_{\bar{k}}, \mathrm{M}_{\chi}^{(a)}))^{-1}$$

where

$$\widehat{\mathrm{L}}_0(\mathrm{M}^{(a)}, \mathrm{T}) = \prod_{\chi \notin \mathscr{S}} \det(1 - \mathrm{T}^{\deg(\chi)}\,\mathrm{Fr}_{k_{\deg(\chi)}} \mid \mathrm{H}^0(\mathrm{A}_{\bar{k}}, \mathrm{M}_{\chi}^{(a)}))^{-1}.$$

Note that $\widehat{\mathrm{L}}_0(\mathrm{M}^{(a)}, \mathrm{T})$ is a rational function since $\widehat{\mathrm{L}}(\mathrm{M}^{(a)}, \mathrm{T})$ is one (Proposition 5.2).

The quasi-unipotence property of $\mathrm{M}^{(a)}$ shows that the infinite product $\widehat{\mathrm{L}}_0(\mathrm{M}^{(a)}, \mathrm{T})$ can be viewed as a formal power series in $\mathscr{O}[[\mathrm{T}]]$ for some cyclotomic order $\mathscr{O}$. We can apply Lemma 5.11 to any non-archimedean place λ of $\mathscr{O}$, since the eigenvalues of Frobenius on $\mathrm{H}^0(\mathrm{A}_{\bar{k}}, \mathrm{M}_{\chi}^{(a)})$ are roots of unity of bounded order for all unramified characters χ. This implies that, for any non-archimedean place λ, the infinite product $\widehat{\mathrm{L}}_0(\mathrm{M}^{(a)}, \mathrm{T})$ converges in the disc defined by $|\mathrm{T}|_{\lambda} < 1$. Taking λ to correspond to places above the characteristic of k, this implies that the rational function $\widehat{\mathrm{L}}_0(\mathrm{M}^{(a)}, \mathrm{T})$ cannot have a zero or pole which is a $|k|$-Weil number of positive weight.

Suppose that $a \in (\mathrm{A} - \mathrm{S})(\bar{k})$. Then $\mathrm{M}_e^{(a)} = 0$. Hence we deduce that

$$1 = \prod_{\chi \in \mathscr{S}} \mathrm{wt}_r(\det(1 - \mathrm{T}\,\mathrm{Fr}_k \mid \mathrm{H}^*(\mathrm{A}_{\bar{k}}, \mathrm{M}_{\chi}^{(a)}))). \tag{5.4}$$

Thus, the Frobenius automorphism has no eigenvalue of weight $-r$ acting on any of the cohomology spaces $\mathrm{H}^i(\mathrm{A}_{\bar{k}}, \mathrm{M}_{\chi}^{(a)})$. By purity, this translates to the condition

$$\mathrm{H}^{-r}(\mathrm{A}_{\bar{k}}, \mathrm{M}_{\chi}^{(a)}) = 0$$

for all $\chi \in \mathscr{S}$.

On the other hand, suppose that $a \in \mathrm{U}(\bar{k})$. Then $\mathrm{M}_e^{(a)} = \mathrm{M}_a$ is pure of weight $-r$. From the above and Proposition 5.13, (2), we deduce that

$$\widehat{\mathrm{L}}(\mathrm{M}^{(a)}, \mathrm{T}) = \mathrm{wt}_r(\widehat{\mathrm{L}}(\mathrm{M}^{(a)}, \mathrm{T})) = \prod_{\chi \in \mathscr{S}} \mathrm{wt}_r(\det(1 - \mathrm{T}\,\mathrm{Fr}_k \mid \mathrm{H}^*(\mathrm{A}_{\bar{k}}, \mathrm{M}_{\chi}^{(a)}))), \tag{5.5}$$

and since the left-hand side is not 1, there exists (by purity again) at least one $\chi \in \mathscr{S}$ such that

$$\mathrm{H}^{-r}(\mathrm{A}_{\bar{k}}, \mathrm{M}_{\chi}^{(a)}) \neq 0.$$

If we combine these two statements, we conclude that $\mathrm{S} = \mathrm{A}$. Indeed, the spaces $\mathrm{H}^{-r}(\mathrm{A}_{\bar{k}}, \mathrm{M}_{\chi}^{(a)})$ are independent of $a \in \mathrm{A}(\bar{k})$ up to isomorphism. Hence, since there exists some $a_0 \in \mathrm{U}(\bar{k})$, if one of these spaces is non-zero, then no $a \in \mathrm{A}(\bar{k})$ can satisfy the condition required to have $a \notin \mathrm{S}(\bar{k})$.

Fixing again $a_0 \in \mathrm{U}(\bar{k})$, let $\chi \in \mathscr{S}$ be such that

$$\mathrm{H}^{-r}(\mathrm{A}_{\bar{k}}, \mathrm{M}_{\chi}^{(a_0)})$$

is non-zero. Since $\mathrm{M}^{(a_0)}$ is a simple perverse sheaf supported on $\mathrm{S} = \mathrm{A}$, and $r = \dim(\mathrm{S}) = \dim(\mathrm{A})$, it follows from Lemma A.15 that $\mathrm{M}_{\chi}^{(a_0)}$ is geometrically trivial. This implies that M is negligible.

5.4. The general case

In this section, we prove Theorem 5.7 in the general case. Thus let M be an arithmetically semisimple perverse sheaf of weight zero in $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{A})$, which we assume is quasi-unipotent and not punctual. We will show that M is negligible.

It suffices to treat the case of a simple perverse sheaf M (Lemma 1.28).

We denote by S the support of M and by r its dimension; we have $r \geqslant 1$ by our assumption that M is not punctual. Let U be an open dense subset of S contained in the smooth locus of S such that M is lisse on U.

Let $(\mathscr{S})_{i \in \mathrm{I}}$ be a finite family of tacs such that the set of ramified characters is contained in the union $\mathscr{S}$ of the $\mathscr{S}_i$. After a finite extension of k, we may assume that each $\mathscr{S}_i$ is defined by a quotient morphism $\pi_i \colon \mathrm{A} \to \mathrm{A}_i$ defined over k and a character $\chi_i \in \widehat{\mathrm{A}}(k)$.

For any subset J of I, we denote by $\mathscr{S}_{\mathrm{J}}$ the intersection of $\mathscr{S}_i$ for $i \in \mathrm{J}$; this is either empty or a tac of A, also defined over k (Lemma 1.25), in which case we denote by $\pi_{\mathrm{J}} \colon \mathrm{A} \to \mathrm{A}_{\mathrm{J}}$ and χ_{J} the corresponding quotient morphism and character; these are all defined over k. From Lemma 1.25, it follows also that $\ker(\pi_{\mathrm{J}})$ is the algebraic subgroup of A generated by the family of subgroups $(\ker(\pi_i))_{i \in \mathrm{I}}$. We write $[\mathscr{S}_{\mathrm{J}}]$ for the set of classes in $[\widehat{\mathrm{A}}]$ of characters in $\mathscr{S}_{\mathrm{J}}$.

Let $a \in \mathrm{A}(k)$. We denote $\mathrm{M}^{(a)} = [x \mapsto x+a]^*\mathrm{M}$, so that the stalk $\mathrm{M}_e^{(a)}$ is canonically isomorphic to the stalk M_a of M at a. The ramified characters for $\mathrm{M}^{(a)}$ are also contained in the tac $\mathscr{S}$, and the perverse sheaf $\mathrm{M}^{(a)}$ is quasi-unipotent (see Remark 5.6, (3)).

We define

$$\widehat{\mathrm{L}}_0(\mathrm{M}^{(a)}, \mathrm{T}) = \prod_{\chi \in [\widehat{\mathrm{A}}] - \mathscr{S}} \det(1 - \mathrm{T}^{\deg(\chi)} \operatorname{Fr}_{k_{\deg(\chi)}} \mid \mathrm{H}^0(\mathrm{A}_{\bar{k}}, \mathrm{M}_{\chi}^{(a)}))^{-1}.$$

By an application of inclusion–exclusion, we have

$$\widehat{\mathrm{L}}(\mathrm{M}^{(a)}, \mathrm{T}) = \widehat{\mathrm{L}}_0(\mathrm{M}^{(a)}, \mathrm{T}) \prod_{\substack{\varnothing \neq \mathrm{J} \subset \mathrm{I} \\ \mathscr{S}_{\mathrm{J}} \neq \varnothing}} \prod_{\chi \in [\mathscr{S}_{\mathrm{J}}]} \det(1 - \mathrm{T}^{\deg(\chi)} \operatorname{Fr}_{k_{\deg(\chi)}} \mid \mathrm{H}^*(\mathrm{A}_{\bar{k}}, \mathrm{M}_{\chi}^{(a)}))^{(-1)^{|\mathrm{J}|}}.$$

For any $\mathrm{J} \subset \mathrm{I}$ such that $\mathscr{S}_{\mathrm{J}}$ is not empty, we denote

$$\mathrm{Q}_{\mathrm{J}}^{(a)} = \mathrm{R}\pi_{\mathrm{J}*}\mathrm{M}_{\chi_{\mathrm{J}}}^{(a)}.$$

Proposition 5.4 implies the formula

$$\prod_{\chi\in[\mathscr{S}_\mathrm{J}]} \det(1-\mathrm{T}^{\deg(\chi)}\,\mathrm{Fr}_{k_{\deg(\chi)}} \mid \mathrm{H}^*(\mathrm{A}_{\bar{k}}, \mathrm{M}^{(a)}_{\chi}))^{-1} = \widehat{\mathrm{L}}(\mathrm{Q}^{(a)}_{\mathrm{J}}, \mathrm{T}), \tag{5.6}$$

so that we can rewrite the above expression as

$$\widehat{\mathrm{L}}(\mathrm{M}^{(a)}, \mathrm{T}) = \widehat{\mathrm{L}}_0(\mathrm{M}^{(a)}, \mathrm{T}) \prod_{\substack{\varnothing\neq \mathrm{J}\subset \mathrm{I}\\ \mathscr{S}_\mathrm{J}\neq\varnothing}} \widehat{\mathrm{L}}(\mathrm{Q}^{a}_{\mathrm{J}}, \mathrm{T})^{(-1)^{|\mathrm{J}|+1}}. \tag{5.7}$$

By Proposition 5.2 (2), this shows in particular that $\widehat{\mathrm{L}}_0(\mathrm{M}^{(a)}, \mathrm{T})$ is a rational function. The quasi-unipotence property of $\mathrm{M}^{(a)}$ shows that the infinite product $\widehat{\mathrm{L}}_0(\mathrm{M}^{(a)}, \mathrm{T})$ can be viewed as a formal power series in $\mathscr{O}[[\mathrm{T}]]$ for some cyclotomic order $\mathscr{O}$. We can apply Lemma 5.11 to any non-archimedean place λ of $\mathscr{O}$, since the eigenvalues of Frobenius on $\mathrm{H}^0(\mathrm{A}_{\bar{k}}, \mathrm{M}^{(a)}_{\chi})$ are roots of unity of bounded order for all unramified characters χ. This implies that, for any non-archimedean place λ, the infinite product $\widehat{\mathrm{L}}_0(\mathrm{M}^{(a)}, \mathrm{T})$ converges in the disc defined by $|\mathrm{T}|_\lambda < 1$. Taking λ to correspond to places above the characteristic of k, this implies that the rational function $\widehat{\mathrm{L}}_0(\mathrm{M}^{(a)}, \mathrm{T})$ cannot have a zero or pole which is a $|k|$-Weil number of positive weight.

Since $r \geqslant 1$, the formula (5.7) therefore implies the formula

$$\mathrm{wt}_r(\widehat{\mathrm{L}}(\mathrm{M}^{(a)}, \mathrm{T})) = \prod_{\substack{\varnothing\neq \mathrm{J}\subset \mathrm{I}\\ \mathscr{S}_\mathrm{J}\neq\varnothing}} \mathrm{wt}_r\left(\widehat{\mathrm{L}}(\mathrm{Q}^{a}_{\mathrm{J}}, \mathrm{T})^{(-1)^{|\mathrm{J}|+1}}\right). \tag{5.8}$$

Let $\mathrm{J}\subset\mathrm{I}$. By proper base change, we have a canonical isomorphism

$$\mathrm{Q}^{(a)}_{\mathrm{J},e} = (\mathrm{R}\pi_{\mathrm{J}*}\mathrm{M}^{(a)}_{\chi_\mathrm{J}})_e \simeq \mathrm{H}^*(\ker(\pi_\mathrm{J})_{\bar{k}}, \mathrm{M}^{(a)}_{\chi_\mathrm{J}}).$$

Since $\mathrm{M}^{(a)}$, hence also $\mathrm{M}^{(a)}_{\chi_\mathrm{J}}$, is a perverse sheaf, the complex $\mathrm{M}^{(a)}_{\chi_\mathrm{J}}$ is concentrated in degrees between $-r$ and r. Its support is $\mathrm{S}-a$, and consequently, the cohomology group

$$\mathrm{H}^i(\ker(\pi_\mathrm{J})_{\bar{k}}, \mathrm{M}^{(a)}_{\chi_\mathrm{J}}) = \mathrm{H}^i((\ker(\pi_\mathrm{J})\cap(\mathrm{S}-a))_{\bar{k}}, \mathrm{M}^{(a)}_{\chi_\mathrm{J}})$$

vanishes unless $0\leqslant i+r\leqslant 2\dim(\ker(\pi_\mathrm{J})\cap(\mathrm{S}-a))$. Since $\mathrm{M}^{(a)}$ has weight 0, this space has weight i when it is non-zero. Using the formula

$$\widehat{\mathrm{L}}(\mathrm{Q}^{(a)}_{\mathrm{J}}, \mathrm{T}) = \exp\Bigl(\sum_{n\geqslant 1} |\mathrm{A}_\mathrm{J}(k_n)| t_{\mathrm{Q}^{(a)}_\mathrm{J}}(e;k_n)\frac{\mathrm{T}^n}{n}\Bigr)$$

of Proposition 5.2, this means that

$$\begin{aligned}
\mathrm{wt}_r(\widehat{\mathrm{L}}(\mathrm{Q}^{(a)}_{\mathrm{J}}, \mathrm{T})) &= \det(1-\mathrm{T}\,\mathrm{Fr}_k \mid \mathrm{H}^{-r}((\ker(\pi_\mathrm{J})\cap(\mathrm{S}-a))_{\bar{k}}, \mathrm{M}^{(a)}_{\chi_\mathrm{J}}))^{-1}\\
&= \det(1-\mathrm{T}\,\mathrm{Fr}_k \mid \mathrm{H}^{-r}(((a+\ker(\pi_\mathrm{J}))\cap\mathrm{S})_{\bar{k}}, \mathrm{M}_{\chi_\mathrm{J}}))^{-1}\\
&= \det(1-\mathrm{T}\,\mathrm{Fr}_k \mid (\mathrm{R}^{-r}\pi_{\mathrm{J}*}\mathrm{M}_{\chi_\mathrm{J}})_{\pi_\mathrm{J}(a)})^{-1} = \det(1-\mathrm{T}\,\mathrm{Fr}_k \mid (\pi_\mathrm{J}^*\mathrm{R}^{-r}\pi_{\mathrm{J}*}\mathrm{M}_{\chi_\mathrm{J}})_a)^{-1}.
\end{aligned}$$

Let $\mathrm{X}=\mathrm{S}-\mathrm{U}$, so that $\mathrm{A}-\mathrm{X}=\mathrm{U}\cup(\mathrm{A}-\mathrm{S})$. If $a\in(\mathrm{A}-\mathrm{X})(k)$, the left-hand side of (5.8) is the part of weight $-r$ of

$$\widehat{\mathrm{L}}(\mathrm{M}^{(a)}, \mathrm{T}) = \exp\Bigl(\sum_{n\geqslant 1} |\mathrm{A}(k_n)| t_\mathrm{M}(a;k_n)\frac{\mathrm{T}^n}{n}\Bigr).$$

Since M_a is $|k|$-pure of weight $-r$ (either because $a \in \mathrm{U}(k)$, so that M is lisse and of weight $-r$ at a, or because $a \in (\mathrm{A} - \mathrm{S})(k)$, so that M_a is zero, hence pure of any weight), we deduce that the equality

$$\det(1 - \mathrm{T}\,\mathrm{Fr}_k \mid \mathrm{M}_a) = \prod_{\substack{\varnothing \neq \mathrm{J} \subset \mathrm{I} \\ \mathscr{S}_\mathrm{J} \neq \varnothing}} \det(1 - \mathrm{T}\,\mathrm{Fr}_k \mid (\pi_\mathrm{J}^* \mathrm{R}^{-r} \pi_{\mathrm{J}*} \mathrm{M}_{\chi_\mathrm{J}})_a)^{(-1)^{|\mathrm{J}|+1}} \tag{5.9}$$

holds. In particular, this gives the equality

$$t_\mathrm{M}(a;k) = \sum_{\substack{\varnothing \neq \mathrm{J} \subset \mathrm{I} \\ \mathscr{S}_\mathrm{J} \neq \varnothing}} (-1)^{|\mathrm{J}|+1} t_{\pi_\mathrm{J}^* \mathrm{R}^{-r} \pi_{\mathrm{J}*} \mathrm{M}_{\chi_\mathrm{J}}}(a;k)$$

of values of trace functions for $a \in (\mathrm{A} - \mathrm{X})(k)$.

Let $n \geqslant 1$. Applying this argument to the base change of M to k_n, we see that the formula

$$t_\mathrm{M}(a;k_n) = \sum_{\substack{\varnothing \neq \mathrm{J} \subset \mathrm{I} \\ \mathscr{S}_\mathrm{J} \neq \varnothing}} (-1)^{|\mathrm{J}|+1} t_{\pi_\mathrm{J}^* \mathrm{R}^{-r} \pi_{\mathrm{J}*} \mathrm{M}_{\chi_\mathrm{J}}}(a;k_n)$$

holds for $a \in (\mathrm{A} - \mathrm{X})(k_n)$. By the injectivity of trace functions (see [98, Th. 1.1.2]), this means that we have an equality

$$\mathrm{M} = \sum_{\substack{\varnothing \neq \mathrm{J} \subset \mathrm{I} \\ \mathscr{S}_\mathrm{J} \neq \varnothing}} (-1)^{|\mathrm{J}|+1} \pi_\mathrm{J}^* \mathrm{R}^{-r} \pi_{\mathrm{J}*} \mathrm{M}_{\chi_\mathrm{J}} \tag{5.10}$$

in the Grothendieck group $\mathrm{K}(\mathrm{A} - \mathrm{X})$.

If $\mathrm{U} = \mathrm{S}$ (e.g. if M is the extension by zero of a lisse sheaf of weight 0 placed in degree $-r$ on a smooth closed subvariety S, which will be the case in the applications of Theorem 5.7 in Chapter 11), then X is empty, so this equality holds in $\mathrm{K}(\mathrm{A})$. The right-hand side is a linear combination of negligible objects (see Example 3.5) so we deduce that M is negligible by taking the Euler–Poincaré characteristic (see Corollary 3.22).

We now consider the general case. Let j be the open immersion of $\mathrm{A} - \mathrm{X}$ in A. Recall that the classes of simple perverse sheaves form a basis of the $\mathbf{Z}$-module $\mathrm{K}(\mathrm{A} - \mathrm{X})$ (see Proposition A.22). Thus, the equality (5.10) implies that there exists some J such that the simple perverse sheaf $j^*\mathrm{M}$ appears in the decomposition in simple perverse sheaves of the class of $j^*\mathrm{N}$ in $\mathrm{K}(\mathrm{A} - \mathrm{X})$, where

$$\mathrm{N} = \pi_\mathrm{J}^* \mathrm{R}^{-r} \pi_{\mathrm{J}*} \mathrm{M}_{\chi_\mathrm{J}}.$$

Furthermore, this means that there exists $i \in \mathbf{Z}$ such that $j^*\mathrm{M}$ occurs in the decomposition of the perverse sheaf ${}^{\mathfrak{p}}\mathscr{H}^i(j^*\mathrm{N})$, since

$$j^*\mathrm{N} = \sum_{i \in \mathbf{Z}} (-1)^i \, {}^{\mathfrak{p}}\mathscr{H}^i(j^*\mathrm{N})$$

in $\mathrm{K}(\mathrm{A} - \mathrm{X})$.

The functor j^* is t-exact (since j is smooth of relative dimension 0) so that there exists a canonical isomorphism ${}^{\mathfrak{p}}\mathscr{H}^i(j^*\mathrm{N}) \to j^* {}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{N})$. Since $j^*\mathrm{M}$ and $j^* {}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{N})$ are pure, hence geometrically semisimple, this implies the existence of an injective morphism

$$f \colon j^*\mathrm{M} \to j^* {}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{N})$$

of perverse sheaves. Applying the functor $j_{!*}$, which preserves injectivity (e.g., by [70, §. 2.17]) and satisfies $j_{!*} \circ j^* = \mathrm{Id}$ on perverse sheaves, we deduce that there exists an injective morphism $j_{!*}f \colon \mathrm{M} \to {}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{N})$. Since N is negligible, so is ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{N})$, and hence also M.

Remark 5.15. A similar argument leads to a proof of the following fact: if M is a negligible arithmetically simple perverse sheaf of weight zero in $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{A})$, and if $\mathscr{N}(\mathrm{M})$ is contained in a finite union of tacs of A, then there exists a morphism $\pi\colon \mathrm{A} \to \mathrm{B}$ of abelian varieties with $\dim(\ker(\pi)) \geqslant 1$, a character $\chi \in \widehat{\mathrm{A}}$ and an object N of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{B})$ such that M is geometrically isomorphic to $(\pi^*\mathrm{N})_\chi$.

This fact is equivalent (for pure perverse sheaves of weight 0) to the characterization of negligible objects by Weissauer [127, Th. 3], since it is known that the assumption on $\mathscr{N}(\mathrm{M})$ is always true (Corollary 3.23). However, the proof that this is so relies on the generic vanishing theorem (Theorem 2.16), which appeals to this result of Weissauer, so this remark does not provide a different proof of this characterization.

We sketch the argument nevertheless for the sake of illustration. It is relatively elementary that it suffices to prove that the isomorphism class of M is invariant under translation by a non-trivial abelian subvariety (this is [127, Lemma 6]), and we will establish this fact.

To simplify matters, we assume that S = U in the notation of the previous proof. Since M is negligible, it is quasi-unipotent; arguing as in the previous proof, we obtain a finite decomposition

$$\mathrm{M} = \sum_{i\in \mathrm{I}} n_i \pi_i^* \mathrm{M}_i$$

in K(A) for some morphisms $\pi_i\colon \mathrm{A} \to \mathrm{A}_i$ with $\dim(\ker(\pi_i)) \geqslant 1$, some objects $\mathrm{M}_i \in \mathrm{D}^{\mathrm{b}}_c(\mathrm{A}_i)$ and some non-zero $n_i \in \mathbf{Z}$.

Since the classes of simple perverse sheaves form a basis of the $\mathbf{Z}$-module K(A), there exists some $i \in \mathrm{I}$ such that

$$\pi_i^* \mathrm{M}_i = m\mathrm{M} + \sum_{j\in \mathrm{J}} m_j \mathrm{M}_{i,j}$$

in K(A) for some non-zero integers m and m_j and some simple perverse sheaves $\mathrm{M}_{i,j}$ not isomorphic to M.

The isomorphism class of the complex $\pi_i^*\mathrm{M}_i$ is invariant under translation by elements of $\ker(\pi_i)$, and a fortiori by the abelian subvariety $\mathrm{A}' = \ker(\pi_i)^\circ$. We claim that this implies that the same property holds for M and the other constituents $\mathrm{M}_{i,j}$. Indeed, the $\bar{k}$-valued points of A' act on the finite set of isomorphism classes of the simple perverse sheaves $(\mathrm{M}, \mathrm{M}_{i,j})$, and thus the stabilizer of any of them is a finite index subgroup. Since it is also an algebraic subgroup, it is equal to $\mathrm{A}'(\bar{k})$, and the assertion follows. Thus the isomorphism class of M is invariant under translation by the non-trivial abelian variety $\ker(\pi_i)^\circ$, as desired.

CHAPTER 6

Stratification and generic Fourier invertibility

As usual, k is a finite field, with an algebraic closure $\bar{k}$ and finite extensions k_n of k in $\bar{k}$ of degree n. We fix a prime ℓ distinct from the characteristic of k.

Let G be a connected commutative algebraic group over k, with dimension d. Given an object M of $\mathrm{D}_c^{\mathrm{b}}(\mathrm{G})$, an integer $n \geqslant 1$ and a character $\chi \in \widehat{\mathrm{G}}(k_n)$, we set

$$\mathrm{S}(\mathrm{M}, \chi) = \sum_{x \in \mathrm{G}(k_n)} \chi(x) t_{\mathrm{M}}(x; k_n).$$

6.1. Stratification for exponential sums

The results of this section are straightforward consequences of Theorem 2.3 and Deligne's Riemann Hypothesis. We spell them out since some of them are likely to be useful for applications to analytic number theory.

Theorem 6.1. *Let* M *be an object of* $\mathrm{D}_c^{\mathrm{b}}(\mathrm{G})$. *Assume that* M *is semiperverse and mixed of weights* $\leqslant 0$. *There exist subsets* $\mathscr{S}_d \subset \cdots \subset \mathscr{S}_0 = \widehat{\mathrm{G}}$ *such that*

(1) *For* $0 \leqslant i \leqslant d$, *the estimate*

$$|\mathscr{S}_i(k_n)| \ll |k|^{n(d-i)}$$

holds for $n \geqslant 1$.

(2) *The set* $\mathscr{S}_d$ *is empty if* M *belongs to the category* $\mathbf{P}_{\mathrm{int}}^{\mathrm{ari}}(\mathrm{G})$.

(3) *For any* $n \geqslant 1$, *any integer* i *with* $1 \leqslant i \leqslant d$ *and any* $\chi \in \widehat{\mathrm{G}}(k_n) - \mathscr{S}_i(k_n)$, *the estimate*

$$\sum_{x \in \mathrm{G}(k_n)} \chi(x) t_{\mathrm{M}}(x; k_n) \ll c_u(\mathrm{M}) |k|^{n(i-1)/2},$$

holds, where the implied constant is independent of M.

(4) *If* G *is either a torus or an abelian variety, then* $\mathscr{S}_i$ *is a finite union of tacs of* G *of dimension* $\leqslant d - i$.

(5) *If* G *is a unipotent group, then* $\mathscr{S}_i$ *is the set of closed points of a closed subvariety of dimension* $\leqslant d - i$ *of the Serre dual* $\mathrm{G}^{\vee}$.

Proof. For $1 \leqslant i \leqslant d$, let $\mathscr{S}_i$ be the set of characters such that there exists some $l \geqslant i$ with $\mathrm{H}_c^l(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi) \neq 0$.

If $\chi \in \widehat{\mathrm{G}} - \mathscr{S}_i$, then we deduce from the trace formula and the Riemann Hypothesis of Deligne, combined with Lemma 1.3, that the estimate

$$\sum_{x \in \mathrm{G}(k_n)} \chi(x) t_{\mathrm{M}}(x; k_n) \ll c_u(\mathrm{M}) |k|^{n(i-1)/2}$$

holds for $n \geqslant 1$, which is (3). We will check that these sets also satisfy conditions (1) and (2).

Fix an integer i with $1 \leqslant i \leqslant d$. For any l and j, we have the perverse spectral sequence

$$\mathrm{H}^j_c(\mathrm{G}_{\bar{k}}, {}^{\mathfrak{p}}\mathscr{H}^{l-j}(\mathrm{M}_\chi)) \implies \mathrm{H}^j_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_\chi)$$

(see (A.3)) so that the condition $\chi \in \mathscr{S}_i$ implies that

$$\mathrm{H}^j_c(\mathrm{G}_{\bar{k}}, {}^{\mathfrak{p}}\mathscr{H}^{l-j}(\mathrm{M}_\chi)) \neq 0$$

for some $l \geqslant i$. Since M is semiperverse, so is M_χ, which means that this condition implies $j \geqslant l \geqslant i$.

Thus, if we denote by $(\mathscr{S}_{j,i})_{0 \leqslant i \leqslant d}$ the sets provided by the Stratified Vanishing Theorem 2.3 applied to ${}^{\mathfrak{p}}\mathscr{H}^j(\mathrm{M})$, we have shown that

$$\mathscr{S}_i \subset \bigcup_{i \leqslant j \leqslant d} \bigcup_{i \leqslant l \leqslant j} \mathscr{S}_{l-j,j}.$$

The set $\mathscr{S}_{l-j,j}$ has character codimension at least j, so that $\mathscr{S}_i$ has the same property, establishing (1). Point (2) follows from the fact that

$$\mathrm{H}^d_c(\mathrm{G}_{\bar{k}}, \mathrm{N}_\chi) = 0$$

for a geometrically simple perverse sheaf N which is not geometrically isomorphic to $\mathscr{L}_{\chi^{-1}}$.

Points (4) and (5) follow from the strengthened versions of the Stratified Vanishing Theorem for tori, abelian varieties and unipotent groups, which are stated in Theorem 2.3, (4) or Proposition 2.7. ☐

Remark 6.2. The following elementary estimate can also sometimes be useful. Fix a locally-closed immersion $u\colon \mathrm{G} \to \mathbf{P}^m$ for some integer $m \geqslant 1$. Let M be an ℓ-adic perverse sheaf on G that is pure of weight zero. Then by orthogonality of characters, we derive that the formula

$$\frac{1}{|\mathrm{G}(k_n)|}\Big|\sum_{\chi \in \widehat{\mathrm{G}}(k_n)} t_{\mathrm{M}}(x; k_n)\chi(x)\Big|^2 = \sum_{x \in \mathrm{G}(k_n)} |t_{\mathrm{M}}(x; k_n)|^2$$

holds for $n \geqslant 1$. By the Riemann Hypothesis, it follows that the estimate

$$\frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi \in \widehat{\mathrm{G}}(k_n)} \Big|\sum_{\chi \in \widehat{\mathrm{G}}(k_n)} t_{\mathrm{M}}(x; k_n)\chi(x)\Big|^2 \ll c_u(\mathrm{M})$$

holds for $n \geqslant 1$ (see Theorem 1.10). Fix then a sequence $\mathrm{T} = (\mathrm{T}_n)$ of positive real numbers, and let $\mathscr{X}_{\mathrm{T}} \subset \widehat{\mathrm{G}}$ be the set such that $\chi \in \mathscr{X}_{\mathrm{T}}(k_n)$ if and only if

$$\Big|\sum_{x \in \mathrm{G}(k_n)} t_{\mathrm{M}}(x; k_n)\chi(x)\Big| > \mathrm{T}_n.$$

Then we find by positivity that

$$|\mathscr{X}_{\mathrm{T}}(k_n)| \ll c_u(\mathrm{M})|\mathrm{G}(k_n)|\mathrm{T}_n^{-2}.$$

Corollary 6.3. *Let k be a finite field, and let* G *be a connected commutative algebraic group of dimension d over k. Let ℓ be a prime distinct from the characteristic of k and let* M *be an ℓ-adic perverse sheaf on* G *which is pure of weight zero.*

For any generic subsets $\mathscr{X}$ and $\mathscr{Y}$ of $\widehat{\mathrm{G}}$, the estimate

$$\frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi \in (\mathscr{X} - \mathscr{Y})(k_n)} \Big|\sum_{x \in k_n} \chi(x) t_{\mathrm{M}}(x; k_n)\Big| \ll \frac{1}{|k|^{n/2}}$$

holds for all $n \geqslant 1$.

Proof. We may assume that $\mathscr{X} = \widehat{\mathrm{G}}$. Let $(\mathscr{S}_i)_{0\leqslant i\leqslant d}$ be sets of characters as in Theorem 6.1. We have

$$\frac{1}{|\mathrm{G}(k_n)|}\sum_{\chi\in(\widehat{\mathrm{G}}-\mathscr{Y})(k_n)}\Bigl|\sum_{x\in k_n}\chi(x)t_{\mathrm{M}}(x;k_n)\Bigr| = \mathrm{S}_0+\cdots+\mathrm{S}_{d-1},$$

where for each integer i with $0\leqslant i<d$, we put

$$\mathrm{S}_i = \frac{1}{|\mathrm{G}(k_n)|}\sum_{\chi\in(\mathscr{S}_i-(\mathscr{S}_{i+1}\cup\mathscr{Y}))(k_n)}\Bigl|\sum_{x\in k_n}\chi(x)t_{\mathrm{M}}(x;k_n)\Bigr|.$$

For $i=0$, the exponential sums in the inner sum are $\ll 1$, and since $\mathscr{Y}$ is generic, the set $\mathscr{S}_0-(\mathscr{S}_1\cup\mathscr{Y})$ has character codimension at least 1, so that

$$\mathrm{S}_0 \ll |k|^{n(-d+(d-1))} = |k|^{-n}.$$

For $i\leqslant i<d$, we have $\mathscr{S}_i-(\mathscr{S}_{i+1}\cup\mathscr{Y})\subset\mathscr{S}_i-\mathscr{S}_{i+1}$, so that by Theorem 6.1, (1) (for the size of $\mathscr{S}_i$) and (3) (estimating the exponential sums for $\chi\notin\mathscr{S}_{i+1}$), we obtain

$$\mathrm{S}_i \ll |k|^{-nd+n((d-i)+i/2)} = |k|^{-ni/2}.$$

$\square$

The next corollary states, intuitively, that for the purpose of computing the arithmetic Fourier transform of a semiperverse complex (mixed of weights $\leqslant 0$), the contribution of any closed (suitably "transverse") subvariety is negligible.

Corollary 6.4. *Let k be a finite field, and let G be a connected commutative algebraic group of dimension d over k. Let ℓ be a prime distinct from the characteristic of k and let M be an object of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$. Assume that M is semiperverse and mixed of weights $\leqslant 0$.*

Let $\mathrm{X}\subset\mathrm{G}$ be a closed subvariety of G and let $i\colon\mathrm{X}\to\mathrm{G}$ be the corresponding closed immersion.

Let $m\geqslant 0$ be an integer such that for each $j\in\mathbf{Z}$, the estimate

$$\dim(\mathrm{X}\cap\mathrm{Supp}(\mathscr{H}^j(\mathrm{M})))\leqslant\dim\mathrm{Supp}(\mathscr{H}^j(\mathrm{M}))-m$$

holds.

There exists a generic subset $\mathscr{X}$ of $\widehat{\mathrm{G}}$ such that the estimate

$$\sum_{x\in\mathrm{X}(k_n)}\chi(x)t_{\mathrm{M}}(x;k_n)\ll\frac{c_u(\mathrm{M})}{|k_n|^{m/2}}$$

holds for all $n\geqslant 1$ and all $\chi\in\mathscr{X}(k_n)$.

Alternatively, we have

$$\mathrm{S}(\mathrm{M},\chi)=\sum_{x\in(\mathrm{G}-\mathrm{X})(k_n)}\chi(x)t_{\mathrm{M}}(x;k_n)+\mathrm{O}(|k_n|^{-m/2}),$$

which explains the interpretation that X does not contribute "systematically" to the arithmetic Fourier transform.

Proof. The assumption implies that the complex $\mathrm{N}=i_!i^*\mathrm{M}[-m](-m/2)$ is semiperverse on G, since M is semiperverse and, for any $j\in\mathbf{Z}$, the support of $\mathscr{H}^j(\mathrm{N})$ is $\mathrm{X}\cap\mathrm{Supp}(\mathscr{H}^{j-m}(\mathrm{M}))$ so that

$$\begin{aligned}\dim(\mathrm{Supp}(\mathscr{H}^j(\mathrm{N})))=\dim(\mathrm{X}\cap\mathrm{Supp}(\mathscr{H}^{j-m}(\mathrm{M})))&\leqslant\dim(\mathrm{Supp}(\mathscr{H}^{j-m}(\mathrm{M})))-m\\&\leqslant-(j-m)-m=-j.\end{aligned}$$

Moreover, the complex N has weights $\leqslant 0$. Thus we may apply Theorem 6.1 to N. Let $\mathscr{S}_0, \ldots, \mathscr{S}_d$ be the corresponding sets of characters, and let $\mathscr{X} = \widehat{\mathrm{G}} - \mathscr{S}_1$. This is a generic subset of $\widehat{\mathrm{G}}$, and for $n \geqslant 1$ and $\chi \in \mathscr{X}(k_n)$, we have

$$(-1)^m |k_n|^{m/2} \sum_{x \in \mathrm{X}(k_n)} \chi(x) t_{\mathrm{M}}(x; k_n) = \sum_{x \in \mathrm{G}(k_n)} \chi(x) t_{\mathrm{N}}(x; k_n) \ll c_u(\mathrm{M}),$$

hence the result. □

Example 6.5. Let $\mathscr{F}$ be a non-zero lisse sheaf on G, pure of weight 0, and let $\mathrm{M} = \mathscr{F}[d](d/2)$. We then have $\mathrm{Supp}(\mathscr{H}^j(\mathrm{M})) = \varnothing$ except when $j = -d$, in which case the support of $\mathscr{H}^{-d}(\mathrm{M})$ is G. We can therefore apply the corollary to any closed subvariety X of G of codimension at least m. In particular, for any closed subvariety $\mathrm{X} \neq \mathrm{G}$, hence of codimension at least 1, there exists a generic set of characters $\mathscr{X}$ for which the estimate

$$\sum_{x \in \mathrm{X}(k_n)} \chi(x) t_{\mathrm{M}}(x; k_n) \ll \frac{c_u(\mathrm{M})}{|k_n|^{1/2}}$$

holds for $n \geqslant 1$ and $\chi \in \mathscr{X}(k_n)$.

A uniform version of the stratified vanishing theorem, as in Remark 2.4, would be especially welcome for stratification estimates, as it would lead to strong potential applications in analytic number theory (compare with the results of Fouvry and Katz [39] based on stratification for the additive Fourier transform). We state a conditional result of this kind for emphasis.

Theorem 6.6. *Let ℓ be a prime number. Let $\mathrm{N} \geqslant 1$ be an integer and let (G, u) be a quasi-projective commutative group scheme over $\mathbf{Z}[1/\ell\mathrm{N}]$.*

Assume that, for all primes $p \nmid \ell\mathrm{N}$, the fiber G_p of G over $\mathbf{F}_p$ is a connected commutative algebraic group such that Theorem 2.3 *holds uniformly with respect to the complexity $c_{u_p}(\mathrm{M})$ where u_p is the locally-closed immersion of G_p deduced from u, i.e., such that for a perverse sheaf* M *on G_p, the sets $\mathscr{S}_i$ in loc. cit. satisfy*

$$|\mathscr{S}_i(k_n)| \ll |k|^{n(d-i)}$$

where the implied constant depends only on $c_{u_p}(\mathrm{M})$.

Let $(\mathrm{M}_p)_{p \nmid \mathrm{N}\ell}$ be a sequence of arithmetically semisimple sheaves on G_p, pure of weight zero, such that $c_u(\mathrm{M}_p) \ll 1$ for all p.

For each prime p, there exist subsets $\mathscr{S}_d(\mathbf{F}_p) \subset \cdots \subset \mathscr{S}_0(\mathbf{F}_p) = \widehat{\mathrm{G}}_p(\mathbf{F}_p)$ such that

(1) *For $0 \leqslant i \leqslant d$ and p prime, we have*

$$|\mathscr{S}_i(\mathbf{F}_p)| \ll p^{d-i}.$$

(2) *The set $\mathscr{S}_d(\mathbf{F}_p)$ is empty if M_p belongs to the category $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{G}_p)$.*

(3) *For any prime p, any integer i with $0 \leqslant i \leqslant d$ and any $\chi \in \widehat{\mathrm{G}}_p(\mathbf{F}_p) - \mathscr{S}_i(\mathbf{F}_p)$, we have*

$$\sum_{x \in \mathrm{G}(\mathbf{F}_p)} \chi(x) t_p(x) \ll p^{(i-1)/2},$$

where t_p is the trace function of M_p over $\mathbf{F}_p$.

(4) *If* G *is a torus or an abelian variety, then the sets $\mathscr{S}_i(\mathbf{F}_p)$ are contained in the union of a bounded number of tacs of $\mathrm{G}_{\mathbf{F}_p}$ of dimension $\leqslant d - i$.*

Remark 6.7. For $\mathrm{G} = \mathbf{G}_a^d$, results of this kind are unconditional; see for instance [39, Th. 1.1, Th. 3.1] (note that there the sets $\mathscr{S}_i$ are points of subschemes defined over $\mathbf{Z}$, which we cannot hope in the general situation where M_p is allowed to vary with p).

In the case of $\mathbf{G}_m^d$ (which is currently conditional), this would give (for instance) stratification and generic square-root cancellation for sums of the type

$$\sum_{x_1,\ldots,x_d\in\mathbf{F}_p^\times} \chi_1(x_1)\cdots\chi_d(x_d) e\Big(\frac{f(x_1,\ldots,x_d)}{p}\Big),$$

where $f \in \mathbf{Z}[\mathrm{X}_1,\ldots,\mathrm{X}_d]$ is a polynomial and $\chi_1,\ldots,\chi_d$ are Dirichlet characters modulo p, together with an a priori algebraic description of the sets of characters where the sum has size $\asymp p^{i/2}$.

Over finite fields, we can still derive some applications, such as the following proposition, similar to [39, Cor. 1.4] (although the vertical direction means that equidistribution is only in the finite set $(1/p\mathbf{Z})/\mathbf{Z} \subset \mathbf{R}/\mathbf{Z}$, or equivalently modulo p, as we phrase it.)

Proposition 6.8. *Let p be the characteristic of k. Let $d \geqslant 1$ and $r \leqslant d$ be integers and let $f = (f_i)\colon \mathbf{G}_m^d \to \mathbf{A}^r$ be a morphism whose image is not contained in an affine hyperplane. For a sequence (w_n) such that $0 \leqslant w_n < |k|^n - 1$ and $w_n/(|k_n|^{1/2}\log|k_n|) \to +\infty$, and for an arbitrary generator y_n of $k_n^\times$, the family of residue classes*

$$\mathrm{Tr}_{k_n/\mathbf{F}_p}(f(y_n^{v_1},\ldots,y_n^{v_d}))\,(\mathrm{mod}\,p),\qquad 0\leqslant v_i\leqslant w_n \text{ for all } i$$

become uniformly distributed in $(\mathbf{Z}/p\mathbf{Z})^r$.

Proof. Let $\mathrm{G} = \mathbf{G}_m^d$ and $q = |k|$; for $n \geqslant 1$, denote by ψ_n the additive character $x \mapsto e(\mathrm{Tr}_{k_n/\mathbf{F}_p}(x)/p)$ of k_n. Using the generator y_n, we can identify the group $\mathrm{G}(k_n) = (k_n^\times)^d$ with the group $(\mathbf{Z}/(q^n-1)\mathbf{Z})^d$ and we also identify $\widehat{\mathrm{G}}(k_n)$ with $(\mathbf{Z}/(q^n-1)\mathbf{Z})^d$, the element $\beta \in (\mathbf{Z}/(q^n-1)\mathbf{Z})^d$ corresponding to the character χ such that

$$\chi(y_n^{v_1},\ldots,y_n^{v_d}) = e\Big(\frac{1}{q^n-1}(\beta_1 v_1+\cdots+\beta_d v_d)\Big).$$

By the Weyl Criterion, we need to prove that

$$\lim_{n\to+\infty}\frac{1}{w_n^d}\sum_{0\leqslant v_i\leqslant w_n}\psi_n\Big(\sum_{i=1}^d h_i f_i(y_n^{v_1},\ldots,y_n^{v_d})\Big) = 0$$

for any $h \in (\mathbf{Z}/p\mathbf{Z})^r - \{0\}$. Detecting the interval $0 \leqslant v \leqslant w_n$ by Fourier expansion, we have to study the limit of

$$\frac{1}{w_n^d}\sum_{\chi\in\widehat{\mathrm{G}}(k_n)}\widehat{\alpha}_n(\chi)\sum_{x\in\mathrm{G}(k_n)}\chi(x)\psi_n\Big(\sum_{i=1}^d h_i f_i(x)\Big)$$

where

$$\widehat{\alpha}_n(\chi) = \frac{1}{(q^n-1)^d}\sum_{0\leqslant v_i\leqslant w_n}\overline{\chi(y_n^{v_1},\ldots,y_n^{v_d})}.$$

Define $g_h\colon \mathrm{G}\to\mathbf{A}^1$ by

$$g_h(x) = \sum_{i=1}^d h_i f_i(x).$$

We can write

$$\frac{1}{w_n^d}\sum_{\chi\in\widehat{\mathrm{G}}(k_n)}\widehat{\alpha}_n(\chi)\sum_{x\in\mathrm{G}(k_n)}\chi(x)\psi_n\Big(\sum_{i=1}^r h_i f_i(x)\Big) = \frac{1}{w_n^d}\sum_{\chi\in\widehat{\mathrm{G}}(k_n)}\widehat{\alpha}_n(\chi) q^{nd/2}\mathrm{S}(\mathrm{M},\chi)$$

for the complex $\mathrm{M} = g_h^*\mathscr{L}_{\psi_1}[d](d/2)$ on G, which is a simple perverse sheaf, pure of weight 0, on G.

We apply Theorem 6.1 to M. Let $(\mathscr{S}_i)$ be the subsets described there. We have $\mathscr{S}_d = \varnothing$ because the image of f is not contained in an affine hyperplane, which implies that g_h is non-constant, and hence M is non-trivial, from which the fact that it does not coincide with $\mathscr{L}_\chi$ for any character $\chi \in \widehat{\mathrm{G}}$ follows. Moreover, we also know that each $\mathscr{S}_i$ is a finite union of tacs of G of dimension $\leqslant d - i$.

The contribution of all $\chi \in (\mathscr{S}_0 - \mathscr{S}_1)(k_n)$ to the previous sum satisfies the bound

$$\frac{1}{w_n^d} \sum_{\chi \in (\widehat{\mathrm{G}} - \mathscr{S}_1)(k_n)} \widehat{\alpha}_n(\chi) q^{nd/2} \mathrm{S}(\mathrm{M}, \chi) \ll \frac{q^{nd/2}}{w_n^d} \sum_{\chi \in \widehat{\mathrm{G}}(k_n)} |\widehat{\alpha}_n(\chi)|.$$

It is well-known that the bound

$$\sum_{\chi \in \widehat{\mathrm{G}}(k_n)} |\widehat{\alpha}_n(\chi)| \ll (\log q)^d \tag{6.1}$$

holds for all $n \geqslant 1$ (see Remark 6.10 below), where the implied constant depends on d, so that

$$\frac{1}{w_n^d} \sum_{\chi \in (\widehat{\mathrm{G}} - \mathscr{S}_1)(k_n)} \widehat{\alpha}_n(\chi) q^{nd/2} \mathrm{S}(\mathrm{M}, \chi) \ll \Big(\frac{q^{n/2} \log(q)}{w_n}\Big)^d,$$

which converges to 0 as $n \to +\infty$ by assumption.

We now handle the remaining terms. Let $1 \leqslant j \leqslant d - 2$. By Theorem 6.1, the estimate

$$\frac{1}{w_n^d} \sum_{\chi \in (\mathscr{S}_j - \mathscr{S}_{j+1})(k_n)} \widehat{\alpha}_n(\chi) q^{nd/2} \mathrm{S}(\mathrm{M}, \chi) \ll \frac{q^{n(d+j)/2}}{w_n^d} \sum_{\chi \in \mathscr{S}_j(k_n)} |\widehat{\alpha}_n(\chi)|$$

holsd for all $n \geqslant 1$. From Lemma 6.9 below and the fact that $\mathscr{S}_j$ is a finite union of tacs of codimension at least j, we deduce that the estimate

$$\sum_{\chi \in \mathscr{S}_j(k_n)} |\widehat{\alpha}_n(\chi)| \ll \Big(\frac{w_n}{q^n}\Big)^j (\log q)^d$$

holds for $n \geqslant 1$. It follows that

$$\frac{1}{w_n^d} \sum_{\chi \in (\mathscr{S}_j - \mathscr{S}_{j+1})(k_n)} \widehat{\alpha}_n(\chi) q^{nd/2} \mathrm{S}(\mathrm{M}, \chi) \ll \frac{q^{n(d+j)/2}}{w_n^d} \Big(\frac{w_n}{q^n}\Big)^j (\log q)^d = \Big(\frac{q^{n/2}}{w_n}\Big)^{d-j} (\log q)^d.$$

The conclusion follows. ∎

Lemma 6.9. *With notation as above, for any tac $\mathscr{S}$ of $\mathbf{G}_m^d$ of dimension $d - j < d$, we have*

$$\sum_{\chi \in \mathscr{S}(k_n)} |\widehat{\alpha}_n(\chi)| \ll \Big(\frac{w_n}{q^n}\Big)^j (\log q)^d.$$

Proof. *Mutatis mutandis*, this is very close to [39, Lemma 9.5], in the (simpler) case where the variety $\mathscr{V}$ of loc. cit. is an affine hyperplane (but with the primes p replaced by the sequence $q^n - 1$). Indeed, let $f \colon \mathbf{G}_m^d \to \mathbf{G}_m^{d-j}$ and $\chi_0 = (\chi_{0,1}, \ldots, \chi_{0,d})$ be the morphism of tori and the character χ_0 defining $\mathscr{S}$. There exists a matrix $m = (m_{k,l})$ of size $d \times (d - j)$ with integral coefficients of rank j such that a character $\eta = (\eta_1, \ldots, \eta_d)$ of $\mathbf{G}_m^d$ belongs to $\mathscr{S}(k_n)$ if and only if

$$\prod_{l=1}^{d} \eta_l^{m_{k,l}} = \chi_{0,k}^{-1}$$

for $1 \leqslant k \leqslant d-j$. When we identify $\widehat{\mathbf{G}}_m^d(k_n)$ with $(\mathbf{Z}/(q^n-1)\mathbf{Z})^d$, this means that $\mathscr{S}(k_n)$ is identified with the set of solutions $(\xi_1,\dots,\xi_d)$ in $(\mathbf{Z}/(q^n-1))^d$ of the linear equation

$$\sum_{l=1}^{d} m_{k,l}\xi_l = y_k$$

for some $y_k \in (\mathbf{Z}/(q^n-1)\mathbf{Z})$. $\square$

Remark 6.10. We recall the proof of (6.1). We denote $\mathrm{N} = q^n - 1$ so that $\widehat{\mathrm{G}}(k_n)$ is isomorphic to $(\mathbf{Z}/\mathrm{N}\mathbf{Z})^d$. The sum to estimate is

$$\begin{aligned}\frac{1}{\mathrm{N}^d}\sum_{\xi\in(\mathbf{Z}/\mathrm{N}\mathbf{Z})^d}\Big|\sum_{0\leqslant v_i\leqslant w_n} e\Big(\frac{\xi_1v_1+\cdots+\xi_dv_d}{\mathrm{N}}\Big)\Big| &= \frac{1}{\mathrm{N}^d}\sum_{\xi\in(\mathbf{Z}/\mathrm{N}\mathbf{Z})^d}\prod_{i=1}^{d}\Big|\sum_{0\leqslant v\leqslant w_n} e\Big(\frac{\xi_i v}{\mathrm{N}}\Big)\Big| \\ &= \frac{1}{\mathrm{N}^d}\prod_{i=1}^{d}\sum_{\xi\in(\mathbf{Z}/\mathrm{N}\mathbf{Z})}\Big|\sum_{0\leqslant v\leqslant w_n} e\Big(\frac{\xi v}{\mathrm{N}}\Big)\Big|,\end{aligned}$$

which shows that it is enough to handle the case $d=1$. In this case, one uses the bound

$$\Big|\sum_{0\leqslant v\leqslant w_n} e\Big(\frac{\xi v}{\mathrm{N}}\Big)\Big| \leqslant \min\Big(w_n+1, \frac{e((w_n+1)\xi/\mathrm{N})-1}{e(\xi/\mathrm{N})-1}\Big) \leqslant \min\Big(w_n+1, \frac{1}{2\|\xi/\mathrm{N}\|}\Big)$$

where $\|\xi/\mathrm{N}\|$ is the distance to the nearest integer of ξ/N (the sum is a finite geometric sum, and in the last step, we used the lower-bound $|\sin(x)| \geqslant 2\|x\|$, valid for $x \in \mathbf{R}$). We then sum over the range $0 \leqslant \xi \leqslant \mathrm{N}-1$; for $\xi = 0$, the bound is $\leqslant (w_n+1) \leqslant \mathrm{N}$, and for $1 \leqslant \xi \leqslant \mathrm{N}-1$, we have

$$\|\xi/\mathrm{N}\| \geqslant \min\Big(\frac{\xi}{\mathrm{N}}, \frac{\mathrm{N}-\xi}{\mathrm{N}}\Big),$$

hence

$$\frac{1}{\mathrm{N}}\sum_{\xi\in(\mathbf{Z}/\mathrm{N}\mathbf{Z})}\Big|\sum_{0\leqslant v\leqslant w_n} e\Big(\frac{\xi v}{\mathrm{N}}\Big)\Big| \leqslant 1 + \sum_{\xi=1}^{(\mathrm{N}-1)/2}\frac{1}{\xi} \ll \log \mathrm{N},$$

as claimed.

6.2. Generic Fourier invertibility

For two semisimple perverse sheaves M and N, Proposition 1.20 implies that if the arithmetic Fourier transforms of M and N coincide, in the sense that $\mathrm{S}(\mathrm{M},\chi) = \mathrm{S}(\mathrm{N},\chi)$ for any $\chi \in \widehat{\mathrm{G}}$, then the trace functions of M and N coincide over k_n for all $n \geqslant 1$, which implies that M and N are isomorphic (Proposition A.22; see also [103, Prop. 4.2.3] for tori).

The stratified vanishing theorem allows us to prove a statement of "generic Fourier invertibility" for pure perverse sheaves, which relaxes the condition of equality of *all* sums $\mathrm{S}(\mathrm{M},\chi)$ and $\mathrm{S}(\mathrm{N},\chi)$ to a condition for a *generic* set of characters.

Theorem 6.11 (Generic Fourier invertibility). *Let* G *be a connected commutative algebraic group over* k. *Let* M *and* N *be arithmetically semisimple* ℓ*-adic perverse sheaves on* G *which are pure of weight zero.*

The perverse sheaves $\mathrm{M}_{\mathrm{int}}$ *and* $\mathrm{N}_{\mathrm{int}}$ *are arithmetically isomorphic if and only if there exists a generic set* $\mathscr{X} \subset \widehat{\mathrm{G}}$ *such that* $\mathrm{S}(\mathrm{M},\chi) = \mathrm{S}(\mathrm{N},\chi)$ *for all* $\chi \in \mathscr{X}$.

Proof. If $\mathrm{M}_{\mathrm{int}}$ is isomorphic to $\mathrm{N}_{\mathrm{int}}$, then the sums $\mathrm{S}(\mathrm{M},\chi)$ and $\mathrm{S}(\mathrm{N},\chi)$ coincide for a generic set of characters because $\mathrm{S}(\mathrm{P},\chi)$ vanishes generically for a negligible object P.

To prove the converse, we may assume that $\mathrm{M}=\mathrm{M}_{\mathrm{int}}$ and $\mathrm{N}=\mathrm{N}_{\mathrm{int}}$, i.e., that M and N are objects of $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{G})$. We then argue by induction on the sum m of the lengths of M and N.

If $m=0$, then the perverse sheaves M and N are both zero.

Suppose now that $m\geqslant 1$ and that the statement holds for all pairs $(\mathrm{M}_1,\mathrm{N}_1)$ of perverse sheaves in $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{G})$ such that the sum of the lengths of M_1 and N_1 is $\leqslant m-1$. One at least of the perverse sheaves M and N is non-zero, and (up to exchanging M and N) we may assume that M is non-zero. Let $(\mathrm{Q}_i)_{i\in\mathrm{I}}$ be the simple components (without multiplicity) of the perverse sheaf $\mathrm{M}\oplus\mathrm{N}$, and for $i\in\mathrm{I}$, let $\mu_{\mathrm{M}}(i)$ (resp. $\mu_{\mathrm{N}}(i)$) be the multiplicity of Q_i in $\mathrm{M}_{\mathrm{int}}$ (resp. $\mathrm{N}_{\mathrm{int}}$).

Let $\mathscr{Y}$ be the set of Frobenius-unramified characters for the perverse sheaf

$$(\mathrm{M}*_{\mathrm{int}}\mathrm{N}^{\vee})\oplus(\mathrm{M}*_{\mathrm{int}}\mathrm{M}^{\vee}),$$

viewed as an object of $\langle\mathrm{M}\oplus\mathrm{N}\rangle^{\mathrm{ari}}$.

For any integer $n\geqslant 1$, we consider the sum

$$\mathrm{T}_n=\frac{1}{|\mathrm{G}(k_n)|}\sum_{\chi\in\mathscr{Y}(k_n)}\mathrm{S}(\mathrm{M}*_{\mathrm{int}}\mathrm{N}^{\vee},\chi).$$

Applying Corollary 4.6 after decomposing M and N in terms of the simple perverse sheaves Q_i, we obtain the formula

$$\lim_{\mathrm{N}\to+\infty}\frac{1}{\mathrm{N}}\sum_{n\leqslant\mathrm{N}}\mathrm{T}_n=\sum_{i\in\mathrm{I}}\mu_{\mathrm{M}}(i)\mu_{\mathrm{N}}(i).$$

On the other hand, we can write

$$\mathrm{T}_n=\frac{1}{|\mathrm{G}(k_n)|}\sum_{\chi\in\mathscr{Y}(k_n)}\mathrm{S}(\mathrm{M}*_{\mathrm{int}}\mathrm{M}^{\vee},\chi)+\frac{1}{|\mathrm{G}(k_n)|}\sum_{\chi\in\mathscr{Y}(k_n)}\Big(\mathrm{S}(\mathrm{M}*_{\mathrm{int}}\mathrm{N}^{\vee},\chi)-\mathrm{S}(\mathrm{M}*_{\mathrm{int}}\mathrm{M}^{\vee},\chi)\Big)$$

for any $n\geqslant 1$. For χ in the generic set $\mathscr{X}\cap\mathscr{Y}$, the assumption implies that

$$\mathrm{S}(\mathrm{M}*_{\mathrm{int}}\mathrm{N}^{\vee},\chi)=\mathrm{S}(\mathrm{M}*_{\mathrm{int}}\mathrm{M}^{\vee},\chi).$$

Thus, using Corollary 6.3, the assumption implies that the bound

$$\frac{1}{|\mathrm{G}(k_n)|}\sum_{\chi\in\mathscr{Y}(k_n)}\Big(\mathrm{S}(\mathrm{M}*_{\mathrm{int}}\mathrm{N}^{\vee},\chi)-\mathrm{S}(\mathrm{M}*_{\mathrm{int}}\mathrm{M}^{\vee},\chi)\Big)\ll|k_n|^{-1/2}$$

holds for $n\geqslant 1$. Applying Corollary 4.6 once more and comparing with the previous computation, we deduce that

$$\sum_{i\in\mathrm{I}}\mu_{\mathrm{M}}(i)\mu_{\mathrm{N}}(i)=\sum_{i\in\mathrm{I}}\mu_{\mathrm{M}}(i)^2.$$

The right-hand side is $\geqslant 1$ since M is non-zero. Hence, there exists i such that $\mu_{\mathrm{M}}(i)\mu_{\mathrm{N}}(i)\geqslant 1$, which means that Q_i appears with positive multiplicity in both M and N. Removing one occurrence of Q_i from M and N, we obtain perverse sheaves M_1 and N_1 in $\mathbf{P}^{\mathrm{ari}}_{\mathrm{int}}(\mathrm{G})$ for which we can apply the induction hypothesis, so that M_1 is isomorphic to N_1, and adding the simple perverse sheaf Q_i to both sides, we deduce that M is isomorphic to N. ∎

Remark 6.12. In the case of tori, this theorem can be compared with a conditional result of Loeser [103, Prop. 4.2.5].

CHAPTER 7

Independence of ℓ

We consider in this section a connected commutative algebraic group G over a finite field k. Let p be the characteristic of k. Since we will vary the prime $\ell \neq p$, we will indicate it in the notation. For an object M of $\mathrm{D}_c^{\mathrm{b}}(\mathrm{G}, \overline{\mathbf{Q}}_\ell)$, we will now denote by $t_{\mathrm{M}}(x; k_n)$ the $\overline{\mathbf{Q}}_\ell$-valued trace function of M, and we will also specify explicitly the isomorphisms ι used to define their complex-valued analogues. In particular, we write $\widehat{\mathrm{G}}^{(\ell)}$ for the set of ℓ-adic characters.

We recall (see, e.g, [48, Def. 1.2] with $\mathrm{E} = \mathbf{C}$) that if A is a set of pairs (ℓ, ι) consisting of a prime number ℓ different from the characteristic of k and an isomorphism $\iota\colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$, a family $(\mathrm{M}_\alpha)_{\alpha \in \mathrm{A}}$ of objects of $\mathbf{Perv}(\mathrm{G}, \overline{\mathbf{Q}}_\ell)$ is said to be a *compatible system* if for any $n \geqslant 1$ and $x \in \mathrm{G}(k_n)$, the complex numbers $\iota(t_{\mathrm{M}_\alpha}(x; k_n))$ are independent of $\alpha = (\ell, \iota) \in \mathrm{A}$. This is equivalent to asking that the eigenvalues of Frobenius for the stalk of M_α at x are independent of α.

The question we wish to address is the following:

Question. Suppose that we have a compatible system $(\mathrm{M}_\alpha)_{\alpha \in \mathrm{A}}$; to what extent are the arithmetic and geometric tannakian groups of M_α independent of α?

We note that the analogue question for the monodromy groups of a compatible system of lisse sheaves on an algebraic variety X over k (especially a curve) has been considered in depth by, among others, Serre [120, p. 1–21], Larsen–Pink [97] and Chin [20]. Using Deligne's Fourier transform, this gives corresponding answers to our question in the case of the group $\mathbf{G}_a$. We note also that the deepest results (such as that of Chin) depend on the global Langlands correspondance over functions fields.

In this section, we take a first step in addressing the question. We will only compare two objects, so for the remainder of this section, we let (ℓ_1, ι_1) and (ℓ_2, ι_2) be pairs of primes and isomorphisms $\iota_j\colon \overline{\mathbf{Q}}_{\ell_j} \to \mathbf{C}$. For $j = 1, 2$, we fix an ℓ_j-adic arithmetically semisimple perverse sheaf M_j on G which is pure of ι_j-weight zero. We assume that M_1 and M_2 are compatible, that is we assume that the system with $\mathrm{A} = \{(\ell_1, \iota_1), (\ell_2, \iota_2)\}$ is compatible.

Lemma 7.1. *The following properties hold*:

1. *For any $n \geqslant 1$, the map $\eta\colon \chi \mapsto \iota_2^{-1} \circ \iota_1 \circ \chi$ is a bijection from $\widehat{\mathrm{G}}^{(\ell_1)}(k_n)$ to $\widehat{\mathrm{G}}^{(\ell_2)}(k_n)$ such that $\iota_1(\mathrm{S}(\mathrm{M}_1, \chi)) = \iota_2(\mathrm{S}(\mathrm{M}_2, \eta(\chi)))$ for all χ.*
2. *For any $\chi \in \widehat{\mathrm{G}}^{(\ell_1)}$, the objects $(\mathrm{M}_1)_\chi$ and $(\mathrm{M}_2)_{\eta(\chi)}$ are compatible.*
3. *The set of weakly unramified characters $\chi \in \widehat{\mathrm{G}}^{(\ell_1)}$ for M_1 such that $\eta(\chi)$ is weakly unramified for M_2 is generic.*

Proof. This boils down to the computation

$$\begin{aligned}\iota_1(t_{(\mathrm{M}_1)_\chi}(x; k_n)) &= \iota_1(t_{\mathrm{M}_1}(x; k_n)\chi(\mathrm{N}_{k_n/k}(x))) \\ &= \iota_1(t_{\mathrm{M}_1}(x; k_n))\iota_1(\chi(\mathrm{N}_{k_n/k}(x))) \\ &= \iota_2(t_{\mathrm{M}_2}(x; k_n))\iota_2(\eta(\chi))(\mathrm{N}_{k_n/k}(x)) = \iota_2(t_{(\mathrm{M}_2)_{\eta(\chi)}}(x; k_n))\end{aligned}$$

for any $n \geqslant 1$ and $x \in \mathrm{G}(k_n)$, which follows from the definitions, and the fact that $\mathscr{X}_w(\mathrm{M}_1)$ and $\eta^{-1}(\mathscr{X}_w(\mathrm{M}_2))$ are both generic, and hence so is their intersection in $\widehat{\mathrm{G}}^{(\ell_1)}$. ☐

LEMMA 7.2. *The tannakian dimensions of* M_1 *and* M_2 *coincide.*

PROOF. By Proposition 3.16, the tannakian dimension of M_1 is equal to the Euler–Poincaré characteristic of $(\mathrm{M}_1)_\chi$ and the tannakian dimension of M_2 is equal to the Euler–Poincaré characteristic of $(\mathrm{M}_2)_{\eta(\chi)}$, provided the character χ is weakly-unramified as well as $\eta(\chi)$. The set of such characters is generic and in particular not empty, and the result then follows from the fact that, since $(\mathrm{M}_1)_\chi$ and $(\mathrm{M}_2)_{\eta(\chi)}$ are compatible, they have the same Euler–Poincaré characteristic (see, e.g., [115, Lemma 6.38]). ☐

We denote from now on by r the common tannakian dimension of M_1 and M_2. We further by K_j a maximal compact subgroup of $\iota_j(\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_1})(\mathbf{C})$, and by μ_j the probability Haar measure on K_j. We define the measures $\nu_{cp,j}$ on $\mathbf{U}_r(\mathbf{C})^\sharp$ and the measure ν_j on $\mathbf{C}$ as in Theorem 4.4 and 4.8, respectively (the latter is the Sato–Tate measure of M_j).

The basic information we have is the following consequence of equidistribution.

PROPOSITION 7.3. *With notation and assumptions as above, we have* $\nu_{cp,1} = \nu_{cp,2}$ *and* $\nu_1 = \nu_2$.

PROOF. It suffices to prove the equality $\nu_{cp,1} = \nu_{cp,2}$, and this is essentially because the measures $\nu_{cp,j}$ are both determined by equidistribution of "the same data".

To be precise, we first note that by the Peter–Weyl Theorem, it is enough to prove that

$$\int \mathrm{Tr}(\varrho(g))d\nu_{cp,1}(g) = \int \mathrm{Tr}(\varrho(g))d\nu_{cp,2}(g)$$

for all finite-dimensional representations ϱ of $\mathbf{U}_r(\mathbf{C})$. By Theorem 4.4, applied to the bounded test function $f = \mathrm{Tr}(\varrho)$, the equality

$$\int \mathrm{Tr}(\varrho(g))d\nu_{cp,j}(g) = \lim_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}} \sum_{n\leqslant \mathrm{N}} \frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi\in\mathscr{X}_j(k_n)} \mathrm{Tr}(\varrho(\Theta_{\mathrm{M}_j,k_n}(\chi)))$$

holds, where $\mathscr{X}_j$ is the set of characters in $\widehat{\mathrm{G}}^{(\ell_j)}$ which are weakly unramified for M_j.

The compatibility assumption implies that $\varrho(\mathrm{M}_1)$ and $\varrho(\mathrm{M}_2)$ are also compatible (since the character $\mathrm{Tr}(\varrho)$ of ϱ is an integral symmetric function of the eigenvalues of its argument), and hence by Lemma 7.1 applied to these two perverse sheaves, we have

$$\mathrm{Tr}(\varrho(\Theta_{\mathrm{M}_1,k_n}(\chi))) = \mathrm{Tr}(\varrho(\Theta_{\mathrm{M}_2,k_n}(\eta(\chi)))) \tag{7.1}$$

if $\chi \in \mathscr{X}_1(k_n)$ is such that $\eta(\chi) \in \mathscr{X}_2(k_n)$. Therefore, by the previous lemma, the difference

$$\int \mathrm{Tr}(\varrho(g))d\nu_{cp,1}(g) - \int \mathrm{Tr}(\varrho(g))d\nu_{cp,2}(g)$$

is equal to

$$\lim_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}} \sum_{n\leqslant \mathrm{N}} \frac{1}{|\mathrm{G}(k_n)|}\Bigl(\sum_{\chi\in\mathscr{Y}_1(k_n)} \mathrm{Tr}(\varrho(\Theta_{\mathrm{M}_1,k_n}(\chi))) - \sum_{\chi\in\mathscr{Y}_2(k_n)} \mathrm{Tr}(\varrho(\Theta_{\mathrm{M}_2,k_n}(\chi)))\Bigr),$$

where $\mathscr{Y}_1$ (resp. $\mathscr{Y}_2$) is the set of $\chi \in \mathscr{X}_1$ such that $\eta(\chi) \notin \mathscr{X}_2$ (resp. the set of $\chi \in \mathscr{X}_2$ such that $\eta^{-1}(\chi) \notin \mathscr{X}_1$).

Both of the sets $\mathscr{Y}_1$ and $\mathscr{Y}_2$ have positive character codimension, and hence we deduce that

$$\int \mathrm{Tr}(\varrho(g))d\nu_{cp,1}(g) - \int \mathrm{Tr}(\varrho(g))d\nu_{cp,2}(g) = 0,$$

which implies the theorem. ∎

The equality of the characteristic polynomial measure or of the Sato–Tate measures of objects in a compatible system can provide a considerable amount of information. In ideal cases, this equality may be enough to imply that $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_1}$ and $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_2}$ are isomorphic. This does happen, but it is far from being always the case.

EXAMPLE 7.4. Let H be a finite group and $\mathrm{H} \subset \mathbf{U}_{|\mathrm{H}|}(\mathbf{C})$ be its regular representation. Then the Sato–Tate measure is

$$\Bigl(1 - \frac{1}{|\mathrm{H}|}\Bigr)\delta_0 + \frac{1}{|\mathrm{H}|}\delta_{|\mathrm{H}|},$$

where δ_z denotes a Dirac mass at a point $z \in \mathbf{C}$. Thus the Sato–Tate measure only determines the order of H in that case.

For characteristic polynomials, Sutherland [123, Remark 1.5] gives examples of non-isomorphic transitive finite permutation groups with the same distributions of characteristic polynomials. We refer to Sutherland's survey [123] for more examples and discussion of Sato–Tate measures in a more traditional context.

COROLLARY 7.5. *We continue with the notation and assumptions above.*

(1) *The reductive ranks of the reductive groups $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_1}$ and $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_2}$ are the same.*
(2) *Tthe group $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_1}$ is finite if and only if the group $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_2}$ is finite, and in this case, both groups have the same order.*

PROOF. (1) The reductive rank of $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_j}$ is the dimension of the space of characteristic polynomials of $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_j}$ (see [120, p. 17] for this fact), hence is equal to the dimension of the support of the measure $\nu_{cp,j}$ on $\mathbf{U}_r(\mathbf{C})^\sharp$. The result therefore follows from the proposition.

(2) This holds because the group $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_j}$ is finite if and only if the measure ν_j is a finite sum of Dirac masses (for the "if" direction, one can use the same result of Serre as in (1)), and if that is true, then the size of $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_j}$ is determined by

$$|\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_j}| = \frac{1}{\nu_j(\{r\})}.$$

∎

REMARK 7.6. In the next chapter, we will also see results which imply that M_1 has tannakian group containing $\mathbf{SL}_r$ if and only M_2 has the same property, and some related statements, following from Larsen's Alternative (see Theorem 8.5 or Proposition 8.17 below).

The next result goes a little bit beyond the equality of Sato–Tate measures, under the assumption that the objects are generically unramified.

PROPOSITION 7.7. *We continue with the notation and assumptions above. If M_1 and M_2 are generically unramified, then the exponent of the groups of connected components of $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_1}$ and $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_2}$ are equal.*

The key ingredient of the proof is a version of the "zero-one law" used by Serre [120, Th., p. 18] for the analogue statement for compatible systems of Galois representations over $\mathbf{Q}$.

LEMMA 7.8 (Zero-one law). *Let M be a generically unramified ℓ-adic perverse sheaf on G of tannakian dimension r, for some prime $\ell \neq p$. Denote by $\mathbf{G}^0$ the neutral component of $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$. For a polynomial $f \in \mathbf{Z}[a_1, \ldots, a_r]$ and an integer $m \geqslant 1$, we denote by $\mathscr{X}_{f,m}$ the set of weakly unramified* *characters* $\chi \in \mathscr{X}_w(\mathrm{M})$ *such that* $\chi \in \mathscr{X}_{f,m}(k_n)$ *if and only if* $f(\mathbf{Fr}_{\mathrm{M},k_n}(\chi)^m) = 0$*, where we define* $f(g)$ *for* $g \in \mathbf{GL}_r$ *by interpreting the variables* a_i *as the coefficients of the characteristic polynomial* $\det(\mathrm{T} - g)$ *of an element* g *of* $\mathbf{GL}_r$*.*

Let $m \geqslant 1$ *be an integer. Then all connected components* H *of* $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ *satisfy* $\mathrm{H}^m = \mathbf{G}^0$ *if and only if for all* $f \in \mathbf{Z}[a_1, \ldots, a_r]$*, the limit*

$$\lim_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}} \sum_{1\leqslant n\leqslant \mathrm{N}} \frac{|\mathscr{X}_{f,m}(k_n)|}{|\mathscr{X}_w(k_n)|} \tag{7.2}$$

exists and is equal to either 0 *or* 1*.*

Proof. We can follow Serre's argument very closely.

First, assume that $\mathrm{H}^m = \mathbf{G}^0$ for all connected components. It follows in particular that $\mathbf{Fr}_{\mathrm{M},k_n}(\chi)^m$ belongs to $\mathbf{G}^0$ for any unramified character χ.

Let F denote the set of $g \in \mathbf{G}_{\mathrm{M}}^{\mathrm{ari}} \subset \mathbf{GL}_r$ such that $f(g) = 0$, so that F is a Zariski-closed subvariety of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ which is conjugacy-invariant. We have then

$$\mathscr{X}_{f,m}(k_n) = \{\chi \in \mathscr{X}_w(\mathrm{M})(k_n) \mid \mathbf{Fr}_{\mathrm{M},k_n}(\chi)^m \in \mathrm{F} \cap \mathbf{G}^0\}$$

for all $n \geqslant 1$ by the previous observation.

If F contains the neutral component, we get $\mathscr{X}_{f,m} = \mathscr{X}_w(\mathrm{M})$; since M is assumed to be generically unramified, we get

$$\lim_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}} \sum_{1\leqslant n\leqslant \mathrm{N}} \frac{|\mathscr{X}_{f,m}(k_n)|}{|\mathscr{X}_w(k_n)|} = 1.$$

On the other hand, if F does not contain the neutral component of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$, then $\mathrm{F} \cap \mathbf{G}^0$ corresponds to a closed set of measure 0 in the set of conjugacy classes of a maximal compact subgroup of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}(\mathbf{C})$, hence

$$\lim_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}} \sum_{1\leqslant n\leqslant \mathrm{N}} \frac{|\mathscr{X}_{f,m}(k_n)|}{|\mathscr{X}_w(\mathrm{M})|} = 0$$

follows from Theorem 4.11 by standard properties of convergence of probability measures.

In any case, the limit (7.2) exists and is equal to either 0 or 1.

Conversely, suppose that there exists a connected component H of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ such that $\mathrm{H}^m \neq \mathbf{G}^0$; we can find some $n \geqslant 1$ and some unramified character $\chi \in \widehat{\mathrm{G}}(k_n)$ such that $\mathbf{Fr}_{\mathrm{M},k_n}(\chi) \in \mathrm{H}$, so that $\mathbf{Fr}_{\mathrm{M},k_n}(\chi)^m$ is not in the neutral component. In fact, for the same reason as above, we have

$$\lim_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}} \sum_{1\leqslant n\leqslant \mathrm{N}} \frac{1}{|\mathscr{X}_w(\mathrm{M})|} \sum_{\substack{\chi\in\mathscr{X}(\mathrm{M})(k_n) \\ \mathbf{Fr}_{\mathrm{M},k_n}(\chi)\in\mathrm{H}}} 1 = \frac{1}{\pi_0},$$

where π_0 is the number of connected components. Applying [120, Lemme 1, p.17], we find an f such that $\mathscr{X}_{f,m}$ has "density" different from 0 and 1 in the above sense. ∎

Proof of Proposition 7.7. We use the notation of Lemma 7.1, and denote by $\mathscr{X}$ the set of weakly unramified characters $\chi \in \widehat{\mathrm{G}}^{(\ell_1)}$ for M_1 such that $\eta(\chi)$ is weakly unramified for M_2; this is a generic set. Let $f \in \mathbf{Z}[a_1, \ldots, a_r]$. By compatibility, the equality

$$\iota_1(f(\mathrm{Fr}_{\mathrm{M}_1,k_n}(\chi)^m)) = \iota_2(f(\mathrm{Fr}_{\mathrm{M}_2,k_n}(\eta(\chi))^m))$$

holds for any character $\chi \in \mathscr{X}(k_n)$ and any integer $m \geqslant 1$, since f has integral coefficients. Hence, for a given m, the statement of Lemma 7.8 holds for M_2 if and only if it does for M_1. The equality of the exponents of the groups of connected components follows by looking at the smallest $m \geqslant 1$ for which the zero-one law holds. $\square$

Remark 7.9. It is of course natural to expect that the groups of connected components are in fact isomorphic.

Finally, under rather strong "connectedness" assumptions, we can get a definitive answer by exploiting deep results of Larsen and Pink [96].

Proposition 7.10. *We continue with the notation and assumptions above. Denote by $\mathbf{G}_j$ the connected derived subgroup of $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_j}$. Assume that for all representations $\varrho\colon \mathbf{GL}_r \to \mathbf{GL}(\mathrm{V})$, the multiplicity of the trivial representation in the restrictions of ϱ to $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_j}$ and to $\mathbf{G}_j$ are equal.*

Then the complex semisimple Lie groups $\iota_1(\mathbf{G}_1)$ and $\iota_2(\mathbf{G}_2)$ are isomorphic.

Moreover, if M_1 or M_2 is arithmetically simple, then the groups $\iota_1(\mathbf{G}_1)$ and $\iota_2(\mathbf{G}_2)$ are conjugate in $\mathbf{GL}_r(\mathbf{C})$.

Proof. Let $\varrho\colon \mathbf{GL}_r \to \mathbf{GL}(\mathrm{V})$ be any finite-dimensional representation of $\mathbf{GL}_r$ and let $\mu_j(\varrho)$ be the multiplicity of the trivial representation in the restriction of ϱ to $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_j}$. By Proposition 7.3, we have

$$\mu_1(\varrho) = \int_{\mathbf{U}_r(\mathbf{C})^\sharp} \mathrm{Tr}(\varrho) d\nu_{cp,1} = \int_{\mathbf{U}_r(\mathbf{C})^\sharp} \mathrm{Tr}(\varrho) d\nu_{cp,2} = \mu_2(\varrho).$$

By our assumption, the multiplicity $\mu_j(\varrho)$ is also the multiplicity $\nu_j(\varrho)$ of the trivial representation in the restriction of ϱ to $\mathbf{G}_j$, and thus we have $\nu_1(\varrho) = \nu_2(\varrho)$.

Since this equality holds for all representations ϱ, and $\mathbf{G}_1$ and $\mathbf{G}_2$ are connected semisimple algebraic groups, a theorem of Larsen and Pink [96, Th. 1] implies that $\iota_1(\mathbf{G}_1)$ and $\iota_2(\mathbf{G}_2)$ are isomorphic.

Assume now that M_1 is arithmetically simple. Denoting by Ad the adjoint representation of $\mathbf{GL}_r$, this is equivalent to $\mu_1(\mathrm{Ad}) = 1$ by Schur's Lemma, hence we also have $\nu_1(\mathrm{Ad}) = 1$ and $\nu_2(\mathrm{Ad}) = 1$ by the previous results. The result then follows from another theorem of Larsen and Pink [96, Th. 2]. $\square$

Remark 7.11. (1) Proposition 7.10 applies for instance if one knows that, for $j = 1$ and $j = 2$, the groups $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_j}$ are connected and semisimple.

However, it doesn't apply in a situation where, say $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_1} = \mathbf{SL}_r$ and $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_2} = \mathbf{GL}_r$, since the determinant is an example of a representation where the multiplicities for $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_2}$ and for its derived connected subgroup are not the same. (We thank one referee for pointing out this issue in our previous version.)

(2) Larsen and Pink [96, Th. 3 and § 3] give examples showing that, in general, the assumption that M_1 (or M_2) is simple cannot be omitted in the second part of the proposition.

CHAPTER 8

Diophantine group theory

In order to determine the tannakian (or monodromy) group associated to a perverse sheaf, Katz has developed essentially two different sets of methods. The first one (see, e.g., [68, 69]) relies on *local monodromy information*, and applies mostly to the additive group, although there is also a weaker analogue for the multiplicative group (see [74, Ch. 16] and Corollary 3.46). However, we are not currently aware of any similar tools for other groups. The second method, expounded in [72], is much more global, and exploits the diophantine potential of the equidistribution of exponential sums to reveal properties of the underlying group. It turns out that this global method adapts very well to the tannakian framework, and this will be our fundamental tool.

We denote as usual by k a finite field, with an algebraic closure $\bar{k}$, and finite extensions k_n of degree n in $\bar{k}$. We fix a prime ℓ different from the characteristic of k.

8.1. The diophantine irreducibility criterion

We first state Katz's criterion for a perverse sheaf to be geometrically simple in terms of its trace functions.

PROPOSITION 8.1. *Let* X *be a quasi-projective algebraic variety over* k*, and* M *an* ℓ*-adic perverse sheaf on* X *which is pure of weight zero. Then the equality*

$$\lim_{n\to+\infty} \sum_{x\in \mathrm{X}(k_n)} |t_{\mathrm{M}}(x;k_n)|^2 = 1 \tag{8.1}$$

holds if and only if M *is geometrically simple.*

See [72, Th. 1.7.2 (3)] for the proof.

REMARK 8.2. This can be seen as a version of Schur's Lemma (compare with Corollary 4.6): intuitively, by equidistribution, the limit in the proposition should converge to the multiplicity of the trivial representation in the representation End(Std), where Std is the standard representation of the (usual) geometric monodromy group of the lisse sheaf on an open dense subset of the support of M that is associated to M. The classical version of Schur's Lemma states that this multiplicity is equal to 1 if and only if the standard representation is irreducible.

8.2. The Frobenius–Schur indicator

Recall that if $\mathbf{G}$ is an arbitrary group and $\varrho\colon \mathbf{G}\to\mathbf{GL}_r(\mathbf{C})$ is a finite-dimensional representation, one says that ϱ is of *orthogonal type* (resp. of *symplectic type*) if there exists a G-invariant non-degenerate symmetric (resp. alternating) bilinear form on $\mathbf{C}^r$. Suppose that ϱ is irreducible. The Frobenius–Schur indicator $\mathrm{FS}(\varrho)$ is defined to be 1 if ϱ is of orthogonal type, -1 if ϱ is of symplectic type, and 0 otherwise. If $\mathbf{G}=\mathrm{K}$ is a compact group, with probability Haar measure μ_{K}, and if ϱ is irreducible and continuous, then one has an integral formula

$$\mathrm{FS}(\varrho)=\int_{\mathrm{K}} \mathrm{Tr}(\varrho(g^2))d\mu_{\mathrm{K}}(g)$$

(see, e.g., [86, Th. 6.2.3]).

As in previous works of Katz (see, e.g., [74, Th. 9.1] or [72, Th. 1.9.6]), there is a diophantine interpretation of the Frobenius–Schur indicator.

Proposition 8.3. *Let* G *be a connected commutative algebraic group over* k*, and let* M *be an arithmetically irreducible* ℓ*-adic perverse sheaf on* G *which is pure of weight zero. Let* $\mathscr{X} = \mathscr{X}_w(\mathrm{M})$ *be the set of weakly unramified characters for* M.

The Frobenius–Schur indicator of M*, viewed as a representation of the arithmetic tannakian group* $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$*, is given by the formula*

$$\mathrm{FS}(\mathrm{M}) = \lim_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}} \sum_{\substack{1\leqslant n\leqslant \mathrm{N}\\ \mathscr{X}(k_n)\neq\varnothing}} \frac{1}{|\mathscr{X}(k_n)|} \sum_{\chi\in\mathscr{X}(k_n)} \mathrm{Tr}(\Theta_{\mathrm{M},k_n}(\chi)^2).$$

The proof is straightforward using the integral formula above and the equidistribution theorem (Theorem 4.4).

8.3. Larsen's Alternative

In this section, $r \geqslant 1$ is an integer and $\mathbf{G}$ is a reductive algebraic subgroup of $\mathbf{GL}_r$ over an algebraically closed field of characteristic zero (recall that reductive groups are not required to be connected). For each integer $m \geqslant 1$, the *absolute* $2m$*-th moment* of an algebraic representation V of $\mathbf{G}$ is defined as

$$\mathrm{M}_{2m}(\mathbf{G}, \mathrm{V}) = \dim(\mathrm{V}^{\otimes m} \otimes (\mathrm{V}^{\vee})^{\otimes m})^{\mathbf{G}}.$$

When V is the "standard" r-dimensional representation given by the inclusion $\mathbf{G} \subset \mathbf{GL}_r$ (also denoted by Std), we will simply write $\mathrm{M}_{2m}(\mathbf{G})$.

If the base field is $\mathbf{C}$, so that $\mathbf{G}$ is a reductive subgroup of $\mathbf{GL}_r(\mathbf{C})$, the moments can be written as integrals over a maximal compact subgroup K of $\mathbf{G}$ with Haar probability measure μ_{K}. Namely, for all $m \geqslant 1$, they are given by the integral expression

$$\mathrm{M}_{2m}(\mathbf{G}) = \int_{\mathrm{K}} |\operatorname{Tr}(g)|^{2m} d\mu_{\mathrm{K}}(g). \tag{8.2}$$

We first note some elementary properties of the moments.

(1) Given a surjective homomorphism $f\colon \mathbf{H} \to \mathbf{G}$ and a representation $\varrho\colon \mathbf{G} \to \mathbf{GL}(\mathrm{V})$, the equality

$$\mathrm{M}_{2m}(\mathbf{H}, \varrho\circ f) = \mathrm{M}_{2m}(\mathbf{G}, \varrho)$$

holds for all $m \geqslant 1$ (since $(\varrho^{\otimes m} \otimes (\varrho^{\vee})^{\otimes m})^{\mathbf{G}} = ((\varrho\circ f)^{\otimes m} \otimes (\varrho\circ f)^{\vee})^{\otimes m})^{\mathbf{H}}$ by definition).

(2) For groups $\mathbf{G}_1$ and $\mathbf{G}_2$ with representations V_1 and V_2, the equality

$$\mathrm{M}_{2m}(\mathbf{G}_1 \times \mathbf{G}_2, \mathrm{V}_1 \boxtimes \mathrm{V}_2) = \mathrm{M}_{2m}(\mathbf{G}_1, \mathrm{V}_1)\mathrm{M}_{2m}(\mathbf{G}_2, \mathrm{V}_2) \tag{8.3}$$

holds for all $m \geqslant 1$ (this might be easiest to see using the integral expression (8.2)).

(3) If $\mathbf{G} \subset \mathbf{GL}(\mathrm{V})$, and Z is a subgroup of scalar matrices in $\mathbf{GL}(\mathrm{V})$, then the equality

$$\mathrm{M}_{2m}(\mathbf{G}, \mathrm{V}) = \mathrm{M}_{2m}(\mathrm{Z}\mathbf{G}, \mathrm{V})$$

holds (because Z acts trivially on the whole space $\mathrm{V}^{\otimes m} \otimes (\mathrm{V}^{\vee})^{\otimes m}$).

(4) If there exists a $\mathbf{G}$-invariant decomposition

$$\mathrm{V}^{\otimes m} = \bigoplus_i n_i \mathrm{V}_i,$$

then the $2m$-th moment satisfies the inequality

$$\mathrm{M}_{2m}(\mathbf{G},\mathrm{V}) \geqslant \sum_i n_i^2, \tag{8.4}$$

with equality if and only if the V_i are pairwise non-isomorphic irreducible representations (see [71, 1.1.4]).

(5) If there exists a $\mathbf{G}$-invariant decomposition

$$\mathrm{End}(\mathrm{V}) = \bigoplus_i m_i \mathrm{W}_i,$$

then the fourth moment satisfies

$$\mathrm{M}_4(\mathbf{G},\mathrm{V}) \geqslant \sum_i m_i^2, \tag{8.5}$$

with equality if and only if the W_i are pairwise non-isomorphic irreducible representations (see [71, 1.1.5]).

Since the tensor constructions involved in the definition of the moments are representations of the ambient group $\mathbf{GL}(\mathrm{V})$, Theorem 4.4 immediately yields a diophantine interpretation of the moments of the arithmetic tannakian group of a perverse sheaf.

Proposition 8.4. *Let* G *be a connected commutative algebraic group over* k*, and let* M *be an arithmetically semisimple* ℓ*-adic perverse sheaf on* G *which is pure of weight zero. For each character* $\chi \in \widehat{\mathrm{G}}(k_n)$*, consider the sum*

$$\mathrm{S}(\mathrm{M},\chi) = \sum_{x\in\mathrm{G}(k_n)} t_{\mathrm{M}}(x;k_n)\chi(x).$$

Let $\mathscr{X} = \mathscr{X}_w(\mathrm{M})$ *be the set of weakly unramified characters for* M *and let* $m \geqslant 0$ *be an integer.*

The absolute moments of M*, viewed as a representation of the arithmetic tannakian group* $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$*, satisfy the following:*

$$\mathrm{M}_{2m}(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}},\mathrm{M}) = \lim_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}} \sum_{\substack{1\leqslant n\leqslant \mathrm{N}\\ \mathscr{X}(k_n)\neq\varnothing}} \frac{1}{|\mathscr{X}(k_n)|} \sum_{\chi\in\mathscr{X}(k_n)} |\mathrm{S}(\mathrm{M},\chi)|^{2m}, \tag{8.6}$$

$$\mathrm{M}_{2m}(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}},\mathrm{M}) \leqslant \liminf_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}} \sum_{1\leqslant n\leqslant \mathrm{N}} \frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi\in\widehat{\mathrm{G}}(k_n)} |\mathrm{S}(\mathrm{M},\chi)|^{2m}. \tag{8.7}$$

Moreover, if *the limit*

$$\lim_{n\to+\infty} \frac{1}{|\mathscr{X}(k_n)|} \sum_{\chi\in\mathscr{X}(k_n)} |\mathrm{S}(\mathrm{M},\chi)|^{2m} \tag{8.8}$$

exists, then it is equal to the $2m$*-th moment* $\mathrm{M}_{2m}(\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}},\mathrm{M})$ *of* M*, viewed as a representation of the geometric tannakian group* $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$*, and we have*

$$\mathrm{M}_{2m}(\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}},\mathrm{M}) = \mathrm{M}_{2m}(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}},\mathrm{M}).$$

Proof. We use the integral expression

$$\mathrm{M}_{2m}(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}},\mathrm{M}) = \int_{\mathrm{K}} |\operatorname{Tr}(g)|^{2m} d\mu_{\mathrm{K}}(g),$$

where $\mathrm{K} \subset \mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}(\mathbf{C})$ is a maximal compact subgroup with Haar probability measure μ_{K}. Recall that to each weakly unramified character $\chi \in \mathscr{X}(k_n)$ is associated the unitary conjugacy class $\Theta_{\mathrm{M},k_n}(\chi)$

such that the equality $\mathrm{S}(\mathrm{M},\chi) = \mathrm{Tr}(\Theta_{\mathrm{M},k_n}(\chi))$ holds, and that these conjugacy classes become equidistributed on average as $n \to +\infty$ by Theorem 4.4. The first formula (8.6) follows from this result applied to the test function $g \mapsto |\,\mathrm{Tr}(g)|^{2m}$.

Moreover, the inequality

$$\frac{1}{\mathrm{N}}\sum_{1\leqslant n\leqslant \mathrm{N}}\frac{1}{|\mathrm{G}(k_n)|}\sum_{\chi\in\mathscr{X}(k_n)}|\mathrm{S}(\mathrm{M},\chi)|^{2m} \leqslant \frac{1}{\mathrm{N}}\sum_{1\leqslant n\leqslant \mathrm{N}}\frac{1}{|\mathrm{G}(k_n)|}\sum_{\chi\in\widehat{\mathrm{G}}(k_n)}|\mathrm{S}(\mathrm{M},\chi)|^{2m}$$

holds by positivity of $|\mathrm{S}(\mathrm{M},\chi)|^{2m}$. Taking the equivalence $|\mathrm{G}(k_n)| \sim |\mathscr{X}(k_n)|$ as $n \to +\infty$ from the generic vanishing theorem into account, we deduce the second formula (8.7).

Finally, the last assertion follows from Proposition 4.17, applied to the representation

$$\varrho = \mathrm{Std}^{\otimes m} \otimes (\mathrm{Std}^{\vee})^{\otimes m},$$

and from the fact that if the limit (8.8) exists, then its value is the same as the limit in (8.6). ☐

We can combine this computation with Larsen's Alternative, a remarkable criterion that ensures that a reductive subgroup $\mathbf{G} \subset \mathbf{GL}_r$ is either finite or contains one of the standard classical groups, provided it has the correct fourth or eighth moment.

Theorem 8.5 (Larsen's Alternative). *Let* V *be a vector space of dimension $r \geqslant 2$ over an algebraically closed field of characteristic zero and let $\mathbf{G} \subset \mathbf{GL}(\mathrm{V})$ be a reductive algebraic subgroup. Let* Z *denote the center of $\mathbf{GL}(\mathrm{V})$ and $\mathbf{G}^\circ$ the connected component of the identity of $\mathbf{G}$.*

(1) *The fourth moment satisfies $\mathrm{M}_4(\mathbf{G},\mathrm{V}) \geqslant 2$. Furthermore, if* V *is self-dual and $r \geqslant 3$, then $\mathrm{M}_4(\mathbf{G},\mathrm{V}) \geqslant 3$.*
(2) *If $\mathrm{M}_4(\mathbf{G},\mathrm{V}) \leqslant 5$, then the representation of $\mathbf{G}$ on* V *is irreducible.*
(3) *If $\mathrm{M}_4(\mathbf{G},\mathrm{V}) = 2$, then either $\mathbf{SL}(\mathrm{V}) \subset \mathbf{G}$ or $\mathbf{G}/(\mathbf{G}\cap \mathrm{Z})$ is finite. If $\mathbf{G}\cap\mathrm{Z}$ is finite, for instance if $\mathbf{G}$ is semisimple, then either $\mathbf{G}^\circ = \mathbf{SL}(\mathrm{V})$ or $\mathbf{G}$ is finite.*
(4) *Assume $r \geqslant 5$. If $\mathrm{M}_4(\mathbf{G},\mathrm{V}) = 2$ and $\mathrm{M}_8(\mathbf{G},\mathrm{V}) = 24$, then $\mathbf{SL}(\mathrm{V}) \subset \mathbf{G}$.*
(5) *Assume that there exists a non-degenerate symmetric bilinear form* B *on* V *such that $\mathbf{G}$ lies in $\mathbf{O}(\mathrm{B})$. If $\mathrm{M}_4(\mathbf{G},\mathrm{V}) = 3$, then either $\mathbf{G} = \mathbf{SO}(\mathrm{B})$, or $\mathbf{G} = \mathbf{O}(\mathrm{B})$, or $\mathbf{G}$ is finite. If r is 2 or 4, then $\mathbf{G}$ is not contained in $\mathbf{SO}(\mathrm{B})$.*
(6) *Assume that there exists a non-degenerate alternating bilinear form* B *on* V *such that $\mathbf{G}$ lies in $\mathbf{Sp}(\mathrm{B})$. If $r \geqslant 4$ and $\mathrm{M}_4(\mathbf{G},\mathrm{V}) = 3$, then either $\mathbf{G} = \mathbf{Sp}(\mathrm{B})$ or $\mathbf{G}$ is finite.*

Proof. The first statement concerning the fourth moment is a straightforward consequence of the inequality (8.4). Indeed, since $\mathrm{V}^{\otimes 2} \otimes (\mathrm{V}^{\vee})^{\otimes 2}$ always contains a trivial one-dimensional subrepresentation, the fourth moment can only be 1 for V of dimension 1. Moreover, there is a $\mathbf{GL}(\mathrm{V})$-invariant (and hence $\mathbf{G}$-invariant) decomposition

$$\mathrm{V}^{\otimes 2} = \mathrm{Sym}^2 \mathrm{V} \oplus \bigwedge^2 \mathrm{V},$$

where the factors are distinct and non-trivial, and of dimension $\geqslant 2$ if $r \geqslant 3$. If V is self-dual, one of the two summands contains a proper one-dimensional $\mathbf{G}$-invariant subspace, so that the fourth moment is at least 3 using (8.4) again.

The other statements concerning the fourth moment are proved by Katz in [71, Th. 1.1.6]. The statement about the eighth moment was conjectured by Katz in [72, 2.3], and proved by Guralnick and Tiep in [55, Th. 1.4]. Indeed, according to loc. cit., a reductive subgroup $\mathbf{G}$ of $\mathbf{GL}(\mathrm{V})$ either satisfies $\mathrm{M}_8(\mathbf{G}) > \mathrm{M}_8(\mathbf{GL}(\mathrm{V}))$ or contains the commutator subgroup $[\mathbf{GL}(\mathrm{V}), \mathbf{GL}(\mathrm{V})] = \mathbf{SL}(\mathrm{V})$, and the eighth moment of $\mathbf{GL}(V)$ is equal to 24 for $r \geqslant 4$, for instance in view of the $\mathbf{GL}(V)$-invariant decomposition

$$V^{\otimes 4} = \mathrm{Sym}^4 V \oplus \bigwedge^4 V \oplus 3S^{(3,1)}V \oplus 2S^{(2,2)}V \oplus 3S^{(2,1,1)}V$$

into pairwise non-isomorphic irreducible representations (see e.g. [49, Ex. 6.5]), where S^λ denotes the Schur functor associated to a partition λ of 4. ☐

In practice, computing a given moment of the arithmetic tannakian group $\mathbf{G}_M^{\mathrm{ari}}$ by means of the limit (8.6) is feasible if there are sufficiently many independent variables of summation, corresponding to the characters of G, in comparison with the number of variables involved in the object M, that is, the dimension of its support. It is then possible, at least in some cases, to detect a diagonal behavior that can lead to the asymptotic formula for the moment. This limitation explains why it is difficult to apply Larsen's alternative when G is one-dimensional, but starting from two-dimensional groups it can be sometimes implemented for objects supported on curves.

REMARK 8.6. (1) Using typical terminology from geometric group theory, it is convenient to summarize the third part of Theorem 8.5 by saying that if $\mathbf{G} \subset \mathbf{GL}(V)$ has fourth moment equal to 2, then either $\mathbf{G} \supset \mathbf{SL}(V)$ or $\mathbf{G}$ is *virtually central* in $\mathbf{GL}(V)$.

(2) The book [72] of Katz develops applications of Larsen's alternative which involve sums of the type

$$S(f) = \sum_{x \in X(k)} t_1(x) t_2(f(x)),$$

for suitable trace functions t_1 and t_2 (on X and some affine space $\mathbf{A}^r$, respectively), parameterized by elements $f\colon X \to \mathbf{A}^r$ of a "function space" $\mathscr{F}$. One of the conditions that are shown by Katz to ensure that the $2m$-th moment can be computed is that the evaluation maps

$$f \mapsto (f(x_1), \dots, f(x_{2m}))$$

be surjective for distinct x_i in $X(k)$ (see [72, §1.15, Th. 1.20.2] for a precise and more general statement).

8.4. Sidon morphisms

DEFINITION 8.7 (Sidon sets and Sidon morphisms). Let A be an abelian group. A subset $S \subset A$ is called a *Sidon set* if all solutions x_1, x_2, x_3, x_4 in S of the equation $x_1x_2 = x_3x_4$ satisfy $x_1 \in \{x_3, x_4\}$.

More generally, let $r \geqslant 2$ be an integer. We say that S is an *r-Sidon set* if all tuples $(x_i)_{1\leqslant i\leqslant r}$ and $(y_i)_{1\leqslant i\leqslant r}$ in S^r such that the equality

$$x_1 \cdots x_r = y_1 \cdots y_r$$

holds satisfy $\{x_1, \dots, x_r\} = \{y_1, \dots, y_r\}$. A Sidon set is thus the same as a 2-Sidon set.

Let $\alpha \in A$. A subset $S \subset A$ is called an *α-symmetric Sidon set* if $S = \alpha S^{-1}$ and all solutions $x_1, x_2, x_3, x_4 \in S$ of the equation $x_1x_2 = x_3x_4$ satisfy $x_1 \in \{x_3, x_4\}$ or $x_2 = \alpha x_1^{-1}$.

Let G be a connected commutative algebraic group over a field k, and let $s\colon X \to G$ be a locally-closed immersion of k-schemes. We say that s is a *Sidon morphism*, or that $s(X)$ is a *Sidon subvariety of* G if, for any extension k' of k, the subset $s(X)(k') \subset G(k')$ is a Sidon set. We define similarly r-Sidon morphisms for any $r \geqslant 2$.

Let i be an involution on X and $a \in G$. We say that s is an *i-symmetric Sidon morphism* if the product morphism $(s \circ i) \cdot s\colon X \to G$ is a constant morphism, say equal to $\alpha \in G(k)$, and if, for any extension k' of k, the set $s(X(k'))$ is an α-symmetric Sidon set in $G(k')$.

The interest of a Sidon morphism $\mathrm{X} \to \mathrm{G}$ is that it leads to computations of the fourth moment for objects M on G that are pushed from X. We have two versions, depending on whether we have a Sidon morphism or a symmetric Sidon morphism.

PROPOSITION 8.8. *Let* G *be a connected commutative algebraic group over a finite field* k *and let* $s\colon \mathrm{X} \to \mathrm{G}$ *be a closed immersion of* k*-schemes. Let* N *be a geometrically simple* ℓ*-adic perverse sheaf on* X *which is pure of weight* 0*, so that the object* $\mathrm{M} = s_*\mathrm{N} = s_!\mathrm{N}$ *on* G *is a geometrically simple perverse sheaf on* G *and is pure of weight* 0.

(1) *If* s *is a Sidon morphism, then the equality*

$$\mathrm{M}_4(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}, \mathrm{M}) = 2$$

holds unless M *has tannakian dimension* $\leqslant 1$.

(2) *If* X *is a curve and* $s\colon \mathrm{X} \to \mathrm{G}$ *is a* 4*-Sidon morphism, then the equality*

$$\mathrm{M}_8(\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}, \mathrm{M}) = \mathrm{M}_8(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}, \mathrm{M}) = 24$$

holds unless N *is geometrically isomorphic to* $s^*\mathscr{L}_\chi[1]$ *for some character* $\chi \in \widehat{\mathrm{G}}$.

PROOF. Let $n \geqslant 1$ be an integer. The formula

$$\frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi \in \widehat{\mathrm{G}}(k_n)} |\mathrm{S}(\mathrm{M}, \chi)|^4 = \sum_{\substack{y_1, \ldots, y_4 \in \mathrm{X}(k_n) \\ s(y_1)s(y_2) = s(y_3)s(y_4)}} t_{\mathrm{N}}(y_1, k_n) t_{\mathrm{N}}(y_2; k_n) \overline{t_{\mathrm{N}}(y_3; k_n) t_{\mathrm{N}}(y_4; k_n)}$$

holds by orthogonality of characters. If s is a Sidon morphism, then we obtain by definition

$$\frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi \in \widehat{\mathrm{G}}(k_n)} |\mathrm{S}(\mathrm{M}, \chi)|^4 = 2\Big(\sum_{y \in \mathrm{X}(k_n)} |t_{\mathrm{N}}(y, k_n)|^2\Big)^2 - \sum_{y \in \mathrm{X}(k_n)} |t_{\mathrm{N}}(y, k_n)|^4,$$

where the second term accounts for the double-counting of the solutions of the equation $s(y_1)s(y_2) = s(y_3)s(y_4)$ where $y_1 = y_2 = y_3 = y_4$. In particular, we deduce that the inequality

$$\frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi \in \widehat{\mathrm{G}}(k_n)} |\mathrm{S}(\mathrm{M}, \chi)|^4 \leqslant 2\Big(\sum_{y \in \mathrm{X}(k_n)} |t_{\mathrm{N}}(y, k_n)|^2\Big)^2$$

holds for all $n \geqslant 1$. Since N is geometrically simple, the right-hand side of this expression converges to 2 as $n \to +\infty$ by Proposition 8.1. Using the inequality (8.7) from Proposition 8.4, we deduce that

$$\mathrm{M}_4(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}, \mathrm{M}) \leqslant 2$$

in the setting of (1). Hence, the fourth moment is either $\leqslant 1$ or equal to 2. By Theorem 8.5 (1), the former is only possible if $\mathrm{M}_{\mathrm{int}}$ is of tannakian dimension $\leqslant 1$.

Now we assume that s is a 4-Sidon morphism. We obtain similarly

$$\frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi \in \widehat{\mathrm{G}}(k_n)} |\mathrm{S}(\mathrm{M}, \chi)|^8 \leqslant 24\Big(\sum_{y \in \mathrm{X}(k_n)} |t_{\mathrm{N}}(y, k_n)|^2\Big)^4$$

where the right-hand side converges to 24 for the same reason as before.

Assume now that X is a curve and s is a 4-Sidon morphism. We apply the Riemann Hypothesis (Theorem 1.10) to the simple perverse sheaves $s^*\mathscr{L}_{\chi^{-1}}[1](1/2)$ (of weight 0) and to N. By assumption, these are not geometrically isomorphic, and therefore the estimate

$$\mathrm{S}(\mathrm{M}, \chi) = \sum_{y \in \mathrm{X}(k_n)} \chi(s(y)) t_{\mathrm{N}}(y; k_n) \ll 1$$

holds for all characters χ. We deduce then that the formula

$$\lim_{n\to+\infty}\frac{1}{|\mathrm{G}(k_n)|}\sum_{\chi\notin\mathscr{X}_w(\mathrm{M})(k_n)}|\mathrm{S}(\mathrm{M},\chi)|^{2m}=0$$

holds for any integer $m\geqslant 1$; we finally conclude from the previous computations and the last assertion of Proposition 8.4 that $\mathrm{M}_8(\mathbf{G}^{\mathrm{geo}}_{\mathrm{M}},\mathrm{M})=\mathrm{M}_8(\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}},\mathrm{M})=24$. ☐

We now state the version involving symmetric Sidon morphisms.

Proposition 8.9. *Let* G *be a connected commutative algebraic group over a finite field* k. *Let* X *be a smooth irreducible algebraic variety over* k *and* i *an involution on* X. *Let* $s\colon\mathrm{X}\to\mathrm{G}$ *be an* i*-symmetric Sidon morphism which is a closed immersion. Let* α *be the constant value of the morphism* $(s\circ i)\cdot s$.

Let N *be a geometrically simple* ℓ*-adic perverse sheaf on* X *which is pure of weight* 0, *so that the object* $s_*\mathrm{N}=s_!\mathrm{N}$ *on* G *is a geometrically simple perverse sheaf on* G *and is pure of weight* 0.

(1) *If* $i^*\mathrm{N}$ *is isomorphic to* $\mathrm{D}(\mathrm{N})$, *then we have* $(s_*\mathrm{N})^\vee=[\times\alpha^{-1}]^*(s_*\mathrm{N})$, *and*

$$\mathrm{M}_4(\mathbf{G}^{\mathrm{ari}}_{s_*\mathrm{N}},s_*\mathrm{N})=3,$$

unless $(s_*\mathrm{N})_{\mathrm{int}}$ *has tannakian dimension* $\leqslant 2$.

(2) *If* $i^*\mathrm{N}$ *is not isomorphic to* $\mathrm{D}(\mathrm{N})$, *then*

$$\mathrm{M}_4(\mathbf{G}^{\mathrm{ari}}_{s_*\mathrm{N}},s_*\mathrm{N})=2,$$

unless $(s_*\mathrm{N})_{\mathrm{int}}$ *has tannakian dimension* $\leqslant 2$.

Proof. Let $\mathrm{M}=s_*\mathrm{N}$. In the situation of (1), the definition of α means that there is an equality $s\circ i=[\times\alpha]\circ(\mathrm{inv}\circ s)$. Therefore, we obtain canonical isomorphisms

$$\begin{aligned}(8.9)\quad \mathrm{M}^\vee&=\mathrm{inv}^*(\mathrm{D}(s_*\mathrm{N}))=\mathrm{inv}^*(s_*(\mathrm{D}(\mathrm{N})))=\mathrm{inv}^*((s\circ i)_*\mathrm{N})\\&=\mathrm{inv}^*([\times\alpha]_*(\mathrm{inv}\circ s)_*\mathrm{N})=(\mathrm{inv}\circ[\times\alpha]\circ\mathrm{inv})_*(s_*\mathrm{N})=[\times\alpha^{-1}]^*\mathrm{M}.\end{aligned}$$

We go back to the general case. Arguing as in the proof of the previous proposition, we obtain the inequality

$$\begin{aligned}\frac{1}{|\mathrm{G}(k_n)|}\sum_{\chi\in\widehat{\mathrm{G}}(k_n)}|\mathrm{S}(\mathrm{M},\chi)|^4\leqslant 2\Big(\sum_{y\in\mathrm{X}(k_n)}|t_{\mathrm{N}}(y,k_n)|^2\Big)^2\\+\sum_{(y,z)\in\mathrm{X}(k_n)^2}t_{\mathrm{N}}(y,k_n)t_{\mathrm{N}}(i(y);k_n)\overline{t_{\mathrm{N}}(z;k_n)t_{\mathrm{N}}(i(z);k_n)}\end{aligned}$$

for all $n\geqslant 1$, by the definition of symmetric Sidon sets. The second sum is equal to the quantity

$$\Big|\sum_{y\in\mathrm{X}(k_n)}t_{\mathrm{N}}(y,k_n)t_{\mathrm{N}}(i(y);k_n)\Big|^2,$$

which converges to 1 under the assumption (1) (using (A.6)), by Proposition 8.1, and to 0 under the assumption (2), by the Riemann Hypothesis. Thus we deduce from Proposition 8.4 that

$$\mathrm{M}_4(\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}},\mathrm{M})\leqslant 3,\qquad\text{resp.}\qquad\mathrm{M}_4(\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}},\mathrm{M})\leqslant 2,$$

in case (1) (resp. (2)), and we conclude as before from Theorem 8.5, (1). ☐

REMARK 8.10. The caveats concerning the tannakian dimension of $s_*\mathrm{N}$ in these statements are necessary. We will indeed see concrete examples (see Example 9.12, (1) and Remark 11.2, (1)) where the fourth moment does *not* coincide with the limit

$$\lim_{n\to+\infty} \frac{1}{|\mathrm{G}(k_n)|} \sum_{\chi\in\widehat{\mathrm{G}}(k_n)} |\mathrm{S}(\mathrm{M},\chi)|^4$$

(althouth the latter exists) because of the contribution of some special ramified characters.

The result of Propositions 8.8 and 8.9 will be the basis of applications in Chapters 9, 10 and 11. Here are the relevant cases of Sidon morphisms, together with some further elementary examples.

PROPOSITION 8.11. *Let k be a field, not necessarily finite.*

(1) *For any $\alpha \in k^\times$, the embedding $x \mapsto (x, \alpha x)$ of $\mathbf{G}_m$ in $\mathbf{G}_m \times \mathbf{G}_a$ is a Sidon morphism.*
(2) *Let C be a smooth projective connected algebraic curve of genus $g \geqslant 2$ over k. Let D be a divisor of degree 1 on C, and let $\mathrm{A} = \mathrm{Jac}(\mathrm{C})$ be the jacobian of C. The closed immersion $s\colon x \mapsto x - \mathrm{D}$ of C in A is a Sidon morphism unless C is hyperelliptic, in which case it is an i-symmetric Sidon morphism, where i is the hyperelliptic involution.*
(3) *With notation as in the previous item, if the gonality of C is at least 5, then s is a 4-Sidon morphism.*
(4) *Let $d \geqslant 1$ be an integer and let f be a separable polynomial of degree d over k. Let Z be the set of zeros of f. The closed immersion $x \mapsto (z - x)_{z\in\mathrm{Z}}$ of $\mathbf{A}^1[1/f]$ in $\mathbf{G}_m^{\mathrm{Z}}$ is a Sidon morphism if $d \geqslant 2$. It is a 4-Sidon morphism if $d \geqslant 4$.*
(5) *Suppose that the characteristic of k is not 3. The graph $s\colon x \mapsto (x, x^3)$ from $\mathbf{G}_a$ to $\mathbf{G}_a^2$ is an i-symmetric Sidon morphism, where i is the involution $x \mapsto -x$.*
(6) *The morphism $x \mapsto (x, 1 - x)$ from $\mathbf{G}_m - \{1\}$ to $\mathbf{G}_m \times \mathbf{G}_m$ is a Sidon morphism.*

PROOF. (1) For $x_1, \ldots, x_4$ in $\mathbf{G}_m$, the equation

$$(x_1, \alpha x_1) \cdot (x_2, \alpha x_2) = (x_3, \alpha x_3) \cdot (x_4, \alpha x_4)$$

in $\mathbf{G}_a \times \mathbf{G}_m$ means that $x_1 + x_2 = x_3 + x_4$ and $x_1 x_2 = x_3 x_4$, which implies that $\{x_1, x_2\} = \{x_3, x_4\}$, both sets being the solutions of the same quadratic equation.

(2) Let $x_1, \ldots, x_4$ in C be solutions of

$$s(x_1) + s(x_2) = s(x_3) + s(x_4).$$

Assume $x_1 \notin \{x_3, x_4\}$. Then the equation implies the existence of a rational function on C with zeros $\{x_1, x_2\}$ and poles $\{x_3, x_4\}$, which corresponds to a morphism $f\colon \mathrm{C} \to \mathbf{P}^1$ of degree at most 2. This is not possible unless C is hyperelliptic.

With the same notation, if C is hyperelliptic with hyperelliptic involution i, then the uniqueness of the morphism $f\colon \mathrm{C} \to \mathbf{P}^1$ of degree 2 up to automorphisms (see, e.g., [100, Rem. 4.30]) shows that i exchanges the points of the fibers of f, or in other words, that the equalities $x_2 = i(x_1)$ and $x_4 = i(x_3)$ hold.

(3) The argument is similar: the equation

$$s(x_1) + s(x_2) + s(x_3) + s(x_4) = s(x_5) + s(x_6) + s(x_7) + s(x_8)$$

where $\{x_i\} \neq \{y_i\}$ implies the existence of a non-constant morphism $f\colon \mathrm{C} \to \mathbf{P}^1$ of degree at most 4, and hence implies that C has gonality at most 4.

(4) Suppose that $x_1, \ldots x_4$ satisfy $s(x_1)s(x_2) = s(x_3)s(x_4)$. Then we get

$$(x_1 - z)(x_2 - z) = (x_3 - z)(x_4 - z)$$

for all $z \in \mathrm{Z}$, i.e., the monic polynomials $(x_1 - \mathrm{X})(x_2 - \mathrm{X})$ and $(x_3 - \mathrm{X})(x_4 - \mathrm{X})$ take the same values at the points of Z. By interpolation, they are equal if $|\mathrm{Z}| = d \geqslant 2$. The case of the 4-Sidon property is analogous with polynomials of degree 4.

(5) Suppose that $x_1, \ldots, x_4 \in \mathbf{G}_a^4$ satisfy

$$\begin{cases} x_1 + x_2 = x_3 + x_4 \\ x_1^3 + x_2^3 = x_3^3 + x_4^3. \end{cases}$$

If $x_2 \neq -x_1$, then these imply that

$$(x_1 + x_2)^2 - 3x_1x_2 = (x_3 + x_4)^2 - 3x_3x_4,$$

and therefore $x_1x_2 = x_3x_4$ when the characteristic is not 3. Now we conclude as in (1).

(6) This is again about quadratic equations: let $x_1, \ldots, x_4$ in $\mathbf{G}_m - \{1\}$ be such that

$$\begin{cases} x_1x_2 = x_3 \\ (1 - x_1)(1 - x_2) = (1 - x_3)(1 - x_4). \end{cases}$$

Then we get further $x_1 + x_2 = x_3 + x_4$, and conclude as before. □

Remark 8.12. (1) Example (1) is classical: it is often attributed to Ruzsa [111], but it was pointed out by Eberhard and Manners [34] that it occurs previously in a paper of Ganley [52, p. 323], where it is attributed to Spence.

Example (5) was also indicated to us by Eberhard and Manners.

(2) There is much work in combinatorics in trying to find the largest possible Sidon sets in a finite abelian group A (for instance, see the classification in [34] of known examples of size $\sim |\mathrm{A}|^{1/2}$, which they show are all related to finite projective planes). A natural analogue geometric question is to classify the Sidon morphisms $s\colon \mathrm{X} \to \mathrm{G}$ such that $\dim(\mathrm{X})$ is maximal. The best possible value for a given group G is $\dim(\mathrm{X}) = \lfloor \frac{\dim(\mathrm{G})}{2} \rfloor$. When can this be achieved?

Note that a subset S of an abelian group A is a Sidon set if and only if the induced map $\mathrm{S}^2/\mathfrak{S}_2 \to \mathrm{A}$ defined by $(x, y) \mapsto x+y$ is injective (where $\mathfrak{S}_2$ acts by permuting the two coordinates). Consider the variant definition of a Sidon morphism $s\colon \mathrm{X} \to \mathrm{G}$ where we ask that s be a morphism such that the induced map $s^{(2)}\colon \mathrm{X}^{(2)} \to \mathrm{G}$ from the symmetric square of X to G is a closed immersion. Again we have $2\dim(\mathrm{X}) \leqslant \dim(\mathrm{G})$, but we can see in this case that if G is an abelian variety, then equality is not possible. Indeed, this would imply that $s^{(2)}$ is an isomorphism, which is impossible (if $\dim(\mathrm{X}) \geqslant 2$, because $\mathrm{X}^{(2)}$ is then singular, and if X is a curve, because it would have to be smooth of genus 2, so G is an abelian surface, but for instance the second cohomology groups do not have the same dimension).

The result concerning jacobians of smooth projective curves can be generalized by considering either Rosenlicht's generalized jacobians (which appear in geometric class field theory, see the book of Serre [118]), or the Picard group of certain singular curves. The case of generalized jacobians is analyzed in complete generality in our paper [37]. We state the result here (see [37, Th. 1]).

Proposition 8.13. *Let k be a (not necessarily finite) field and let* C *be a smooth projective geometrically connected curve of genus g over k. Let $\mathfrak{m}$ be an effective divisor on* C *and* $\mathrm{J}_{\mathfrak{m}}$ *the associated generalized jacobian, which is a commutative algebraic group of dimension* $g + \max(\deg(\mathfrak{m}) - 1, 0)$. *Let δ be a divisor of degree* 1 *on* C *whose support does not intersect that of* $\mathfrak{m}$. *Let* $s\colon \mathrm{C} - \mathfrak{m} \to \mathrm{J}_{\mathfrak{m}}$ *be the morphism induced by the map* $x \mapsto (x) - \delta$ *on divisors.*

If $\dim(\mathrm{J}_{\mathfrak{m}}) \geqslant 2$, *then s is either a Sidon morphism or a symmetric Sidon morphism.*

If, moreover, $(\mathrm{C} - \mathfrak{m})(k)$ *is non-empty, then it is a symmetric Sidon set if and only if one of the following conditions hold:*

(1) $g = 1$ *and* $\deg(\mathfrak{m}) = 2$*; in this case, writing* $\mathfrak{m} = (p) + (q)$ *(where* p *and* q *are not necessarily* k*-points of* C*, but the divisor* $\mathfrak{m}$ *is assumed to be defined over* k*), the value* α *of* $s(x) + s(p + q - x)$ *for* $x \in (\mathrm{C} - \mathfrak{m})(k)$ *is independent of* x *and* $s((\mathrm{C} - \mathfrak{m})(k))$ *is an* α*-symmetric Sidon set.*

(2) $g \geqslant 2$*, the curve* C *is hyperelliptic, and either* $\deg(\mathfrak{m}) \leqslant 1$ *or* $\mathfrak{m} = (p) + (i(p))$ *for some* $p \in \mathrm{C}$*, where* i *is the hyperelliptic involution on* C*. In both of these cases, the value* α *of* $s(x) + s(i(x))$ *for* $x \in (\mathrm{C} - \mathfrak{m})(k)$ *is independent of* x *and* $s((\mathrm{C} - \mathfrak{m})(k))$ *is an* α*-symmetric Sidon set.*

Remark 8.14. (1) Both the generalized jacobians $\mathrm{J_S}$ and the Picard group scheme of an irreducible curve are connected commutative algebraic groups over k which may involve all types of groups (unipotent groups, abelian varieties and tori).

More precisely, the following results hold:

(1) Let C be a smooth projective curve of genus $g \geqslant 0$ over k and $\mathfrak{m}$ an effective divisor on C. Write $\mathfrak{m}$ in the form

$$\mathfrak{m} = \sum_{x \in \mathrm{Supp(S)}} n_x\,(x)$$

with $n_x \geqslant 1$. The generalized jacobian $\mathrm{J}_{\mathfrak{m}}$ is an extension

$$0 \to \mathrm{L}_{\mathfrak{m}} \to \mathrm{J}_{\mathfrak{m}} \to \mathrm{Jac(C)} \to 0$$

of the (usual) jacobian of C, with kernel $\mathrm{L}_{\mathfrak{m}} = \mathrm{R}_{\mathfrak{m}}/\mathbf{G}_m$, where $\mathrm{R}_{\mathfrak{m}}$ is isomorphic to a product

$$\mathrm{R}_{\mathfrak{m}} = \prod_{x \in \mathrm{Supp}(\mathfrak{m})} (\mathbf{G}_m \times \mathrm{V}_x)$$

with V_x unipotent of dimension $n_x - 1$, and with $\mathbf{G}_m$ embedded diagonally in $\mathrm{R}_{\mathfrak{m}}$ (see, e.g., [**118**, p. 2 and V.13, V.14]).

In particular, assuming that $g \geqslant 1$, the group $\mathrm{J}_{\mathfrak{m}}$ has non-trivial abelian, toric and unipotent parts as soon as the support of $\mathfrak{m}$ contains two distinct points, one of which at least has coefficient $\geqslant 2$.

(2) Let C be an irreducible projective curve C over an algebraically closed field. Let $\widetilde{\mathrm{C}} \to \mathrm{C}$ be the normalization of C, and for $x \in \mathrm{C}(k)$, define m_x to be the cardinality of the fiber of $\widetilde{\mathrm{C}} \to \mathrm{C}$ over x. Then $\mathrm{Pic}^0(\mathrm{C})$ has dimension $\dim \mathrm{H}^1(\mathrm{C}, \mathscr{O}_{\mathrm{C}})$, and it is an extension

$$0 \to \mathrm{K_C} \to \mathrm{Pic}^0(\mathrm{C}) \to \mathrm{Jac}(\widetilde{\mathrm{C}}) \to 0$$

of the jacobian of the normalization $\widetilde{\mathrm{C}}$, with kernel $\mathrm{K_C}$ which is an extension of a torus of dimension

$$\sum_{x \in \mathrm{C}-\mathrm{U}} (m_x - 1)$$

by a unipotent group, of dimension therefore equal to

$$\dim \mathrm{H}^1(\mathrm{C}, \mathscr{O}_{\mathrm{C}}) - g(\widetilde{\mathrm{C}}) - \sum_{x \in \mathrm{C}-\mathrm{U}} (m_x - 1)$$

(see, e.g., [**100**, Def. 5.13, Th. 7.5.19, Lemma 5.18]).

Note furthermore that these two classes of algebraic groups are closely related (e.g., any generalized jacobian $\mathrm{J}_{\mathfrak{m}}$ is the Picard group of some singular curve).

(2) All the examples of Sidon morphisms in Proposition 8.11 can be interpreted in terms of generalized jacobians. For instance, consider the curve $\mathrm{C} = \mathbf{P}^1$ over k, and the effective divisor $\mathrm{S} = (0) + 2(\infty)$, so that $\mathrm{U} = \mathbf{P}^1 - \{0, \infty\} = \mathbf{G}_m$. According to the above, the generalized jacobian J_S is isomorphic to $\mathrm{G} = (\mathbf{G}_m \times (\mathbf{G}_m \times \mathbf{G}_a))/\mathbf{G}_m^\Delta$, where the subgroup $\mathbf{G}_m^\Delta$ is embedded diagonally by $x \mapsto (x, (x, 0))$. An isomorphism $\varphi\colon \mathrm{J}_\mathrm{S} \to \mathrm{G}$ is given as follows: given a divisor E of degree 0 on $\mathbf{P}^1$, represent it as the divisor of a rational function $g\colon \mathbf{P}^1 \to \mathbf{P}^1$, and let

$$\varphi(\mathrm{E}) = (g(0), (g(\infty), \frac{g'}{g}(\infty)))$$

(this can be checked from the description in [118, p. 2 and V.13, V.14]). The morphism $\mathrm{G} \to \mathbf{G}_m \times \mathbf{G}_a$ given by $(x, (y, a)) \mapsto (xy^{-1}, a)$ is an isomorphism, and using it to identify G with $\mathbf{G}_m \times \mathbf{G}_a$, the formula above becomes $\varphi(\mathrm{E}) = (\frac{g(1)}{g(\infty)}, \frac{g'}{g}(\infty))$.

Consider then the morphism $\mathrm{U} = \mathbf{G}_m \to \mathrm{J}_\mathrm{S}$ defined using the divisor $\mathrm{D} = (1)$. Then the morphism $s_\mathrm{D}\colon x \mapsto (x) - (1)$ is given by $s_\mathrm{D}(x) = (x, 1-x)$ (take $g(t) = (t-x)/(t-1)$ to compute $\varphi((x)-(1))$). This is a Sidon morphism, the argument for this being identical with Proposition 8.11, (1).

We refer again to [37, § 2] for more discussion, in particular in comparison with the paper of Eberhard and Manners.

8.5. Gabber's torus trick

We discuss here another criterion to have a large tannakian group that also involves Sidon sets, but in a very different manner from their appearance in the previous sections. This criterion is difficult to apply for an individual object, but it leads to simple specialization results.

We use a version of Gabber's "torus trick" (see [69, Th. 1.0]). The following statement is specialized to the case of $\mathbf{SL}_r$ and written in the language of compact Lie groups.

THEOREM 8.15 (Gabber). *Let* V *be a finite-dimensional complex vector space of dimension* $r \geqslant 1$*, and let* $\mathbf{G}$ *be a connected semisimple compact subgroup of* $\mathbf{GL}(\mathrm{V})$ *that acts irreducibly on* V*. Let* D *be the subgroup consisting of the elements of* $\mathbf{GL}(\mathrm{V})$ *that are diagonal with respect to some basis, and let* $\chi_1, \ldots, \chi_r$ *be the characters* $\mathrm{D} \to \mathbf{C}^\times$ *giving the coefficients of the elements of* D*.*

Let $\mathrm{A} \subset \mathrm{D}$ *be a subgroup of the normalizer of* $\mathbf{G}$ *in* $\mathbf{GL}(\mathrm{V})$*. Let* $\mathrm{S} \subset \widehat{\mathrm{A}}$ *be the subset of the group of characters of* A *given by the restrictions to* A *of the diagonal characters* χ_i*. If* $|\mathrm{S}| = r$ *and* S *is a Sidon set in* $\widehat{\mathrm{A}}$*, then* $\mathbf{G} = \mathbf{SU}(\mathrm{V})$*.*

REMARK 8.16. Properly speaking, Gabber's original result implies here that G contains a maximal torus of $\mathbf{SU}(\mathrm{V})$, and the fact that G is semisimple and connected then implies that G is $\mathbf{SU}(\mathrm{V})$ (see, e.g., [13, p. 36, prop. 13]).

We emphasize that the subgroup A can be arbitrary: it may be finite, and need not be closed.

We can then deduce the following criterion.

PROPOSITION 8.17. *Let* G *be a connected commutative algebraic group over the finite field* k*. Let* M *be a simple perverse sheaf on* G *which is pure of weight* 0 *and of tannakian dimension* $r \geqslant 1$*. Assume that* M *is generically unramified.*

The geometric tannakian group $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ contains $\mathbf{SL}_r$ if and only there exists an unramified character $\chi \in \widehat{\mathrm{G}}(k_n)$ for some integer $n \geqslant 1$ such that the eigenvalues of $\mathbf{\Theta}_{\mathrm{M},k_n}(\chi)$ are distinct and form a Sidon set in $\mathbf{C}^\times$.

PROOF. Suppose that $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ contains $\mathbf{SL}_r$. Let $\mathrm{U} \subset \mathbf{SU}_r(\mathbf{C})$ be the set of matrices whose eigenvalues are distinct and form a Sidon set in $\mathbf{C}^\times$. This is an open set (for the Lie group topology), so that equidistribution implies

$$\liminf_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}} \sum_{1\leqslant n\leqslant \mathrm{N}} \frac{1}{|\mathrm{G}(k_n)|} \sum_{\mathbf{\Theta}_{\mathrm{M},k_n}(\chi)\in \mathrm{U}} 1 > 0,$$

and hence there exists $n \geqslant 1$ and $\chi \in \widehat{\mathrm{G}}(k_n)$ such that $\mathbf{\Theta}_{\mathrm{M},k_n}(\chi) \in \mathrm{U}$.

Conversely, if an unramified character $\chi \in \widehat{\mathrm{G}}(k_n)$ exists with $\mathbf{\Theta}_{\mathrm{M},k_n}(\chi) \in \mathrm{U}$, then we can apply Theorem 8.15 to the group A generated by a fixed element in the conjugacy class $\mathbf{\Theta}_{\mathrm{M},k_n}(\chi)$, and to the neutral component of the geometric tannakian group of M (which is normalized by $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$, since $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ is normal in $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ by Proposition 3.39 and its neutral component is a characteristic subgroup). ☐

In general, we do not have robust methods to check the existence of a character with the desired properties. However, we may combine this with a specialization argument.

PROPOSITION 8.18. *Let G be a connected commutative algebraic group over the finite field k. Let M be a simple perverse sheaf on G which is pure of weight 0 and of tannakian dimension $r \geqslant 1$. Assume that M is generically unramified. Let $f\colon \mathrm{G} \to \mathrm{H}$ be a morphism of commutative algebraic groups over k.*

Suppose that the object $\mathrm{N} = \mathrm{R}f_!\mathrm{M}$ is a geometrically simple perverse sheaf on H that is pure of weight 0, and suppose that $\chi \circ f$ is unramified for M whenever χ is unramified for N.

If the geometric tannakian group $\mathbf{G}_{\mathrm{N}}^{\mathrm{geo}}$ contains $\mathbf{SL}_r$, then $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ contains $\mathbf{SL}_r$.

PROOF. By Proposition 8.17, the assumption implies that there exists a character $\chi \in \widehat{\mathrm{H}}(k_n)$ unramified for N for which $\mathbf{\Theta}_{\mathrm{N},k_n}(\chi)$ has distinct eigenvalues forming a Sidon set. Since $\mathbf{\Theta}_{\mathrm{M},k_n}(\chi\circ f)$ has the same characteristic polynomial, the character $\chi \circ f \in \widehat{\mathrm{G}}(k_n)$ has the same property; by Proposition 8.17 again, it follows that $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ contains $\mathbf{SL}_r$. ☐

8.6. Recognition criteria for $\mathbf{E}_6$

We include here a criterion of Krämer to recognize the exceptional group $\mathbf{E}_6$ in one of its 27-dimensional faithful representations (we always mean by $\mathbf{E}_6$ the simply-connected form).

PROPOSITION 8.19 (Krämer). *Let $\mathbf{G}$ be a connected semisimple linear algebraic group over $\overline{\mathbf{Q}}_\ell$ or $\mathbf{C}$ and ϱ an irreducible faithful 27-dimensional representation of $\mathbf{G}$. If the 729-dimensional representation $\mathrm{End}(\varrho)$ of $\mathbf{G}$ contains an irreducible 78-dimensional subrepresentation, then $\mathbf{G}$ is isomorphic to the exceptional group $\mathbf{E}_6$ and ϱ is one of its two fundamental 27-dimensional representations.*

See [91, Lemma 4] for the proof. We will apply this in Section 11.2, although somewhat differently than we use Larsen's Alternative. The following criterion is closer to the spirit of the latter, and might have interesting applications (see again Section 11.2 for an attempt).

PROPOSITION 8.20. *Let* $\mathbf{G}$ *be a connected semisimple linear algebraic group over* $\overline{\mathbf{Q}}_\ell$ *or* $\mathbf{C}$*, and let* ϱ *be a faithful representation of* $\mathbf{G}$ *of dimension* 27*. Then* $\mathbf{G}$ *is isomorphic to the exceptional group* $\mathbf{E}_6$ *and* ϱ *to one of the two fundamental* 27*-dimensional representations of* $\mathbf{G}$ *if and only if* $\mathrm{M}_4(\mathbf{G}, \varrho) = 3$ *and* ϱ *is not self-dual.*

PROOF. Suppose first that $\mathbf{G} = \mathbf{E}_6$ and ϱ is one of its fundamental representations of dimension 27. These representations are not self-dual (see [11, Table 1, p. 213]). Using the Weyl dimension formula (see [12, Th. 2, p. 151] and [11, Pl. V, p. 260]), we see that the dimensions of the irreducible representations of $\mathbf{E}_6$ that may possibly occur in the 729-dimensional representation on $\mathrm{End}(\varrho)$ are 1, 27, 78, 351, 650. We know that the trivial representation appears once in $\mathrm{End}(\varrho)$, and that the 78-dimensional adjoint representation Ad appears at least once. But the equation

$$729 - 79 = 650 = 27a + 78b + 351c + 650d$$

has the unique non-negative integral solution $(a, b, c, d) = (0, 0, 0, 1)$ (looking modulo 3, it becomes $d \equiv 1 \pmod 3$). So we must have an isomorphism

$$\mathrm{End}(\varrho) \simeq \mathbf{1} \oplus \mathrm{Ad} \oplus \varrho_{650},$$

where ϱ_{650} has dimension 650, and hence the fourth moment $\mathrm{M}_4(\mathbf{E}_6, \varrho)$ is equal to 3. (This is also noted without proof by Katz [71, Rem. 1.2.3].)

We now prove the converse, and assume that $\mathrm{M}_4(\mathbf{G}, \varrho) = 3$ and ϱ is not self-dual. Since the fourth moment is $\leqslant 5$, the representation ϱ is irreducible (Theorem 8.5, (1)). Now let

$$\mathbf{G}_1 \times \cdots \times \mathbf{G}_k \to \mathbf{G} \xrightarrow{\varrho} \mathbf{GL}_{27}$$

be the representation obtained from the decomposition of the algebraic universal covering of $\mathbf{G}$ in product of almost simple groups. This composition decomposes as an external tensor product

$$\varrho_1 \boxtimes \cdots \boxtimes \varrho_k$$

of irreducible representations of $\mathbf{G}_i$. We then have

$$3 = \mathrm{M}_4(\mathbf{G}, \varrho) = \prod_{i=1}^{k} \mathrm{M}_4(\mathbf{G}_i, \varrho_i)$$

by (8.3). The condition $\mathrm{M}_4(\mathbf{G}_i, \varrho_i) = 1$ is impossible (since it implies that $\dim(\varrho_i) = 1$, and hence ϱ_i would be trivial and this contradicts the faithfulness assumption), so we have a single factor $\mathbf{G}_1$.

The representation ϱ_1 is not self-dual, which implies that the root system of $\mathbf{G}_1$ (and hence of $\mathbf{G}$) can only be of type E_6, or A_l for $l \geqslant 2$ or D_l with $l \geqslant 3$ odd (see, e.g. [12, p. 132, prop. 12], combined with the fact that the longest element of the Weyl group acts by $-\mathrm{Id}$ for the other simple root systems).

The groups of type A_l with $l \geqslant 2$ which have a 27-dimensional irreducible representation are of type A_2 (the representation with highest weight $2\varpi_1 + 2\varpi_2$, in the standard notation of Bourbaki) or A_{26} (the standard representation). In the first case, the representation is actually self-dual, and in the second case, the fourth moment is equal to 2, so these are excluded (in particular, groups of type $\mathrm{D}_3 = \mathrm{A}_2$ are also excluded).

Let $l \geqslant 5$ be an odd integer. The representations of groups of type D_l which are not self-dual and have smallest possible dimension are the half-spin representations of dimension 2^{l-1} (see [12, p. 210]). Thus only D_5 could possibly give rise to a representation of dimension 27; but one can check that there is no representation of this dimension of a group of type D_5 (e.g., because of the Weyl Dimension Formula, see [12, Th. 2, p. 151]).

We conclude that the group $\mathbf{G}_1$ must be E_6; since its 27-dimensional representations are faithful, the projection $\mathbf{G}_1 \to \mathbf{G}$ is an isomorphism. $\square$

Remark 8.21. This criterion also shows that it may happen that the fourth moment $\mathrm{M}_4(\mathbf{G}, \mathrm{V})$ of a representation of a group $\mathbf{G}$ is equal to 3, but the representation V is not self-dual.

8.7. Finiteness of tannakian groups on abelian varieties

The following result strenghtens Theorem 5.7 in situations when one can apply Larsen's Alternative to the fourth moment on abelian varieties.

Proposition 8.22. *Let* M *be a geometrically simple perverse sheaf of weight zero on a simple abelian variety* A *over* k. *Let* d *be the tannakian dimension of* M. *If the group* $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ *is virtually central, then the object* $\mathrm{End}(\mathrm{M})$ *in* $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$ *is punctual and the fourth moment of* $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ *is equal to* d^2.

Proof. We observe that $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}/(\mathrm{Z}\cap\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}})$ is the arithmetic tannakian group of the arithmetically semisimple object $\mathrm{End}(\mathrm{M})$, and apply Theorem 5.7 to obtain the first conclusion.

In particular, this implies that $\mathrm{End}(\mathrm{M})$, as a representation of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$, is a direct sum of characters. From (8.4), applied to a decomposition in sum of characters, it follows that

$$\mathrm{M}_4(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}) \geqslant d^2.$$

On the other hand, let K be a maximal compact subgroup of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}(\mathbf{C})$, and μ its Haar probability measure. By (8.2) and Schur's Lemma, we derive the inequality

$$\mathrm{M}_4(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}) = \int_{\mathrm{K}} |\operatorname{Tr}(g)|^4 d\mu(g) \leqslant d^2 \int_{\mathrm{K}} |\operatorname{Tr}(g)|^2 d\mu(g) = d^2,$$

which concludes the proof. $\square$

Remark 8.23. There may exist irreducible subgroups $\mathbf{G}$ of $\mathbf{GL}(\mathrm{V})$ with fourth moment equal to $\dim(\mathrm{V})^2$. Indeed, this is the case, for instance, of any group which has the property that all irreducible representations with trivial central character have dimension 1, since only such representations can appear in the decomposition of $\mathrm{End}(\mathrm{V})$. A concrete example is given by finite Heisenberg groups (see, e.g., the paper [53] of Gérardin for the relevant facts).

CHAPTER 9

The product of the additive and the multiplicative groups

9.1. Introduction

In this chapter, we consider what is perhaps the simplest case of our equidistribution results beyond those of the additive group and the multiplicative group, namely the case of $\mathrm{G} = \mathbf{G}_m \times \mathbf{G}_a$. Concretely, this means that we are looking at the distribution of two-parameter exponential sums of the type

$$\frac{1}{p} \sum_{(x,y)\in \mathbf{F}_p^\times \times \mathbf{F}_p} \chi(x) e\Big(\frac{ay}{p}\Big) t(x,y), \tag{9.1}$$

where p is a prime number, χ a complex-valued multiplicative character of the finite field $\mathbf{F}_p$, and the function t is a trace function on $\mathbf{G}_m \times \mathbf{G}_a$ over $\mathbf{F}_p$. In practice, we mostly consider the analogues over extensions of $\mathbf{F}_p$ of degree $n \to +\infty$, but we will also discuss an horizontal statement in Corollary 9.16.

Throughout this chapter, we denote by k a finite field with an algebraic closure $\bar{k}$, and by ℓ a prime different from the characteristic of k. We also fix a non-trivial additive character $\psi\colon k \to \overline{\mathbf{Q}}_\ell^\times$. For every $n \geqslant 1$, we define $\psi_n = \psi \circ \mathrm{Tr}_{k_n/k}$, a non-trivial additive character of the extension k_n of k of degree n in $\bar{k}$.

We always denote by G the group $\mathbf{G}_m \times \mathbf{G}_a$, and we will denote by p_1 and p_2 the projections $\mathrm{G} \to \mathbf{G}_m$ and $\mathrm{G} \to \mathbf{G}_a$. For any $n \geqslant 1$ and any pair (χ, a) of an ℓ-adic character of $k_n^\times$ and an element of k_n, we will sometimes denote by $\langle \chi, a\rangle$ the character $(x,y) \mapsto \chi(x)\psi_n(ay)$ of $\mathrm{G}(k_n)$, and by $\mathscr{L}_{\chi,a}$ the corresponding ℓ-adic character sheaf.

We first state the specialization of Theorem 4.8 to this case, showing that there is always *some* equidistribution statement for the sums (9.1) in the vertical direction.

Theorem 9.1. *Let* M *be an arithmetically semisimple ℓ-adic perverse sheaf on $\mathbf{G}_m \times \mathbf{G}_a$ over k, with trace function over k_n denoted $t(x,y;k_n)$. Assume that* M *is pure of weight zero.*

There exist an integer $r \geqslant 0$ and a reductive subgroup $\mathbf{G} \subset \mathbf{GL}_r$ such that the sums

$$\mathrm{S}_n(a,\chi) = \sum_{(x,y)\in k_n^\times \times k_n} \chi(x)\psi_n(ay) t(x,y;k_n),$$

where (a,χ) are pairs of an element of k_n and a multiplicative character of $k_n^\times$, become equidistributed on average as $n \to +\infty$, with limit measure the image under the trace of the Haar probability measure on a maximal compact subgroup of $\mathbf{G}(\mathbf{C})$.

With $\mathrm{G} = \mathbf{G}_m \times \mathbf{G}_a$, $\mathbf{G} = \mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ and r the tannakian dimension of M, this is Theorem 4.8 for the object M.

The remainder of this chapter will be dedicated to the exploration of special examples. We consider in particular examples where the object M (and hence the trace function in (9.1)) is supported on the “diagonal” $y = x$. Larsen's Alternative will allow us to prove, with surprisingly little computation, that in this case the group $\mathbf{G}$ in Theorem 9.1 is always essentially as large as possible.

More precisely, we first define $\Delta\colon \mathbf{G}_m \to \mathbf{G}_m \times \mathbf{G}_a$ to be the diagonal embedding $x \mapsto (x, x)$; this is a closed immersion. Define the diagonal in $\mathbf{G}_m \times \mathbf{G}_a$ to be the image of Δ, and $j\colon \mathbf{G}_m \to \mathbf{G}_a$ to be the open immersion.

For any morphism $\lambda\colon \mathbf{G}_m \to \mathrm{G}$, for an integer $n \geqslant 1$ and a pair $(\chi, a) \in \widehat{\mathbf{G}}_m(k_n) \times k_n$, we denote by $\mathscr{L}^{\lambda}_{\chi,a}$ the sheaf $\lambda_*(\mathscr{L}_\chi \otimes j^*\mathscr{L}_{\psi(ay)})$ on $\mathrm{G}_{\bar{k}}$.

Theorem 9.2. *With notation as in Theorem 9.1, suppose that the input object* M *is geometrically simple and supported on the diagonal. Suppose that* M *is not punctual and not geometrically isomorphic to* $\mathscr{L}^{\Delta}_{\eta,b}[1]$ *for some* $\langle \eta, b\rangle \in \widehat{\mathrm{G}}(k_n)$*. Then the integer* r *is* $\geqslant 2$ *and the group* $\mathbf{G}$ *contains* $\mathbf{SL}_r$.

We will see that we can in fact fairly often show that $\mathbf{G} = \mathbf{GL}_r$, and in that setting the sums

$$\mathrm{S}_n(\chi, a) = \sum_{x \in k_n^\times} \chi(x)\psi_n(ax)t_{\mathrm{M}}(x, x; k_n)$$

tend to be distributed like the trace of a random matrix in $\mathbf{U}_r(\mathbf{C})$, almost independently of the input object M.

Example 9.3. Let $a \in k$, and let χ be a multiplicative character of k. The *Kloosterman–Salié sums* over k are defined by

$$\mathrm{Kl}(\chi, a; k) = \frac{1}{\sqrt{|k|}} \sum_{x \in k^\times} \chi(x)\psi(ax + x^{-1}).$$

These sums have been studied extensively, in particular because of their applications in the analytic theory of modular forms (see the surveys [85] or [59]). We may fix χ, obtaining a family of exponential sums parameterized by a: this is the discrete additive Fourier transform of the function which is 0 at $x = 0$ and otherwise maps x to $\chi(x)\psi(x^{-1})$. Alternatively, we may fix a, and then we are considering the discrete Mellin transform of the function $x \mapsto \psi(ax + x^{-1})$. These are both well-known examples of their respective theories, and their distribution properties are as follows:

– If χ is the trivial character, we have *Kloosterman sums*, which are equidistributed with respect to the Sato–Tate measure, that is, to the image of the Haar probability measure on the space of conjugacy classes of $\mathbf{SU}_2(\mathbf{C})$; this reflects the fact that the geometric and arithmetic monodromy groups for the ℓ-adic Fourier transform of the extension by zero of $\mathscr{L}_{\psi(x^{-1})}$ are both equal to $\mathbf{SL}_2$, by work of Katz [68, Thm. 11.1].
– If the characteristic p of k is odd and χ is the character of order 2, then we have *Salié sums*, whose arithmetic monodromy group is a finite subgroup of $\mathbf{SL}_2$, isomorphic to a semi-direct product of $\mathbf{F}_p^\times$ and $\mathbf{Z}/2\mathbf{Z}$ (this can be deduced from [69, Cor. 8.9.2], which shows that the corresponding sheaf is Kummer-induced). The finiteness of the group reflects the fact that Salié sums can be computed elementarily (see, e.g., [14, p. 288, Exerc. 50]), and is also an analogue of the fact that Bessel functions with half-integral index are elementary functions (see, e.g., [14, p. 269, Exerc. 20]).
– If $p \geqslant 7$ and χ is fixed, but χ^2 is non-trivial, then the neutral component of the geometric monodromy group is $\mathbf{SL}_2$, but the determinant of geometric monodromy group is not trivial, and more precisely has order equal to the order of χ; see [69, Th. 8.11.3, Lemma 8.11.6].

– If instead we fix $a \in \mathbf{F}_p^\times$ and vary the multiplicative character χ, then the geometric tannakian group (which coincides with the one associated by Katz's theory in [74], see Appendix B) contains $\mathbf{SL}_2$ for all a. Indeed, the sheaf $\mathscr{L}_{\psi(ax+x^{-1})}$ on $\mathbf{G}_m$ is not geometrically isomorphic to any of its non-trivial multiplicative translates by the same argument as in the proof of [74, Th. 14.2], and hence the characterisation in [74, Cor. 8.3] shows that this tannakian group is Lie-irreducible; since it is a subgroup of $\mathbf{GL}_2$, it necessarily contains $\mathbf{SL}_2$. If $a = -1$, then the arithmetic and geometric tannakian groups are both equal to $\mathbf{SL}_2$ (this is the case of the Evans sums in [74, Th. 14.2]). In general, the tannakian determinant is geometrically isomorphic to the skyscraper sheaf at $\alpha = -1/a$ (so its Mellin transform is proportional to $\chi \mapsto \chi(\alpha)$) by [74, Th. 21.1]. Letting n denote the order of α in the finite group $\mathbf{F}_p^\times$, it follows that the geometric tannakian group consists of those matrices whose determinant is an nth root of unity.

– If $a = 0$ and we vary χ, we have Gauss sums; the arithmetic and tannakian groups are equal to $\mathbf{GL}_1$.

The relation with Theorem 9.1 is the following: we are considering the finite field k, and the perverse sheaf M of weight zero is $\mathrm{M} = \Delta_*\mathscr{L}[1](1/2)$, where $\mathscr{L}$ is the lisse sheaf $\mathscr{L} = \mathscr{L}_{\psi(x^{-1})}$ of rank one; it is geometrically simple, and perverse since Δ is a closed immersion (Corollary A.8). The group $\mathbf{G}$ of Theorem 9.1 is then $\mathbf{GL}_2$ (as follows from Theorem 9.2).

Note that when we specialize to a fixed character χ or a fixed a, we obtain a monodromy group or a tannakian group that is a subgroup of $\mathbf{G}$ (as seems natural), which has the following property: the identity component of the derived group $\mathbf{G}'$ is independent of χ (resp. a), except for a finite exceptional set. In fact, the exceptional set for fixed χ contains only the Legendre character (if p is odd), and the exceptional set for fixed a contains only $a = 0$.

Note also that when we vary χ for a fixed, only the neutral component of the identity of the geometric tannakian group is independent of χ, but the tannakian group is usually not connected.

Finally, observe that here *none* of the "specialized" geometric tannakian groups for either $\mathbf{G}_a$ or $\mathbf{G}_m$ coincides with the geometric tannakian group $\mathbf{G} = \mathbf{GL}_2$. However, in an intuitive sense, the collection of all of them "generate" this group.

We expect these phenomena to be very general, and we will consider such questions in greater generality in later works.

Remark 9.4. (1) Theorem 9.2 applies for instance to one-variable exponential sums of the form

$$\frac{1}{\sqrt{|k|}} \sum_{x \in k^\times} \chi(x)\eta(g(x))\psi(ax + f(x))$$

for suitable polynomials f and g and for a multiplicative character η.

It is worth noting that, even if we are only interested in the distribution of these one-variable sums (and not in the more general sums allowed by Theorem 9.1 with a two-variable trace function), the *proof* of Theorem 9.1, passing through the tannakian machinery, requires the consideration of objects supported on all of $\mathbf{G}_m \times \mathbf{G}_a$, simply because the convolution of two objects on $\mathbf{G}_m \times \mathbf{G}_a$ that are supported on the diagonal Δ will be supported on the product set $\Delta \cdot \Delta = \mathbf{G}_m \times \mathbf{G}_a$.

(2) Remark 8.14, (2), suggests a different interpretation of Theorem 9.2. Indeed, using this remark, we can view $\mathbf{G}_m \times \mathbf{G}_a$ as a generalized jacobian of $\mathrm{C} = \mathbf{P}^1$ and the diagonal morphism $\mathbf{G}_m \to \mathbf{G}_m \times \mathbf{G}_a$ as a morphism of the type $x \mapsto (x) - (\mathrm{D})$ for a suitable divisor D on $\mathbf{G}_m$. For a

perverse sheaf of weight zero on $\mathbf{G}_m$, we have the arithmetic Mellin transform

$$\chi \mapsto \sum_{x\in k_n^\times} t_{\mathrm{M}}(x;k_n)\chi(x)$$

as in the work of Katz, which may have a variety of tannakian groups (see [74, Ch. 14 to 27] for examples involving for instance $\mathbf{SL}_n$, $\mathbf{GL}_n$, $\mathbf{O}_{2n}$, $\mathbf{SO}_n$, $\mathbf{Sp}_{2g}$ and G_2). Then the further operation of twisting by an additive character ψ leads to the sums

$$(\chi,\psi) \mapsto \sum_{x\in k_n^\times} t_{\mathrm{M}}(x;k_n)\chi(x)\psi(x)$$

which correspond to the diagonal object $\Delta_*\mathrm{M}$ on the generalized jacobian $\mathbf{G}_m\times\mathbf{G}_a$. The theorem is then, analytically, an instance of the common situation where twisting an exponential sum by a generic additive character leads to "more random" exponential sums (here, replacing a potentially complicated tannakian group on $\mathbf{G}_m$ by one that in almost all cases contains the special linear group). Note however that the tannakian dimension may change when adding this extra twist.

EXAMPLE 9.5. The following case of two-variable equidistribution has been studied "by hand" by Kowalski and Nikeghbali [89, §4.1, Th. 11]. Let $d>5$ be a fixed integer, and consider the sums

$$\mathrm{S}(\chi,a)=\frac{1}{\sqrt{|k|}}\sum_{t\in k}\chi(t^d-dt-a)$$

where the character χ is extended by $\chi(0)=0$ if χ is non-trivial and $\chi(0)=1$ if χ is trivial.

We can express these sums as Mellin transforms, namely

$$\mathrm{S}(\chi,a)=\sum_{x\in k^\times}\sum_{y\in k}\widehat{\mathrm{S}}(x,y)\chi(x)\psi(ay)$$

where

$$\widehat{\mathrm{S}}(x,y)=\frac{1}{|\mathrm{G}(k)|}\sum\sum_{\langle\chi,a\rangle\in\widehat{\mathrm{G}}(k)}\overline{\chi(x)}\psi(-ay)\mathrm{S}(\chi,a).$$

We compute then

$$\begin{aligned}\widehat{\mathrm{S}}(x,y)&=\frac{1}{\sqrt{|k|}}\frac{1}{|\mathrm{G}(k)|}\sum\sum_{\langle\chi,a\rangle\in\widehat{\mathrm{G}}(k)}\overline{\chi(x)}\psi(-ay)\sum_{t\in k}\chi(t^d-dt-a)\\&=\frac{1}{\sqrt{|k|}}\frac{1}{|\mathrm{G}(k)|}\sum_{t\in k}\sum_{a\in k}\psi(-ay)\sum_{\chi}\overline{\chi(x)}\chi(t^d-dt-a)\\&=\frac{1}{|k|^{3/2}}\sum\sum_{\substack{t\in k,a\in k\\t^d-dt-a=x}}\psi(-ay)=\frac{1}{|k|^{3/2}}\sum_{t\in k}\psi(-y(t^d-dt-x)).\end{aligned}$$

Note that this trace function is not of diagonal type. It was proved however in [89] that when $|k|\to+\infty$ (including the horizontal case where $k=\mathbf{F}_p$ with $p\to+\infty$), the sums $\mathrm{S}(\chi,a)$ become equidistributed like the trace of random matrices in the unitary group $\mathbf{U}_{d-1}(\mathbf{C})$. This was done by applying Deligne's equidistribution theorem, and the computation of the relevant monodromy group by Katz, for each fixed χ, and then averaging over χ.

It would be interesting to recover this result directly from Theorem 9.1 (with $\mathbf{G}=\mathbf{GL}_{d-1}$), but it is not obvious how to do so: the reader can check that the computation of the fourth moment, for instance, is not at all straightforward.

Remark 9.6. Finally we remark that since the key tool to compute tannakian groups for objects supported on the diagonal will be Larsen's Alternative combined with the Sidon property of the diagonal, one can prove similar results for objects of the form $[x \mapsto (x, x^3)]_*\mathrm{M}$ on $\mathbf{G}_a^2$, for M on $\mathbf{G}_a$ (in characteristic $\neq 3$), and objects of the form $[x \mapsto (x, 1-x)]_*\mathrm{M}$ on $\mathbf{G}_m^2$ for M on $\mathbf{G}_m - \{1\}$ (see Proposition 8.11, (5) and (6)). The corresponding exponential sums are of the form

$$\sum_{x \in k_n} t_{\mathrm{M}}(x; k_n)\psi_n(ax + bx^3)$$

and

$$\sum_{x \in k_n^\times - \{1\}} t_{\mathrm{M}}(x; k_n)\chi_1(x)\chi_2(1-x).$$

respectively.

9.2. Tannakian group for diagonal objects

We first compute the tannakian dimension r for a perverse sheaf on $\mathrm{G} = \mathbf{G}_m \times \mathbf{G}_a$ which is supported on the diagonal.

Lemma 9.7. *Let $\mathrm{M} = \Delta_*(\mathscr{M})[1]$ for some geometrically irreducible middle extension sheaf $\mathscr{M}$ on $\mathbf{G}_m$.*

(1) *The tannakian dimension r of the object M is given by the formula*

$$r = \sum_\lambda \max(0, \lambda - 1) + \sum_{x \in \bar{k}^\times} (\mathrm{swan}_x(\mathscr{M}) + \mathrm{drop}_x(\mathscr{M})) + \mathrm{rank}(\mathscr{M}) + \mathrm{swan}_0(\mathscr{M}), \tag{9.2}$$

where λ runs over the breaks of $\mathscr{M}$ at infinity, in the sense of [68, Ch. 1], counted with multiplicity.

(2) *We have $r = 1$ if and only if $\mathrm{M} = \mathscr{L}_{\eta,b}^{\Delta}[1]$ for some $\langle \eta, b\rangle \in \widehat{\mathrm{G}}$.*

(3) *For all but finitely many $a \in \bar{k}$, the tannakian dimension of $\mathrm{M}_a = p_{1,*}\mathrm{M} \otimes j^*\mathscr{L}_{\psi(ax)}$ on $\mathbf{G}_{m,k(a)}$ is equal to r.*

Proof. (1) By Proposition 3.16, it is enough to determine the "generic" value of the dimension of the cohomology space

$$\mathrm{H}_c^0(\mathrm{G}_{\bar{k}}, \mathrm{M} \otimes p_1^*\mathscr{L}_\chi \otimes p_2^*\mathscr{L}_{\psi(ay)})$$

as χ varies in $\widehat{\mathbf{G}}_m$ and a in $\bar{k}$. We have a canonical isomorphism

$$\mathrm{H}_c^0(\mathrm{G}_{\bar{k}}, \mathrm{M} \otimes p_1^*\mathscr{L}_\chi \otimes p_2^*\mathscr{L}_{\psi(ay)}) = \mathrm{H}_c^1(\mathbf{G}_{m,\bar{k}}, \mathscr{M} \otimes \mathscr{L}_\chi \otimes j^*\mathscr{L}_{\psi(ax)}),$$

If χ is non-trivial, this space is also isomorphic to

$$\mathrm{H}_c^1(\mathbf{A}_{\bar{k}}^1, j_!(\mathscr{M} \otimes \mathscr{L}_\chi) \otimes \mathscr{L}_{\psi(ax)}).$$

For all but at most one value of χ, the sheaf $j_!(\mathscr{M} \otimes \mathscr{L}_\chi)$ is a Fourier sheaf in the sense of [69, (7.3.5)] (i.e., a middle extension sheaf $\mathscr{F}$ such that Deligne's Fourier transform is also a middle extension sheaf). Hence, the space $\mathrm{H}_c^1(\mathbf{A}_{\bar{k}}^1, j_!(\mathscr{M} \otimes \mathscr{L}_\chi) \otimes \mathscr{L}_{\psi(ax)})$ is the stalk at a of the Fourier transform of $j_!(\mathscr{M} \otimes \mathscr{L}_\chi)$, and its generic value r_χ as a varies in $\bar{k}$ is computed in [69, Lemma 7.3.9, (2)], namely

$$r_\chi = \sum_\lambda \max(0, \lambda - 1) + \sum_{x \in \bar{k}} (\mathrm{swan}_x(j_!(\mathscr{M} \otimes \mathscr{L}_\chi)) + \mathrm{drop}_x(j_!(\mathscr{M} \otimes \mathscr{L}_\chi)),$$

where λ runs over the breaks at ∞ of $j_!(\mathscr{M}\otimes\mathscr{L}_\chi)$, counted with multiplicity. Since $\mathscr{L}_\chi$ is lisse on $\mathbf{G}_m$, the formulas

$$\mathrm{swan}_x(j_!(\mathscr{M}\otimes\mathscr{L}_\chi))=\mathrm{swan}_x(\mathscr{M})\qquad \mathrm{drop}_x(j_!(\mathscr{M}\otimes\mathscr{L}_\chi))=\mathrm{drop}_x(\mathscr{M})$$

hold for any $x\in\bar{k}^\times$. Since $\mathscr{L}_\chi$ is tamely ramified at 0 for χ non-trivial, we have

$$\mathrm{swan}_0(j_!(\mathscr{M}\otimes\mathscr{L}_\chi))=\mathrm{swan}_0(\mathscr{M})\qquad \mathrm{drop}_0(j_!(\mathscr{M}\otimes\mathscr{L}_\chi))=\mathrm{rank}(\mathscr{M})$$

for χ non-trivial, which leads to (9.2).

(2) Since $\mathrm{rank}(\mathscr{M})\geqslant 1$, and all terms in the sum (9.2) are non-negative, we deduce that the condition $r=1$ may hold only if $\mathscr{M}$ has rank 1 and $\mathscr{M}$ is lisse on $\mathbf{G}_m$, tame at 0, and has (unique) break at most 1 at ∞. Twisting by a suitable Kummer sheaf, we may then assume that $\mathscr{M}$ is lisse on $\mathbf{A}^1$, and it must then be geometrically isomorphic to an Artin–Schreier sheaf, which by untwisting implies that M is geometrically isomorphic to some $\mathscr{L}^{\Delta}_{\eta,b}$.

(3) For the object $\mathrm{M}_a=\mathscr{M}[1]\otimes j^*\mathscr{L}_{\psi(ax)}$ on $\mathbf{G}_{m,k(a)}$, the tannakian dimension is its compactly-supported Euler–Poincaré characteristic, which is equal to

$$r_a=\mathrm{swan}_0(\mathscr{M}\otimes j^*\mathscr{L}_{\psi(ax)})+\mathrm{swan}_\infty(\mathscr{M}\otimes j^*\mathscr{L}_{\psi(ax)})\\ +\sum_{x\in\bar{k}^\times}(\mathrm{swan}_x(\mathscr{M}\otimes j^*\mathscr{L}_{\psi(ax)})+\mathrm{drop}_x(\mathscr{M}\otimes j^*\mathscr{L}_{\psi(ax)}))\tag{9.3}$$

(see (C.12)). Since $\mathscr{L}_{\psi(ax)}$ is lisse on $\mathbf{G}_a$, the formulas

$$\mathrm{swan}_x(\mathscr{M}\otimes j^*\mathscr{L}_{\psi(ax)})=\mathrm{swan}_x(\mathscr{M}),\qquad \mathrm{drop}_x(\mathscr{M}\otimes j^*\mathscr{L}_{\psi(ax)}))=\mathrm{drop}_x(\mathscr{M})$$

hold for $x\in\bar{k}$.

Assume that $a\neq 0$. Let λ be a break of $\mathscr{M}$ at infinity, and V_λ the corresponding break-space. Then $\mathrm{V}_\lambda\otimes\mathscr{L}_{\psi(ax)}$ coincides with the μ-break-space W_μ of $\mathscr{M}\otimes j^*\mathscr{L}_{\psi(ax)}$ where $\mu=\max(1,\lambda)$, except possibly if $\lambda=1$ and $\mathscr{L}_{\psi(-ax)}$ occurs in V_λ. Thus, for all but finitely many a, we have

$$\begin{aligned}\mathrm{swan}_\infty(\mathscr{M}\otimes j^*\mathscr{L}_{\psi(ax)})=\sum_\mu\mu\dim\mathrm{W}_\mu&=\sum_\lambda\dim\mathrm{V}_\lambda+\sum_{\lambda>1}(\lambda-1)\dim\mathrm{V}_\lambda\\&=\mathrm{rank}(\mathscr{M})+\sum_\lambda\max(0,\lambda-1),\end{aligned}$$

which leads to $r_a=r$ by comparing (9.3) with (9.2). ☐

Remark 9.8. We will classify all objects of tannakian dimension 1 in Section 9.4, and the diagonal objects of tannakian dimension 2 in Section 9.3.

We continue with a lemma to exclude finite tannakian groups in the diagonal situation. The first step is to exploit the specific shape of G to understand the structure of the set of characters which are not Frobenius unramified for suitable objects (or which are ramified, for objects which are generically unramified).

Lemma 9.9. *Let* M *be a perverse sheaf on* $\mathrm{G}=\mathbf{G}_m\times\mathbf{G}_a$ *and* N *an object of* $\langle\mathrm{M}\rangle^{\mathrm{ari}}$ *which is arithmetically semisimple and pure of weight* 0. *For all but finitely many* $a\in\bar{k}$, *the set of* $\chi\in\widehat{\mathbf{G}}_m$ *such that* $\langle\chi,a\rangle$ *is not Frobenius-unramified for* N *is finite.*

In particular, if M *has finite arithmetic tannakian group, then for all but finitely many* $a\in\bar{k}$, *the set of* $\chi\in\widehat{\mathbf{G}}_m$ *such that* $\langle\chi,a\rangle$ *is ramified is finite.*

Proof. The first statement follows immediately from the proof of Proposition 3.37 combined with Theorem 2.30.

The last statement follows from the first as in the proof of Corollary 3.38. ☐

Lemma 9.10. *Let* $\mathrm{C} \subset \mathrm{G} = \mathbf{G}_m \times \mathbf{G}_a$ *be a line given by* $y = \alpha x$ *where* $\alpha \in k^\times$.

Let M *be a geometrically simple perverse sheaf on* $\mathbf{G}_m \times \mathbf{G}_a$ *supported on* C *and of weight zero. Assume that the arithmetic tannakian group* $\mathbf{G}$ *of* M *is finite. Then* M *is punctual.*

Proof. The assumption implies that M is generically unramified by Corollary 3.38.

We assume that M is not punctual to get a contradiction. Then M is, up to twist and shift, the pushforward to G of a middle extension sheaf $\mathscr{M}$ on $\mathbf{G}_m \simeq \mathrm{C}$.

For all a, we denote $\mathscr{M}_a = \mathscr{M} \otimes j^*\mathscr{L}_{\psi(ax)}$; then $\mathrm{M}_a = \mathscr{M}_a[1](1/2)$ is a perverse sheaf on $\mathbf{G}_m$.

By Lemma 9.9, there exists $n \geqslant 1$ and $a \in k_n$ such that for all but finitely many $\chi \in \widehat{\mathbf{G}}_m$, the character $\langle \chi, a\rangle$ is unramified for M. The action of the Frobenius automorphism of k_n on the space

$$\mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M} \otimes \mathscr{L}_{\chi,a}) = \mathrm{H}^0_c(\mathbf{G}_{m,\bar{k}}, \mathrm{M}_a \otimes \mathscr{L}_\chi)$$

is then by assumption of finite order bounded independently of χ. The corresponding unitary Frobenius elements $\mathbf{\Theta}_{\mathrm{M}_a, k_{nm}}(\chi)$, for $m \geqslant 1$, are then dense in a maximal compact subgroup K of the complex points of the arithmetic tannakian group of the perverse sheaf M_a on $\mathbf{G}_m$ by Corollary 4.14. It follows that K, and hence also $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}_a}$, is a finite group since a compact real Lie group has no non-trivial small subgroup. By Katz's results on finite tannakian groups on $\mathbf{G}_m$ (see Theorem B.2), this would imply that the perverse sheaf M_a is punctual, which is a contradiction. ☐

We will now prove a slightly more general statement than Theorem 9.2.

Theorem 9.11. *Let* $\lambda\colon \mathbf{G}_m \to \mathbf{G}_m \times \mathbf{G}_a$ *be the closed embedding* $\lambda(x) = (x, \alpha x)$ *for some* $\alpha \in k^\times$ *and let* C *be its image.*

Let M *be a geometrically simple perverse sheaf on* $\mathbf{G}_m \times \mathbf{G}_a$ *supported on* C *and of weight zero. Assume that* M *is not punctual, and that* M *is not geometrically isomorphic to* $\mathscr{L}^\lambda_{\eta,b}[1](1/2)$ *for some* $\langle \eta, b\rangle \in \widehat{\mathrm{G}}(k)$.

Let $r \geqslant 0$ *be the tannakian dimension of* M *and denote* $\mathbf{G} = \mathbf{G}^{\mathrm{ari}}_{\mathrm{M}} \subset \mathbf{GL}_r$.

We then have $r \geqslant 2$, *the group* $\mathbf{G}$ *contains* $\mathbf{SL}_r$ *and the standard representation of* $\mathbf{G}$ *in* $\mathbf{GL}_r$ *is not self-dual.*

Note that the last item implies in particular that $\mathbf{G}$ cannot be equal to $\mathbf{SL}_2$ or $\pm\mathbf{SL}_2 = \mathbf{SL}_2 \cup \{-\mathrm{Id}\}\,\mathbf{SL}_2$.

Proof. We may assume that $\alpha = 1$. We first note that our assumptions and Lemma 9.7 imply that $r \geqslant 2$ (otherwise, M would be punctual or geometrically isomorphic to some perverse sheaf $\mathscr{L}^\lambda_{\eta,b}[1](1/2)$).

We will apply Larsen's Alternative. The closed immersion λ is a Sidon morphism (Proposition 8.11, (1)), and therefore we have $\mathrm{M}_4(\mathbf{G}) = 2$ by Proposition 8.8 (since $r \geqslant 2$).

Our assumptions therefore imply that $\mathrm{M}_4(\mathbf{G}) = 2$. By Larsen's Alternative (Theorem 8.5, (3)), it follows that either $\mathbf{G}$ contains $\mathbf{SL}_r$, or $\mathbf{G}/\mathbf{G} \cap \mathrm{Z}$ is finite, where $\mathrm{Z} \subset \mathbf{GL}_r$ is the group of scalar matrices. We must show that this second case actually does not arise. We proceed by contradiction, assuming therefore that $\mathbf{G}/\mathbf{G} \cap \mathrm{Z}$ is finite.

The intersection $\mathbf{G} \cap \mathrm{Z}$ is either finite or equal to Z. In the first case, the group $\mathbf{G}$ would be finite, so that the object M would be punctual by Lemma 9.10, which contradicts our assumptions.

So we are left with the case $\mathbf{G} \cap \mathrm{Z} = \mathrm{Z}$. The object $\mathrm{End}(\mathrm{M})$ of $\langle \mathrm{M}\rangle^{\mathrm{ari}}$ has tannakian group $\mathbf{G}/\mathbf{G} \cap \mathrm{Z}$, which is then finite. In particular, this object is generically unramified (Corollary 3.38).

Let $n \geqslant 1$. For $a \in k_n$, the complex $\mathrm{M}_a = p_{1,*}\mathrm{M} \otimes j^*\mathscr{L}_{\psi(ax)}$ on $\mathbf{G}_{m,k_n}$ is a perverse sheaf, geometrically simple and of weight 0, since the restriction of p_1 to C is an isomorphism. For all but a bounded number of $a \in k_n$, Lemma 9.9 implies that M_a has the property that

$$\mathrm{S}(\mathrm{End}(\mathrm{M}_a), \chi) = |\mathrm{S}(\mathrm{M}_a, \chi)|^2 = |\mathrm{S}(\mathrm{M}, \langle \chi, a\rangle)|^2$$

take only finitely many values as $\chi \in \widehat{\mathbf{G}}_{m,k_n}$ varies. By equidistribution, this is only possible if the arithmetic tannakian group of the object $\mathrm{End}(\mathrm{M}_a) \in \langle \mathrm{M}_a\rangle^{\mathrm{ari}}$ on $\mathbf{G}_{m,k_n}$ is finite. By Katz's Theorem B.2, this implies that $\mathrm{End}(\mathrm{M}_a)$ is punctual, say

$$\mathrm{End}(\mathrm{M}_a) = \bigoplus_{s \in \mathrm{S}_a} n(a, s)\gamma_{a,s}^{\deg} \otimes \delta_s$$

for a subset $\mathrm{S}_a \subset k_n^\times$, integers $n(a, s) \geqslant 1$ and unitary scalars $\gamma_{a,s}$. For all but finitely many $a \in \bar{k}$, we know also from Lemma 9.7 (3) that

$$r^2 = \dim \mathrm{End}(\mathrm{M}) = \dim \mathrm{End}(\mathrm{M}_a) = \sum_{s \in \mathrm{S}_a} n(a, s).$$

Since all $\chi \in \widehat{\mathrm{G}}(k_n)$ are unramified for $\mathrm{End}(\mathrm{M}_a)$ and $|\gamma_{a,s}| = 1$, we compute

$$\begin{aligned} \frac{1}{|\widehat{\mathbf{G}}_m(k_n)|} \sum_{\chi \in \widehat{\mathbf{G}}_m(k_n)} |\mathrm{S}(\mathrm{M}_a, \chi)|^4 &= \frac{1}{|\widehat{\mathbf{G}}_m(k_n)|} \sum_{\chi \in \widehat{\mathbf{G}}_m(k_n)} |\mathrm{S}(\mathrm{End}(\mathrm{M}_a), \chi)|^2 \\ &= \frac{1}{|\widehat{\mathbf{G}}_m(k_n)|} \sum_{\chi \in \widehat{\mathbf{G}}_m(k_n)} \Big| \sum_{s \in \mathrm{S}_a} n(a, s)\gamma_{a,s}^n \chi(s)\Big|^2 = \sum_{s \in \mathrm{S}_a} n(a, s)^2 |\gamma_{a,s}^n|^2 \geqslant r^2. \end{aligned}$$

Averaging over $a \in k_n$, then letting $n \to +\infty$, it follows that $\mathrm{M}_4(\mathbf{G}) \geqslant r^2 \geqslant 4$, which is a contradiction.

Finally, we note that the tannakian dual of M is supported on the image of the diagonal under the inversion map of $\mathbf{G}_m \times \mathbf{G}_a$, namely on the hyperbola

$$\{(x^{-1}, -x) \mid x \in \mathbf{G}_m\} \subset \mathbf{G}_m \times \mathbf{G}_a.$$

Since this is not a translate of the diagonal, the tannakian dual of M cannot be geometrically isomorphic to M. □

EXAMPLE 9.12. (1) Suppose that $\mathrm{M} = \mathscr{L}_{\eta,b}^{\Delta}[1](1/2)$ for some $(\eta, b) \in \widehat{\mathrm{G}}(k)$, which corresponds to the case excluded in Theorem 9.11. For $n \geqslant 1$, denote by η_n the character $\eta \circ \mathrm{N}_{k_n/k}$ of $k_n^\times$. Then the sums $\mathrm{S}_n(\chi, a)$ are essentially Gauss sums, namely

$$\mathrm{S}_n(\chi, a) = \frac{1}{|k|^{n/2}} \sum_{x \in k_n^\times} (\chi\eta_n)(x)\psi_n((a + b)x) = \frac{1}{|k|^{n/2}} \overline{(\chi\eta_n)}(a + b)\tau(\chi\eta_n, \psi_n)$$

(see (3) for the normalization).

The equidistribution properties of the Gauss sums are well-known (see for instance [68, Th. 9.5]), and one deduces easily that the arithmetic tannakian group of M is equal to $\mathbf{GL}_1$. The fourth moment of all sums $\mathrm{S}_n(\chi, a)$ converges to 2, as we saw in the previous proof, but the single contribution to the fourth moment of the (ramified) character $\langle \eta^{-1}, -b\rangle$ is $(|k_n| - 1)^4/|k_n|^4 \to 1$. (See Proposition 9.20 for the classification of objects of tannakian dimension 1 in general.)

(2) Let $\mathrm{M}_0 = \mathscr{K}\ell_{2,\psi}(1/2)$ be the Kloosterman complex of rank 2 on $\mathbf{G}_m$ (see (B.2)) associated to ψ, twisted to be pure of weight 0 (see [69, Th. 8.4.13]). It is of the form $\mathscr{M}_0[1](1/2)$ for some middle extension sheaf $\mathscr{M}_0$, pure of weight 0 as lisse sheaf on $\mathbf{G}_m$.

The object M_0 has tannakian dimension 1 and geometric tannakian group equal to $\mathbf{GL}_1$ as a $\mathbf{G}_m$-object (since it is a hypergeometric complex, see Theorem B.4). On the other hand, the object $\mathrm{M} = \Delta_*\mathrm{M}_0 = \Delta_*\mathscr{M}_0[1](1/2)$ on G has tannakian dimension 2, and arithmetic tannakian group $\mathbf{GL}_2$ by Lemma 9.7 and Theorem 9.11.

We compute the corresponding exponential sums to see the concrete meaning of the theorem in this case. For $n \geqslant 1$ and $\langle \chi, a\rangle \in \widehat{\mathrm{G}}(k_n)$, we have the formula

$$\begin{aligned}\mathrm{S}_n(\chi, a) &= \frac{1}{|k_n|}\sum_{x\in k_n^\times}\Big(\sum_{y\in k_n^\times}\psi_n(xy+1/y)\Big)\chi(x)\psi_n(ax)\\ &= \frac{1}{|k_n|}\sum_{y\in k_n^\times}\psi_n(1/y)\sum_{x\in k^\times}\psi_n((a+y)x)\chi(x).\end{aligned}$$

For χ non-trivial, extended by $\chi(0)=0$, this is equal to

$$\mathrm{S}_n(\chi, a) = \frac{\tau(\chi,\psi_n)}{|k_n|}\sum_{y\in k_n^\times}\overline{\chi(a+y)}\psi_n(1/y).$$

In order to complete the determination of the tannakian group in the situation of Theorem 9.11, we need to compute the tannakian determinant of M. There are various tools to do this:

(1) one can attempt to compare the tannakian determinants for M (supported on a line) with those on $\mathbf{G}_m$, which can often be computed using the results of Katz [74];

(2) one can use the relation between the tannakian determinant at $\langle\chi, a\rangle$ and the determinant of Frobenius acting on the cohomology group

$$\mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \mathrm{M}_{\langle\chi,a\rangle}) \simeq \mathrm{H}^0_c(\mathbf{G}_{m,\bar{k}}, \mathrm{M}_a\otimes\mathscr{L}_\chi)$$

(with notation as above). The latter determinant (on a curve) may often be computed using the theory of local epsilon factors of Deligne and Laumon (see Appendix C). We will not give explicit examples here, but we perform a computation of this kind in Chapter 10 (see Proposition 10.10).

As an example of the first approach, we have for instance the following criterion:

Proposition 9.13. *Let $\mathrm{C}\subset\mathrm{G} = \mathbf{G}_m\times\mathbf{G}_a$ be a line defined by $y=\alpha x$ where $\alpha\in k^\times$. Let* M *be a geometrically simple perverse sheaf on $\mathbf{G}_m\times\mathbf{G}_a$ supported on* C *and of weight zero. Assume that* M *is not punctual, and that the restriction of* M *to* C *is not geometrically isomorphic to $\mathscr{L}_{\eta,b}[1]$ for some multiplicative character η and some b. Let $r\geqslant 0$ be the tannakian dimension of* M.

Suppose that for all but finitely many a, the tannakian determinant of $p_{1,*}\mathrm{M}_{\langle 1,a\rangle}$ on $\mathbf{G}_m$ is geometrically of infinite order. Then we have $\mathbf{G}=\mathbf{GL}_r$.

Proof. Since $\mathbf{G}$ contains $\mathbf{SL}_r$, it suffices to prove that the determinant of $\mathbf{G}$ is arithmetically of infinite order.

Since $p_1\colon \mathrm{C}\to\mathbf{G}_m$ is an isomorphism, it follows that for any $a\in\mathbf{G}_a$, the object $\mathrm{N}_a = p_{1,*}\mathrm{M}_{\langle 1,a\rangle}$ on $\mathbf{G}_m$ is a perverse sheaf, and is arithmetically semisimple and pure of weight 0.

We claim that the assumption implies that the determinants of $\Theta_{\mathrm{M},k_n}(\langle\chi,a\rangle)$ are equidistributed on average on the unit circle, where $\langle\chi,a\rangle$ vary among Frobenius-unramified classes for the determinant. Indeed, denoting $\mathscr{X}$ this set of characters, we have for any non-zero integer $h\in\mathbf{Z}$ the relation

$$\frac{1}{|\mathrm{G}(k_n)|}\sum_{\langle\chi,a\rangle\in\mathscr{X}(k_n)}\det(\Theta_{\mathrm{M},k_n}(\langle\chi,a\rangle))^h=\frac{1}{|k_n|}\sum_{a\in k_n}\frac{1}{|k_n^\times|}\sum_{\substack{\chi\in\widehat{\mathbf{G}}_m(k_n)\\ \langle\chi,a\rangle\in\mathscr{X}(k_n)}}\det(\Theta_{\mathrm{N}_a,k_n}(\chi))^h.$$

The contribution of those finitely many a such that N_a has geometrically finite-order determinant tends to 0. For the other values of a, we have

$$\lim_{\mathrm{N}\to+\infty}\frac{1}{\mathrm{N}}\sum_{n\leqslant\mathrm{N}}\frac{1}{|k_n^\times|}\sum_{\substack{\chi\in\widehat{\mathbf{G}}_m(k_n)\\ \langle\chi,a\rangle\in\mathscr{X}(k_n)}}\det(\Theta_{\mathrm{N}_a,k_n}(\chi))^h=0$$

by equidistribution, in fact uniformly with respect to a since the complexity of $\det(\mathrm{N}_a)$ is bounded independently of a. We deduce that

$$\lim_{\mathrm{N}\to+\infty}\frac{1}{\mathrm{N}}\sum_{n\leqslant\mathrm{N}}\frac{1}{|k_n|}\sum_{a\in k_n}\frac{1}{|k_n^\times|}\sum_{\substack{\chi\in\widehat{\mathbf{G}}_m(k_n)\\ \langle\chi,a\rangle\in\mathscr{X}(k_n)}}\det(\Theta_{\mathrm{N}_a,k_n}(\chi))^h=0,$$

which proves the claim.

But by Theorem 4.4, the determinants of $\Theta_{\mathrm{M},k_n}(\langle\chi,a\rangle)$ are known to be equidistributed on average on the subset of the unit circle corresponding to the determinant of the arithmetic tannakian group of M; if the latter were finite, this would be a finite group of roots of unity. By contraposition, the result follows. ☐

Remark 9.14. If $\det(\mathrm{M})$ is known to be generically unramified, then it suffices to assume that the tannakian determinant of $p_{1,*}\mathrm{M}$ on $\mathbf{G}_m$ is geometrically of infinite order, since in this case we can apply Proposition 3.44 to some twist $\mathrm{M}_{\langle\chi_1,a_1\rangle}$ such that the set of characters χ for which the character $\langle\chi_1,a_1\rangle\langle\chi,0\rangle$ is unramified is generic.

Example 9.15. Proposition 9.13 applies for instance to objects of the form

$$\mathrm{M}=\mathscr{L}_{\eta(f)}[1](1/2)$$

where η is a non-trivial multiplicative character of k, and $f\in k[\mathrm{X}]$ is a polynomial such that $f(0)\neq 0$ with degree $d\geqslant 2$ such that η^d is non-trivial, as explained by Katz in [74, Th. 17.5]. Indeed, in this case, the assumption of the proposition holds for all $a\neq 0$.

The dimension formula (9.2) shows that the tannakian dimension is $d+1$. Note that [74, Th. 17.5] provides the equidistribution for the subfamily with $a=0$, under the assumption that f is not of the form $g(\mathrm{X}^b)$ for some $b\geqslant 2$, but as traces of matrices in $\mathbf{U}_d(\mathbf{C})$, because the corresponding object on $\mathbf{G}_m$ has tannakian dimension d. This means that the characters $\langle\chi,0\rangle$ are examples of weakly-unramified characters for M which are not unramified (since they do not give the "right" dimension).

As explained in Remark 4.20, (2), we expect that we can apply Theorem 4.19 unconditionally to G. Thus this proposition should imply the following result:

Corollary 9.16. *Let ℓ be a prime number. Assume that* Theorem 4.19 *holds for* G. *For all $p\neq\ell$, let M_p be a geometrically simple perverse sheaf of weight zero on $(\mathbf{G}_m\times\mathbf{G}_a)_{\mathbf{F}_p}$ supported on the diagonal with $c_u(\mathrm{M}_p)\ll 1$, where u is the natural locally-closed immersion $\mathbf{G}_m\times\mathbf{G}_a\hookrightarrow\mathbf{A}^2\hookrightarrow\mathbf{P}^2$. Suppose that the tannakian dimension r of M_p is independent of p and that M_p satisfies the* *assumption of Proposition 9.13 for $k = \mathbf{F}_p$. Then the sums*

$$\mathrm{S}(\chi, a; p) = \sum_{x \in \mathbf{F}_p^\times} t_{\mathrm{M}_p}(x)\chi(x)e\Big(\frac{ax}{p}\Big),$$

for χ a multiplicative character of $\mathbf{F}_p$ and $a \in \mathbf{F}_p$, become equidistributed according to the trace of a random unitary matrix in $\mathbf{U}_r(\mathbf{C})$.

9.3. Diagonal objects of dimension 2

The computation of Lemma 9.7 allow us, for instance, to classify those sheaves $\mathscr{M}$ which give rise to geometrically simple perverse sheaves on the diagonal with tannakian dimension $r = 2$. Indeed, the (usual) rank of $\mathscr{M}$ must be either 1 or 2.

In the first case, one and only one of the following conditions must be true:

(1) $\mathscr{M}$ is lisse on $\mathbf{G}_m$, tamely ramified at 0 and has (unique) break at ∞ equal to 2; if the characteristic of k is not equal to 2, then the only such sheaves are isomorphic to

$$\mathscr{L}_{\psi(ax^2+bx)} \otimes \mathscr{L}_\eta$$

where $a \neq 0$ and η is a multiplicative character. The corresponding exponential sums are "twisted quadratic Gauss sums".

(2) $\mathscr{M}$ is lisse on $\mathbf{G}_m$ and has Swan conductor 1 at 0 and unique break $\leqslant 1$ at ∞; the only such sheaves are isomorphic to

$$\mathscr{L}_{\psi(a/x+bx)} \otimes \mathscr{L}_\eta$$

where $a \neq 0$ and η is a multiplicative character (we recover the example of Kloosterman–Salié sums).

(3) there exists a unique $\beta \in \bar{k}^\times$ such that $\mathscr{M}$ is lisse on $\mathbf{A}^1 - \{\beta\}$, it has unique break $\leqslant 1$ at ∞ and is tamely ramified at 0 and β; the only such sheaves are isomorphic to

$$\mathscr{L}_{\eta(x-\beta)} \otimes \mathscr{L}_{\xi(x)} \otimes \mathscr{L}_{\psi(\alpha x)}$$

where $\beta \neq 0$, $\alpha \in \mathbf{G}_a$ and η and ξ are multiplicative characters. The corresponding exponential sums are

$$\frac{1}{|k_n|^{1/2}} \sum_{x \in k_n^\times} \eta(x-\beta)(\chi\xi)(x)\psi((a+\alpha)x),$$

which can be seen as twisted Jacobi sums.

On the other hand, if $\mathscr{M}$ has rank 2, then it must be lisse on $\mathbf{G}_m$, tamely ramified at 0 and have breaks $\leqslant 1$ at ∞. Up to twist by a multiplicative character, we obtain a sheaf lisse on $\mathbf{A}^1$ with breaks $\leqslant 1$ at ∞. Since we assume $\mathscr{M}$ to be geometrically irreducible, the two breaks must be equal, say equal to λ. Their sum is the Swan conductor at ∞, which is also the Euler–Poincaré characteristic (since $\mathscr{M}$ is lisse on $\mathbf{G}_m$ and tame at 0, see (C.12)); thus either $\lambda = 1/2$ or $\lambda = 1$. The first case gives Euler–Poincaré characteristic equal to 1, so we have a hypergeometric sheaf of rank 2 by Katz's classification (see Theorem B.4, e.g., a Kloosterman sheaf of rank 2, with the corresponding sums described in Example 9.12, (2)). In the second case, we may have a pullback of such a sheaf by $x \mapsto x^2$. For the pullback of the Kloosterman sheaf, the exponential sums are then given by the formulas

$$\begin{aligned}
\mathrm{S}_n(\chi, a) &= \frac{1}{|k_n|}\sum_{x\in k_n^\times}\Bigl(\sum_{y\in k_n^\times}\psi_n(xy+xy^{-1})\Bigr)\chi(x)\psi_n(ax)\\
&= \frac{1}{|k_n|}\sum_{y\in k_n^\times}\sum_{x\in k_n^\times}\chi(x)\psi_n(x(a+y+y^{-1}))\\
&= \frac{\tau(\chi,\psi_n)}{|k_n|}\sum_{\substack{y\in k_n^\times\\ a+y+y^{-1}\neq 0}}\overline{\chi(a+y+y^{-1})},
\end{aligned}$$

for χ non-trivial.

9.4. Negligible objects and objects of dimension one

We conclude our discussion of the group $\mathrm{G} = \mathbf{G}_m \times \mathbf{G}_a$ by classifying the negligible objects as well as the objects of tannakian dimension 1. This may be helpful for further investigations (e.g., to compute the determinant of the tannakian group in some cases, or to apply the Goursat–Kolchin–Ribet criterion, see [69, Prop. 1.8.2]).

We will denote by $\mathrm{FT}_{\psi/\mathbf{G}_m}$ the relative Fourier transform functor $\mathrm{D}_c^{\mathrm{b}}(\mathrm{G}) \to \mathrm{D}_c^{\mathrm{b}}(\mathrm{G})$, defined by

$$\mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{M}) = \mathrm{R}q_{2,!}(q_1^*\mathrm{M}\otimes \mathscr{L}_{\psi(ay)})$$

where q_1 and q_2 are the two projections $\mathbf{G}_m \times \mathbf{G}_a \times \mathbf{G}_a \to \mathbf{G}_m \times \mathbf{G}_a$, and we use coordinates (x, y, a) on $\mathrm{G} \times \mathbf{G}_a = \mathbf{G}_m \times \mathbf{G}_a \times \mathbf{G}_a$. This functor satisfies the same basic properties as the Fourier transform over base fields (see, e.g., [77, § 2]), and in particular $\mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{M})[1]$ is perverse if M is perverse.

Proposition 9.17. *Let* M *be a simple perverse sheaf on* G *over* $\bar{k}$.

The perverse sheaf M *is negligible if and only if* M *is isomorphic to an object of the form*

$$p_1^*(\mathrm{N}) \otimes \mathscr{L}_{\psi(ay)}[1] \tag{9.4}$$

for some perverse sheaf N *on* $\mathbf{G}_m$ *and some* a*, or to an object of the form*

$$\mathscr{L}_\chi[1] \otimes p_2^*(\mathrm{M}), \tag{9.5}$$

for some perverse sheaf M *on* $\mathbf{G}_a$ *and some multiplicative character* χ.

Proof. It is elementary that the objects of the two forms in the statement are negligible (see Example 3.5), so we need to prove the converse.

Let M be a simple negligible perverse sheaf on G. We consider the (shifted) Fourier transform

$$\mathrm{F} = \mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{M})[1]$$

of M relative to $\mathbf{G}_m$; this is a perverse sheaf on G. For $a \in \mathbf{G}_a$, the restriction F_a of this complex to $\mathbf{G}_m \times \{a\}$ is isomorphic to $p_{1!}(\mathrm{M}_{\psi(ay)})$. Hence, for a generic, the object $\mathrm{F}_a = (\mathrm{F}|\mathbf{G}_m \times \{\, a \,\})[-1]$ is a perverse sheaf by Proposition 1.1; moreover, if a is such that $\mathbf{G}_m \times \{a\}$ intersects a dense open subset where M is lisse, the generic rank of F_a is still zero.

We now distinguish cases according to the dimension d of the support of F.

(1) If $d = 0$, then F is supported on finitely many points. Since M is simple and the Fourier transform preserves simple perverse sheaves, F is also simple. This implies that the support of F is irreducible, and hence it is a single point (x, a). The point a correspond to the character $\psi(ay)$ via

inverse Fourier transform. Hence, M is of the form $p_1^*(\mathrm{N}) \otimes \mathscr{L}_{\psi(ay)}$, where N is a sheaf with finite support in $\mathbf{G}_m$, which is an object of the form (9.4).

(2) If $d = 1$, then the support of F is a curve $\mathrm{C} \subset \mathbf{G}_m \times \mathbf{G}_a$. If, for generic $a \in \mathbf{G}_a$, the intersection of C with $\mathbf{G}_m \times \{a\}$ is non-empty, then the support of F_a is finite and non-empty, contradicting the fact that this sheaf is of generic rank zero. Hence, for generic $a \in \mathbf{G}_a$, the intersection of C with $\mathbf{G}_m \times \{a\}$ is empty. We then deduce that $\mathrm{F}_a = 0$ for generic a. Hence, C is a finite union of horizontal lines. As in (1), C is irreducible, and hence is of the form $\mathbf{G}_m \times \{ a \}$ for some $a \in \mathbf{G}_a$. Hence, M is of the form $p_1^*(\mathrm{N}) \otimes \mathscr{L}_{\psi(ay)}$ for some perverse sheaf N on $\mathbf{G}_m$; this is again of the form (9.4).

(3) Finally, assume that $d = 2$. Let η be the generic point of $\mathbf{G}_a$. Then F_η is a perverse sheaf with Euler–Poincaré characteristic zero on $\mathbf{G}_m$ over $k(\eta)$. By Proposition B.3, it follows that F_η, viewed as a perverse sheaf on $\mathbf{G}_m$ over $k(\eta)$, is geometrically isomorphic to a Kummer perverse sheaf $\mathscr{L}_\chi[1]$ for some multiplicative character χ. Hence, F is of the form $p_2^*(\mathrm{N}') \otimes \mathscr{L}_\chi[1]$ for some perverse sheaf N' on $\mathbf{G}_a$. Taking the relative inverse (shifted) Fourier transform, we find that there exists some object N of $\mathrm{D}_c^{\mathrm{b}}(\mathbf{G}_a)$ such that M is isomorphic to $p_2^*(\mathrm{N}) \otimes \mathscr{L}_\chi[1]$. ☐

We will now classify the objects of tannakian dimension one.

By Proposition 3.47, the most obvious objects of tannakian dimension one on $\mathbf{G}_m \times \mathbf{G}_a$ are those of the form $\mathrm{H} \boxtimes \mathrm{N}$, for some simple hypergeometric complex H on $\mathbf{G}_m$ and some simple perverse sheaf N on $\mathbf{G}_a$ with Fourier transform of rank one (we refer again to Section B.4 for reminders concerning hypergeometric complexes, which are the objects of tannakian rank 1 on the multiplicative group). The next lemma provides another class of such objects.

Lemma 9.18. *Let $f \in \bar{k}(x)^\times$ be a rational function and U a dense open set of $\mathbf{G}_a$ where f is defined and non-zero. Let $\mathrm{C} \subset \mathrm{V} = \mathbf{G}_m \times \mathrm{U}$, with coordinates (x, a), be the curve with equation $f(a) = x$. Let $\overline{\mathbf{Q}}_{\ell,f}$ be the intermediate extension to $\mathbf{G}_m \times \mathbf{G}_a$ of the constant sheaf on C shifted by 1, and let M_f be the inverse relative Fourier transform of $\overline{\mathbf{Q}}_{\ell,f}$, also shifted by 1.*

(1) *Write $f = f_1/f_2$ with $f_i \in \bar{k}[x]$ coprime. Let $\widetilde{\mathrm{C}} \subset \mathrm{G}$ be the curve with equation*
$$f_1(a) = f_2(a)x,$$
and let $i\colon \widetilde{\mathrm{C}} \to \mathrm{G}$ be the closed immersion. We then have isomorphisms
$$\overline{\mathbf{Q}}_{\ell,f} \simeq i_*\overline{\mathbf{Q}}_\ell[1] \simeq i_!\overline{\mathbf{Q}}_\ell[1].$$
(2) *The perverse sheaf M_f on G has tannakian dimension one.*
(3) *For any $y_0 \in \mathbf{G}_a$, the restriction of M_f to $\mathbf{G}_m \times \{y_0\} \subset \mathrm{G}$ is of the form $\mathscr{G}_{y_0}[2]$ for some sheaf $\mathscr{G}_{y_0}$ on $\mathbf{G}_m$, identified to a sheaf on $\mathbf{G}_m \times \{y_0\}$, of generic rank bounded by $\max(\deg(f_1), \deg(f_2))$.*
(4) *If $f \in k(x)$, then for $n \geqslant 1$, the Fourier transform on $\mathrm{G}(k_n)$ of the trace function of M_f is given by*
$$(\chi, b) \mapsto |k_n|\chi(f(b))$$
for $\langle \chi, b\rangle \in \widehat{\mathrm{G}}(k_n)$ in a generic set.

Proof. The curve $\widetilde{\mathrm{C}}$ contains C, and the assumption that f_1 and f_2 are coprime implies that $\widetilde{\mathrm{C}}$ is smooth (since the partial derivative with respect to x is $f_2(a)$, which is non-zero on $\widetilde{\mathrm{C}}$). It is irreducible since it is isomorphic to the open subset of $\mathbf{G}_a$ where $f_1 f_2$ is non-zero by the projection $(x, a) \mapsto a$ with inverse $a \mapsto f_1(a)/f_2(a)$. Since $i_* = i_!$ for a closed immersion, it follows that $i_*\overline{\mathbf{Q}}_\ell[1] = i_!\overline{\mathbf{Q}}_\ell[1]$ is a perverse sheaf, and since it restricts to $\overline{\mathbf{Q}}_{\ell,f}$ on V, these perverse sheaves are isomorphic.

We next show that for generic $\langle \chi, b\rangle \in \widehat{\mathrm{G}}$, we have

$$\dim \mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, (\mathrm{M}_f)_{\langle \chi, b\rangle}) = 1,$$

which will prove (2).

This cohomology group can be computed by first taking the relative additive Fourier transform F of M_f, restricting it to the line $\mathbf{G}_m \times \{\, b\,\}$, then taking the cohomology of $\mathrm{F} \otimes \mathscr{L}_\chi$ on $\mathbf{G}_m \times \{\, b\,\}$. Since the Fourier transform F of M_f is $\overline{\mathbf{Q}}_{\ell,f}$, there exists a dense open set U of $\mathbf{G}_a$ such that for $b \in \mathrm{U}$, the restriction of F to $\mathbf{G}_m \times \{b\}$ is a rank one skyscraper sheaf supported on $f(b)$. Such a sheaf, tensored with any character $\mathscr{L}_\chi$, has its 0-th cohomology group of dimension 1.

In fact, the same argument shows that if $\langle \chi, b\rangle$ is defined over k_n, then the action of Frobenius on the one-dimensional space $\mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, (\mathrm{M}_f)_{\langle \chi, b\rangle})$ is $|k_n|\chi(f(b))$, which proves the last statement.

To prove (3), we observe that, by definition of the Fourier transform, yet another description of M_f is

$$\mathrm{M}_f = \mathrm{R}\phi_* \mathscr{L}_{\psi(-ay)}[2],$$

where ϕ is the restriction of the projection $(x, y, a) \mapsto (x, y)$ to the subvariety

$$\mathrm{Z} = \{(x, y, a) \in \mathbf{G}_m \times \mathbf{G}_a \times \mathbf{G}_a \mid f_1(a) = x f_2(a)\}$$

of $\mathbf{G}_m \times \mathbf{G}_a^2$. Since ϕ is an affine quasi-finite morphism, we obtain (3) with

$$\mathscr{G} = \widetilde{\phi}_* \mathscr{L}_{\psi(-ay_0)},$$

where $\widetilde{\phi}$ is the restriction of ϕ to Z_{y_0}. This sheaf has generic rank bounded by the size of the fibers of ϕ, and is $\leqslant \max(\deg(f_1), \deg(f_2))$. $\square$

Remark 9.19. The last statement amounts to the following computation of Fourier transform on $\mathrm{G}(k_n)$: by the first part, writing $f = f_1/f_2$, where f_i are polynomials without common factor, the perverse sheaf $\overline{\mathbf{Q}}_{\ell,f}$ is the shifted constant sheaf on the smooth irreducible curve defined by $f_1(a) = x f_2(a)$ in G. Therefore, the trace function of $\overline{\mathbf{Q}}_{\ell,f}$ at $(x, a) \in \mathrm{G}(k_n)$ is equal to 1 if $f_1(a) = x f_2(a)$, and 0 otherwise, so the trace function of M_f has the value

$$\sum_{\substack{a \in k \\ f_1(a) = x f_2(a)}} \psi_n(-ay)$$

at $(x, y) \in k_n^\times \times k_n$. For $(\chi, b) \in \widehat{\mathrm{G}}(k_n)$, we get the Fourier transform

$$\sum_{(x,y) \in \mathrm{G}(k_n)} \chi(x)\psi_n(by) \sum_{\substack{a \in k_n \\ f_1(a) = x f_2(a)}} \psi(-ay) = \sum_{\substack{a \in k_n \\ f_1(a) f_2(a) \neq 0}} \sum_{y \in k_n} \chi(f(a))\psi_n((b-a)y)$$

and this is equal to $|k_n|\chi(f(b))$ if $f_1(b) f_2(b) \neq 0$, and 0 otherwise.

The basic classes of objects of tannakian dimension 1 we have just described turn out to be sufficient to obtain all of them.

Proposition 9.20. *Let* M *be a simple perverse sheaf on* G *over* $\bar{k}$. *Assume that* M *is in* $\mathbf{P}_{\mathrm{int}}(\mathrm{G})$. *Then* M *has tannakian dimension one if and only if there exist a rational function* f, *a simple hypergeometric complex* H *on* $\mathbf{G}_m$ *and a perverse sheaf* N *on* $\mathbf{G}_a$ *with Fourier transform of rank one such that* M *is isomorphic to the convolution*

$$\mathrm{M}_f *_{\mathrm{int}} (\mathrm{H} \boxtimes \mathrm{N}).$$

Proof. Since the tannakian dimension is multiplicative in convolutions, the "if" assertion follows from Lemma 9.18 and the fact that $\mathrm{H} \boxtimes \mathrm{N}$ has tannakian dimension 1 by an application of Proposition 3.47.

Conversely, let M be a simple perverse sheaf on G of tannakian dimension one. As in the dimension zero case, we consider the shifted Fourier transform $\mathrm{F} = \mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{M})[1]$ of M relative to $\mathbf{G}_m$. For generic $a \in \mathbf{G}_a$, the object $\mathrm{F}_a = \mathrm{F}|\mathbf{G}_m \times \{ a \}$ on $\mathbf{G}_m \times \{a\}$ is perverse of generic rank one, as in the beginning of the proof of Proposition 9.17.

In particular, for the generic point η of $\mathbf{G}_a$, the object F_η, viewed as a perverse sheaf on $\mathbf{G}_{m,k(\eta)}$, is of tannakian dimension one. By Theorem B.4, (2), it is isomorphic to a hypergeometric complex multiplicatively translated by a non-zero rational function $f(\eta)$ of η, and tensored by a rank one object on $k(\eta)$. Thus, there exists a dense open subset W of $\mathbf{G}_m \times \mathbf{G}_a$ and an isomorphism

$$\mathrm{F}|\mathrm{W} \simeq p_2^*(\mathrm{N})|\mathrm{W} \otimes m_f^*(\mathrm{H}), \tag{9.6}$$

where N is a perverse sheaf on $\mathbf{G}_a$ of generic rank one, H is a hypergeometric complex on $\mathbf{G}_m$ and $m_f \colon \mathrm{W} \to \mathbf{G}_m$ is the morphism

$$(x, a) \mapsto x/f(a)$$

(in particular, $x/f(a)$ is defined and non-zero for $(x, a) \in \mathrm{W}$). Using [69, Th. 8.4.10] and Proposition 9.17, we may assume that H is simple.

Let $\widehat{\mathrm{N}}$ be the Fourier transform of N. We claim that there are isomorphisms

$$\mathrm{M} \simeq (\mathrm{H} \boxtimes \mathbf{1}) *_{\mathrm{int}} \mathrm{M}_f *_{\mathrm{int}} (\mathbf{1} \boxtimes \widehat{\mathrm{N}}) \simeq \mathrm{M}_f *_{\mathrm{int}} (\mathrm{H} \boxtimes \widehat{\mathrm{N}}), \tag{9.7}$$

which will conclude the proof. The second isomorphism follows from commutativity and associativity of the convolution combined with the isomorphism of Proposition 3.47, hence we need only check the first.

Let $\mathrm{P} = (\mathrm{H} \boxtimes \mathbf{1}) *_{\mathrm{int}} \mathrm{M}_f$; we need to show that M is isomorphic to $\mathrm{P} *_{\mathrm{int}} (\mathbf{1} \boxtimes \widehat{\mathrm{N}})$. We will do this by showing that the restriction to W of their relative Fourier transforms are isomorphic; since both objects involved are perverse sheaves, this will give the result.

Precisely, denote

$$\mathrm{P}_! = (\mathrm{H} \boxtimes \mathbf{1}) *_! \mathrm{M}_f, \qquad \mathrm{P}_* = (\mathrm{H} \boxtimes \mathbf{1}) *_* \mathrm{M}_f.$$

We claim that there are isomorphisms

$$\mathrm{P}_! \simeq \mathrm{P} \simeq \mathrm{P}_*, \tag{9.8}$$

$$\mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{P})|\mathrm{W} \simeq m_f^*(\mathrm{H}), \tag{9.9}$$

$$\mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{P} *_{\mathrm{int}} (\mathbf{1} \boxtimes \widehat{\mathrm{N}})) \simeq p_2^*(\mathrm{N}) \otimes \mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{P}), \tag{9.10}$$

where M_f is the complex in Lemma 9.18, and $\mathbf{1}$ denotes the unit object on $\mathbf{G}_a$.

Assuming these to be true, it follows by combining (9.6), (9.9) and (9.10) that

$$\mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{M})|\mathrm{W} \simeq p_2^*(\mathrm{N})|\mathrm{W} \otimes m_f^*(\mathrm{H}) \simeq p_2^*(\mathrm{N})|\mathrm{W} \otimes \mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{P})|\mathrm{W} \simeq \mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{P} *_! (\mathbf{1} \boxtimes \widehat{\mathrm{N}}))|\mathrm{W},$$

proving the first part of (9.7), and thereby concluding the proof.

We will begin with the rigorous sheaf-theoretic computations, but we include afterwards the (potentially more enlightening) computations of trace functions (assuming all objects to be defined over k).

Proof of (9.8). Since G is affine, it suffices to prove that $\mathrm{P}_!$ is semiperverse: indeed, it first follows that $\mathrm{P}_!$ is perverse by [70, Lemma 2.6.7]; since the dual of P_* is $(\mathrm{D}(\mathrm{H}) \boxtimes \mathbf{1}) *_! \mathrm{D}(\mathrm{M}_f)$, this first fact (applied to the duals of H of M_f, which are of the same type) also implies that P_* is perverse, and (9.8) follows.

To prove that $\mathrm{P}_!$ is semiperverse, we need to estimate the dimension of the support of the cohomology sheaves $\mathscr{H}^i(\mathrm{P}_!)$. Let $i \in \mathbf{Z}$ and $(x_0, y_0) \in \mathrm{G}$. By the definition of convolution and the proper base change theorem, we have an isomorphism

$$\mathscr{H}^i(\mathrm{P}_!)_{(x_0,y_0)} \simeq \mathrm{H}^i_c(\mathrm{G}_{\bar{k}}, [(x,y) \mapsto (x_0 x, y_0 + y)]^*(\mathrm{H} \boxtimes \mathbf{1}) \otimes [(x,y) \mapsto (x^{-1}, -y)]^* \mathrm{M}_f).$$

Let $\mathrm{H}_{x_0} = [x \mapsto x_0 x]^* \mathrm{H}$. The last cohomology group is isomorphic to

$$\mathrm{H}^i_c(\mathrm{G}_{\bar{k}}, p_1^* \mathrm{H}_{x_0} \otimes p_2^*[y \mapsto y + y_0]^* \mathbf{1} \otimes [(x,y) \mapsto (x^{-1}, -y)]^* \mathrm{M}_f).$$

If we denote by $\widetilde{\mathrm{M}}_{f,y_0}$ the complex on $\mathbf{G}_m$ given by $[x \mapsto (x^{-1}, -y_0)]^* \mathrm{M}_f$, then the projection formula shows that the cohomology group is isomorphic to

$$\mathrm{H}^i_c(\mathbf{G}_{m,\bar{k}}, \mathrm{H}_{x_0} \otimes \widetilde{\mathrm{M}}_{f,y_0}).$$

By Lemma 9.18 (3) the object $\widetilde{\mathrm{M}}_{f,y_0}$ is of the form $\mathscr{G}_{y_0}[2]$ for some sheaf on $\mathbf{G}_m$.

(1) If H_{x_0} is of the form $\mathscr{F}_{x_0}[1]$ for some simple middle extension hypergeometric sheaf on $\mathbf{G}_m$, then $\mathscr{H}^i(\mathrm{P}_!)_{(x_0,y_0)}$ is isomorphic to

$$\mathrm{H}^{3+i}_c(\mathbf{G}_{m,\bar{k}}, \mathscr{F}_{x_0} \otimes \mathscr{G}_{y_0}).$$

It follows immediately that $\mathscr{H}^i(\mathrm{P}_!)$ is zero for $i \geqslant 0$.

For $i = -1$, since $\mathscr{F}_{x_0}$ and $\mathscr{G}_{y_0}$ are lisse on dense open subsets of $\mathbf{G}_m$ and $\mathscr{F}_{x_0}$ is simple, the stalk at (x_0, y_0) is non-trivial if and only if $\mathscr{F}_{x_0}$ is a Jordan–Hölder factor of the dual of $\mathscr{G}_{y_0}$ (on such an open set).

But given y_0, Lemma 9.18 (3) shows that the sheaf $\mathscr{G}_{y_0}$ has generic rank at most $\max(\deg(f_1), \deg(f_2))$, hence has at most as many Jordan–Hölder factors. The hypergeometric sheaves $\mathscr{F}_{x_0}$ are pairwise non-isomorphic by [69, 8.5.6], since they are all multiplicative translates of a fixed hypergeometric sheaf; for a given y_0, there are therefore at most $\max(\deg(f_1), \deg(f_2))$ values of x_0 for which the stalk of $\mathscr{H}^1(\mathrm{P}_!)$ at (x_0, y_0) is non-zero. This implies that the support of $\mathscr{H}^1(\mathrm{P}_!)$ is of dimension $\leqslant 1$, so that $\mathrm{P}_!$ is semiperverse.

(2) If H is isomorphic to δ_{x_1} for some $x_1 \in \mathbf{G}_m$, then $\mathrm{H}_{x_0} \simeq \delta_{x_0 x_1}$. Then $\mathscr{H}^i(\mathrm{P}_!)_{(x_0,y_0)}$ is isomorphic to

$$\mathrm{H}^{2+i}_c(\mathbf{G}_{m,\bar{k}}, \delta_{x_0 x_1} \otimes \mathscr{G}_{y_0}),$$

which is zero if $i \neq -2$. Thus $\mathrm{P}_!$ is also semiperverse in that case.

Proof of (9.9). We compute $\mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{P}_!)$, which is the same as $\mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{P})$ by the previous result. Consider

$$\mathrm{X} = \{(x, u, v, y, c, d, a) \in \mathbf{G}_m^3 \times \mathbf{G}_a^4 \mid uv = x,\ c + d = y\},$$

and the morphism $\sigma_{x,a} \colon (x, u, v, y, c, d, a) \mapsto (x, a)$ from X to G. Then

$$\mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{P}_!) = \mathrm{R}\sigma_{x,a,!}(\mathscr{L}_{\psi(ay)} \otimes \mathrm{H}(u) \otimes \mathbf{1}(c) \otimes \mathrm{M}_f(v,d)),$$

where we use a shorthand notation for pullbacks, where, e.g.

$$\mathrm{H}(u) = [(x, u, v, c, d, y) \mapsto u]^* \mathrm{H}.$$

Denoting

$$\mathrm{Y} = \{(x, u, v, y, c, d, a) \in \mathrm{X} \mid c = 0\}, \qquad \mathrm{Z} = \{(x, u, v, y, a) \in \mathbf{G}_m^3 \times \mathbf{G}_a^2 \mid uv = x\},$$

and noting the isomorphism $\mathrm{Y} \to \mathrm{Z}$ given by

$$(x, u, v, y, c, d, a) \mapsto (x, u, v, y, a)$$

with inverse

$$(x, u, v, y, a) \mapsto (u, x, v, y, 0, y, a),$$

this becomes

$$\mathrm{R}\sigma_{x,a,!}(\mathscr{L}_{\psi(ay)} \otimes \mathrm{H}(u) \otimes \mathrm{M}_f(v, y)),$$

with an abuse of notation involving in using the notation $\sigma_{x,a}$ again for the projection from points on Z to (x, a). Factoring $\sigma_{x,a}$ into $(x, u, v, y, a) \mapsto (x, u, v, a) \mapsto (x, a)$, we recognize

$$\mathrm{R}\sigma_{x,a,!}(\mathrm{H}(u) \otimes \mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{M}_f)(v, a)) = \mathrm{R}\sigma_{x,a,!}(\mathrm{H}(u) \otimes \overline{\mathbf{Q}}_{\ell,f}(v, a)),$$

again with some abuse of notation.

We write $f = f_1/f_2$ as in Lemma 9.18. Let

$$\mathrm{Z}_f = \{(x, u, v, a) \mid uv = x,\ vf_2(a) = f_1(a)\}.$$

By Lemma 9.18, we have an isomorphism

$$\mathrm{R}\sigma_{x,a,!}(\mathrm{H}(u) \otimes \overline{\mathbf{Q}}_{\ell,f}(v, a)) \simeq \mathrm{R}\sigma_{x,a,!}(\mathrm{H}(u)),$$

and since $\sigma_{x,a}\colon \sigma_{x,a}^{-1}(\mathrm{W}) \to \mathrm{W} \subset \mathbf{G}_m \times \mathbf{G}_a$ is an isomorphism with inverse

$$(x, a) \mapsto (x, x/f(a), f(a), a),$$

this is simply $\mathrm{H}(x/f(a)) = m_f^*(\mathrm{H})$ on W. In other words, we have proved that there is an isomorphism

$$\mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{P}_!)|\mathrm{W} \simeq m_f^*(\mathrm{H}),$$

as desired.

(Formally, we can also show that the trace functions on both sides of (9.9) coincide, disregarding the difference between the various convolutions. For $(x, a) \in k^\times \times k$, the object $\mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{P})$ has trace function at (x, a) equal to

$$\sum_{y\in k}\sum_{u\in k^\times} t_{\mathrm{H}}(u)t_{\mathrm{M}_f}(x/u, y)\psi(ay) = \sum_{u\in k^\times} t_{\mathrm{H}}(u)\sum_{y\in k} t_{\mathrm{M}_f}(x/u, y)\psi(ay).$$

The inner sum over y is the value at $(x/u, a)$ of the trace function of the relative Fourier transform of M_f, hence by definition it is the trace function at $(x/u, a)$ of the object $\overline{\mathbf{Q}}_{\ell,f}$. Writing $f = f_1/f_2$ where f_1 and f_2 are polynomials without common zeros, this value is 1 if $f_1(a) = xu^{-1}f_2(a)$, and 0 otherwise. Thus, provided $f_1(a)f_2(a) \neq 0$, the above expression is

$$\sum_{u\in k^\times} t_{\mathrm{H}}(u)\sum_{y\in k} t_{\mathrm{M}_f}(x/u, y)\psi(ay) = \sum_{\substack{u\in k^\times\\ f_1(a)=xu^{-1}f_2(a)}} t_{\mathrm{H}}(u) = t_{\mathrm{H}}(x/f(a)).$$

If $(x, a) \in \mathrm{W}$, this is the same as the trace function of $m_f^*(\mathrm{H})$, as desired.)

Proof of (9.10). The argument is similar. Consider

$$\mathrm{X} = \{(x, u, v, y, c, d, a) \in \mathbf{G}_m^3 \times \mathbf{G}_a^4 \mid uv = x,\ c + d = y\},$$

and the morphism $\sigma_{x,a}\colon (x, u, v, y, c, d, a) \mapsto (x, a)$ from X to G. Then

$$\mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{P}_! *_! (\mathbf{1} \boxtimes \widehat{\mathrm{N}})) \simeq \mathrm{R}\sigma_{x,a,!}(\mathscr{L}_{\psi(ay)} \otimes \mathrm{P}_!(u, c) \otimes \mathbf{1}(v) \otimes \widehat{\mathrm{N}}(d)),$$

where we use the shorthand notation for pullbacks. Denoting

$$\mathrm{Y} = \{(x, u, v, y, c, d, a) \in \mathrm{X} \mid u = x,\ v = 1,\ c + d = y\} \simeq \mathrm{Z} = \{(x, c, d, y, a) \mid c + d = y\},$$

this is

$$\mathrm{R}\sigma_{x,a,!}(\mathscr{L}_{\psi(ay)} \otimes \mathrm{P}_!(x, c) \otimes \widehat{\mathrm{N}}(d))$$

(with again an abuse of notation involved in the notation $\sigma_{x,a}$). From the isomorphism

$$\mathrm{Z} \simeq \{(x,c,d,a) \in \mathbf{G}_m \times \mathbf{G}_a^3\}$$

with inverse $(x,c,d,a) \mapsto (x,c,d,c+d,a)$, we get

$$\mathrm{R}\sigma_{x,a,!}(\mathscr{L}_{\psi(ac)} \otimes \mathscr{L}_{\psi(ad)} \otimes \mathrm{P}_!(x,c) \otimes \widehat{\mathrm{N}}(d)).$$

Factor $\sigma_{x,a}$ as $(x,c,d,a) \mapsto (x,c,a) \mapsto (x,a)$; by Fourier inversion, we obtain

$$\mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{P}_! *_! (\mathbf{1} \boxtimes \widehat{\mathrm{N}})) \simeq \mathrm{R}\sigma_{x,a,!}(\mathrm{P}_!(x,c) \otimes \mathrm{N}(a) \otimes \mathscr{L}_{\psi(ac)}) = \mathrm{N}(a) \otimes \mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{P}_!).$$

We know that $\mathrm{P}_! \simeq \mathrm{P}_* \simeq \mathrm{P}$ (see (9.8)); the argument above can be repeated with the Fourier transform and convolution defined by direct images, and we also obtain an isomorphism

$$\mathrm{FT}_{\psi/\mathbf{G}_m,*}(\mathrm{P}_* *_* (\mathbf{1} \boxtimes \widehat{\mathrm{N}})) \simeq \mathrm{N}(a) \otimes \mathrm{FT}_{\psi/\mathbf{G}_m,*}(\mathrm{P}_*).$$

Again because $\mathrm{P}_! \simeq \mathrm{P}_* \simeq \mathrm{P}$ and the two Fourier transforms coincide, we conclude that

$$\mathrm{M} \simeq \mathrm{P} *_! (\mathbf{1} \boxtimes \widehat{\mathrm{N}}) \simeq \mathrm{P} *_* (\mathbf{1} \boxtimes \widehat{\mathrm{N}}) \simeq \mathrm{P} *_{\mathrm{int}} (\mathbf{1} \boxtimes \widehat{\mathrm{N}}),$$

which establishes (9.10).

(Here also we illustrate the result by computing trace functions. The value at (x,a) of the trace function of the relative Fourier transform of $\mathrm{P} *_{\mathrm{int}} (\mathbf{1} \boxtimes \widehat{\mathrm{N}})$ is equal to

$$\sum_{y \in k} \psi(ay) \Bigl(\sum_{v \in k} t_{\mathrm{P}}(x,v) t_{\widehat{\mathrm{N}}}(y-v) \Bigr) = \sum_{v \in k} t_{\mathrm{P}}(x,v) \sum_{w \in k} \psi(a(v+w)) t_{\widehat{\mathrm{N}}}(w).$$

By Fourier inversion, this is the same as

$$t_{\mathrm{N}}(a) \sum_{v \in k} t_{\mathrm{P}}(x,v) \psi(av),$$

which is the value at (x,a) of the trace function of $p_2^*(\mathrm{N}) \otimes \mathrm{FT}_{\psi/\mathbf{G}_m}(\mathrm{P})$. By (9.9), this coincides on W with the trace function of $p_2^*(\mathrm{N})|\mathrm{W} \otimes m_f^*(\mathrm{H})$.) $\square$

Remark 9.21. The trace functions (over k) of simple negligible objects are of the form

$$(x,y) \mapsto t(x)\psi(by)$$

for some trace function t on $\mathbf{G}_m$ and some $b \in k$, or

$$(x,y) \mapsto \chi(x)t(y)$$

for some trace function t on $\mathbf{G}_a$ and some multiplicative character χ.

The trace functions of simple objects of tannakian dimension one are convolutions of functions of the three types

$$(x,y) \mapsto \sum_{\substack{z \in k \\ f(z)=x}} \psi(-yz), \qquad (x,y) \mapsto \mathscr{H}(x), \qquad (x,y) \mapsto t(y),$$

where f is a non-zero rational function, $\mathscr{H}$ is the trace function of a hypergeometric sheaf and t is the trace function of an object on $\mathbf{G}_a$ whose Fourier transform has generic rank one. The associated exponential sums are (up to normalization by powers of $|k|$) of the form

$$\mathrm{S}(\chi,a) = \chi(f(a)) \widehat{\mathscr{H}}(\chi) \widehat{t}(a),$$

where $\widehat{t}$ is the trace function of an ℓ-adic character, and $\widehat{\mathscr{H}}$ is a product of monomials in Gauss sums (see (B.1)).

CHAPTER 10

Variance of arithmetic functions in arithmetic progressions

10.1. Introduction

In this chapter, we will consider some of the first natural concrete applications of our results to problems which, as stated, do not seem to refer to algebraic groups, or equidistribution statements of any kind. These problems are related to one of the most essential questions of modern analytic number theory, namely the study of arithmetic functions in arithmetic progressions to large moduli.

Concretely, this means that we are given an arithmetic function f (i.e., a complex-valued function defined on the set of positive integers), an integer $q \geqslant 1$ (the "modulus") and $x \geqslant 2$, and we seek to understand the quantities

$$\sum_{\substack{n \leqslant x \\ n \equiv a \,(\mathrm{mod}\, q)}} f(n)$$

for a varying among residue classes modulo q, or only for a coprime to q. The focus is on these sums in settings where both x and q are large, and the goal is often to obtain asymptotic formulas valid for q as large as possible in comparison with x.

The literature on this topic is enormous, and the applications cover almost all of analytic number theory: indeed, this subject encompasses, almost by definition, all of sieve theory and its applications (see [46]), and it is in particular at the source of most of the recent developments in prime number theory, going back to the Bombieri–Vinogradov Theorem (see, e.g., [64, Ch. 17]), and including such celebrated results as the Green–Tao Theorem, or Zhang's Theorem [128], or the Maynard–Tao method (see, e.g., [87]).

The problems that we consider here are the analogue for polynomials over finite fields, and in the limit when the size of the field tends to infinity, of questions related to the distribution of the quantities above, and especially of their *variance*, as functions of a. In other words, we are interested in

$$\sum_{a \,(\mathrm{mod}\, q)} \Bigl| \sum_{\substack{n \leqslant x \\ n \equiv a \,(\mathrm{mod}\, q)}} f(n) - \frac{1}{q} \sum_{n \leqslant x} f(n) \Bigr|^2$$

or (often more naturally for applications) the variant

$$\sum_{\substack{a \,(\mathrm{mod}\, q) \\ (a,q)=1}} \Bigl| \sum_{\substack{n \leqslant x \\ n \equiv a \,(\mathrm{mod}\, q)}} f(n) - \frac{1}{\varphi(q)} \sum_{n \leqslant x} f(n) \Bigr|^2 \tag{10.1}$$

where the sum covers only invertible residue classes. (In both cases, the choice of "expected main term" is natural, but might require adjustments, depending on the arithmetic function involved.)

The serious study of these function field analogues has been initiated especially by Keating and Rudnick and a number of collaborators (see for instance [80], in the case where f is the von Mangoldt function, using results of Katz [75], which themselves relied on his work on the Mellin transform over finite fields [74]).

It is quite easy to understand the link between a quantity like (10.1), in the function field case, and equidistribution problems of the type considered in Chapter 4. Indeed, we are then in the situation where q is a polynomial in $k[t]$ for some finite field k, and the sum over $n \leqslant x$ is replaced by the sum over monic polynomials $g \in k[t]$ of degree m. Then for any complex-valued function f defined for polynomials in $k[t]$, we see (using orthogonality of characters, or the discrete Plancherel formula) that the formula

$$\sum_{a \in (k[t]/qk[t])^\times} \Bigl| \sum_{\substack{\deg(g)=m \\ g \equiv a \,(\mathrm{mod}\, q)}} f(g) - \frac{1}{|(k[t]/qk[t])^\times} \sum_{\deg(g)=m} f(g) \Bigr|^2 =$$

$$|(k[t]/qk[t])^\times| \quad \sum_{\chi \neq 1} \Bigl| \sum_{\deg(g)=m} \chi(g) f(g) \Bigr|^2$$

holds, where χ runs over non-trivial characters of the group $(k[t]/qk[t])^\times$. These characters can be identified with the characters of $\mathrm{G}(k)$ for some commutative algebraic group G (by a simple special case of geometric class-field theory; in the case which we will consider, when q is squarefree, it will be a very explicit torus). Moreover, for many natural arithmetic functions, the inner sum over g monic of degree m in $k[t]$ can be interpreted as the value at χ of the arithmetic Fourier transform of some object on this group G. In the limit where k is replaced by its extensions k_n of degree $n \to +\infty$ (and m is fixed), we can therefore expect to determine the asymptotic behavior of this variance from our equidistribution theorems.

We will now consider in detail the version of this question when f is the von Mangoldt function associated to a higher-degree L-function (the classical von Mangoldt function being related to the Riemann zeta function, which has degree 1), in which case Hall, Keating, and Roddity-Gershon [56] have shown that new phenomena appear (again relying on [74]). These are conjectured to correspond to new behavior also in the (currently inaccessible) situation over number fields. We refer the reader to the introductions of both papers [80] and [56] for extensive discussions of these motivating conjectures, and for additional references to other papers.

We will see that, as suggested by the discussion above, the equidistribution theory for arithmetic Fourier transforms on higher-dimensional tori leads to generalizations, strengthenings, and better understanding, of these previous results. This leads in particular to Theorem 3 in the Introduction, but the method is suitable for the proof of many similar statements.

In the remainder of this chapter, as before, we denote by k a finite field, with an algebraic closure $\bar{k}$, and for each $n \geqslant 1$ by k_n the extension of degree n of k in $\bar{k}$. We fix a prime ℓ distinct from the characteristic of k, and all complexes are understood to be ℓ-adic complexes.

10.2. Equidistribution on tori associated to polynomials

In what follows, we fix a square-free monic polynomial $f \in k[t]$ of degree $d \geqslant 2$. We denote by B the (étale) k-algebra $\mathrm{B} = k[t]/fk[t]$ of degree d over k (in spite of the notation, B depends on f), by Z the zero locus of f, and by $\mathbf{A}^1_k[1/f]$ the complement of Z in the affine line over k.

We begin with a result of Katz [75].

Proposition 10.1 (Katz). *The functor* $\mathrm{A} \mapsto (\mathrm{B} \otimes_k \mathrm{A})^\times$ *on k-algebras is represented by a torus* T *defined over k. This torus splits over any extension of k where f splits in linear factors.*

Moreover, the map $x \mapsto t - x$ defines a closed immersion

$$i_f \colon \mathbf{A}^1_k[1/f] \longrightarrow \mathrm{T},$$

and there exists a morphism of algebraic groups

$$p\colon \mathrm{T} \longrightarrow \mathbf{G}_m$$

satisfying $p \circ i_f = (-1)^{\deg(f)} f$, *where we view* f *as defining a morphism* $\mathbf{A}^1[1/f] \to \mathbf{G}_m$.

REMARK 10.2. As noted by Katz [75, p. 3224], the torus T is isomorphic to a generalized jacobian associated to $\mathbf{P}^1$ with divisor $(\infty) + \mathrm{Z}$ (compare Remark 8.14).

We call the morphism p the *norm*. If f splits completely over k, say

$$f = \prod_{z \in \mathrm{Z}} (t - z),$$

then the torus T is split by the morphism sending g to $(g(z))_{z \in \mathrm{Z}}$. The norm is then given by

$$p(g) = \prod_{z \in \mathrm{Z}} g(z),$$

and in particular one has

$$p(i_f(x)) = \prod_{z \in \mathrm{Z}} (z - x) = (-1)^{\deg(f)} f(x).$$

We denote by $\widehat{\mathrm{B}}^\times$ (resp. by $\widehat{k}^\times$) the group of ℓ-adic characters of the finite group $\mathrm{B}^\times$ (resp. of $k^\times$). We extend characters of $\mathrm{B}^\times$ to $k[t]$ by putting $\chi(g) = 0$ if g is not coprime to f. Since $\mathrm{B}^\times = \mathrm{T}(k)$, the group $\widehat{\mathrm{B}}^\times$ of characters of $\mathrm{B}^\times$ is also equal to the group $\widehat{\mathrm{T}}(k)$ of characters of $\mathrm{T}(k)$ (although we will sometimes distinguish them to avoid confusion between characters of B, operating on polynomials, and characters of T).

If f splits over k as above, then the Chinese Remainder Theorem induces an isomorphism $(\widehat{k}^\times)^{\mathrm{Z}} \to \widehat{\mathrm{B}}^\times$, under which an element $(\chi_z)_{z \in \mathrm{Z}} \in (\widehat{k}^\times)^{\mathrm{Z}}$ corresponds to the character χ of $\mathrm{B}^\times$ that maps $g \in k[t]$ to

$$\chi(g) = \prod_{z \in \mathrm{Z}} \chi_z(g(z)).$$

Let M be a perverse sheaf on $\mathbf{A}^1_k[1/f]$ which is pure of weight zero. We are interested in the distribution properties of families of one-variable exponential sums of the type

$$\sum_{x \in k - \mathrm{Z}} t_{\mathrm{M}}(x) \chi(t - x) \tag{10.2}$$

for $\chi \in \widehat{\mathrm{B}}^\times$, or of the underlying L-functions (recall that t is an indeterminate).

We start by interpreting these sums as Mellin transforms on T in order to apply our general equidistribution results. Let $\chi \in \widehat{\mathrm{B}}^\times$. Let $\widetilde{\chi}$ be the character of $\mathrm{T}(k)$ corresponding to χ. The sum (10.2) takes the form

$$\sum_{x \in k} t_{\mathrm{M}}(x) \chi(t - x) = \sum_{x \in \mathbf{A}^1[1/f](k)} t_{\mathrm{M}}(x) \widetilde{\chi}(i_f(x)) = \sum_{y \in \mathrm{T}(k)} t_{i_{f*}\mathrm{M}}(y) \widetilde{\chi}(y). \tag{10.3}$$

Note also that by adapting the argument of [75, Lem. 1.1], for any $n \geqslant 1$, we have

$$\sum_{x \in k_n} t_{\mathrm{M}}(x; k_n) \chi(\mathrm{N}_{k_n/k}(t - x)) = \sum_{y \in \mathrm{T}(k_n)} t_{i_{f*}\mathrm{M}}(y; k_n) \widetilde{\chi}(\mathrm{N}_{k_n/k}(y)). \tag{10.4}$$

The variation with $\chi \in \widehat{\mathrm{B}}^\times$ of the sums (10.2) is therefore governed by the tannakian group of the perverse sheaf $i_{f*}\mathrm{M}$ on T. By Theorem 3.26, this perverse sheaf is generically unramified.

We first compute the tannakian dimension of the object $i_{f*}\mathrm{M}$, in the most important cases.

Lemma 10.3. *Let $\mathscr{F}$ be a middle extension sheaf on $\mathbf{A}^1_k[1/f]$ which is pure of weight zero.*[1] *Define* $\mathrm{M} = \mathscr{F}[1](1/2)$*, which is a perverse sheaf of weight zero on* $\mathbf{A}^1_k[1/f]$*. The tannakian dimension r of $i_{f*}\mathrm{M}$ is given by*

$$r = (\deg(f) - 1)\,\mathrm{rank}(\mathscr{F}) + \sum_{x \in \mathbf{P}^1(\bar{k})} \mathrm{swan}_x(\mathscr{F}) + \sum_{x \in \bar{k}} \mathrm{drop}_x(\mathscr{F}) \geqslant (\deg(f) - 1)\,\mathrm{rank}(\mathscr{F}).$$

Proof. The object M is a perverse sheaf and so is $i_{f*}\mathrm{M}$ because i_f is a closed immersion (see Corollary A.8). The tannakian dimension is the Euler–Poincaré characteristic $\chi_c(\mathrm{T}_{\bar{k}}, (i_{f*}\mathrm{M})_\chi)$ for a generic character $\chi \in \widehat{\mathrm{T}}$ (Proposition 3.16).

For any integer i, we have natural isomorphisms

$$\mathrm{H}^i_c(\mathrm{T}_{\bar{k}}, (i_{f*}\mathrm{M})_\chi) \simeq \mathrm{H}^i_c(\mathbf{A}^1[1/f]_{\bar{k}}, \mathrm{M} \otimes i_f^*\mathscr{L}_\chi) \simeq \mathrm{H}^i_c(\mathbf{A}^1[1/f]_{\bar{k}}, \mathscr{F}[1] \otimes i_f^*\mathscr{L}_\chi).$$

As explained in [75, p. 3227], the pullback $i_f^*\mathscr{L}_\chi$ is geometrically isomorphic to the tensor product

$$\mathscr{L} = \bigotimes_{z \in \mathrm{Z}} \mathscr{L}_{\chi_z(z-x)}$$

where x is the coordinate on $\mathbf{A}^1[1/f]$ and χ corresponds to the tuple (χ_z) of characters of $k^\times$ as above.

Now using the Euler–Poincaré formula on a curve (see Theorem C.2), we obtain

$$r = -\chi_c(\mathbf{A}^1[1/f]_{\bar{k}}, \mathscr{F} \otimes \mathscr{L}) = -\,\mathrm{rank}(\mathscr{F})\chi_c(\mathbf{A}^1[1/f]_{\bar{k}}) \\ + \sum_{x \in \mathbf{P}^1} \mathrm{swan}_x(\mathscr{F} \otimes \mathscr{L}) + \sum_{x \in \mathbf{A}^1[1/f]} \mathrm{drop}_x(\mathscr{F} \otimes \mathscr{L}).$$

The first term is equal to $\mathrm{rank}(f)(\deg(f) - 1)$ since f is square-free, and the second is the sum of Swan conductors of $\mathscr{F}$, since the sheaf $\mathscr{L}$ is everywhere tame. The third is the sum of the drops of $\mathscr{F}$ on $\mathbf{A}^1[1/f]$, since $\mathscr{L}$ is lisse on $\mathbf{A}^1[1/f]$. ☐

We now apply Larsen's Alternative to compute the tannakian group of such perverse sheaves.

Proposition 10.4. *Let $\mathscr{F}$ be a middle extension sheaf on $\mathbf{A}^1_k[1/f]$ which is pure of weight zero and irreducible of rank at least 2. Let* $\mathrm{M} = \mathscr{F}[1](1/2)$*. Assume that* M *is not geometrically isomorphic to $i_f^*\mathscr{L}_\chi[1]$ for some character χ of* G.

Then $i_{f*}\mathrm{M}$ is a geometrically simple perverse sheaf, pure of weight zero and of tannakian dimension at least 2.

Moreover, if $\deg(f) \geqslant 2$, then the fourth moment of the tannakian group $\mathbf{G}^{\mathrm{ari}}_{i_{f*}\mathrm{M}}$ of $i_{f*}\mathrm{M}$ is equal to 2*, and if $\deg(f) \geqslant 4$, then the eighth moment is equal to* 24.

Proof. The previous lema implies that $i_{f*}\mathrm{M}$ has tannakian dimension $\geqslant 2$. It is geometrically simple since M is.

One argument to obtain the result is to observe that i_f is a Sidon morphism when $\deg(f) \geqslant 2$, and a 4-Sidon morphism when $\deg(f) \geqslant 4$ (by Proposition 8.11, (4), since these properties can be checked after a finite extension), so that the result follows from Proposition 8.8 since the tannakian dimension is $\geqslant 2$, and the assumption on M).

[1] Recall (see Example A.18 (3)) that this means that the restriction of $\mathscr{F}$ to any dense open set where it is lisse is punctually pure of weight 0.

For the sake of concreteness, we show also how to perform the computation of the eighth moment using the interpretation of the sums in terms of Dirichlet characters. The eighth moment of the full family of exponential sums over k is equal to

$$\frac{1}{|\mathrm{B}^\times|}\sum_{\chi\in\widehat{\mathrm{B}}^\times}\Bigl|\sum_{x\in k}t_{\mathrm{M}}(x)\chi(t-x)\Bigr|^8=\sum_{x_1,\ldots,x_8}\prod_{i=1}^{4}t_{\mathrm{M}}(x_i)\prod_{i=5}^{8}\overline{t_{\mathrm{M}}(x_i)}\times$$
$$\frac{1}{|\mathrm{B}^\times|}\sum_{\chi\in\widehat{\mathrm{B}}^\times}\chi((t-x_1)\cdots(t-x_4))\overline{\chi((t-x_5)\cdots(t-x_8))}.$$

By orthogonality, the inner sum is 0 unless

$$(t-x_1)\cdots(t-x_4)\equiv(t-x_5)\cdots(t-x_8)\pmod{f},$$

in which case it is equal to $|\mathrm{B}^\times|$. Since the degree of f is at least 4, this congruence can only occur when

$$(t-x_1)\cdots(t-x_4)=(t-x_5)\cdots(t-x_8)$$

in $k[t]$. We then distinguish according to the size of $\{x_1,\ldots,x_4\}$. If this set has four elements, then so does $\{x_5,\ldots,x_8\}$, and the two sets are equal. The contribution arising from this case is

$$\sum_{x_1,\ldots,x_4}\sum_{\sigma\in\mathfrak{S}_4}t_{\mathrm{M}}(x_1)\cdots t_{\mathrm{M}}(x_4)\overline{t_{\mathrm{M}}(x_{\sigma(1)})\cdots t_{\mathrm{M}}(x_{\sigma(4)})}=24\Bigl(\sum_{x\in k}|t_{\mathrm{M}}(x)|^2\Bigr)^4.$$

On the other hand, if the set $\{x_1,\ldots,x_4\}$ has three elements, say x, y and z, then so does $\{x_5,x_6,x_7,x_8\}$, and there are an absolutely bounded number of possibilities for (x_1,x_2,x_3,x_4) given x, y and z. A similar result holds for two or one elements, and since $t_{\mathrm{M}}(x)\ll|k|^{-1/2}$, one sees that these altogether contribute at most

$$\frac{1}{|k|^4}\sum_{x,y,z\in k}1\ll\frac{1}{|k|}.$$

These computations can be repeated over k_n for $n\geqslant 1$ using (10.4), and using Proposition 8.1, we deduce by letting $n\to+\infty$ that

$$\frac{1}{|\mathrm{B}^\times|}\sum_{\chi\in\widehat{\mathrm{B}}^\times}\Bigl|\sum_{x\in k}t_{\mathrm{M}}(x)\chi(t-x)\Bigr|^8\to 24$$

as $|k|\to+\infty$.

Finally, the usual argument using the definition of generic sets of characters together with (10.3) and the Riemann Hypothesis imply that

$$\frac{1}{|\mathrm{B}^\times|}\sum_{\substack{\chi\in\widehat{\mathrm{B}}^\times\\ \widetilde{\chi}\text{ ramified}}}\Bigl|\sum_{x\in k}t_{\mathrm{M}}(x)\chi(t-x)\Bigr|^8\to 0,$$

so that Proposition 8.4 gives the result. ∎

Corollary 10.5. *Under the assumptions of the proposition, the tannakian group of $i_{f*}\mathrm{M}$ contains $\mathbf{SL}_r$, where r is the tannakian dimension of $i_{f*}\mathrm{M}$, if $\deg(f)\geqslant 4$.*

Proof. By Lemma 10.3, the assumption implies $r\geqslant 4$, and the result follows from Larsen's Alternative, in the form of the eighth moment theorem of Guralnick and Tiep (see Theorem 8.5 (4)). ∎

10.3. Application to von Mangoldt functions

Suppose again that M is of the form $\mathscr{F}[1](1/2)$ for some middle extension sheaf $\mathscr{F}$ on $\mathbf{A}^1_k[1/f]$ which is pure of weight zero and geometrically irreducible of rank at least 2.

The statement of equidistribution on average for the object $i_{f*}\mathrm{M}$ leads automatically to distribution statements of any "continuous" function of the polynomials in the variable T which are the twisted L-functions of M, namely

$$\det(1-\mathrm{Fr}_k\,\mathrm{T}\mid \mathrm{H}^0_c(\mathbf{A}^1[1/f]_{\bar{k}},\mathrm{M}\otimes\mathscr{L}_{\widetilde{\chi}}))=\det(1-\mathrm{Fr}_k\,\mathrm{T}\mid \mathrm{H}^0_c(\mathrm{T}_{\bar{k}},(i_{f*}\mathrm{M})_{\chi}))$$

as $\chi\in\widehat{\mathrm{B}}^\times$ varies, where $\widetilde{\chi}$ is now the character of the fundamental group of $\mathbf{A}^1_k[1/f]$ that corresponds to χ by class-field theory, and $\mathscr{L}_{\widetilde{\chi}}$ is the associated rank one sheaf.

For instance, this leads to statements concerning the variance of von Mangoldt functions in arithmetic progressions, as we now explain.

Write

$$\mathrm{L}(\mathrm{M},\mathrm{T})=\det(1-\mathrm{Fr}_k\,\mathrm{T}\mid \mathrm{H}^0_c(\mathbf{A}^1[1/f]_{\bar{k}},\mathrm{M}))=\prod_x\det(1-\mathrm{Fr}_{k_{\deg(x)}}\mathrm{T}^{\deg(x)}\mid\mathscr{F}_x)^{-1},$$

where x runs over the set of closed points of $\mathbf{A}^1_k[1/f]$, which may be identified with the set of irreducible monic polynomials in $k[t]$ which are coprime to f. Expanding the logarithmic derivative of the local factor at a closed point x, corresponding to an irreducible monic polynomial $\pi\in k[t]$, we have

$$-\mathrm{T}d\log(\det(1-\mathrm{Fr}_{k_{\deg(x)}}\mathrm{T}^{\deg(x)}\mid\mathscr{F}_x)^{-1})=\sum_{\nu\geqslant 1}\Lambda_{\mathrm{M}}(\pi^\nu)\mathrm{T}^{\nu\deg(\pi)},$$

which defines the von Mangoldt function $\Lambda_{\mathrm{M}}(\pi^\nu)$ for any monic irreducible polynomial π coprime to f and any $\nu\geqslant 1$. We further define $\Lambda_{\mathrm{M}}(g)=0$ if $g\in k[t]$ is not a power of such an irreducible polynomial. The full logarithmic derivative then has the formal power series expansion

$$-\mathrm{T}\frac{\mathrm{L}'(\mathrm{M},\mathrm{T})}{\mathrm{L}(\mathrm{M},\mathrm{T})}=\sum_g\Lambda_{\mathrm{M}}(g)\mathrm{T}^{\deg(g)}$$

over all monic polynomials $g\in k[t]$.

For an integer $m\geqslant 1$ and a polynomial $a\in k[t]$, we then define

$$\psi_{\mathrm{M}}(m;f,a)=\sum_{\substack{\deg(g)=m\\ g\equiv a\,(\mathrm{mod}\,f)}}\Lambda_{\mathrm{M}}(g).$$

We consider the average

$$\mathrm{A}_{\mathrm{M}}(m;f)=\frac{1}{|\mathrm{B}^\times|}\sum_{a\in\mathrm{B}^\times}\psi_{\mathrm{M}}(m;f,a)$$

and the variance

$$\mathrm{V}_{\mathrm{M}}(m;f)=\frac{1}{|\mathrm{B}^\times|}\sum_{a\in\mathrm{B}^\times}\left|\psi_{\mathrm{M}}(m;f,a)-\mathrm{A}_{\mathrm{M}}(m;f)\right|^2.$$

These are related to exponential sums as follows.

Proposition 10.6. *With assumptions and notation as above, we have*

$$\mathrm{V}_{\mathrm{M}}(m;f)=\frac{1}{|\mathrm{B}^\times|^2}\sum_{\substack{\chi\in\widehat{\mathrm{B}}^\times\\ \chi\neq 1}}\mathrm{V}_{\mathrm{M}}(m;\chi)$$

where

$$\mathrm{V_M}(m;\chi) = \Big| \sum_{x\in k_m} t_{\mathrm{M}}(x;k_m)\chi(\mathrm{N}_{k_m/k}(t-x))\Big|^2.$$

In particular, if χ is weakly unramified for $i_{f*}\mathrm{M}$, then we have $\mathrm{V_M}(m;\chi) = |\operatorname{Tr}(\Theta_{\mathrm{M}}(\chi)^m)|^2$.

Proof. The first part is proved, using the orthogonality of characters, exactly like [56, §6, (6.3.4)]. The second assertion then follows from Lemma 3.34 and (10.4). ☐

Remark 10.7. The von Mangoldt function can be replaced by many other arithmetic functions in this argument; we refer to the discussion by Sawin in [113] (which proves analogue equidistribution statements to ours for the case of “short intervals”, which amounts to considering a unipotent group instead of a torus) and to [114] for a discussion of how classical arithmetic functions which are related to “factorization functions” (functions of polynomials g that depend only on the factorization type of g) can be interpreted as trace functions using representation theory of the symmetric groups.

We now obtain a formula for the variance, with some additional assumption.

Corollary 10.8. *In addition to the assumptions of this section, assume that $m \geqslant 2$, and that the tannakian determinant of M is geometrically of infinite order. Then*

$$\lim_{|k|\to+\infty} |\mathrm{B}^\times|^2 \mathrm{V_M}(m;f) = \min(m,r),$$

where r is the tannakian dimension of $i_{f*}\mathrm{M}$.

Proof. Combined with Corollary 10.5, the assumption implies that the arithmetic and geometric tannakian groups of $i_{f*}\mathrm{M}$ are both equal to $\mathbf{GL}_r$. Thus the limit exists by Theorem 4.15 and is equal to

$$\int_{\mathbf{U}_r(\mathbf{C})} |\operatorname{Tr}(g^m)|^2 d\mu(g)$$

where μ is the Haar probability measure. This matrix integral is equal to $\min(m,r)$ by work of Diaconis and Evans [33, Th. 2.1]. ☐

To check the assumption on the tannakian determinant, we have a first general criterion, which is however quite restricted.

Proposition 10.9. *With notation and assumptions as above, suppose that there exists $z \in \mathrm{Z}$ such that the local monodromy representation of $\mathscr{F}$ at z has a non-zero unipotent tame component while the local monodromy at infinity has no unipotent tame component. Then the tannakian determinant of $i_{f*}\mathrm{M}$ is geometrically of infinite order.*

Proof. We apply Corollary 3.46 to the norm morphism $p\colon \mathrm{T} \to \mathbf{G}_m$. Indeed, $p \circ i_f$ coincides with the finite morphism $\varepsilon f\colon \mathbf{A}^1[1/f] \to \mathbf{G}_m$, where $\varepsilon = (-1)^{\deg(f)}$ (Proposition 10.1), so that the equalities $\mathrm{R}p_!(i_{f*}\mathrm{M}) = \mathrm{R}p_!(i_{f!}\mathrm{M}) = (\varepsilon f)_*\mathrm{M} = ((\varepsilon f)_!\mathscr{F})[1](1/2)$ hold, and the sheaf $((\varepsilon f)_!\mathscr{F})[1](1/2) = ((\varepsilon f)_*\mathscr{F})[1](1/2)$ has no tame unipotent local monodromy at infinity, but has some non-trival tame unipotent monodromy at 0 in view of the canonical isomorphism

$$((\varepsilon f)_*\mathscr{F})_0 \simeq \bigoplus_{z\in \mathrm{Z}} \mathscr{F}_z.$$

Hence, the tannakian determinant of the object $i_{f*}\mathrm{M}$ is geometrically of infinite order. ☐

We now explain the proof of Theorem 3, where we will also use a different approach to checking that the tannakian determinant has infinite order, which may be useful in other contexts.

Let $\pi\colon \mathscr{E} \to \mathbf{P}^1$ be the morphism which "is" the Legendre elliptic curve. We start with the sheaf

$$\mathscr{F} = \mathrm{R}^1\pi_*\overline{\mathbf{Q}}_\ell(1/2).$$

This is a middle extension sheaf on $\mathbf{A}^1_k$. It is pure of weight zero and geometrically irreducible of rank 2 (in particular, its H^2_c vanishes), and is tamely ramified at 0, 1 and ∞, with drop equal to 1 at 0 and 1. Using Lemma 10.3, we compute that the tannakian dimension is $r = 2\deg(f) - 2 + a$, where a is the degree of the gcd of f and $t(t-1)$.

Now the pullback of $\mathscr{F}$ to $\mathbf{A}^1_k[1/f]$ is a middle extension sheaf, geometrically irreducible of rank 2 and pure of weight 0, for which we keep the same notation. We can then apply Corollary 10.8 to $\mathscr{F}$, using the following proposition. In order to conclude after doing so, we check that the contribution of the local factors at $z \in \mathrm{Z}$ to the L-functions (which might not be of weight 0) is negligible (compare [56, Prop. 6.5.3]).

Proposition 10.10. *Let $\mathrm{M} = \mathscr{F}[1](1/2)$. The tannakian determinant of $i_{f*}\mathrm{M}$ is geometrically of infinite order.*

Proof. If f is not coprime to $t(t-1)$, then we can apply Proposition 10.9, since $\mathscr{F}$ has non-trivial tame unipotent monodromy at 0 and 1, and none at infinity. So we assume that f is coprime with $t(t-1)$.

We may assume that the polynomial f splits in linear factors over k and that $k \neq \mathrm{Z} \cup \{0, 1\}$. Fix a non-trivial additive character ψ of k. We will then prove in Proposition 10.11 below, using the theory of local constants, that there exists a generic set of characters $\mathscr{X} \subset \mathscr{X}(\mathrm{M})$ and elements $\xi_z \in \mathbf{A}^1[1/f]$ such that for $n \geqslant 1$ and $\chi \in \mathscr{X}(k_n)$, the equality

$$\det(\Theta_{\mathrm{M},k_n}(\chi)) = \gamma^n\, \mathrm{H}_1(\chi)^{-1}\, \mathrm{H}_2\Big(\prod_{z\in\mathrm{Z}} \chi_z^{-1}\Big)^{-1}$$

holds, for some number γ independent of χ and n, where the functions H_1 and H_2 are products of Gauss sums described in (10.6) and (10.7) below.

On $\mathbf{G}_m$, the function

$$\chi_z \mapsto \chi_z(\xi_z)\frac{1}{|k|}\Big(\sum_{y\in k^\times} \chi_z(y)\psi(y)\Big)^2$$

coincides for χ_z non-trivial with the arithmetic Mellin transform of the multiplicative translated hypergeometric complex $\mathrm{Hyp}_{\xi_z}(!, \psi, 1, 1; \varnothing)(1/2)$ (see (B.1) for this; in this case, this is a shifted and translated Kloosterman sheaf). Since the function $\chi \mapsto \mathrm{H}_1(\chi)^{-1}$ is the product of these functions over $z \in \mathrm{Z}$, it coincides generically with the Mellin transform on T of the tensor product

$$\bigotimes_{z\in\mathrm{Z}} p_z^*\mathrm{Hyp}_{\xi_z^{-1}}(!; \psi, 1, 1; \varnothing)(1/2),$$

where p_z is the projection from T to the z-component in the splitting $g \mapsto (g(z))$ of the torus T. (Indeed, this reflects the formula

$$\sum_{x\in\mathrm{T}(k)} \chi(x) \prod_{z\in\mathrm{Z}} f_z(p_z(x)) = \sum_{(x_z)\in(k^\times)^{\mathrm{Z}}} \prod_{z\in\mathrm{Z}} \chi_z(x_z) \prod_{z\in\mathrm{Z}} f_z(x_z) = \prod_{z\in\mathrm{Z}} \sum_{x\in k^\times} \chi_z(x) f_z(x)$$

for arbitrary functions f_z on $k^\times$.)

Similarly, the function $\chi \mapsto \mathrm{H}_2(\prod \chi_z^{-1})^{-1}$, which only depends on the product η of the component characters (χ_z), coincides (for η non-trivial) with the arithmetic Mellin transform of the object $\Delta_*\mathrm{L}$, where $\mathrm{L} = \mathrm{Hyp}(!, \psi, \lambda_2, \lambda_2; \varnothing)(1/2)$ and

$$\Delta \colon \mathbf{G}_m \to \mathbf{G}_m^{\mathrm{Z}} \simeq \mathrm{T}$$

is the closed immersion $x \mapsto (x^{-1}, \ldots, x^{-1})$. This reflects the fact that Δ is a morphism of algebraic groups, and that the dual $\widehat{\Delta}$ on $\widehat{\mathrm{T}}(k)$ is given by

$$(\chi_z)_{z\in\mathrm{Z}} \mapsto \prod_{z\in\mathrm{Z}} \chi_z^{-1}.$$

By Theorem 6.11, the formula (10.5) therefore implies that the tannakian determinant of M is geometrically isomorphic in $\overline{\mathbf{P}}(\mathrm{T})$ to the perverse sheaf

$$\mathrm{D} = (\Delta_*\mathrm{L}) * \Big(\bigotimes_{z\in\mathrm{Z}} p_z^*\mathrm{Hyp}_{\xi_z}(!; \psi, 1, 1; \varnothing)(1/2)\Big).$$

The object D visibly has infinite geometric tannakian group since for any $m \geqslant 1$, we have

$$\mathrm{D}^{*m} = (\Delta_*\mathrm{L})^{*m} * \Big(\bigotimes_{z\in\mathrm{Z}} p_z^*\mathrm{Hyp}_{\xi_z^{-1}}(!; \psi, 1, 1; \varnothing)^{*m}(1/2)\Big),$$

in $\overline{\mathbf{P}}(\mathrm{T})$, and the m-th convolution powers on $\mathbf{G}_m$ of the hypergeometric complexes that appear are not geometrically trivial (see Theorem B.4). ☐

We complete this section by proving the formula for the determinant.

Proposition 10.11. *Suppose that f splits in linear factors over k. For $z \in \mathrm{Z}$, define*

$$\xi_z = z(z-1) \prod_{x\in\mathrm{Z}-\{z\}} (z-x)^2 \in k^\times.$$

There exist numbers ε_0, ε_1 with the following property. For a character $\chi \in \mathscr{X}(\mathrm{M})$ such that all components χ_z for $z \in \mathrm{Z}$ are non-trivial, and such that the product of the components is not of order at most 2, we have

$$\det(\mathbf{\Theta}_{\mathrm{M}}(\chi))^{-1} = (-1)^r |k|\, \varepsilon_0\varepsilon_1\, \mathrm{H}_1(\chi)\, \mathrm{H}_2\Big(\prod_{z\in\mathrm{Z}} \chi_z^{-1}\Big) \tag{10.5}$$

where

$$\mathrm{H}_1(\chi) = \prod_{z\in\mathrm{Z}} \chi_z(\xi_z^{-1}) |k| \Big(\sum_{y\in k^\times} \chi_z(y)\psi(y)\Big)^{-2}, \tag{10.6}$$

$$\mathrm{H}_2(\chi) = |k| \Big(\sum_{y\in k^\times} (\lambda_2\chi)(y)\psi(y)\Big)^{-2}. \tag{10.7}$$

Proof. Let $j \colon \mathbf{A}^1[1/f] \to \mathbf{P}^1$ be the open immersion. Let $\chi \in \widehat{\mathrm{B}}$ be a Dirichlet character and $\mathscr{L}_\chi$ the lisse rank 1 sheaf on $\mathbf{A}^1[1/f]$ that corresponds to it. The L-function of $j_!(\mathrm{M} \otimes \mathscr{L}_\chi)$ satisfies a functional equation of the form

$$\mathrm{L}(j_!(\mathrm{M} \otimes \mathscr{L}_\chi), \mathrm{T}) = \varepsilon(\chi)\mathrm{T}^a \mathrm{L}(\mathrm{D}(j_!(\mathrm{M} \otimes \mathscr{L}_\chi)), \mathrm{T}^{-1})$$

where $a = -\chi(j_!(\mathrm{M} \otimes \mathscr{L}_\chi)) = -r$ is an integer and

$$\varepsilon(\chi) = \det(-\mathrm{Fr}_k \mid \mathrm{H}^0(\mathbf{P}^1_{\bar{k}}, j_!(\mathrm{M} \otimes \mathscr{L}_\chi)))^{-1} = \det(-\mathrm{Fr}_k \mid \mathrm{H}^0_c(\mathbf{A}^1_{\bar{k}}[1/f], \mathrm{M} \otimes \mathscr{L}_\chi))^{-1}$$

(see, e.g., [98, (3.1.1.3), (3.1.1.5)] or the reminder in Section C.1).

By Lemma 3.34, if $\chi \in \widehat{\mathrm{T}}$ is unramified for M, then we deduce that

$$\det(\mathbf{\Theta}_{\mathrm{M}}(\widetilde{\chi})) = (-1)^r \varepsilon(\chi)^{-1}, \tag{10.8}$$

where r is the tannakian dimension of $i_{f*}\mathrm{M}$. By a theorem of Laumon,[2] we can express the constant $\varepsilon(\chi)$ as a product over closed points

$$\varepsilon(\chi) = |k|^{-2} \prod_{x \in |\mathbf{P}^1|} \varepsilon_x(\chi)$$

of local constants, previously defined by Deligne [26] and characterized by the properties of [98, Th. 3.1.5.4]. Precisely, fixing a non-trivial additive character ψ of k and a non-zero meromorphic differential 1-form ω on $\mathbf{P}^1$, we can then define

$$\varepsilon_x(\chi) = \varepsilon(\mathbf{P}^1_{(x)}, j_*(\mathrm{M} \otimes \mathscr{L}_\chi)|\mathbf{P}^1_{(x)}, \omega \mid \mathbf{P}^1_{(x)})$$

with the notation of loc. cit. See again Section C.1; in particular the factor $|k|^{-2}$ above is given by (C.1), namely the exponent is obtained by the computation

$$-2 = 1 \cdot (1-0) \cdot (-2),$$

where -2 is the generic rank of the object M (a sheaf of rank 2 in degree -1).

We take $\omega = dt$, where t is the standard coordinate on $\mathbf{P}^1$. The data of ψ and ω allows us to define non-trivial additive characters ψ_x of the completed local field at any closed point $x \in |\mathbf{P}^1|$ by the recipe in [98, Th. 3.1.5.4, (v)]. For all closed points $x \in \mathbf{A}^1$, the character ψ_x is of conductor zero since ω is regular at x (see [98, 3.1.3.6]). For $x = \infty$, we have $c(\psi_\infty) = -2$ since ω has a double pole at ∞.

The main tool to compute the local constants is the formula (C.7) for twisting by a lisse sheaf: for any closed point x, if K is an ℓ-adic complex on the *trait* $\mathbf{P}^1_{(x)}$ and F is a lisse $\overline{\mathbf{Q}}_\ell$-sheaf on $\mathbf{P}^1_{(x)}$ of rank $r(\mathrm{F})$, then we have

$$\varepsilon(\mathbf{P}^1_{(x)}, (\mathrm{K} \otimes \mathrm{F})|\mathbf{P}^1_{(x)}, \omega|\mathbf{P}^1_{(x)}) = \det(\mathrm{Fr}_x \mid \mathrm{F})^{a(\mathbf{P}^1_{(x)}, \mathrm{K}, \omega|\mathbf{P}^1_{(x)})} \varepsilon(\mathbf{P}^1_{(x)}, \mathrm{K}, \omega|\mathbf{P}^1_{(x)})^{r(\mathrm{F})}, \tag{10.9}$$

where the local exponent $a(\mathbf{P}^1_{(x)}, \mathrm{K}, \omega|\mathbf{P}^1_{(x)})$ is defined in (C.4) and (C.3). Moreover, we will often use the formula

$$\varepsilon(\mathbf{P}^1_{(x)}, \mathrm{K}[1], \omega) = \varepsilon(\mathbf{P}^1_{(x)}, \mathrm{K}, \omega)^{-1}$$

(see (C.10)).

Let $(\chi_z)_{z \in \mathrm{Z}}$ be the tuple of characters corresponding to χ. We recall that $\mathscr{L}_\chi$ is isomorphic to $\bigotimes_{z \in \mathrm{Z}} \mathscr{L}_{\chi_z(z-t)}$.

We now compute the local constants, distinguishing between the cases $x \in \mathbf{A}^1 - (\{0,1\} \cup \mathrm{Z})$, $x \in \{0,1\}$, $x \in \mathrm{Z}$ and $x = \infty$.

Case 1. Let $x \in \mathbf{A}^1$ and $x \notin \mathrm{Z} \cup \{0,1\}$. In this case, $\mathrm{M} \otimes \mathscr{L}_\chi$ is a lisse sheaf shifted by 1, and since $c(\psi_x) = 0$, we find

$$\varepsilon_x(\chi) = 1 \tag{10.10}$$

by (10.9).

Case 2. Let $x \in \{0,1\}$. Then $\mathscr{L}_\chi$ is a lisse sheaf at x, since we assumed that f is coprime with $t(t-1)$. We find

$$\varepsilon_x(\chi) = \varepsilon_x\, t_{\mathscr{L}_\chi}(x)^{a(\mathbf{P}^1_{(x)}, \mathrm{M}|\mathbf{P}^1_{(x)}, dt)}$$

[2] Which, in the case we use it, goes back to Deligne [26, Th. 9.3]; see [98, 3.2.1.9] for references.

by (10.9) with $\mathrm{F} = \mathscr{L}_\chi$, where $\varepsilon_x = \varepsilon(\mathbf{P}^1_{(x)}, \mathrm{M}, dt)$, which is independent of χ. We further compute that

$$a(\mathbf{P}^1_{(x)}, \mathrm{M}|\mathbf{P}^1_{(x)}, dt) = -a(\mathbf{P}^1_{(x)}, \mathscr{F}(1/2)|\mathbf{P}^1_{(x)}, dt) = -(2-1+0) = -1$$

by (C.4) and (C.3), since $\mathscr{F}$ has drop 1 at x (see, e.g., [74, p. 73]) and dt is regular at x. Hence,

$$\varepsilon_x(\chi) = \varepsilon_x \prod_{z\in \mathrm{Z}} \chi_z(z-x)^{-1}. \tag{10.11}$$

Case 3. Let $x \in \mathrm{Z}$. Then we can write

$$\mathrm{M} \otimes \mathscr{L}_\chi = \mathscr{F}[1](1/2) \otimes \mathscr{L}^{(x)} \otimes \mathscr{L}_{\chi_x(t-x)} = (\mathscr{F}(1/2) \otimes \mathscr{L}^{(x)} \otimes \mathscr{L}_{\chi_x(t-x)})[1],$$

where $\mathscr{F}$ and $\mathscr{L}^{(x)}$ are both lisse sheaves at x. Applying (10.9) after an inversion due to the shift, we get

$$\begin{aligned}\varepsilon_x(\chi) &= \varepsilon(\mathbf{P}^1_{(x)}, \mathscr{F}[1](1/2) \otimes \mathscr{L}^{(x)} \otimes \mathscr{L}_{\chi_x(t-x)}, dt)^{-1}\\ &= \det(\mathrm{Fr}_x \mid \mathscr{F}(1/2) \otimes \mathscr{L}^{(x)})^{-a} \varepsilon(\mathbf{P}^1_{(x)}, \mathscr{L}_{\chi_x(t-x)}, dt)^{-2}\end{aligned}$$

where

$$a = a(\mathbf{P}^1_{(x)}, \mathscr{L}_{\chi_x(t-x)}, dt) = 1 + 0 - 0 = 1$$

if χ_x is non-trivial by (C.4) and (C.3) again.

We have

$$\det(\mathrm{Fr}_x \mid \mathscr{F}(1/2) \otimes \mathscr{L}^{(x)}) = \frac{1}{|k|} \prod_{\substack{z\in \mathrm{Z}\\ z\neq x}} \chi_z(z-x)^2,$$

and by (C.9), we find that

$$\varepsilon(\mathbf{P}^1_{(x)}, \mathscr{L}_{\chi_x(t-x)}, dt) = \varepsilon_0(\mathbf{P}^1_{(x)}, \mathscr{L}_{\chi_x(t-x)}, dt) = -\chi_x(-1) \sum_{y\in k^\times} \chi(y)\psi(y)$$

if χ_x is not trivial (here we also use the fact that $x \in k$).

These computations imply that

$$\varepsilon_x(\chi) = \prod_{\substack{z\in \mathrm{Z}\\ z\neq x}} \chi_z(z-x)^{-2}\, |k| \Big(\sum_{y\in k^\times} \chi(y)\psi(y) \Big)^{-2}, \tag{10.12}$$

if χ_x is not trivial.

Case 4. Let $x = \infty$. Write $u = 1/t$, a uniformizer at ∞, so that $dt = -u^{-2}du$. Then $\mathscr{L}_\chi = \mathscr{L}^{(\infty)} \otimes \mathscr{L}_{\eta(u)}$ where

$$\mathscr{L}^{(\infty)} = \bigotimes_{z\in \mathrm{Z}} \mathscr{L}_{\chi_z(uz-1)}, \qquad \eta = \prod_{z\in \mathrm{Z}} \chi_z^{-1}.$$

The sheaf $\mathscr{L}^{(\infty)}$ is lisse at ∞ and the local eigenvalue of Frobenius there is equal to $(-1)^{\deg(f)}$. On the other hand, we have $\mathrm{M} = \mathscr{F}[1](1/2)$, and $\mathscr{F}$ is of rank 2, tamely ramified at ∞ with local monodromy isomorphic to $\mathscr{L}_{\lambda_2} \otimes \mathrm{Unip}(2)$, where λ_2 is the Legendre character and $\mathrm{Unip}(2)$ is a unipotent Jordan block of size 2 (see, e.g., [74, p. 73]).

Computing first as in the previous case, we get

$$\begin{aligned}\varepsilon_\infty(\chi) &= \varepsilon(\mathbf{P}^1_{(\infty)}, \mathscr{F}(1/2) \otimes \mathscr{L}^{(\infty)} \otimes \mathscr{L}_{\eta(u)}, -u^{-2}du)^{-1}\\ &= \det(\mathrm{Fr}_\infty \mid \mathscr{L}^{(\infty)})^{-a} \varepsilon(\mathbf{P}^1_{(\infty)}, \mathscr{F}(1/2) \otimes \mathscr{L}_{\eta(u)}, -u^{-2}du)^{-1}\end{aligned}$$

where

$$\begin{aligned}a &= a(\mathbf{P}^1_{(\infty)}, \mathscr{F}(1/2)\otimes\mathscr{L}_{\eta(u)}, -u^{-2}du)\\ &= a(\mathbf{P}^1_{(\infty)}, \mathscr{F}(1/2)\otimes\mathscr{L}_{\eta(u)}) - 2\times 2 = (2+0-2)-4 = -4\end{aligned}$$

if η is non-trivial (see again (C.4) and (C.3)). Note then that

$$\det(\mathrm{Fr}_\infty \mid \mathscr{L}^{(\infty)})^{-a} = \prod_{z\in \mathrm{Z}} \chi_z(-1)^4 = 1.$$

The shape of the local monodromy and the multiplicativity property under extensions shows that if $\lambda_2\eta$ is not trivial, then the formula

$$\varepsilon(\mathbf{P}^1_{(\infty)}, \mathscr{F}(1/2)\otimes\mathscr{L}_{\eta(u)}, -u^{-2}du) = \varepsilon(\mathbf{P}^1_{(\infty)}, \mathscr{L}_{(\lambda_2\eta)(u)}, -u^{-2}du)^2$$

holds. Indeed, in this case, the stalk at ∞ of $\mathscr{F}\otimes\mathscr{L}_\eta$ and of its semisimplification both vanish, so that

$$\varepsilon(\mathbf{P}^1_{(\infty)}, \mathscr{F}(1/2)\otimes\mathscr{L}_{\eta(u)}, -u^{-2}du) = \varepsilon_0(\mathbf{P}^1_{(\infty)}, j_*\mathscr{F}(1/2)\otimes\mathscr{L}_{\eta(u)}, -u^{-2}du),$$

where ε_0 is the local factor defined by (C.5), and j is the inclusion of the generic point of $\mathbf{P}^1_{(\infty)}$, and one can apply (C.6); compare [26, 8.12].

Let β be the character of the local field at infinity associated to $\lambda_2\eta$ by local class field theory. Using (C.8), we derive the formula

$$\varepsilon(\mathbf{P}^1_{(\infty)}, \mathscr{L}_{(\lambda_2\eta)(u)}, -u^{-2}du) = \beta(-u^{-2})|k|^{-2}\varepsilon(\mathbf{P}^1_{(\infty)}, \mathscr{L}_{(\lambda_2\eta)(u)}, du).$$

From (C.9), we deduce further that if $\lambda_2\eta$ is non-trivial, then

$$\varepsilon(\mathbf{P}^1_{(\infty)}, \mathscr{L}_{(\lambda_2\eta)(u)}, -u^{-2}du) = |k|^{-2}\sum_{y\in k^\times}(\lambda_2\eta)(y)\psi(y).$$

The final outcome is that

$$\varepsilon_\infty(\chi) = |k|^4\Bigl(\sum_{y\in k^\times}(\lambda_2\eta)(y)\psi(y)\Bigr)^{-2}, \tag{10.13}$$

if $\eta\notin\{1,\lambda_2\}$.

We now simply combine the formulas (10.10), (10.11), (10.12) and (10.13) to conclude the proof, noting that the contribution of all $x\in\mathrm{Z}$ involves the product

$$\prod_{x\in\mathrm{Z}}\prod_{\substack{z\in\mathrm{Z}\\ z\neq x}}\chi_z(z-x)^{-2} = \prod_{z\in\mathrm{Z}}\chi_z\Bigl(\prod_{\substack{x\in\mathrm{Z}\\ x\neq z}}(z-x)^{-2}\Bigr).$$

□

Remark 10.12. It it also certainly possible to perform this computation by automorphic methods (using the global case of the $\mathbf{GL}_2$-Langlands correspondence over $k(t)$, first proved by Drinfeld). However, more general situations might be easier to handle using these geometric arguments.

Yet another possible approach, which would be well-suited for generalizations, would be to use Loeser's general computation of the tannakian determinant for an arbitrary perverse sheaf on a torus $\mathbf{T}$ (see [103, Th. 3.6.1]), which can be identified with an element of the hypergeometric group $\mathbf{H}_{\mathrm{int}}(\mathbf{T})$ of Gabber and Loeser (see Example 3.51). This group is isomorphic (loc. cit.) to $\mathbf{T}(\bar{k})\times\mathbf{Z}^{\mathrm{S}}$ for some explicit set S (related to sub-tori of dimension 1 in $\mathbf{T}$ and tame ℓ-adic characters of $\mathbf{G}_m$. It would then be enough to show that there exists some $s\in\mathrm{S}$ such that the s-component of $\det(\mathrm{M})$ is non-zero to deduce that $\det(\mathrm{M})$ has infinite order (without computing exactly the determinant).

CHAPTER 11

Equidistribution on abelian varieties

In this chapter, we consider some aspects of equidistribution on abelian varieties. We denote as before by k a finite field, and by $\bar{k}$ an algebraic closure of k. We denote by k_n the extension of degree n in $\bar{k}$. The prime ℓ is different from the characteristic of k.

11.1. Equidistribution in the jacobian of a curve

The main result of this section is a generalization of a theorem announced by Katz during a talk at a workshop held at the University of Zürich in September 2012 [73], answering a question of Tsimerman.

Let C be a smooth projective geometrically connected curve of genus $g \geqslant 2$ over k, and let $\mathrm{A} = \mathrm{Jac}(\mathrm{C})$ be its jacobian. We recall that C may not have k-rational points but always has a k-rational divisor of degree one. We fix such a divisor Δ and we denote by $s_\Delta \colon \mathrm{C} \hookrightarrow \mathrm{A}$ the closed immersion obtained by sending a point x to the class of the divisor $(x) - \Delta$. Recall that the functor $s_{\Delta *} = s_{\Delta !}$ preserves perversity (Corollary A.8).

Theorem 11.1 (Katz). *Let Δ be a divisor of degree one on C. Let M_0 be a geometrically simple perverse sheaf on C of generic rank $r \geqslant 1$ which is pure of weight zero. Let $\mathrm{M} = s_{\Delta *}\mathrm{M}_0$ and let d denote the tannakian dimension of M.*

(1) *We have $d \geqslant (2g-2)r \geqslant 2$.*
(2) *Assume that C is hyperelliptic, that $\Delta = (0_{\mathrm{C}})$ for some k-rational point $0_{\mathrm{C}} \in \mathrm{C}(k)$ fixed by the hyperelliptic involution i, and that $\mathrm{D}(\mathrm{M}_0)$ is geometrically isomorphic to $i^*\mathrm{M}_0$. Then, up to conjugacy, there are inclusions*
$$\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}} = \mathbf{G}_{\mathrm{M}}^{\mathrm{ari}} = \mathbf{Sp}_d \ \text{ or } \ \mathbf{SO}_d \subset \mathbf{G}_{\mathrm{M}}^{\mathrm{geo}} \subset \mathbf{G}_{\mathrm{M}}^{\mathrm{ari}} \subset \mathbf{O}_d .$$
(3) *If C is not hyperelliptic, or if C is hyperelliptic but $\mathrm{D}(\mathrm{M}_0)$ is not geometrically isomorphic to $i^*\mathrm{M}_0$, then there are inclusions*
$$\mathbf{SL}_d \subset \mathbf{G}_{\mathrm{M}}^{\mathrm{geo}} \subset \mathbf{G}_{\mathrm{M}}^{\mathrm{ari}} \subset \mathbf{GL}_d .$$

Proof. We write $s = s_\Delta$ for simplicity. Since A is an abelian variety, the dimension d is the Euler–Poincaré characteristic of M_χ for *any* $\chi \in \widehat{\mathrm{A}}$ (see Proposition 3.21), in particular for the trivial character, which means that $d = \chi(\mathrm{A}_{\bar{k}}, \mathrm{M}) = \chi(\mathrm{C}_{\bar{k}}, \mathrm{M}_0)$. Write $\mathrm{M}_0 = \mathscr{F}_0[1](1/2)$ for some middle extension sheaf $\mathscr{F}_0$ on C of generic rank r; using the Euler–Poincaré characteristic formula on a curve (see (C.11), for instance), it follows that

$$\chi(\mathrm{C}_{\bar{k}}, \mathrm{M}_0) = \chi(\mathrm{C}_{\bar{k}}, \mathscr{F}_0[1]) = (2g-2)r + \sum_{x \in \mathrm{C}(\bar{k})} (\mathrm{swan}_x(\mathscr{F}_0) + \mathrm{drop}_x(\mathscr{F}_0)) \geqslant (2g-2)r. \tag{11.1}$$

According to Proposition 8.11 (2), the embedding s is a Sidon morphism if C is not hyperelliptic, and is an i-symmetric Sidon morphism in the hyperelliptic situation of (2).

Suppose first that C is not hyperelliptic. Using the fact that $d \geqslant 2$, we deduce from Proposition 8.8 that $\mathrm{M}_4(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}) = 2$. Thus, by Larsen's Alternative (Theorem 8.5 (3)), either $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ is virtually central, i.e., $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}/\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}} \cap \mathrm{Z}$ is finite, or $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ contains $\mathbf{SL}_d$. Proposition 8.22 shows that the first case is not possible, since $\mathrm{M}_4(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}) = 2$ is not the square of an integer. Then the fact that $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ contains $\mathbf{SL}_d$ implies that $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ also contains $\mathbf{SL}_d$ (indeed, the intersection $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}} \cap \mathbf{SL}_d$ is a normal subgroup of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ by Proposition 3.39, and hence is a normal subgroup of $\mathbf{SL}_d$; it is therefore either equal to $\mathbf{SL}_d$, or is contained in the center $\boldsymbol{\mu}_d$; but since $d \geqslant 2$, the latter would imply that $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}/\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$ is not abelian).

We now assume that C is hyperelliptic. First we consider the case when $d \geqslant 3$.

If $\mathrm{D}(\mathrm{M}_0)$ is not geometrically isomorphic to $i^*\mathrm{M}_0$, then Proposition 8.9 (2) implies $\mathrm{M}_4(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}) = 2$ since we assume that $d \geqslant 3$; as previously, we then conclude that $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ contains $\mathbf{SL}_d$.

If the conditions of (2) hold, then the constant morphism $(s \circ i) + s$ is given by

$$s(i(x)) + s(x) = (x) + i(x) - 2(0_{\mathrm{C}}) = 0,$$

the identity element of A. Proposition 8.9 (1) implies then that M is self-dual and has $\mathrm{M}_4(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}) = 3$, again from our assumption that $d \geqslant 3$. We conclude in that case by Larsen's Alternative (Theorem 8.5 (5)), combined with the fact that $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ is infinite by Theorem 5.7.

There remains to consider the case when $d = 2$ (and C hyperelliptic). Since $d = \chi(\mathrm{C}_{\bar{k}}, \mathrm{M}_0)$, formula (11.1) shows that this situation can only occur if $(g, r) = (2, 1)$ and if the sheaf $\mathscr{F}_0$ is lisse on C. Thus the curve C has genus 2, and the sheaf $\mathscr{F}_0$ is a rank 1 sheaf corresponding to a character of the fundamental group of C. As we will recall below in general, there exists then a character $\chi_0 \in \widehat{\mathrm{A}}(k)$ such that $\mathscr{F}_0$ is geometrically isomorphic to $s^*\mathscr{L}_{\chi_0}$ on C. The duality condition $\mathrm{M}_0 \simeq i^*\,\mathrm{D}(\mathrm{M}_0)$ is then always satisfied.

We claim that in this situation, the fourth moment $\mathrm{M}_4(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}})$ is still equal to 2. Indeed, from the proof of Proposition 8.9, we know that

$$\frac{1}{|\mathrm{A}(k_n)|} \sum_{\chi \in \widehat{\mathrm{A}}(k_n)} |\mathrm{S}(\mathrm{M}, \chi)|^4$$

converges to 3 as $n \to +\infty$. The contribution of the character χ_0^{-1}, which is the only ramified character, is

$$\frac{1}{|\mathrm{A}(k_n)|}|\mathrm{S}(\mathrm{M}, \chi_0^{-1})|^4 = \frac{1}{|\mathrm{A}(k_n)|}\Big|\sum_{x \in \mathrm{C}(k_n)} t_{\mathrm{M}_0}(x; k_n)\overline{\chi_0(x)}\Big|^4 = \frac{|\mathrm{C}(k_n)|^4}{|k_n|^2|\mathrm{A}(k_n)|}$$

which converges to 1 as $n \to +\infty$. We then conclude from Larsen's Alternative that the group $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ contains $\mathbf{SL}_2 = \mathbf{Sp}_2$. ∎

Remark 11.2. (1) Note that the last case provides a concrete example where the limit

$$\lim_{n \to +\infty} \frac{1}{|\mathrm{A}(k_n)|} \sum_{\chi \in \widehat{\mathrm{A}}(k_n)} |\mathrm{S}(\mathrm{M}, \chi)|^4$$

exists, where the sum ranges over all characters, but its value is *not* the fourth moment of the standard representation of the tannakian group (see Remark 8.10).

(2) If the curve C has gonality at least 5, then the inclusions

$$\mathbf{SL}_d \subset \mathbf{G}_{\mathrm{M}}^{\mathrm{geo}} \subset \mathbf{G}_{\mathrm{M}}^{\mathrm{ari}} \subset \mathbf{GL}_d$$

can be deduced without appealing to Proposition 8.22. Indeed, the immersion s_Δ is then a 4-Sidon morphism by Proposition 8.11 (3), so we deduce from Proposition 8.8 (2) that $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ (and hence also $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}}$, as before) contains $\mathbf{SL}_d$. (Precisely, we are in the excluded case of this statement, but we can observe that there are only finitely many ramified characters here, and that the assumption implies that the genus of C is at least five, so that the contribution to the 8-th moment of the ramified characters is

$$\ll \frac{1}{|k_n|^g}|k_n|^{8/2} \to 0,$$

so that we do obtain the correct 8-th moment.)

Remark 11.3. In characteristic zero, Krämer and Weissauer [92] have obtained closely related results, using more geometric methods in the case of the object $\mathrm{M} = s_{\Delta*}\overline{\mathbf{Q}}_\ell[1]$.

We now explain how Theorem 11.1 answers a question of Tsimerman, which was Katz's original motivation. Let $\varrho\colon \pi(\mathrm{C})^{\mathrm{ab}} \to \mathbf{C}^\times$ be a character of finite order. By the Riemann hypothesis for curves over finite fields, the Artin L-function $\mathrm{L}_\mathrm{C}(\varrho, s)$ is a polynomial of degree $2g-2$ in the variable $\mathrm{T} = q^{-s}$ all of whose reciprocal roots have absolute value $\sqrt{q}$. We can then write

$$\mathrm{L}(\varrho, \mathrm{T}/\sqrt{q}) = \det\left(1 - \mathrm{T}\Theta_{\mathrm{C}/k,\varrho}\right)$$

for a unique conjugacy class $\Theta_{\mathrm{C}/k,\varrho}$ in the unitary group $\mathbf{U}_{2g-2}(\mathbf{C})$.

Question (Tsimerman). How are these conjugacy classes distributed as ϱ varies?

From now on, we shall normalize the characters as follows: we fix a divisor $\Delta = \sum n_i x_i$ of degree one on C and we only consider those characters ϱ satisfying

$$\prod \varrho(\mathrm{Fr}_{\kappa(x_i),x_i})^{n_i} = 1.$$

Through the isomorphism $\pi_1(\mathrm{C})^{\mathrm{ab}} \simeq \pi_1(\mathrm{A})$ induced by $s_\Delta\colon \mathrm{C} \hookrightarrow \mathrm{A} = \mathrm{Jac}(\mathrm{C})$, such normalized characters correspond to characters $\varrho\colon \pi_1(\mathrm{A}) \to \mathbf{C}^\times$ satisfying $\varrho(\mathrm{Fr}_{k,0_\mathrm{A}}) = 1$. Since they are in addition supposed to be of finite order, they arise via the Lang isogeny from the elements of $\widehat{\mathrm{A}}(k)$. Replacing k with k_n, we obtain the corresponding characters in $\widehat{\mathrm{A}}(k_n)$. Thus the following statement answers Tsimerman's question when considering conjugacy classes associated to normalized characters over k_n and taking $n \to +\infty$.

Corollary 11.4. *Let C be a smooth projective geometrically connected curve of genus $g \geqslant 2$ over k with jacobian A.*

(1) *If C is hyperelliptic, the hyperelliptic involution has a fixed k-point $0 \in \mathrm{C}(k)$, and we use this point to define the embedding $\mathrm{C} \to \mathrm{A}$, then the conjugacy classes $(\Theta_{\mathrm{C}/k,\chi})_{\chi\in\widehat{\mathrm{G}}(\mathrm{A}_n),\chi\neq 1}$ are conjugacy classes in $\mathbf{USp}_{2g-2}(\mathbf{C})$ and become equidistributed with respect to the image of the Haar probability measure on the set of conjugacy classes.*
(2) *If C is not hyperelliptic and $(2g-2)\Delta$ is a canonical divisor on C, then the conjugacy classes $(\Theta_{\mathrm{C}/k,\chi})_{\chi\in\widehat{\mathrm{A}}(k_n),\chi\neq 1}$ are conjugacy classes in $\mathbf{SU}_{2g-2}(\mathbf{C})$ and become equidistributed with respect to the image of the Haar probability measure on the set of conjugacy classes.*

Proof. Consider the weight zero perverse sheaf $\mathrm{M}_0 = \overline{\mathbf{Q}}_\ell(1/2)[1]$ on C and set $\mathrm{M} = s_{\Delta*}\mathrm{M}_0$. For each rank one ℓ-adic lisse sheaf $\mathscr{L}$ on A, there are isomorphisms

$$\mathrm{H}^i(\mathrm{A}_{\bar{k}}, \mathrm{M}\otimes\mathscr{L}) \simeq \mathrm{H}^i(\mathrm{A}_{\bar{k}}, s_{\Delta*}(\mathrm{M}_0\otimes s_\Delta^*\mathscr{L})) \simeq \mathrm{H}^i(\mathrm{C}_{\bar{k}}, \mathrm{M}_0\otimes s_\Delta^*\mathscr{L}) \simeq \mathrm{H}^{i+1}(\mathrm{C}_{\bar{k}}, s_\Delta^*\mathscr{L}(1/2))$$

by the projection formula and the exactness of $s_{\Delta*}$. It follows that M has tannakian dimension

$$-\chi(\mathrm{C}_{\bar{k}}, s_\Delta^*\mathscr{L}) = 2g-2,$$

and moreover that all non-trivial characters are unramified for M (since we are considering an abelian variety). By Theorem 4.11, it suffices therefore to prove that the arithmetic and geometric tannakian groups of M coincide and are equal to $\mathbf{Sp}_{2g-2}$ in case (1) and to $\mathbf{SL}_{2g-2}$ in case (2).

Assume C is hyperelliptic with hyperelliptic involution i. Then the sheaf M_0 is geometrically isomorphic to $i^*\,\mathrm{D}(\mathrm{M}_0)$ and, because of the shift by 1 in the definition of M_0, the corresponding self-duality is symplectic. Therefore, if in addition we assume that i has a fixed k-point $0 \in \mathrm{C}(k)$, which we use as divisor Δ, Theorem 11.1 (2) gives the equality $\mathbf{G}_{\mathrm{M}}^{\mathrm{geo}} = \mathbf{G}_{\mathrm{M}}^{\mathrm{ari}} = \mathbf{Sp}_{2g-2}$.

If C is not hyperelliptic and $(2g-2)\Delta$ is a canonical divisor on C, in view of Theorem 11.1 (3), it suffices to show that the arithmetic group $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ lies in $\mathbf{SL}_{2g-2}$. For this, we compute the determinant of the action of Frobenius on $\mathrm{H}^1(\mathrm{C}_{\bar{k}}, \mathscr{L}(1/2))$. Since this cohomology is even-dimensional, this is also the determinant of $-\mathrm{Fr}_k$, which is the constant in the functional equation for the L-function of $\mathscr{L}(1/2)$. By a classical result of Weil [125],[1] in the case of $\mathscr{L}$ this constant is given by $q^{1-g}\varrho_{\mathscr{L}}(\mathrm{can})$ for a canonical divisor can, where $\varrho_{\mathscr{L}}$ is the character associated to $\mathscr{L}$, which factors through the jacobian. Taking the half-Tate twist into account, along with the fact that $\varrho_{\mathscr{L}}(\mathrm{can}) = 1$ since $(2g-2)\Delta$ is a canonical divisor and characters are normalized to take the value 1 at Δ, it follows that the determinant is trivial, as claimed. ☐

We conclude this section by a (partial) generalization of Theorem 11.1 to the setting of generalized jacobians arising in geometric class-field theory. This gives a natural example of an application of our results where the algebraic group G is not restricted to being either a torus, an abelian variety or a unipotent group, but may involve all three of these fundamental building blocks (see Remark 8.14). For simplicity, we will only deal with the case where C is not hyperelliptic.

Theorem 11.5. *Assume that the curve* C *is not hyperelliptic. Let* S *be an effective divisor on the curve* C. *Let* U *be the complement of the support of* S *in* C. *Let* Δ *be a divisor of degree one on* U. *Let* J_{S} *be the generalized jacobian of* C *relative to the divisor* S, *and let* $s_\Delta\colon \mathrm{U} \to \mathrm{J}_{\mathrm{S}}$ *be the natural immersion defined by* $x \mapsto (x) - \Delta$.

Let M_0 *be a semiperverse object on* U, *mixed of weights* $\leqslant 0$ *and put* $\mathrm{M} = s_{\Delta!}\mathrm{M}_0$. *Let* d *be the tannakian dimension of the semisimplification* $\widetilde{\mathrm{M}}$ *of the part of* ${}^{\mathfrak{p}}\mathscr{H}^0(\mathrm{M})$ *which is pure of weight* 0. *Assume that* $\widetilde{\mathrm{M}}$ *is non-zero.*

Then we have $d \geqslant 2$, *and either the arithmetic tannakian group of* $\widetilde{\mathrm{M}}$ *contains* $\mathbf{SL}_d$ *or* $\mathbf{G}_{\widetilde{\mathrm{M}}}^{\mathrm{ari}}$ *is virtually central in* $\mathbf{GL}_d$.

Proof. We note that M is a semiperverse object on J_{S} since s_{D} is quasi-finite, and is mixed of weights $\leqslant 0$ by the Riemann Hypothesis.

To check that $d \geqslant 2$, we use the general Euler–Poincaré characteristic formula (see Theorem C.2) as in (11.1), to conclude. We then need only observe that s_{D} is a Sidon morphism by Proposition 8.13, and apply Larsen's Alternative. ☐

Remark 11.6. (1) Since we do not know in general if perverse sheaves on the group J_{S} are generically unramified, the corresponding equidistribution statement is currently restricted to the distribution of the arithmetic Fourier transforms

$$\sum_{x \in \mathrm{U}(k_n)} t_{\mathrm{M}}(x; k_n)\chi(x)$$

for $\chi \in \widehat{\mathrm{J}}_{\mathrm{S}}(k_n)$.

(2) Again because the group J_{S} is *a priori* fairly arbitrary here, we can not exclude the possibility that $\mathbf{G}_{\widetilde{\mathrm{M}}}^{\mathrm{ari}}$ is virtually central (e.g., finite), since we do not have currently a general version of

[1] Which can also easily be recovered from the theory of local constants, applying the results of Deligne and Laumon (see Appendix C).

Proposition 8.22. (In our case, since the jacobian of C is a non-trivial quotient of J_S, we can expect that the statement should indeed extend.)

(3) It is possible that $\widetilde{M}$ is zero; in this case, we have of course $d = 0$, and the tannakian group is trivial.

11.2. The intermediate jacobian of a cubic threefold

Our second application involving abelian varieties is related to a very classical and important construction in algebraic geometry, that of the intermediate jacobian of a smooth cubic threefold, which was used by Clemens and Griffiths to prove that these threefolds, over $\mathbf{C}$, are not rational (although they are unirational).

The geometric setting, which over finite fields goes back at least to the work of Bombieri and Swinnerton-Dyer [10] (computing the zeta function of smooth cubic threefolds over finite fields) is the following.

Let k be a field of characteristic different from 2, and let $X \subset \mathbf{P}^4_k$ be a smooth cubic threefold over k. We denote by $F(X)$ the Fano scheme of lines in X, which is a smooth projective and geometrically connected surface over k (see, e.g., [5, § 4] or [10, Lem. 3] or [2, Cor. 1.12, Th. 1.16]; this uses the fact that the characteristic is different from 2). Let then $A(X)$ be the Albanese variety of $F(X)$, which is known to be isomorphic to the Picard variety of $F(X)$ (see, e.g. [61, Cor. 4.3.3]). It has dimension 5, and if the base field is contained in $\mathbf{C}$, then the analytification of $A(X)$ is canonically isomorphic to the intermediate jacobian $J(X)$ of Griffiths, which is defined analytically in terms of Hodge theory (this is due to Murre; see [5, Prop. 9]).

The Albanese morphism $s\colon F(X) \to A(X)$ is a closed immersion, according to a theorem of Beauville [5, p. 201, cor.]. If we view $A(X)$ as the Picard variety, then the morphism s can be identified geometrically with the map sending a line $l \in F(X)$ to the divisor defined by the curve C_s which is the Zariski-closure in $F(X)$ of the set of lines $l' \neq l$ such that $l' \cap l$ is not empty.

The problem we consider is then the following: if k is a finite field of odd characteristic, what is the arithmetic tannakian group of the perverse sheaf $M = s_*\overline{\mathbf{Q}}_\ell[2](1)$ on $A(X)$? (It is perverse because s is a closed immersion, as in previous similar examples.) The corresponding exponential sums are then

$$S(M, \chi) = \frac{1}{|k_n|} \sum_{l \in F(X)(k_n)} \chi(s(l))$$

for a character $\chi \in \widehat{A(X)}(k_n)$.

Up to correcting a small oversight, the following answer is the analogue over finite fields of a result of Krämer over $\mathbf{C}$ (see [91, Th. 2]).

Proposition 11.7. *Let k be a finite field of characteristic different from 2. Let X be a smooth cubic threefold over k, and denote by $F(X)$ the Fano scheme of lines in X, by $A(X)$ the Albanese variety of $F(X)$, and by*

$$s\colon F(X) \to A(X)$$

the natural closed immersion.

Let ℓ be a prime different from the characteristic of k, and let M be the object $M = s_*\overline{\mathbf{Q}}_\ell[2](1)$ on $A(X)$. The connected derived subgroup of the arithmetic tannakian monodromy group of the object M of the category $\overline{\mathbf{P}}^{\mathrm{ari}}(A(X))$ is isomorphic to the exceptional group $\mathbf{E}_6$.

For the proof, we will use the following lemma, whose proof was communicated to us by Beauville.

Lemma 11.8 (Beauville). *With notation as above, there is no $x \in \mathrm{A}(\mathrm{X})$ such that $-s(\mathrm{F}(\mathrm{X})) = x + s(\mathrm{F}(\mathrm{X}))$, and there is no non-zero $x \in \mathrm{A}(\mathrm{X})$ such that $s(\mathrm{F}(\mathrm{X})) = x + s(\mathrm{F}(\mathrm{X}))$.*

Proof. We argue by contradiction.

For the first assertion, if x existed such that $-s(\mathrm{F}(\mathrm{X})) = x + s(\mathrm{F}(\mathrm{X}))$, then the involution $a \mapsto -x - a$ of $\mathrm{A}(\mathrm{X})$ would induce an involution i of the variety $\mathrm{F}(\mathrm{X})$ with a finite number of fixed points. The quotient variety $\mathrm{F}(\mathrm{X})/i$ is then a normal variety with only isolated ordinary double points as singularities. In particular, it is Gorenstein (see, e.g., [35, Cor. 21.19]), so its canonical divisor $\mathrm{K}_{\mathrm{F}(\mathrm{X})/i}$, defined as the direct image of the canonical divisor of the smooth locus of $\mathrm{F}(\mathrm{X})/i$, is a Cartier divisor (see e.g [83, p. 79]). Since the projection $p\colon \mathrm{F}(\mathrm{X}) \to \mathrm{F}(\mathrm{X})/i$ is étale outside of the set of fixed points, the canonical divisor of $\mathrm{F}(\mathrm{X})$ is $\mathrm{K} = p^*(\mathrm{K}_{\mathrm{F}(\mathrm{X})/i})$. This implies that $\mathrm{K}^2 = 2(\mathrm{K}^2_{\mathrm{F}(\mathrm{X})/i})$ is even. However, it is known that $\mathrm{K}^2 = 45$, which is odd (see, e.g., [61, Prop. 4.6]).

For the second assertion, note that $s(\mathrm{F}(\mathrm{X})) = x + s(\mathrm{F}(\mathrm{X}))$ would imply that

$$s(\mathrm{F}(\mathrm{X})) - s(\mathrm{F}(\mathrm{X})) = x + s(\mathrm{F}(\mathrm{X})) - s(\mathrm{F}(\mathrm{X})),$$

so that the theta divisor $\Theta(\mathrm{X}) = s(\mathrm{F}(\mathrm{X})) - s(\mathrm{F}(\mathrm{X}))$ satisfies $\Theta(\mathrm{X}) = x + \Theta(\mathrm{X})$. However, Beauville [5, § 3, Prop. 2] showed that $\Theta(\mathrm{X})$ is smooth except for a single singularity, so this equality can only happen if $x = 0$. ☐

Remark 11.9. The cohomological analogue of this proposition is not true: for $\varepsilon \in \{-1, 1\}$, the cohomology class of $\varepsilon s(\mathrm{F}(\mathrm{X}))$ in $\mathrm{H}^6(\mathrm{A}(\mathrm{X}))$ is $\Theta^3/6$, where Θ is the cohomology class of the symmetric theta divisor $s(\mathrm{F}(\mathrm{X})) - s(\mathrm{F}(\mathrm{X}))$ (the fact that $s(\mathrm{F}(\mathrm{X}))$ has the same class as $-s(\mathrm{F}(\mathrm{X}))$ is due to the fact that $x \mapsto -x$ acts trivially on even-degree cohomology groups; the computation in terms of Θ is explained, e.g., in [61, Cor. 5.3.12, (i)]).

We now give a proof of Proposition 11.7 adapting Krämer's argument over $\mathbf{C}$, the key point being the recognition criterion of $\mathbf{E}_6$ in Proposition 8.19.

Proof. Since $\mathrm{F}(\mathrm{X})$ is a smooth, projective and geometrically connected surface, the object M is a simple perverse sheaf on $\mathrm{A}(\mathrm{X})$. The tannakian dimension of M is equal to the Euler–Poincaré characteristic of M over $\bar{k}$ (Proposition 3.21), which is equal to the Euler–Poincaré characteristic of the Fano surface $\mathrm{F}(\mathrm{X})$, which is 27 (a result of Fano, see, e.g., [2, Prop. 1.23]).

Let $\Theta(\mathrm{X})$ be the theta divisor $s(\mathrm{F}(\mathrm{X})) - s(\mathrm{F}(\mathrm{X}))$ in $\mathrm{A}(\mathrm{X})$, and $i\colon \Theta(\mathrm{X}) \to \mathrm{A}(\mathrm{X})$ the closed immersion. The object $\mathrm{M} * \mathrm{M}^\vee$ contains the object $\mathrm{N} = i_*\overline{\mathbf{Q}}_\ell[1]$ by the decomposition theorem (see [91, proof of Th. 2]). This is also a simple perverse sheaf since Θ is a geometrically irreducible divisor (see, e.g., [5, Prop. 2]). The tannakian dimension of N can be computed as in [91, Cor. 6] (or by lifting to characteristic 0, as can be done as in [10, Proof of Lemma 5]), and is equal to 78.

To conclude using Proposition 8.19, applied to the connected derived subgroup $\mathbf{G}$ of $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$, it suffices therefore to check that $\mathbf{G}$ still acts irreducibly on the 27-dimensional representation corresponding to M.

To see this, note that the neutral component $(\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}})^\circ$ acts irreducibly by Corollary 5.9 combined with Lemma 11.8. Then its derived group $\mathbf{G}$ must also act irreducibly since

$$(\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}})^\circ = \mathrm{C} \cdot \mathbf{G}$$

for some torus C, which is central by irreducibility. ☐

It is natural to ask whether this proposition can also be proved using the fourth moment criterion of Proposition 8.20 instead of Krämer's criterion.

We have not fully succeeded in doing so, but we can show that the question translates to an interesting geometric property of the cubic threefolds. Conversely, this property follows in fact from the previous proof, as we will now explain.

In order to apply Proposition 8.20, we need to check that the object M is not self-dual, that it has tannakian dimension 27 and that its fourth moment is $\mathrm{M}_4(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}) = 3$.

Lemma 11.8 implies that M is not self-dual. The second property is derived as in the beginning of the previous proof. Now we attempt to compute the fourth moment.

We write $\mathrm{F} = \mathrm{F}(\mathrm{X})$ and $\mathrm{A} = \mathrm{A}(\mathrm{X})$. We use the diophantine interpretation of the fourth moment. Summing over all characters, we find as usual using orthogonality that for $n \geqslant 1$, the formula

$$\frac{1}{|\mathrm{A}(k_n)|} \sum_{\chi \in \widehat{\mathrm{A}}(k_n)} |\mathrm{S}(\mathrm{M}, \chi)|^4 = \frac{1}{|k_n|^4} \sum_{\substack{(l_1,\ldots,l_4) \in \mathrm{F}(k_n)^4 \\ s(l_1)-s(l_2)=s(l_3)-s(l_4)}} 1$$

holds. We rewrite this in the form

$$\frac{1}{|k_n|^4} \sum_{(l_1,l_2) \in \mathrm{F}(k_n)^2} \mathrm{N}(l_1, l_2)$$

where

$$\mathrm{N}(l_1, l_2) = |\{(l_3, l_4) \in \mathrm{F}(k_n)^2 \mid s(l_1) - s(l_2) = s(l_3) - s(l_4)\}|.$$

For $s \in \mathrm{F}(k_n)$, we have

$$\mathrm{N}(s, s) = |\mathrm{F}(k_n)|,$$

and hence

$$\frac{1}{|k_n|^4} \sum_{(l_1,l_2) \in \mathrm{F}(k_n)^2} \mathrm{N}(l_1, l_2) = \frac{|\mathrm{F}(k_n)|^2}{|k_n|^4} + \frac{1}{|k_n|^4} \sum_{l_1 \neq l_2 \in \mathrm{F}(k_n)} \mathrm{N}(l_1, l_2)$$

Since F is a geometrically irreducible surface over k, the first term converges to 1 as $n \to +\infty$. To handle the second term, consider the morphism given by the first projection

$$f \colon \mathrm{F}^2 \times_{\mathrm{A}} \mathrm{F}^2 \to \mathrm{F}^2,$$

where the fiber product is defined by the morphisms $(l_1, l_2) \mapsto s(l_1) - s(l_2)$ and $(l_3, l_4) \mapsto s(l_3) - s(l_4)$.

We then have

$$\mathrm{N}(l_1, l_2) = |f^{-1}(l_1, l_2)(k_n)| = |f^{-1}(l_1, l_2)(\bar{k})^{\mathrm{Fr}_{k_n}}|,$$

the number of fixed points of the Frobenius of k_n acting on the fiber of f. The fiber product $\mathrm{F}^2 \times_{\mathrm{A}} \mathrm{F}^2$ contains the diagonal $\Delta = \{(l_1, l_2, l_1, l_2)\}$, and we denote by $\widetilde{f}$ the morphism obtained by restriction

$$\widetilde{f} \colon (\mathrm{F}^2 \times_{\mathrm{A}} \mathrm{F}^2) - \Delta \to \mathrm{F}^2.$$

We then deduce that

$$\mathrm{N}(l_1, l_2) = 1 + \widetilde{\mathrm{N}}(l_1, l_2) \quad \text{where} \quad \widetilde{\mathrm{N}}(l_1, l_2) = |\widetilde{f}^{-1}(l_1, l_2)(\bar{k})^{\mathrm{Fr}_{k_n}}|,$$

and hence

$$\frac{1}{|k_n|^4} \sum_{(l_1,l_2) \in \mathrm{F}(k_n)^2} \mathrm{N}(l_1, l_2) = \frac{|\mathrm{F}(k_n)|^2}{|k_n|^4} + \frac{|\mathrm{F}(k_n)|(|\mathrm{F}(k_n)| - 1)}{|k_n|^4} + \frac{1}{|k_n|^4} \sum_{l_1 \neq l_2 \in \mathrm{F}(k_n)} \widetilde{\mathrm{N}}(l_1, l_2).$$

A theorem of Beauville [5, Prop. 8] implies that $\widetilde{f}$ is generically finite of degree 5. By the Chebotarev Density Theorem (see, e.g., [78, Th. 9.7.13]) it follows that

$$\lim_{n\to+\infty} \frac{1}{|k_n|^4} \sum_{l_1\neq l_2\in \mathrm{F}(k_n)} \widetilde{\mathrm{N}}(l_1,l_2)$$

is equal to the number of orbits of the Galois group of the Galois closure of $\widetilde{f}$ in its permutation representation on the generic fiber of $\widetilde{f}$.

The generic point η of F^2 is a pair of two disjoint lines $\eta = (\widetilde{s}_1, \widetilde{s}_2)$. Beauville showed that the points $(\widetilde{s}_3, \widetilde{s}_4) \in \mathrm{F}^2$ such that $(\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3, \widetilde{s}_4)$ is in the fiber over η are such that the lines $\widetilde{s}_3$ and $\widetilde{s}_4$ are contained in the intersection S of X and of the projective 3-space spanned by $(\widetilde{s}_1, \widetilde{s}_2)$. Thus S is a smooth cubic surface, and the lines $\widetilde{s}_3$ and $\widetilde{s}_4$ are elements of the set Λ of the five lines in S intersecting both $\widetilde{s}_1$ and $\widetilde{s}_2$; for these geometric facts, see [5, p. 203, rem. 2] or [61, proof of Cor. 4.3.9].

We claim that the subgroup of the Galois group of the 27 lines which fixes the two lines $\widetilde{s}_1$ and $\widetilde{s}_2$ is the Galois group of the set of seven lines $\{\widetilde{s}_1, \widetilde{s}_2\} \cup \Lambda$ (see Lemma 11.11 below). Now it follows from the work of Harris on Galois groups of enumerative problems (see [58, p. 718]) that this permutation representation is indeed transitive, in fact that it has image isomorphic to the whole symmetric group $\mathfrak{S}_5$, *if we take the base field to be* $\mathbf{C}$ *and the cubic threefold to be general.*

One may expect this to also be true in our situation:

- the restriction of the base field should not be problematic (indeed, the fact that the "generic" Galois group of the 27 lines on a smooth cubic surface is isomorphic to the Weyl group of $\mathbf{E}_6$, which is the starting point of Harris's work, is known in all odd characteristics, by work of Achter [1, Prop. 4.8]);
- the (four-dimensional) family of hyperplane sections that we consider is dominant over the (also four-dimensional) space of hyperplane sections of the cubic hypersurface X (indeed, for any hyperplane H in $\mathbf{P}^4$ intersecting X in a smooth surface, we can pick two distinct lines (l_1, l_2) in $\mathrm{X} \cap \mathrm{H}$, and the corresponding section is $\mathrm{H} \cap \mathrm{X}$);
- and the family of all hyperplane sections of X is probably general enough for the result of Harris to extend. (This is in fact the most delicate point.)

If we assume that this expectation holds for X, then we would deduce that

$$\lim_{n\to+\infty} \frac{1}{|k_n|^4} \sum_{l_1\neq l_2\in \mathrm{F}(k_n)} \widetilde{\mathrm{N}}(l_1,l_2) = 1 \tag{11.2}$$

holds, and hence conclude that

$$\lim_{n\to+\infty} \frac{1}{|k_n|^4} \sum_{(l_1,l_2)\in \mathrm{F}(k_n)^2} \mathrm{N}(l_1,l_2) = 3.$$

Under this assumption, we therefore derive from Proposition 8.4 that $\mathrm{M}_4(\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}) \leqslant 3$. Since M is not of tannakian dimension 1, the fourth moment is equal to either 2 or 3. We can at least partially exclude the first possibility as follows:

(1) For "most" cubic threefolds, the abelian variety A is absolutely simple (see Lemma 11.12 below for a precise statement). In this case, there are only finitely many characters $\chi \in \widehat{\mathrm{A}}$

which are not weakly-unramified, and for which

$$|\mathrm{S}(\mathrm{M},\chi)| = \Big|\frac{1}{|k_n|}\sum_{x\in\mathrm{F}(k_n)}\chi(s(x))\Big| \ll |k_n|,$$

so that

$$\frac{1}{|\widehat{\mathrm{A}}(k_n)|}\sum_{\chi\notin\mathscr{X}_w(\mathrm{A})(k_n)}|\mathrm{S}(\mathrm{M},\chi)|^4 \ll \frac{|k_n|^4}{|k_n|^5}\to 0$$

as $n\to+\infty$, and from Proposition 8.4, the computation we have performed actually means that the fourth moment is equal to 3.

(2) We may use the beginning of Krämer's proof to deduce that $\mathrm{M}*\mathrm{M}^\vee$ contains an irreducible summand of dimension 78, which excludes the possibility that the fourth moment be equal to 2.

So under the above assumptions, we conclude that $\mathrm{M}_4(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}) = 3$ and we can then apply Proposition 8.20 (as in the previous argument, we use Corollary 5.9 to deduce that the neutral component of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ still acts irreducibly).

Now, going backwards, if we use Proposition 11.7, then we *do* know that the fourth moment of $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ is equal to 3, since the tannakian group is $\mathbf{E}_6$. It follows that, at least in the first of the above two situations, the limit formula (11.2) must be true.

Remark 11.10. In contrast with Theorem 11.1, Proposition 11.7 will not extend to compute the fourth moment for perverse sheaves of the form $s_*\mathrm{M}$ for a more general simple perverse sheaf M on F(X). One can expect that, in this case, the fourth moment should be equal to 2, but this seems difficult to prove.

We now state and prove the two lemmas we used above. The first one is certainly a standard fact in the study of the 27 lines.

Lemma 11.11. *Let* S *be a smooth cubic hypersurface in* $\mathbf{P}^3$ *over an algebraically closed field. Let* l_1 *and* l_2 *be two disjoint lines in* S*. Let* Λ *be the set of the five lines in* S *intersecting both* l_1 *and* l_2*. Any Galois-automorphism of the twenty seven lines that fixes the lines in* $\{l_1,l_2\}\cup\Lambda$ *is the identity.*

Proof. The key point in this computation is the fact that no line on S is disjoint from all lines in Λ. More precisely, we use the classical description of S as a blow-up of $\mathbf{P}^2$ in six points which are in general position (see, e.g., [61, Prop. 3.2.3]), and the resulting partition of the 27 lines in subsets

$$\begin{gathered}\mathrm{E}_1,\ldots,\mathrm{E}_6\\ \mathrm{L}_{ij},\quad 1\leqslant i<j\leqslant 6\\ \mathrm{L}_1,\ldots,\mathrm{L}_6,\end{gathered}$$

with incidences described as follows:

$$\begin{gathered}\mathrm{E}_i\cap\mathrm{L}_j\neq\varnothing \text{ if and only if } i\neq j,\\ \mathrm{E}_i\cap\mathrm{L}_{i,j}\neq\varnothing \text{ for any } j,\\ \mathrm{L}_i\cap\mathrm{L}_{i,j}\neq\varnothing \text{ for any } j,\\ \mathrm{L}_{i,j}\cap\mathrm{L}_{k,l}\neq\varnothing \text{ for } \{i,j\}\cap\{k,l\}=\varnothing,\end{gathered}$$

all other pairs of lines being disjoint (see, e.g., [61, Rem. 3.2.4, 3.3.1]).

We choose the blow-up, as we may, so that $l_1=\mathrm{E}_1$ and $l_2=\mathrm{E}_2$ (see [61, 3.3.2]). We then have $\Lambda=\{\mathrm{L}_{12},\mathrm{L}_3,\mathrm{L}_4,\mathrm{L}_5,\mathrm{L}_6\}$.

Let σ be a Galois automorphism which fixes the seven given lines. Since σ respects incidence relations, we see:

(1) For any i, we have $\sigma(\mathrm{E}_i) = \mathrm{E}_i$. Indeed, assume that $i = 3$ for simplicity, since all cases are similar. Then E_3 meets L_4, L_5, L_6, which implies that $\sigma(\mathrm{E}_3)$ also intersects these three lines. But the only lines with this property are E_1, E_2 and E_3; since σ fixes the first two of these, we have $\sigma(\mathrm{E}_3) = \mathrm{E}_3$.

(2) For any i, we have $\sigma(\mathrm{L}_i) = \mathrm{L}_i$. Indeed, assume that $i = 1$; from the previous point, the lines L_{12}, E_2, ..., E_6, which all meet L_1, are fixed by σ, so that $\sigma(\mathrm{L}_1)$ fixes all of them. We see that the only line with this property is L_1, so that $\sigma(\mathrm{L}_1) = \mathrm{L}_1$.

(3) For any $i < j$, we have $\sigma(\mathrm{L}_{i,j}) = \mathrm{L}_{i,j}$. We consider the example of $\mathrm{L}_{1,3}$, the other cases being similar. The lines E_1, E_3, L_1, L_3 all meet $\mathrm{L}_{1,3}$, and hence (by the first two points) also intersect $\sigma(\mathrm{L}_{1,3})$. But this means that $\sigma(\mathrm{L}_{1,3})$ must of one of the $\mathrm{L}_{i,j}$, and the only one that has the desired property is $\mathrm{L}_{1,3}$.

□

The second lemma concerns the "generic" simplicity of the intermediate jacobian. Explicit examples that show that this property is not always valid are given for instance by Debarre, Laface and Roulleau [25, Cor. 4.12]; for the Fermat threefold

$$x_0^3 + \cdots + x_4^3 = 0$$

over $\mathbf{F}_p$, with $p \geqslant 5$, the intermediate jacobian is isogenous to E^5, where E is the Fermat curve $y_0^3 + y_1^3 + y_2^3 = 0$.

Lemma 11.12. *Let k be a finite field of characteristic $p > 11$. Let $\mathscr{M}$ be the coarse moduli space of smooth projective cubic threefolds over k. For any integer $n \geqslant 1$, let $\mathscr{M}_s(k_n)$ be the subset of $\mathrm{X} \in \mathscr{M}(k_n)$ such that the abelian variety $\mathrm{A}(\mathrm{X})$ is simple over k_n.*

There exists $\delta > 0$ such that the asymptotic formula

$$|\mathscr{M}_s(k_n)| = |\mathscr{M}(k_n)|(1 + \mathrm{O}(|k_n|^{-\delta}))$$

holds for $n \geqslant 1$.

Proof. Fix a prime ℓ invertible in k. Let $\mathscr{F}$ be the lisse ℓ-adic sheaf on $\mathscr{M}$ parameterizing the cohomology group $\mathrm{H}^1(\mathrm{A}(\mathrm{X})_{\bar{k}}, \overline{\mathbf{Q}}_\ell)$. The geometric monodromy group of $\mathscr{F}$ is the symplectic group $\mathbf{Sp}_{10}$ by a result of Achter [1, Th. 4.3] (based on semicontinuity of monodromy and the extension to positive odd characteristic of a result of Collino [21] over $\mathbf{C}$, which states that the Zariski-closure of the image of $\mathscr{M}$ in the moduli space $\mathscr{A}_5$ of principally polarized abelian varieties of dimension 5 contains the locus $\mathscr{H}_5$ of jacobians of hyperelliptic curves of genus 5).

Using the method in [84, § 6, § 8], this implies that there exists $\delta > 0$ such that the set $\mathscr{M}_i(k_n)$ of threefold $\mathrm{X} \in \mathscr{M}(k_n)$ for which the characteristic polynomial of Frobenius acting on $\mathrm{H}^1(\mathrm{A}(\mathrm{X})_{\bar{k}}, \overline{\mathbf{Q}}_\ell)$ is irreducible in $\mathbf{Q}[\mathrm{X}]$ satisfies the asymptotic

$$|\mathscr{M}_i(k_n)| = |\mathscr{M}(k_n)|(1 + \mathrm{O}(|k_n|^{-\delta}))$$

for $n \geqslant 1$, and one deduces the lemma since $\mathscr{M}_i(k_n) \subset \mathscr{M}_s(k_n)$. □

Remark 11.13. A qualitative form of the result, namely the equality

$$\lim_{n \to +\infty} \frac{|\mathscr{M}_s(k_n)|}{|\mathscr{M}(k_n)|} = 1,$$

can be proved, *mutatis mutandis*, for finite fields of all odd characteristic. It also possible to improve this estimate to obtain absolute simplicity.

CHAPTER 12

"Much remains to be done"

We conclude this book with a selection of open problems (related to the results of this text) and questions (concerning potential generalizations and more speculative possibilities).

12.1. Problems

(1) Prove a version of the vanishing theorem where the size of the exceptional sets is controlled by the complexity in all cases, and moreover where those sets have a clear algebraic or geometric structure (similar to that of tacs for tori and abelian varieties).

(2) Prove that any object is generically unramified for a general group, or find a counter-example to this statement.

(3) Establish functoriality properties relating tannakian groups of M on G (resp. N on H) with those of $f_*\mathrm{M}$ (resp. of $f^*\mathrm{N}$) when we have a morphism $f\colon \mathrm{G} \to \mathrm{H}$ of commutative algebraic groups.

(4) Study the situation in families over a base like $\mathrm{Spec}(\mathbf{Z})$.

(5) Find additional and robust tools to compute the tannakian group, or at least to determine some of its properties, which are applicable when Larsen's Alternative is not. In particular, find analogues (if they exist) of the local monodromy techniques for the Fourier transform on $\mathbf{G}_a$ (i.e, of the local Fourier transform functors of Laumon).

We note that recent work of Lawrence and Sawin [99] and Javanpeykar, Krämer, Lehn and Maculan [66] computes the tannakian group of many objects of the form $i_*\overline{\mathbf{Q}}_\ell[d]$ on abelian varieties over $\mathbf{C}$, where $i\colon \mathrm{X} \to \mathrm{A}$ is a closed immersion which is either a hypersurface (in the case of [99]) or has dimension $< (\dim(\mathrm{A})-1)/2$ (in the case of [66]). It should be possible to extend their results to the situation over finite fields, and it would be interesting to see if it also leads to more cases with other perverse sheaves with similar support conditions.

(6) Construct interesting concrete perverse sheaves where the tannakian group is an exceptional group. In this direction, we note that automorphic methods have been used by Heinloth, Ngô and Yun [60] to construct sheaves on $\mathbf{G}_a$ with any of the exceptional groups as geometric monodromy groups, hence (taking inverse Fourier transforms of these) also to sheaves with these as tannakian groups. Moreover, Katz has shown the existence of examples involving G_2, for $\mathbf{G}_a$ [68, Th. 11.1], $\mathbf{G}_m$ [74, Ch. 26, 27] and on some elliptic curves [76, Th. 4.1]. In the case of $\mathbf{G}_m$, his result is of a "statistic" nature: in a certain family of objects whose trace functions are related to hypergeometric sums *evaluated at a fixed* a, he shows that for "most" values of a in $\mathbf{G}_m$, the tannakian group is G_2. After the first draft of this book was written, Zurbuchen [129] improved this result by showing that (as expected by Katz) any (non-zero) value of a has the desired property, provided the characteristic of the finite field is large enough.

(7) Find further applications!

12.2. Questions

Many of the following questions are rather speculative and much more open-ended that the problems above. They may not have any interesting answer, but we find them intriguing.

(1) *For a given* G*, what are the tannakian groups that may arise?*

This is motivated in part by the striking difference concerning finite groups between $\mathbf{G}_a$ and $\mathbf{G}_m$ or abelian varieties. In the former case, the solution of Abhyankar's Conjecture gives a characterization of which finite groups will appear, and a recent series of works of Katz, Rojas-Léon and Tiep has shown that there are many possibilities, even when one restricts attention to Fourier transforms of general hypergeometric sheaves (see for instance [79]) and $\mathbf{G}_m$). On the other hand, we have already mentioned that Katz proved that finite cyclic groups are the only possible finite geometric tannakian groups on $\mathbf{G}_m$, and Corollary 5.10 is a statement going in a similar direction for abelian varieties (although not yet as precise since it requires *a priori* that the arithmetic tannakian group be finite).

(2) *For a given group* G*, if* M *is a semisimple perverse sheaf associated to a semisimple lisse sheaf* $\mathscr{F}$ *on an open dense subset of* G*, what (if any) are the relations between the "ordinary" monodromy group* $\mathbf{G}$ *of* $\mathscr{F}$ *and its tannakian groups?*

In particular, suppose that $\mathscr{F}$ has finite monodromy group; what constraints does that impose on the tannakian group of M? We note that there is one "obvious" relation: the tannakian group acts irreducibly on its standard representation if and only if the lisse sheaf $\mathscr{F}$ is irreducible.

Since this last fact amounts to the discrete Plancherel formula, or equivalently to a relation between the second moments of both groups, a more specific question could be: are there non-trivial inequalities between the moments of the monodromy group of $\mathscr{F}$ and those of the tannakian group? For instance, does there exist a constant $c \geqslant 0$, independent of the size of the finite field k, such that

$$\mathrm{M}_4(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}) \leqslant c\mathrm{M}_4(\mathbf{G}), \quad \text{and (or)} \quad \mathrm{M}_4(\mathbf{G}) \leqslant c\mathrm{M}_4(\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}})\,?$$

One can get trivial bounds, similar to the bounds for the norm of the discrete Fourier transform on $\mathrm{G}(k_n)$ when viewed as a map from $\mathrm{L}^{2m}(\mathrm{G}(k_n))$ to $\mathrm{L}^{2m}(\mathrm{G}(k_n))$ for $m > 1$ and n varying, but this norm has been determined by Gilbert and Rzeszotnik [54, Th. 2.1] and depends on n. (On the other hand, a referee has pointed out that certain heuristic examples indicate that the question is most likely "No", and it will be interesting to confirm this rigorously.)

Of course, the meaning of "relation" between the ordinary and tannakian groups could encompass very different aspects, and it is of interest to note that the papers of Lawrence and Sawin and of Javanpeykar, Krämer, Lehn and Maculan use their computations of tannakian groups of certain objects on abelian varieties to study the ordinary monodromy groups of families of subvarieties of abelian varieties (see [99, Th. 5.6] and [66, Th. 4.10] for details).

(3) *Can one construct a "natural" fiber functor* ω *on the tannakian category for* G*, similar to Deligne's fiber functor for* $\mathbf{G}_m$*?*

This would lead to a definition of Frobenius conjugacy classes for all characters (by considering for any χ the conjugacy class in $\mathbf{G}_{\mathrm{M}}^{\mathrm{ari}}$ corresponding to the fiber functor defined by $\mathrm{M} \mapsto \omega(\mathrm{M}_\chi)$), and potentially provide useful extra information to help determine the tannakian group. But note that this is not even clear in the case of $\mathbf{G}_a$.

(4) *Can one find an* a priori *characterization of the families* $(f_n)_{n\geqslant 1}$*, where* $f_n\colon \widehat{\mathrm{G}}(k_n) \to \mathbf{C}$ *is a function, that arise as the arithmetic Fourier transforms of trace functions of complexes, or of perverse sheaves, on* G*?*

More generally, is there a natural "geometric" object, with appropriate notions of sheaves, etc, on the "space" of characters of G? A crucial test for such a geometric interpretation of the discrete Fourier transforms would be the definition of an inverse transform.

This geometric description exists when G is unipotent, since the Serre dual $\mathrm{G}^\vee$ is also a commutative algebraic group, and the Fourier transform is defined as a functor from $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G})$ to $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G}^\vee)$, but such a strong "algebraicity" property does not hold for other commutative algebraic groups (see for instance [**15**, Example 1.8], or Remark 5.3).

There are however some hints in a more positive direction:

(a) Gabber and Loeser [**50**, Th. 3.4.7] have characterized perverse sheaves on tori in terms of the structure of their (coherent) Mellin transforms (which can also be defined for semiabelian varieties), and Loeser [**103**, Ch. 4] has defined a variant over finite fields taking the Frobenius automorphism into account.
It would be of considerable interest to understand better the (essential) image of these Mellin transforms, and to obtain a geometric form of Mellin inversion in this context.

(b) Considering the well-established analogy of ℓ-adic sheaves with $\mathscr{D}$-modules (the basic setup of Katz's work [**69**]), it is well-understood in the complex setting that the Mellin transform of a $\mathscr{D}$-module is a difference equation (e.g., the Mellin transform $\Gamma(s)$ of the exponential satisfies the difference equation $\Gamma(s+1) = s\Gamma(s)$); see for instance the paper of Loeser and Sabbah [**104**].

(5) *Is there an analogous theory for non-commutative algebraic groups?*

For instance, let G be a reductive group over a finite field k, such as $\mathbf{SL}_d(k)$. Deligne–Lusztig Theory parameterizes the irreducible representations of $\mathrm{G}(k_n)$ (or some other more convenient basis of the ℓ-adic representation ring) in terms of pairs $(\mathbf{T}, \theta)$ of a maximal torus of G over k and a character θ of $\mathbf{T}(k)$ (see for instance [**19**, Ch. 7]), and the corresponding series of representations have (essentially) constant dimension as θ varies, so that the character values in such series are suggestively sums of a fixed number of roots of unity. The theory of character sheaves of Lusztig gives a geometric form of this theory.

Are there equidistribution statements for the Fourier coefficients of suitably algebraic conjugacy-invariant functions on $\mathrm{G}(k_n)$? In the case of characteristic functions of conjugacy classes, this might lead to interesting consequences concerning the error term in the Chebotarev Density Theorem for Galois extensions with Galois group of the form $\mathrm{G}(k_n)$.

In the case of (possibly non-commutative) unipotent groups, the Serre dual still exists as a unipotent group; a theory of character sheaves, and of the Fourier transform has been studied by Lusztig and Boyarchenko–Drinfeld (see for instance the survey [**15**]).

(6) *Is there an analogue of automorphic duality for other groups than* $\mathbf{G}_a$*?*

What we mean by this is the following: in the case of a simple middle extension sheaf $\mathscr{F}$ on $\mathbf{G}_a$ over a finite field k that is pure of weight zero and not geometrically isomorphic to an Artin–Schreier sheaf, there is (by the Langlands correspondence, due to Lafforgue in this generality) an automorphic representation π on some general linear group over the adèle ring of $k(t)$ such that (among other properties) the L-functions of (twists of) π coincide (up to normalization) with those of (twists of) the Fourier transform of $\mathscr{F}$. Automorphic methods and results are then available to study the Fourier transform of $\mathscr{F}$.

If G is a commutative algebraic group which is different from $\mathbf{G}_a$, are there objects of a similar nature as automorphic forms and representations that would "correspond" to the arithmetic Fourier transform of suitable perverse sheaves on G? Such objects would presumably have some kind of L-function, which would coincide with the $\widehat{\mathrm{L}}$-function that we have defined. In particular, is there such a theory for $\mathbf{G}_m$?

APPENDIX A

Review of perverse sheaves

In this appendix, we summarize the basic definitions and facts about ℓ-adic perverse sheaves. The fundamental reference for this material is the work of Beillison, Bernstein, Deligne and Gabber [8]. Other useful summaries of perverse sheaves are provided by Katz in [70, §2.1 to 2.3] and in [72, § 1.1, 1.2, 1.5]. For basic material on trace functions in this context, see also [98, § 1.1].

A.1. Complexes of ℓ-adic sheaves

In this appendix, we work over a field k of characteristic p and fix a prime $\ell \neq p$. For X a separated scheme of finite type over k, one can define the triangulated category of complexes of ℓ-adic sheaves $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X}) = \mathrm{D}^{\mathrm{b}}_c(\mathrm{X}, \overline{\mathbf{Q}}_\ell)$.

For $\mathrm{M} \in \mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$, we write $\mathscr{H}^n(\mathrm{M})$ for the n-th cohomology sheaf of M, which is an ℓ-adic constructible sheaf. We denote by $\tau^{\leqslant n}$ and $\tau^{\geqslant n}$ the truncation functors; for every object M, we have canonical maps $\tau^{\leqslant n}(\mathrm{M}) \to \mathrm{M}$ and $\mathrm{M} \to \tau^{\geqslant n}(\mathrm{M})$. The composite functor $\tau^{\geqslant 0} \circ \tau^{\leqslant 0}$ is canonically isomorphic to $\mathrm{M} \mapsto \mathscr{H}^0(\mathrm{M})$.

For varying X, the categories $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ satisfy all the properties of Grothendieck's formalism of the six functors (see [28, 1.12] or [8, 2.2.18] in the case when k is finite or algebraically closed, which suffices for this book).

More precisely, $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ is endowed with two bifunctors

$$(\mathrm{M},\mathrm{N}) \mapsto \mathrm{RHom}(\mathrm{M},\mathrm{N}), \qquad (\mathrm{M},\mathrm{N}) \mapsto \mathrm{M} \otimes \mathrm{N}$$

from $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X}) \times \mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ to $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$, and for a morphism $f\colon \mathrm{X} \to \mathrm{Y}$ of finite type, we have functors

$$\mathrm{M} \mapsto \mathrm{R}f_*\mathrm{M} \qquad \mathrm{M} \mapsto \mathrm{R}f_!\mathrm{M}$$

from $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ to $\mathrm{D}^{\mathrm{b}}_c(\mathrm{Y})$, and functors

$$\mathrm{M} \mapsto f^*\mathrm{M} \qquad \mathrm{M} \mapsto f^!\mathrm{M}$$

from $\mathrm{D}^{\mathrm{b}}_c(\mathrm{Y})$ to $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$. These functors satisfy the usual compatibilities and adjunctions.

The dualizing complex for X is defined to be $s^!\overline{\mathbf{Q}}_\ell$, where $s\colon \mathrm{X} \to \mathrm{Spec}(k)$ is the structure morphism, and the Verdier dual of $\mathrm{M} \in \mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ is $\mathrm{D}(\mathrm{M}) = \mathrm{RHom}(\mathrm{M}, s^!\overline{\mathbf{Q}}_\ell)$. When X is smooth of pure dimension d, there is a canonical isomorphism

$$s^!\overline{\mathbf{Q}}_\ell \simeq \overline{\mathbf{Q}}_\ell(d)[2d]. \tag{A.1}$$

Let $s\colon \mathrm{X} \to \mathrm{Spec}(k)$ be the structure morphism. For any object M of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ and $i \in \mathbf{Z}$, the i-th *cohomology group* of X with coefficients in M (resp. *cohomology group with compact support of* X *with coefficients in* M) is given by

$$\mathrm{H}^i(\mathrm{X},\mathrm{M}) = \mathscr{H}^i(s_*\mathrm{M}), \qquad \mathrm{H}^i_c(\mathrm{X},\mathrm{M}) = \mathscr{H}^i(s_!\mathrm{M}),$$

where we identify ℓ-adic sheaves on $\mathrm{Spec}(k)$ with $\overline{\mathbf{Q}}_\ell$-vector spaces.

When X is a smooth curve, two other important results (the Euler–Poincaré characteristic formula and Laumon's product formula for epsilon factors) which are used in this book will be reviewed in Appendix C.

A.2. Perverse sheaves

Definition A.1. A complex $M \in D_c^b(X)$ is said to be *semiperverse* if its cohomology sheaves satisfy

$$\dim \operatorname{supp}(\mathscr{H}^i(M)) \leqslant -i \text{ for every } i \in \mathbf{Z},$$

and M is said to be *perverse* if both M and D(M) are semiperverse (see [8, (4.0.1')]).

We denote by $\mathbf{Perv}(X)$ the full subcategory of perverse sheaves in $D_c^b(X)$, by ${}^{\mathfrak{p}}D^{\leqslant 0}(X)$ the full subcategory of semiperverse sheaves, and by ${}^{\mathfrak{p}}D^{\geqslant 0}(X)$ the full subcategory of objects M such that D(M) is semiperverse.

Theorem A.2. *The data of* ${}^{\mathfrak{p}}D^{\leqslant 0}(X)$ *and* ${}^{\mathfrak{p}}D^{\geqslant 0}(X)$ *give rise to a t-structure on* $D_c^b(X)$*. Its heart* $\mathbf{Perv}(X) = {}^{\mathfrak{p}}D^{\leqslant 0}(X) \cap {}^{\mathfrak{p}}D^{\geqslant 0}(X)$ *is therefore an abelian category.*

Example A.3. Suppose that X is smooth of pure dimension d, and let $\mathscr{F}$ be a lisse ℓ-adic sheaf on X. Then the complex $\mathscr{F}[d]$ (i.e., the sheaf $\mathscr{F}$ put in degree $-d$) is a perverse sheaf.

Indeed, $\mathscr{F}[d]$ is clearly semiperverse and by (A.1), we see that $D(\mathscr{F}[d]) = \mathscr{F}^\vee(d)[d]$, where $\mathscr{F}^\vee$ is the dual lisse sheaf of $\mathscr{F}$, so the dual of $\mathscr{F}[d]$ is also semiperverse.

If M is a complex in $D_c^b(X)$ with support $Y \subset X$, then there exists an open dense subset U of Y such that the restriction of M to U is lisse, i.e., all of the cohomology sheaves of $M|U$ are lisse sheaves. We then say that M *is lisse on* U.

One also defines

$${}^{\mathfrak{p}}D^{\leqslant n}(X) = {}^{\mathfrak{p}}D^{\leqslant 0}(X)[n] \text{ and } {}^{\mathfrak{p}}D^{\geqslant n}(X) = {}^{\mathfrak{p}}D^{\geqslant 0}(X)[n].$$

The inclusion functors ${}^{\mathfrak{p}}D^{\leqslant n}(X) \subset D_c^b(X)$ and ${}^{\mathfrak{p}}D^{\geqslant n}(X) \subset D_c^b(X)$ admit right and left adjoints, called the *perverse truncation functors*, which are denoted

$${}^{\mathfrak{p}}\tau^{\leqslant n} \colon D_c^b(X) \to {}^{\mathfrak{p}}D^{\leqslant n}(X) \text{ and } {}^{\mathfrak{p}}\tau^{\geqslant n} \colon D_c^b(X) \to {}^{\mathfrak{p}}D^{\geqslant n}(X).$$

Definition A.4. The n-th *perverse cohomology sheaf* of a complex $M \in D_c^b(X)$ is the perverse sheaf

$${}^{\mathfrak{p}}\mathscr{H}^n(M) = \tau^{\leqslant 0}\tau^{\geqslant 0}(M[n]) \in \mathbf{Perv}(X).$$

Given a distinguished triangle $M \to N \to L \to$ in $D_c^b(X)$, we have a long exact sequence

$$\cdots \to {}^{\mathfrak{p}}\mathscr{H}^i(M) \to {}^{\mathfrak{p}}\mathscr{H}^i(N) \to {}^{\mathfrak{p}}\mathscr{H}^i(L) \to {}^{\mathfrak{p}}\mathscr{H}^{i+1}(M) \to \cdots \tag{A.2}$$

of perverse cohomology sheaves.

Let M be a perverse sheaf on X. From general principles, there are convergent spectral sequences

$$E_2^{p,q} = H^p(X, {}^{\mathfrak{p}}\mathscr{H}^q(M)) \Longrightarrow H^{p+q}(X, M), \qquad E_2^{p,q} = H_c^p(X, {}^{\mathfrak{p}}\mathscr{H}^q(M)) \Longrightarrow H_c^{p+q}(X, M), \tag{A.3}$$

which are called the *perverse spectral sequences.*

We also have an equality

$$M = \sum_{j \in \mathbf{Z}} (-1)^j \, {}^{\mathfrak{p}}\mathscr{H}^j(M) \tag{A.4}$$

in the Grothendieck group K(X) (see, e.g., [98, (0.8)]).

As with the standard t-structure, perverse cohomology sheaves give a criterion to check whether a complex is semiperverse.

Lemma A.5. *A complex* $\mathrm{M} \in \mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ *is semiperverse if and only if* ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M}) = 0$ *for all integers* $i \geqslant 1$.

See [8, Prop. 1.3.7] for the proof.

Definition A.6. An exact[1] functor from $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ to $\mathrm{D}^{\mathrm{b}}_c(\mathrm{Y})$ is said to be *left t-exact* (resp. *right t-exact*) if it sends ${}^{\mathfrak{p}}\mathrm{D}^{\geqslant 0}(\mathrm{X})$ to ${}^{\mathfrak{p}}\mathrm{D}^{\geqslant 0}(\mathrm{Y})$ (resp. ${}^{\mathfrak{p}}\mathrm{D}^{\leqslant 0}(\mathrm{X})$ to ${}^{\mathfrak{p}}\mathrm{D}^{\leqslant 0}(\mathrm{Y})$). It is said to be *$t$-exact* if it is both left and right t-exact.

The following important result is a direct consequence of Artin's cohomological vanishing theorem (see [8, Th. 4.1.1]).

Theorem A.7. *Let* $f\colon \mathrm{X} \to \mathrm{Y}$ *be an affine morphism, then* $\mathrm{R}f_*$ *is right t-exact and* $\mathrm{R}f_!$ *is left t-exact.*

Since a closed immersion i is affine and proper (so that $\mathrm{R}i_* = i_!$), we obtain as corollary:

Corollary A.8. *If i is a closed immersion, then i_* is t-exact.*

More generally (see [8, Cor. 4.1.3]), the functors $f_!$ and f_* are t-exact if f is quasi-finite and affine.

A central result is the construction of the intermediate extension, see [8, Cor. 1.4.25].

Proposition A.9. *Let* $j\colon \mathrm{U} \to \mathrm{X}$ *be a locally closed immersion. Let* M *be a perverse sheaf on* U*. Then there exists a unique perverse sheaf* $j_{!*}(\mathrm{M})$ *on* X*, called the* middle extension *or* intermediate extension *of* M*, such that*

- *There exists an isomorphism* $j^* j_{!*}(\mathrm{M}) \simeq \mathrm{M}$.
- *The perverse sheaf* $j_{!*}(\mathrm{M})$ *is supported on the closure* $\overline{\mathrm{U}}$ *of* U.
- *The perverse sheaf* $j_{!*}(\mathrm{M})$ *has no subobject and no quotient supported on* $\overline{\mathrm{U}} - \mathrm{U}$.

The most important example of this construction is when $j\colon \mathrm{U} \to \mathrm{X}$ is a dense open immersion, with U smooth of pure dimension d, and $\mathrm{M} = \mathscr{F}[d]$ for a lisse sheaf $\mathscr{F}$. Note that the uniqueness implies that $\mathrm{D}(j_{!*}\mathscr{F}[d]) = j_{!*}\mathscr{F}^{\vee}(d)[d]$. When $\mathscr{F} = \overline{\mathbf{Q}}_\ell$ is the constant sheaf on U, then $j_{!*}\overline{\mathbf{Q}}_\ell[d]$ is called the *intersection complex* of X.

Example A.10. Let X be a curve, U a dense open subset of X contained in the smooth locus of X and $\mathscr{F}$ a lisse sheaf on U. Then $j_{!*}\mathscr{F}[1] = \mathrm{R}^0 j_*\mathscr{F}[1]$, where $j\colon \mathrm{U} \to \mathrm{X}$ is the open immersion.

The fundamental result concerning the category of perverse sheaves is the following theorem [8, Th. 4.3.1].

Theorem A.11. *The category* $\mathbf{Perv}(\mathrm{X})$ *is artinian and noetherian, i.e., all objects are of finite length. Its simple objects are of the form* $j_{!*}\mathscr{F}[d]$ *where* $j\colon \mathrm{U} \to \mathrm{X}$ *is a locally closed immersion with* U *smooth irreducible of dimension* d *and* $\mathscr{F}$ *is an irreducible lisse sheaf on* U.

Example A.12. Let X be a smooth and geometrically connected curve. Following Katz [69, § 7.3], a constructible sheaf $\mathscr{F}$ on X is called a *middle extension sheaf* if, for any dense open set U of X such that $\mathscr{F}$ is lisse on U, with open immersion $j\colon \mathrm{U} \to \mathrm{X}$, the canonical morphism $\mathscr{F} \to j_*j^*\mathscr{F}$ is an isomorphism.

There is a one-to-one correspondence between irreducible middle extension sheaves and simple perverse sheaves on X with support equal to X; for a middle extension sheaf $\mathscr{F}$, the corresponding

[1] Namely, a functor that commutes with shift and preserves distinguished triangles.

simple perverse sheaf is $\mathscr{F}[1]$. Conversely, for a simple perverse sheaf M with support equal to X, of the form $j_{!*}\mathscr{F}[1]$ as in the theorem, the corresponding (irreducible) middle extension sheaf is $j_*\mathscr{F}$.

For simple perverse sheaves, the bounds on the dimension of the support of the cohomology sheaves have an "automatic improvement" from the bound given by the semi-perversity, except for $\mathscr{H}^{-\dim(\mathrm{X})}$.

Proposition A.13. *Let* M *be a simple perverse sheaf on* X *which is not punctual. Then for any* $i \neq -\dim(\mathrm{supp}(\mathrm{X}))$, *we have*

$$\dim \mathrm{supp}(\mathscr{H}^i(\mathrm{M})) \leqslant -i-1.$$

Proof. This results from the classification of simple perverse sheaves and from the general description of the intermediate extension functor in [8, Prop. 2.1.11]. □

Example A.14. In the case of a curve, this property can be seen from Example A.10, since in that case any simple perverse sheaf which is not punctual is supported on a dense open subset.

We thank S. Morel for communicating us a proof of the following lemma (see also [72, Sub-lemma 1.10.5]).

Lemma A.15. *Let k be an algebraically closed field. Let* X *be an irreducible projective variety of dimension d over k, and let* M *be a simple perverse sheaf on* X *such that* $\mathrm{H}^{-d}(\mathrm{X},\mathrm{M})$ *is non-zero. Then the support of* M *is* X *and there exists an open immersion* $j\colon \mathrm{U} \hookrightarrow \mathrm{X}$ *such that* $\mathrm{M} = j_{!*}\overline{\mathbf{Q}}_\ell[d]$.

Proof. By the classification of simple perverse sheaves in Theorem A.11, there exists a locally closed immersion $j\colon \mathrm{U} \hookrightarrow \mathrm{X}$ and a simple lisse sheaf $\mathscr{F}$ on U such that $\mathrm{M} = j_{!*}\mathscr{F}[\dim(\mathrm{U})]$. The cohomolog $\mathrm{H}^i(\mathrm{X},\mathrm{M})$ vanishes unless $|i| \leqslant \dim(\mathrm{U})$ (see [8, 4.2.4]), so the assumption implies that $\dim(\mathrm{U}) = d$. Besides, the formula for intermediate extensions from [8, 2.1.11] implies the vanishing $\mathscr{H}^i(\mathrm{M}) = 0$ for $i < -d$. From the spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{X}, \mathscr{H}^q(\mathrm{M})) \Longrightarrow \mathrm{H}^{p+q}(\mathrm{X},\mathrm{M}),$$

we then get an isomorphim $\mathrm{H}^{-d}(\mathrm{X},\mathrm{M}) \simeq \mathrm{H}^0(\mathrm{X}, \mathscr{H}^{-d}(\mathrm{M}))$. The non-vanishing of this cohomology group implies that $\mathscr{H}^{-d}(\mathrm{M})$ has a global section. Hence, $\mathscr{F}$ has a global section and is therefore trivial. □

A.3. Weights

In this section we assume that k is a finite field of characteristic p, and denote by $\bar{k}$ an algebraic closure. We also fix an isomorphism $\iota\colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$.

Let q be a prime power and let $w \in \mathbf{Z}$ be an integer. An element $x \in \overline{\mathbf{Q}}_\ell$ is said to be a *q-Weil number of weight w* if it is algebraic over $\mathbf{Q}$, and if all the complex conjugates of $\iota(x)$ are complex numbers with modulus $q^{w/2}$.

Let X be a separated scheme of finite type over k and $\mathscr{F}$ a $\overline{\mathbf{Q}}_\ell$-sheaf on X. Let x be a closed point of X, with residue field $k(x)$. Viewing $k(x)$ as a subfield of the fixed algebraic closure $\bar{k}$ of k defines a geometric point $\widetilde{x}\colon \mathrm{Spec}(\bar{k}) \to \mathrm{X}$ supported at x. The geometric Frobenius automorphism, inverse of $y \mapsto y^{k(x)}$ in $\mathrm{Gal}(\bar{k}/k)$, acts on the stalk $\mathscr{F}_{\widetilde{x}}$ of $\mathscr{F}$ at $\widetilde{x}$. We denote by Fr_x this endomorphism of $\mathscr{F}_{\widetilde{x}}$, which is well-defined up to conjugacy.

Definition A.16 ([28, 1.2], [8, 5.1.5]). Let X be a separated scheme of finite type over k, $\mathscr{F}$ a $\overline{\mathbf{Q}}_\ell$-sheaf on X, and M an object of $\mathrm{D}_c^{\mathrm{b}}(\mathrm{X})$.

(1) The sheaf $\mathscr{F}$ is *punctually pure of weight* w if for every $x \in |\mathrm{X}|$, the eigenvalues of Fr_x are $|k(x)|$-Weil numbers of weight w.

(2) The sheaf $\mathscr{F}$ is *mixed* if it admits a finite filtration with successive quotients that are punctually pure. The weights of the non-zero quotients are called the punctual weights of $\mathscr{F}$.

(3) The complex M is *mixed* if all its cohomology sheaves are mixed. It is *mixed of weights* $\leqslant w$ if for every $i \in \mathbf{Z}$, the sheaf $\mathscr{H}^i(\mathrm{M})$ is mixed with punctual weights $\leqslant w + i$. It is *mixed of weights* $\geqslant w$ if its Verdier dual $\mathrm{D}(\mathrm{M})$ is mixed of weights $\leqslant -w$.

(4) The complex M is *pure of weight* w if it is both mixed of weights $\leqslant w$ and of weights $\geqslant w$.

Remark A.17. Deligne also defines ι-weights and ι-pure or mixed sheaves and complexes for any fixed isomorphism ι; the notion above means that the objects are ι-pure for all ι (see [**28**, 1.2.6]).

We write $\mathrm{D}_{\leqslant w}(\mathrm{X})$ and $\mathrm{D}_{\geqslant w}(\mathrm{X})$ for the full subcategories of $\mathrm{D}_c^{\mathrm{b}}(\mathrm{X})$ of objects mixed of weights $\leqslant w$ and $\geqslant w$. Thanks to the shift in the definition, one has in particular $\mathrm{D}_{\leqslant w}[1] = \mathrm{D}_{\leqslant w+1}$.

Example A.18. (1) Suppose that X is smooth of pure dimension d, and that $\mathrm{M} \in \mathrm{D}_c^{\mathrm{b}}(\mathrm{X})$ is such that all its cohomology sheaves are lisse on X. Then M is pure of weight w if and only if each sheaf $\mathscr{H}^i(\mathrm{M})$ is punctually pure of weight $w + i$.

(2) The characterization of (1) does not apply in general. For instance, let $\mathrm{X} = \mathbf{A}^1$ be the affine line, and $j\colon \mathbf{G}_m \to \mathrm{X}$ the open immersion. Let $\mathrm{M} = (j_* \mathrm{Kl}_2)[1](1/2)$ be the Kloosterman sheaf of rank 2 shifted to be in degree -1 and Tate-twisted to be of weight 0 (see (B.2)). Then M is pure of weight 0. However, the cohomology sheaf $\mathscr{H}^{-1}(\mathrm{M}) = j_* \mathrm{Kl}_2(1/2)$ is not punctually pure of weight -1: indeed, the stalk of this sheaf at 0 has rank 1 with a Frobenius eigenvalue of weight -2.

(3) If $\mathscr{F}$ is a middle extension sheaf on X (see Example A.12 for the definition), we say that $\mathscr{F}$ is pure of weight w if the perverse sheaf $\mathrm{M}[\dim(\mathrm{X})](\dim(\mathrm{X})/2)$ is pre of weight w. This is equivalent to the condition that the restriction of $\mathscr{F}$ to any dense open set where it is lisse is punctually pure of weight w.

Deligne's main theorem in [**28**, 3.3.1, 6.2.3], which directly implies the most general form of the Riemann Hypothesis over finite fields, is the following:

Theorem A.19 (Deligne). *Let* $f\colon \mathrm{X} \to \mathrm{Y}$ *be a separated morphism of schemes of finite type over* k*. Then the functor* $\mathrm{R}f_!$ *sends* $\mathrm{D}_{\leqslant w}(\mathrm{X})$ *to* $\mathrm{D}_{\leqslant w}(\mathrm{Y})$*.*

Using duality, one gets the following list of compatibilities of the different functors on $\mathrm{D}_c^{\mathrm{b}}(\mathrm{X})$ (see [**8**, 5.1.14]):

(1) $\mathrm{R}f_!$ and f^* preserve $\mathrm{D}_{\leqslant w}$;

(2) $\mathrm{R}f_*$ and $f^!$ preserve $\mathrm{D}_{\geqslant w}$;

(3) $\otimes$ sends $\mathrm{D}_{\leqslant w} \times \mathrm{D}_{\leqslant w'}$ to $\mathrm{D}_{\leqslant w+w'}$;

(4) RHom sends $\mathrm{D}_{\leqslant w} \times \mathrm{D}_{\geqslant w'}$ to $\mathrm{D}_{\geqslant -w+w'}$;

(5) Verdier duality exchanges $\mathrm{D}_{\leqslant w}$ and $\mathrm{D}_{\geqslant -w}$.

A.4. Trace functions

We continue with the notation of the previous section, so that X is an algebraic variety over a finite field k.

Let M be a complex in $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$. For any integer $n \geqslant 1$ and $x \in \mathrm{X}(k_n)$, the stalk $\mathrm{M}_{\bar{x}}$ of M at a geometric point $\bar{x}$ above x is a complex of finite-dimensional $\overline{\mathbf{Q}}_\ell$-vector spaces, with only finitely many non-zero cohomology spaces. The geometric Frobenius Fr_{k_n} of k_n (the inverse of the automorphisme $a \mapsto a^{|k_n|}$ of k_n) acts on $\mathrm{M}_{\bar{x}}$, and this action is independent of the choice of $\bar{x}$ up to conjugacy. We denote

$$t_{\mathrm{M}}(x; k_n) = \sum_{i \in \mathbf{Z}} (-1)^i \operatorname{Tr}(\mathrm{Fr}_{k_n} \mid \mathscr{H}^i(\mathrm{M})_{\bar{x}}),$$

which is also independent of $\bar{x}$ above x.

Whenever we have fixed the isomorphism $\iota_0 \colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$ (as in the whole of the main text, see Section 3), we will view the trace function as a function $\mathrm{X}(k_n) \to \mathbf{C}$ whenever convenient.

Definition A.20. The *trace function* t_{M} of M is the data of the whole family of functions $(t_{\mathrm{M}}(\cdot; k_n))_{n \geqslant 1}$.

Remark A.21. We will sometimes write simply $t_{\mathrm{M}}(x)$ for $t_{\mathrm{M}}(x; k)$, when $x \in \mathrm{X}(k)$.

Viewing $\mathrm{X}(k_n)$ as a subset of $\mathrm{X}(\bar{k})$, we will also sometimes denote the stalk of M simply by M_x, instead of introducing explicitly a specific geometric point over x.

Let $f \colon \mathrm{X} \to \mathrm{Y}$ be a morphism of algebraic varieties over k. The following properties holds for objects M_i and M of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ and N of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{Y})$:

$$\begin{aligned}
t_{\overline{\mathbf{Q}}_\ell} &= 1 \quad (\overline{\mathbf{Q}}_\ell \text{ in degree } 0)\\
t_{\mathrm{M}[k]} &= (-1)^k t_{\mathrm{M}}, \qquad t_{\mathrm{M}(w)} = q^{-w/2} t_{\mathrm{M}}\\
t_{\mathrm{M}_2} &= t_{\mathrm{M}_1} + t_{\mathrm{M}_3} \quad \text{ for any distinguished triangle } \mathrm{M}_1 \to \mathrm{M}_2 \to \mathrm{M}_3 \to,\\
t_{\mathrm{M}_1 \otimes \mathrm{M}_2} &= t_{\mathrm{M}_1} t_{\mathrm{M}_2}\\
t_{f^*\mathrm{N}} &= t_{\mathrm{N}} \circ f, \quad \text{ i.e. } \quad t_{f^*\mathrm{N}}(x; k_n) = t_{\mathrm{N}}(f(x); k_n) \text{ for all } n \geqslant 1 \text{ and } x \in \mathrm{X}(k_n)\\
t_{\mathrm{R}f_!\mathrm{M}}(y; k_n) &= \sum_{\substack{x \in \mathrm{X}(k_n)\\ f(x) = y}} t_{\mathrm{M}}(x; k_n).
\end{aligned}$$

The last of these properties is a form of the Grothendieck–Lefschetz trace formula (see [62, Exp. III, §4]). Applied to a complex M and to the structure morphism $\mathrm{X} \to \mathrm{Spec}(k)$, it takes the customary form

$$\sum_{x \in \mathrm{X}(k_n)} t_{\mathrm{M}}(x; k_n) = \sum_{i \in \mathbf{Z}} (-1)^i \operatorname{Tr}(\mathrm{Fr}_{k_n} \mid \mathrm{H}^i_c(\mathrm{X}_{\bar{k}}, \mathrm{M})). \tag{A.5}$$

Suppose that M is a semisimple perverse sheaf which is pure of weight 0. Then by a result of Gabber (see [115, proof of Prop. 6.40]), the equality

$$t_{\mathrm{D}(\mathrm{M})}(x; k_n) = \overline{t_{\mathrm{M}}(x; k_n)} \tag{A.6}$$

holds for all $n \geqslant 1$ and $x \in \mathrm{X}(k_n)$.

We also recall a useful injectivity statement:

Proposition A.22. *Let* M_1 *and* M_2 *be objects of* $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$. *The trace functions of* M_1 *and* M_2 *coincide, in the sense that*

$$t_{\mathrm{M}_1}(x; k_n) = t_{\mathrm{M}_2}(x; k_n)$$

for all $n \geqslant 1$ *and all* $x \in \mathrm{X}(k_n)$, *if and only if the classes of* M_1 *and* M_2 *in the Grothendieck group* $\mathrm{K}(\mathrm{X})$ *are equal. In particular, if* M *and* N *are semisimple perverse sheaves, then* M *and* N *are isomorphic.*

Moreover, the classes of simple perverse sheaves form a basis of the $\mathbf{Z}$*-module* $\mathrm{K}(\mathrm{X})$.

This is proved in [98, Th. 1.1.2].

APPENDIX B

The arithmetic Mellin transform over finite fields

We summarize here the most important results of Katz [74] concerning the arithmetic Mellin transform on $\mathbf{G}_m$. These results are used in various places in the book.

B.1. The category $\mathcal{P}$

Let k be a finite field with algebraic closure $\bar{k}$ and with finite extensions k_n/k for $n \geqslant 1$.

Katz defines a category $\mathcal{P}$ as the full subcategory of the category of perverse sheaves on $\mathbf{G}_m$ over $\bar{k}$ whose objects are perverse sheaves N such that, for any perverse sheaf M on $\mathbf{G}_m$, the objects $\mathrm{M} *_! \mathrm{N}$ and $\mathrm{M} *_* \mathrm{N}$ are both perverse (see [74, Ch. 2] and [70, 2.6.2]). Katz proved that a perverse sheaf N is an object of $\mathcal{P}$ if and only if it admits no shifted Kummer sheaf $\mathscr{L}_\chi[1]$ as either subobject or quotient (this follows, e.g, from the combination of [70, Lemma 2.6.13, Lemma 2.6.14, Cor. 2.6.15]).

The category $\mathcal{P}_{\mathrm{ari}}$ is defined as the full subcategory of perverse sheaves on $\mathbf{G}_m$ over k whose objects are those perverse sheaves N such that the base change of N to $\bar{k}$ is an object of $\mathcal{P}$ ([74, Ch. 4]).

Using the correct notion of exactness from the work of Gabber and Loeser, the categories $\mathcal{P}_{\mathrm{ari}}$ and $\mathcal{P}$ are neutral tannakian categories with the middle convolution

$$\mathrm{M} *_{\mathrm{int}} \mathrm{N} = \mathrm{Im}(\mathrm{M} *_! \mathrm{N} \to \mathrm{M} *_* \mathrm{N})$$

as tensor operation (see [50, p. 535]).

The tannakian dimension of an object of $\mathcal{P}$ is its Euler–Poincaré characteristic.

B.2. Deligne's fiber functor and Frobenius conjugacy classes

One remarkable canonical fiber functor on the tannakian category $\mathcal{P}$ is given by a theorem of Deligne.

Theorem B.1 (Deligne). *Let k be a finite field with algebraic closure $\bar{k}$. Let $j_0\colon \mathbf{G}_m \to \mathbf{A}^1$ be the open immersion. Then the functor*

$$\omega_{\mathrm{Del}}\colon \mathrm{M} \mapsto \mathrm{H}^0(\mathbf{A}^1_{\bar{k}}, j_{0!}\mathrm{M})$$

is a fiber functor on the category $\mathcal{P}$.

This is [74, Th. 3.1 and Appendix].

Let N be an object of $\mathcal{P}_{\mathrm{ari}}$ which is arithmetically semisimple and pure of weight 0. Let $\mathbf{G}_{\mathrm{N}}^{\mathrm{ari}}$ be the tannakian group of the tannakian subcategory of $\mathcal{P}_{\mathrm{ari}}$ generated by N. Using Deligne's fiber functor and the tannakian formalism, Katz defines a Frobenius conjugacy class $\mathbf{Fr}_{\mathrm{N},k_n}(\chi)$ in $\mathbf{G}_{\mathrm{N}}^{\mathrm{ari}}$ for any $n \geqslant 1$ and any ℓ-adic character χ of $k_n^\times$ by considering the fiber functor $\omega_\chi\colon \mathrm{M} \mapsto \omega_{\mathrm{Del}}(\mathrm{M}_\chi)$ (see [74, Ch. 5]).

B.3. Finite tannakian groups

THEOREM B.2 (Katz). *Let k be a finite field with algebraic closure $\bar{k}$. Let* N *be a perverse sheaf in the category $\mathcal{P}_{\mathrm{ari}}$. Assume that* N *is arithmetically semisimple and pure of weight* 0.

(1) *If every Frobenius conjugacy class of* N *is quasi-unipotent, then the object* N *is punctual.*
(2) *If the geometric tannakian group of* N*, i.e., the tannakian group of the tannakian subcategory of $\mathcal{P}$ generated by $\mathrm{N} \otimes \bar{k}$, is* finite*, then the object* N *is punctual.*

These statements are [74, Th 6.2 and Th. 6.4].

B.4. Hypergeometric complexes and sheaves

Katz has also classified the perverse sheaves on $\mathbf{G}_m$ with tannakian dimension 0 and 1. Indeed, since the tannakian dimension is equal to the Euler–Poincaré characteristic in this case, the question is to classify simple perverse sheaves M on $\mathbf{G}_m$ with $\chi(\mathrm{M}) = 0$ or 1.

For Euler–Poincaré characteristic zero, we have:

PROPOSITION B.3. *Let k be an algebraically closed field of characteristic $p > 0$ with $p \neq \ell$. Let* M *be a simple perverse sheaf on $\mathbf{G}_m$ with $\chi(\mathbf{G}_m, \mathrm{M}) = 0$. Then there exists a tame character χ of $\mathbf{G}_m$ such that* M *is isomorphic to $\mathscr{L}_\chi[1]$.*

This is [69, Prop. 8.5.2].

Katz has furthermore shown that the objects with Euler–Poincaré characteristic 1 are exactly the *hypergeometric complexes* on $\mathbf{G}_m$, defined in [69, 8.2, 8.3].

We recall the definition and notation for hypergeometric complexes. Let k be a field of positive characteristic. Fix a pair (m, n) of non-negative integers and a non-trivial ℓ-adic additive character ψ of a finite subfield of k. Denote by $j\colon \mathbf{G}_m \to \mathbf{A}^1$ the open immersion. Let

$$\boldsymbol{\chi} = (\chi_1, \ldots, \chi_n), \qquad \boldsymbol{\varrho} = (\varrho_1, \ldots, \varrho_m)$$

be two tuples of tame ℓ-adic continuous characters $\pi_1^t(\mathbf{G}_m) \to \overline{\mathbf{Q}}_\ell^\times$. Denote by $\bar{\psi}$ the inverse of ψ and write

$$\bar{\boldsymbol{\chi}} = (\chi_1^{-1}, \ldots, \chi_n^{-1}), \qquad \bar{\boldsymbol{\varrho}} = (\varrho_1^{-1}, \ldots, \varrho_m^{-1}).$$

The *hypergeometric complex* $\mathrm{Hyp}(!, \psi, \boldsymbol{\chi}; \boldsymbol{\varrho})$ in $\mathrm{D}_c^{\mathrm{b}}(\mathbf{G}_m)$ is then defined inductively as follows:

(1) If $(m, n) = (0, 0)$ then $\mathrm{Hyp}(!, \psi, \varnothing; \varnothing)$ is the skyscraper sheaf supported at 1.

(2) If $(m, n) = (1, 0)$ then $\mathrm{Hyp}(!, \psi, \chi; \varnothing) = j^*(\mathscr{L}_\psi) \otimes \mathscr{L}_\chi[1]$.

(3) If $(m, n) = (0, 1)$ then $\mathrm{Hyp}(!, \psi, \varnothing, \varrho) = \mathrm{inv}^*(j^*(\mathscr{L}_{\bar{\psi}}) \otimes \mathscr{L}_{\bar{\varrho}})[1]$.

(4) If $(m, n) = (m, 0)$ with $m \geqslant 2$ then $\mathrm{Hyp}(!, \psi, \boldsymbol{\chi}; \varnothing)$ is the convolution
$$\mathrm{Hyp}(!, \psi, \chi_1; \varnothing) *_! \cdots *_! \mathrm{Hyp}(!, \psi, \chi_n; \varnothing).$$

(5) If $(m, n) = (0, n)$ with $n \geqslant 2$ then $\mathrm{Hyp}(!, \psi, \varnothing; \boldsymbol{\varrho})$ is the convolution
$$\mathrm{Hyp}(!, \psi, \varnothing; \varrho_1) *_! \cdots *_! \mathrm{Hyp}(!, \psi, \varnothing; \varrho_n).$$

(6) In the general case, we have
$$\mathrm{Hyp}(!, \psi, \boldsymbol{\chi}; \boldsymbol{\varrho}) = \mathrm{Hyp}(!, \psi, \boldsymbol{\chi}; \varnothing) *_! \mathrm{Hyp}(!, \psi, \varnothing; \boldsymbol{\varrho}).$$

For $\lambda \in k^\times$, define also

$$\mathrm{Hyp}_\lambda(!, \psi, \boldsymbol{\chi}; \boldsymbol{\varrho}) = [x \mapsto \lambda x]_* \mathrm{Hyp}(!, \psi, \boldsymbol{\chi}; \boldsymbol{\varrho}).$$

It follows from these definitions that the general convolution formula

$$\mathrm{Hyp}_\lambda(!,\psi,\boldsymbol{\chi};\boldsymbol{\varrho}) *_! \mathrm{Hyp}_\mu(!,\psi,\boldsymbol{\chi}';\boldsymbol{\varrho}') = \mathrm{Hyp}_{\lambda\mu}(!,\psi,\boldsymbol{\chi},\boldsymbol{\chi}';\boldsymbol{\varrho},\boldsymbol{\varrho}')$$

holds.

Let K be an extension of k. We say that a complex M on $\mathbf{G}_m$ over K is *hypergeometric over* k if there exists $\lambda \in k^\times$, an additive character ψ, and families of tame multiplicative characters $\boldsymbol{\chi}$ and $\boldsymbol{\varrho}$ over k such that $\mathrm{M} \otimes \mathrm{K}$ is isomorphic to $\mathrm{Hyp}_\lambda(!,\psi,\boldsymbol{\chi};\boldsymbol{\varrho})$, where the characters involved are defined on $\mathbf{G}_m$ over K by composition with the canonical morphism $\pi_1^t((\mathbf{G}_m)_{\mathrm{K}}) \to \pi_1^t((\mathbf{G}_m)_k)$.

Before stating some of the main results concerning hypergeometric sheaves, we need a further definition: the tuples $\boldsymbol{\chi}$ and $\boldsymbol{\varrho}$ are said to be *disjoint* if $(n,m) \neq (0,0)$ and $\chi_i \neq \varrho_j$ for all i and j.

Theorem B.4 (Katz). *Assume that k is algebraically closed.*

1. *If the tuples $\boldsymbol{\chi}$ and $\boldsymbol{\varrho}$ are disjoint, then $\mathrm{Hyp}_\lambda(!,\psi,\boldsymbol{\chi};\boldsymbol{\varrho})$ is a simple nonpunctual perverse sheaf of Euler characteristic 1 on $\mathbf{G}_m$.*
2. *Let K be an extension of k and $\bar{\mathrm{K}}$ an algebraic closure of K. Let M be a simple perverse sheaf M on $\mathbf{G}_m$ over K with Euler–Poincaré characteristic equal to 1. Then the base change $\mathrm{M} \otimes \bar{\mathrm{K}}$ of M to $\bar{\mathrm{K}}$ is hypergeometric over K.*
3. *Let k be a finite field and $\bar{k}$ an algebraic closure of k. If the tuples $\boldsymbol{\chi}$ and $\boldsymbol{\varrho}$ are disjoint, then the tannakian group of the hypergeometric object $\mathrm{Hyp}_\lambda(!,\psi,\boldsymbol{\chi};\boldsymbol{\varrho})$ on $\mathbf{G}_m$ over $\bar{k}$ is $\mathbf{GL}_1$.*

Proof. The first statement follows from [69, Th. 8.4.2], and the third is explained, e.g., in [74, proof of Cor. 6.3].

The second statement is [69, Th. 8.5.3] if $\mathrm{K} = \bar{\mathrm{K}} = k$. Applied to $\bar{\mathrm{K}}$ instead of an algebraic closure of k, this gives the result except that we only know *a priori* that $\lambda \in \bar{\mathrm{K}}^\times$. We need to check that in fact $\lambda \in \mathrm{K}^\times$. To do this, we check the steps of the proof of loc. cit., which is easily seen to provide this extra information.

Say that M is of type (m,n) if m is the dimension of the tame part of M at 0 and n the dimension of the tame part at infinity. The strategy of the proof is to reduce by induction to the case $m > n$, then to $n = 0$ and finally to the case $m = n = 0$.

Each of these reduction steps follows a similar pattern. First, up to tensoring M by $\mathscr{L}_\Lambda$ for some tame continuous character Λ of $\pi_1^t(\mathbf{G}_{m,k})$, one can assume that the trivial character occurs in the local monodromy at 0. From Kummer theory, we have an isomorphism $\pi_1^t(\mathbf{G}_{m,k}) \simeq \widehat{\mathbf{Z}}(1)_{p'}$; since M is defined over $k(\eta)$, the character Λ must be of finite order and hence is a character of $k^\times$. All the characters χ and ϱ appear as such Λ.

After this tensoring step, one considers the Fourier transform $\mathrm{FT}_\psi(j_*\mathrm{M})$, and one checks that it is of type $(n, m-1)$, and is still a geometrically simple perverse sheaf of Euler characteristic 1.

At the end of the induction, one is left either with a skyscraper sheaf, which must be supported on some $\lambda \in \mathrm{K}^\times$ since M is geometrically simple over K, or with a perverse sheaf that is geometrically isomorphic to $\mathscr{L}_{\psi(\lambda x)}$ for some $\lambda \in \bar{\mathrm{K}}^\times$, and since this sheaf is defined over K, we must have $\lambda \in \mathrm{K}$, as desired. ∎

Remark B.5. (1) In [69, Ch. 8], Katz has also determined the geometric monodromy group of almost all hypergeometric sheaves. We observe in passing that this computation has recently been used by Fresán and Jossen [44] to construct examples of E-functions that are not related to hypergeometric functions, answering a question raised by Siegel in his fundamental paper [121].

(2) Theorem B.4 is a key ingredient in the proof of the theorem of Gabber and Loeser that determines the group $\mathbf{H}_{\mathrm{int}}(\mathbf{G}^r_{m,\bar{k}})$ of isomorphism classes of objects on $\mathbf{G}^r_{m,\bar{k}}$ with tannakian rank 1,

which is explained in Example 3.51. In fact, it is not very difficult to deduce from Theorem B.4, (3), that the group $\mathbf{H}_{\mathrm{int}}(\mathbf{G}_{m,\bar{k}})$ is isomorphic to

$$\bar{k}^{\times} \times \mathbf{Z}^{\Pi(\mathbf{G}_{m,\bar{k}},\overline{\mathbf{Q}}_{\ell})},$$

where we recall that $\Pi(\mathbf{G}_{m,\bar{k}},\overline{\mathbf{Q}}_{\ell})$ denotes the set of continuous tame characters of $\mathbf{G}_{m,\bar{k}}$ (see Section 1.10).

An isomomorphism Φ between these groups is determined as follows: given $\lambda \in \bar{k}^{\times}$ and a function $f \in \mathbf{Z}^{\Pi(\mathbf{G}_{k,\bar{k}},\overline{\mathbf{Q}}_{\ell})}$, let $\boldsymbol{\chi}$ be the tuple whose distinct elements are the characters χ such that $f(\chi) \geqslant 1$, each repeated with multiplicity $f(\chi)$, and let $\boldsymbol{\varrho}$ be the tuple whose distinct elements are the characters χ such that $f(\chi) \leqslant -1$, each repeated with multiplicity $-f(\chi)$. Then one has

$$\Phi(\lambda, f) = \mathrm{Hyp}_{\lambda}(!, \psi, \boldsymbol{\chi}; \boldsymbol{\varrho}).$$

Conversely, the function f can be recovered from an element M of $\mathbf{H}_{\mathrm{int}}(\mathbf{G}_m)$ by looking at the tame characters appearing in the local monodromy at 0 and ∞, and their multiplicities.

Let now k be a finite field, with ψ a non-trivial additive character of k. Let $\lambda \in k^{\times}$ and let

$$\mathrm{M} = \mathrm{Hyp}_{\lambda}(!, \psi, \boldsymbol{\chi}; \boldsymbol{\varrho})$$

for tuples $\boldsymbol{\chi}$ and $\boldsymbol{\varrho}$ of tame characters associated to multiplicative characters $k^{\times} \to \overline{\mathbf{Q}}_{\ell}^{\times}$ (denoted in the same manner). The trace function of M is then given by

$$t_{\mathrm{M}}(x; k) = (-1)^{m-n} \sum_{\substack{(x_i)\in(k^{\times})^n,\ (y_j)\in(k^{\times})^m \\ x_1\cdots x_n = \lambda^{-1} x y_1 \cdots y_m}} \psi\Bigl(\sum_{i=1}^{n} x_i - \sum_{j=1}^{m} y_j\Bigr) \prod_{i=1}^{n} \chi_i(x_i) \prod_{j=1}^{m} \varrho_j(y_j),$$

with the obvious analogue for finite extensions of k (see [69, (8.2.7)]).

For a multiplicative character $\chi\colon k^{\times} \to \overline{\mathbf{Q}}_{\ell}^{\times}$, let

$$\tau(\psi, \chi) = \sum_{x\in k^{\times}} \psi(x)\chi(x)$$

denote the Gauss sums over k. Then the arithmetic Mellin transform of the hypergeometric complex M is

(B.1)
$$\sum_{x\in k^{\times}} \chi(x) t_{\mathrm{M}}(x; k) = \chi(\lambda) \prod_{i=1}^{m} \tau(\psi, \chi\chi_i) \prod_{j=1}^{n} \tau(\bar{\psi}, \bar{\chi}\bar{\varrho}_j)$$

for $\chi\colon k^{\times} \to \overline{\mathbf{Q}}_{\ell}^{\times}$ (a monomial in Gauss sums; see [69, (8.2.7), (8.2.8)]).

In particular, if $n \geqslant 1$ and $\chi_i = 1$ for all i, and if $\boldsymbol{\varrho}$ is empty and $\lambda = 1$, we obtain the unnormalized *hyper-Kloosterman sums*

$$(-1)^n \sum_{\substack{x_1,\ldots,x_n\in k^{\times} \\ x_1\cdots x_n = x}} \psi(x_1 + \cdots + x_n).$$

The corresponding hypergeometric complex

(B.2)
$$\mathscr{K}\ell_{n,\psi} = \mathrm{Hyp}(!, \psi, (1, \ldots, 1); \varnothing)$$

is called a *Kloosterman complex*; it is of the form $\mathrm{Kl}_n[1]$ for a lisse sheaf Kl_n of rank n on $\mathbf{G}_m$, called the *Kloosterman sheaf* of rank n (see [69, Rem. 8.4.3]).

APPENDIX C

The product formula for epsilon factors

We recall in this Appendix the formula of Laumon [98] for the epsilon factor of an object of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ on a curve X, and recall the main parts of the formalism of local epsilon factors. We also include the general Euler–Poincaré characteristic formula.

C.1. The product formula

The results in this section are quoted directly from [98, § 3].

Let k be a finite field of characteristic p, with k_n/k the extension of k of degree n in an algebraic closure $\bar{k}$ of k.

Let X be a smooth projective curve over k. We denote by [X] the set of closed points of X. For a complex M in $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$, the L-function of M is defined by the product

$$\mathrm{L}(\mathrm{M},\mathrm{T}) = \prod_{x\in[\mathrm{X}]} \det(1-\mathrm{T}^{\deg(x)}\,\mathrm{Fr}_{k_{\deg(x)}} \mid \mathrm{M}_x)^{-1}.$$

It satisfies the relation

$$\mathrm{L}(\mathrm{M},\mathrm{T}) = \det(1-\mathrm{T}\,\mathrm{Fr}_k \mid \mathrm{H}^*(\mathrm{X}_{\bar{k}},\mathrm{M}))^{-1}$$

and the functional equation

$$\mathrm{L}(\mathrm{M},\mathrm{T}) = \varepsilon(\mathrm{M})\mathrm{T}^{a(\mathrm{M})}\mathrm{L}(\mathrm{D}(\mathrm{M}),\mathrm{T}^{-1}),$$

where

$$a(\mathrm{M}) = -\chi(\mathrm{X}_{\bar{k}},\mathrm{M}),\quad \varepsilon(\mathrm{M}) = \det(-\mathrm{Fr}_k \mid \mathrm{H}^*(\mathrm{X}_{\bar{k}},\mathrm{M}))^{-1}.$$

Laumon's product formula, which had been conjectured by Deligne, is an expression for $\varepsilon(\mathrm{M})$ in terms of local epsilon factors.

Consider a fixed non-trivial ℓ-adic additive character ψ of $\mathbf{F}_p$, and denote $\psi_k = \psi\circ\mathrm{Tr}_{k/\mathbf{F}_p}$. Furthermore, consider a fixed non-zero meromorphic 1-form ω on X.

Theorem C.1 (Laumon). *Suppose that* X *is connected. Let g be the common genus of all the connected components of $\mathrm{X}_{\bar{k}}$, and $n\geqslant 1$ the number of these connected components.*

Let M *be an object of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{X})$ of generic rank $r(\mathrm{M})$. There exist specific local constants $\varepsilon_x(\mathrm{M})$, depending on the choice of ω, such that*

$$\varepsilon(\mathrm{M}) = |k|^c \prod_{x\in|\mathrm{X}|}\varepsilon_x(\mathrm{M}) \tag{C.1}$$

where $c = n(1-g)r(\mathrm{M})$.

This is [98, Th. 3.2.1.1], defining (in the notation of loc. cit.) the local factors by

$$\varepsilon_x(\mathrm{M}) = \varepsilon(\mathrm{X}_{(x)},\mathrm{M}|\mathrm{X}_{(x)},\omega|\mathrm{X}_{(x)}). \tag{C.2}$$

C.2. Local epsilon factors

We summarize here the basic identities and formal properties of the local epsilon factors $\varepsilon_x(\mathrm{M})$ in Laumon's Theorem C.1. The existence and uniqueness of these local factors, subject to certain conditions, are given precisely by Laumon in [98, Th 3.1.5.4]; they were defined earlier by Deligne [26].

The local epsilon factors are attached to a triple $(\mathrm{T}, \mathrm{M}, \omega)$, where T is a strictly henselian local ring of equal characteristic with residue field containing k, M is an object of $\mathrm{D}_c^{\mathrm{b}}(\mathrm{T})$ and ω is a non-zero meromorphic 1-form on T.

The notation $\varepsilon(\mathrm{X}_{(x)}, \mathrm{M}|\mathrm{X}_{(x)}, \omega|\mathrm{X}_{(x)})$ in (C.2) refers to these factors with the subscript (x) referring to strict localization at x.

We now recall the local exponents $a(\mathrm{T}, \mathrm{M}, \omega)$ and $a(\mathrm{T}, \mathrm{M})$, which require additional notation (see [98, 3.1.5]):

(1) We denote by v the valuation of T, extended to 1-forms by $v(adb) = v(a)$ if $v(b) = 1$.

(2) We denote by t the closed point of T and by η the generic point.

(3) We denote by $\bar{t}$ (resp. $\bar{\eta}$) a geometric generic point of T above t (resp. above η).

(4) We denote by k_t the residue field of T at t.

(5) For an object M of $\mathrm{D}_c^{\mathrm{b}}(\mathrm{T})$, we denote by $r(\mathrm{M}_{\bar{\eta}})$ (resp. $r(\mathrm{M}_{\bar{t}})$) the generic rank of M (resp. the rank of the stalk at the closed point) and by $s(\mathrm{M}_{\bar{\eta}})$ the Swan conductor; all of these are defined for an étale sheaf first and extended by additivity, see [98, § 2.2.1].

With these notation, the local conductor exponents are defined by the formulas

$$a(\mathrm{T}, \mathrm{M}) = r(\mathrm{M}_{\bar{\eta}}) + s(\mathrm{M}_{\bar{\eta}}) - r(\mathrm{M}_{\bar{t}}), \tag{C.3}$$

$$a(\mathrm{T}, \mathrm{M}, \omega) = a(\mathrm{T}, \mathrm{M}) + r(\mathrm{M}_{\bar{\eta}})v(\omega). \tag{C.4}$$

(see [98, (3.1.5.1), (3.1.5.2)]).

In the global case, we will denote

$$a_x(\mathrm{M}, \omega) = a(\mathrm{X}_{(x)}, \mathrm{M}|\mathrm{X}_{(x)}, \omega|\mathrm{X}_{(x)}).$$

Furthermore, for a lisse $\overline{\mathbf{Q}}_\ell$-sheaf $\mathscr{F}$ on the generic point η of T, one defines

$$\varepsilon_0(\mathrm{T}, \mathscr{F}, \omega) = \varepsilon(\mathrm{T}, j_!\mathscr{F}, \omega), \tag{C.5}$$

where $j\colon \{\eta\} \to \mathrm{T}$ is the open immersion (see [98, 3.1.5.6, p. 187]).

For a short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, we have

$$\varepsilon_0(\mathrm{T}, \mathscr{F}, \omega) = \varepsilon_0(\mathrm{T}, \mathscr{F}', \omega)\varepsilon_0(\mathrm{T}, \mathscr{F}'', \omega). \tag{C.6}$$

The local epsilon factors satisfy (among other things) the following properties (see, respectively, formula (3.1.5.6), formula (3.1.5.5) and section 3.5.3.1 in [98]):

(1) For any lisse $\overline{\mathbf{Q}}_\ell$-sheaf $\mathscr{F}$ of rank r on T, the formula

$$\varepsilon(\mathrm{T}, \mathrm{M} \otimes \mathscr{F}, \omega) = \det(\mathrm{Fr} \mid \mathscr{F})^{a(\mathrm{T},\mathrm{M},\omega)}\varepsilon(\mathrm{T}, \mathrm{M}, \omega)^r \tag{C.7}$$

holds, where Fr denotes the geometric Frobenius automorphism at the closed point t of T.

(2) For a non-zero rational function a on T, the formula

$$\varepsilon(\mathrm{T}, \mathrm{M}, a\omega) = \chi(a)|k_t|^{r(\mathrm{M}_{\bar{\eta}})v(a)}\varepsilon(\mathrm{T}, \mathrm{M}, \omega) \tag{C.8}$$

holds, where χ is the character of the completion of the residue field at η associated, by local class field theory,[1] to the lisse sheaf $\det(\mathrm{M})|\eta$ on η, viewed as a character of the local Galois group.

(3) For a non-trivial multiplicative character χ of the residue field k_t and the corresponding lisse Kummer sheaf $\mathscr{L}_\chi$ on $\{\eta\}$, and for a uniformizer π at x, we have

$$\varepsilon_0(\mathrm{T}, \mathscr{L}_\chi, d\pi) = \chi(-1) \sum_{a \in k_t^\times} \chi(a)\psi(\mathrm{Tr}_{k_t/\mathbf{F}_p}(a)). \tag{C.9}$$

We also have the elementary shift formula

$$\varepsilon(\mathrm{T}, \mathrm{M}[1], \omega) = \varepsilon(\mathrm{T}, \mathrm{M}, \omega)^{-1}. \tag{C.10}$$

C.3. The Euler–Poincaré characteristic formula

We keep the notation of Section C.1. In particular, X is a smooth projective curve over a finite field k with algebraic closure $\bar{k}$. We assume that X is geometrically connected, and denote by g the genus of X.

Let M be a complex in $\mathrm{D}_c^{\mathrm{b}}(\mathrm{X})$. For any point $x \in \mathrm{X}(\bar{k})$, the Swan conductor $\mathrm{swan}_x(\mathrm{M})$ is defined by additivity from the case of a $\overline{\mathbf{Q}}_\ell$-sheaf (in which case, it is defined for instance in [98, (2.1.2.5)] or [68, Ch. 1]). Similarly, the drop $\mathrm{drop}_x(\mathrm{M})$ is defined by additivity from the drop

$$\mathrm{drop}_x(\mathscr{F}) = \mathrm{rank}(\mathscr{F}) - \dim(\mathscr{F}_x)$$

of a $\overline{\mathbf{Q}}_\ell$-sheaf $\mathscr{F}$.

THEOREM C.2 (Grothendieck–Ogg–Shafarevich). *Let* $\mathrm{U} \subset \mathrm{X}$ *be an open dense subset. Let* M *be a complex in* $\mathrm{D}_c^{\mathrm{b}}(\mathrm{U})$*, let* V *be an open dense subset of* U *on which* M *is lisse of generic rank* $r(\mathrm{M})$*.*

We have

$$\chi(\mathrm{U}_{\bar{k}}, \mathrm{M}) = \chi(\mathrm{U}_{\bar{k}}, \overline{\mathbf{Q}}_\ell) r(\mathrm{M}) - \sum_{x \in \mathrm{X}(\bar{k})} \mathrm{swan}_x(\mathrm{M}) - \sum_{x \in \mathrm{U}(\bar{k})} \mathrm{drop}_x(\mathrm{M}),$$

where $\chi(\mathrm{U}_{\bar{k}}, \overline{\mathbf{Q}}_\ell) = (2-2g) - |(\mathrm{X} - \mathrm{U})|$.

This statement follows from [98, Th. 2.2.1.2], which corresponds to X = U (up to changes in notation) by applying this result to $j_*\mathrm{M}$, where $j\colon \mathrm{U} \to \mathrm{X}$ is the open immersion, and using the additivity of the Euler–Poincaré characteristic, in the sense that

$$\chi(\mathrm{X}_{\bar{k}}, j_*\mathrm{M}) = \chi(\mathrm{U}_{\bar{k}}, \mathrm{M}) + \chi((\mathrm{X} - \mathrm{U})_{\bar{k}}, i^* j_* \mathrm{M})$$

with i the closed immersion of $\mathrm{X} - \mathrm{U}$ in X.

For the case of a $\overline{\mathbf{Q}}_\ell$-sheaf, the statement is also given for instance in [74, Ch. 14].

We consider some special cases that appear in this book.

(1) If U = X and $\mathrm{M} = \mathscr{F}[1]$ for some $\overline{\mathbf{Q}}_\ell$-sheaf $\mathscr{F}$ of generic rank r on X, then the formula becomes

$$\chi(\mathrm{X}_{\bar{k}}, \mathrm{M}) = (2g-2)r + \sum_{x \in \mathrm{X}(\bar{k})} (\mathrm{swan}_x(\mathscr{F}) + \mathrm{drop}_x(\mathscr{F})). \tag{C.11}$$

(2) If $\mathrm{U} = \mathbf{G}_m \subset \mathrm{X} = \mathbf{P}^1$ and $\mathrm{M} = \mathscr{F}[1]$ for some $\overline{\mathbf{Q}}_\ell$-sheaf $\mathscr{F}$ of generic rank r on $\mathbf{G}_m$, then

$$\chi((\mathbf{G}_m)_{\bar{k}}, \mathrm{M}) = \mathrm{swan}_0(\mathscr{F}) + \mathrm{swan}_\infty(\mathscr{F}) + \sum_{x \in \bar{k}^\times} (\mathrm{swan}_x(\mathscr{F}) + \mathrm{drop}_x(\mathscr{F})). \tag{C.12}$$

[1] Normalized as explained in [98, (3.1.4)].

APPENDIX D

Deligne's letter to Kazhdan

We reproduce below the content of Deligne's letter to Kazhdan, in which the ℓ-adic Fourier transform was defined for the first time (the typography is not faithfully reproduced).

29-11-76

Dear Каждан,

This is perhaps a partial answer to an old letter of yours. I thought to the matter again because of some estimations of trigonometrical sums Hooley asked me about. As I am in a hurry to continue writing up Weil II, I will leave many open ends and soon turn to French.

Theme: many functions correspond to sheaves, and operations on functions to operations on sheaves. What about harmonic analysis on $\mathbb{G}_a$?

ⓐ If X is a scheme $/\mathbb{F}_q$, we will consider

α) objects of the derived category $\mathrm{D}^b(\mathrm{X}, \overline{\mathbb{Q}}_\ell)$
$\downarrow$ by $\sum(-1)^i\mathrm{H}^i$

β) virtual ℓ-adic sheaves [this means either: elements of the Grothendieck group of the abelian category of constructible sheaves — or if possible and useful, objects of some Picard category having this K^0 as set of isomorphism classes of objects]
$\downarrow$ by $\mathrm{Tr}(\mathrm{F}_x^*, \mathscr{F}_{\bar{x}})$ (this map is injective)

γ) "functions": a system of functions on the $\mathrm{X}(\mathbb{F}_{q^n})$

ⓑ Here are corresponding operations:

On functions: $+, \cdot, \sum$ On $\alpha), \beta)$: $\oplus, \otimes, \mathrm{R}\pi_!$

Convolution of functions: if G is a group, and $\mathrm{K}, \mathrm{L} \in \mathrm{D}^b(\mathrm{X}, \overline{\mathbb{Q}}_\ell)$, one considers the product

$$\pi\colon \mathrm{G} \times \mathrm{G} \to \mathrm{G}, \quad \text{and} \quad \mathrm{K} \boxtimes \mathrm{L} = \mathrm{pr}_1^*\mathrm{K} \otimes \mathrm{pr}_2^*\mathrm{L}, \text{ and}$$
$$\mathrm{K} * \mathrm{L} = \mathrm{R}\pi_!(\mathrm{K} \boxtimes \mathrm{L})$$

Kernel: given $\mathrm{Z} \to \mathrm{X} \times \mathrm{Y}$ and $\mathrm{K} \in \mathrm{D}^b(\mathrm{Z}, \overline{\mathbb{Q}}_\ell)$, this defines an operation $\mathrm{D}^b(\mathrm{X}) \to \mathrm{D}^b(\mathrm{Y})$
$$\mathrm{L}_\mathrm{X} \mapsto \mathrm{Rp}_{2!}(\mathrm{K} \otimes \mathrm{Rp}_1^*\mathrm{L}_\mathrm{X}).$$

ⓒ Now I want to consider Fourier transform.

Let us choose $\psi\colon \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^*$. If f is a function on X, we get a sheaf $\mathscr{F}(\psi f)$.[1] Fourier transform, on $\mathbb{G}_a$, is given by the kernel $\mathscr{F}(\psi(xy))$ on $\mathbb{G}_a \times \mathbb{G}_a$.

Definition: $\underline{\mathrm{F}}(\mathrm{K}) = \mathrm{Rpr}_{2!}(\mathscr{F}(\psi(xy)) \otimes \mathrm{Rpr}_1^*\mathrm{K})$

Proposition 1: $\underline{\mathrm{F}}(\mathrm{K} * \mathrm{L}) = \underline{\mathrm{F}}(\mathrm{K}) \otimes \underline{\mathrm{F}}(\mathrm{L})$ (from $\mathscr{F}(\psi(x(y' + y''))) = \mathscr{F}(\psi(xy')) \otimes \mathscr{F}(\psi(xy'')))$

[1] Pull back by f of the sheaf on $\mathbb{G}_a$, rank 1, defined by ψ and Artin–Schreier $\mathrm{T}^p - \mathrm{T} = \mathrm{X}$.

Proposition 2: $\underline{\mathrm{FF}}(\mathrm{K}) = \mathrm{K}^\vee(-1)[-2]$: $\quad\vee$ is for "image by $x \mapsto -x$", (-1) for a Tate twist, and $[-]$ for décalage.

Kernels compose like expected: we have to compute $\mathrm{R}\pi_! \mathscr{F}(\psi(x+z)y)$ for $\pi\colon \mathbb{G}_a \times \mathbb{G}_a \times \mathbb{G}_a \xrightarrow{(13)} \mathbb{G}_a \times \mathbb{G}_a$, one gets

$$\begin{cases} \mathbb{Q}_\ell(-1) & \text{on the diagonal, in degree 2} \\ 0 & \text{elsewhere} \end{cases}$$

hence the result.

It is convenient in such computations to forget writing ψ and writing $\int \cdots dy$ for a $\mathrm{R}\pi_!$.

Remark: this defines, via prop 1, an isomorphism $\underline{\mathrm{F}}(\mathrm{K} \otimes \mathrm{L})(-1)[-2] = \underline{\mathrm{F}}(\mathrm{K}) * \underline{\mathrm{F}}(\mathrm{L})$.

For Plancherel formula, one suffer somewhat of not having complex conjugation. Let $\overline{\mathrm{F}}$ be F defined using $\psi(-x)$. Then

a) inner product: $\langle \mathrm{K}, \mathrm{L} \rangle = \mathrm{R}\Gamma(\mathrm{K} \otimes \mathrm{L})$

b) Proposition $\langle \mathrm{FK}, \overline{\mathrm{F}}\mathrm{L} \rangle = \langle \mathrm{K}, \mathrm{L} \rangle(-1)[-2]$.

This boils down to the usual

$$\int \psi((x'-x'')y)\mathrm{K}(x)\mathrm{L}(x'')dx'dx''dy \underset{\substack{\uparrow \\ \text{by } \int_y}}{=} \int \delta^{(-1)[-2]}(x'-x'')\mathrm{K}(x')\mathrm{L}(x'')dx'dx''.$$

Everything done above can be generalized to any abelian connected unipotent group U. The dual U^* is to be taken in Serre's sense (it is natural only up to inseparable isogenies, but this does not matter. For n large enough, one has a pairing

$$\mathrm{U} \times \mathrm{U}^* \xrightarrow{\cdot} \mathrm{W}_n$$

(better: the pairing is in the cowitt vectors $\mathrm{W}_{-\infty} = \varinjlim_{\text{by V}} \mathrm{W}_n$). Given $\psi\colon \mathbb{Q}_p/\mathbb{Z}_p = \mathrm{W}_{-\infty}(\mathbb{F}_p) \to \overline{\mathbb{Q}}_\ell^*$, and using the sheaf given by the Lang covering of $\mathrm{W}_{-\infty}/\mathbb{F}_p$ and ψ, everything can be repeated, with $(-1)[-2]$ replaced by $(-d)[-2d]$ where d is the dimension.

This requires to be careful if one wants to consider $\mathbb{Q}_p$ as a (ind pro quasi) unipotent algebraic group $/\mathbb{F}_p$.

d Where $\underline{\mathrm{F}}$ is, there should also be an action of the metaplectic group! (here symplectic). Let me work for $\mathbb{G}_a$, and for $p \neq 2$. The most precise way of speaking I see is working over $\mathbb{F}_p$, with kernels. [It gives more than actions of $\mathrm{SL}(2,k)$ on $\mathrm{D}^b(\mathbb{G}_a, \mathbb{Q}_\ell), k/\mathbb{F}_q$.]

Wanted: $\mathrm{P} \in \mathrm{D}^b(\mathrm{SL}(2) \times \mathbb{G}_a \times \mathbb{G}_a)$, viewed as a family of kernels on $\mathbb{G}_a \times \mathbb{G}_a$ parametrized by SL(2). Plus "$\mathrm{P}_{g'} \cdot \mathrm{P}_{g''} = \mathrm{P}_{g' \cdot g''}$"

We know what is wanted for generators:

U^-	$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$	$\mapsto$	$(\ \ \otimes \mathscr{F}(\psi(\frac{ax^2}{2}))$	(noyau sur la diagonale)
H	$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$	$\mapsto$	$(x \to \lambda x)_*(\ \ )$	(noyau sur $y = \lambda x$)
$a \neq 0$ U_0^+	$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$	$\mapsto$	$\left(\int_x \mathscr{F}\psi(\frac{a^{-1}x^2}{2})\right)^{-1} \mathscr{F}\psi(\frac{a^{-1}x^2}{2}) *$	(noyau: faisceau loc $\mathrm{c}^{\underline{t}}$, de rg 1, en degré -1)

An explanation: $\mathrm{R}\Gamma\mathscr{F}\psi(a^{-1}\frac{x^2}{2})$ is of dimension 1, and degree 1, and I take the dual one dimensional vector space [in a: a sheaf], in degree -1.

En français:

Raisonnons un peu a priori. Comme “fonctions”, on sait ce que sont les noyaux cherchés. On cherche des faisceaux leur donnant naissance. Sur $\mathrm{U}^- \times \mathrm{H} \times \mathrm{U}_0^+ \times \mathrm{U}^- \times \mathbb{G}_a \times \mathbb{G}_a$, composant les générateurs, on trouve un faisceau localement constant de rang 1, placé en degré -1, qui convient. En chaque point de $\mathrm{U}^- \times \mathrm{H} \times \mathrm{U}_0^+ \times \mathrm{U}^-$, comme fonction de x, y, il est de la forme ψf, pour f une fonction qui, en x, y (sur $\mathbb{G}_a \times \mathbb{G}_a$) est quadratique homogène. Regardons la surjection

$$\mathrm{U}^- \times \mathrm{H} \times \mathrm{U}_0^+ \times \mathrm{U}^- \longrightarrow \mathrm{G} - \mathrm{B}^- \qquad\qquad (\mathrm{G} = \mathrm{SL}(2), \mathrm{B}^- = \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right))$$

Puisque comme “fonctions” ce que nous cherchons existe, le faisceau obtenu est constant sur les fibres de (cette application $\times \mathbb{G}_a \times \mathbb{G}_a$).

<u>Obtenu</u>: un faisceau de rang 1, en degré -1, localement constant, sur $(\mathrm{G} - \mathrm{B}^-) \times \mathbb{G}_a \times \mathbb{G}_a$.

Pour compléter ce tableau, il est bon de comprendre en quel sens, pour $a \to 0$, on a

$$\Big(\int \mathscr{F}\psi(a^{-1}\frac{x^2}{2})dx\Big)^{-1} \cdot \quad \mathscr{F}\psi(a^{-1}\frac{x^2}{2}) \quad \longrightarrow \quad \delta \quad (\text{faisceau } \mathbb{Q}_\ell \text{ en } x = 0)$$

[où]

$$\int \mathscr{F}\psi(a^{-1}\frac{x^2}{2})dx$$

est un faisceau de rang 1 (degré -1) sur la droite de a; ce faisceau se trivialise sur le revêtement de la droite de a donné par $\sqrt{a}$, car

$$\int \mathscr{F}\psi(a^{-2}\frac{x^2}{2})dx = \int \mathscr{F}\psi(\frac{(a^{-1}x)^2}{2}dx = \int \mathscr{F}\psi(\frac{x^2}{2})dx \quad \text{par ch}^{\underline{\text{nt}}} \text{ de variable}$$

Il correspond à une somme de Gauss; sur $\int \cdots$, $\quad |\text{Frobenius}| = q^{1/2}$.
Traçons le plan a, x ; le faisceau considéré est défini pour $a \neq 0$; il se ramifie (sauvagement) le long de $a = 0$, et la ramification est équisingulière pour $x \neq 0$. Si j est l’inclusion de $a \neq 0$ dans le plan, on a

$$\begin{cases} j_*(\text{faisceau}) = j_!(\text{faisceau}) & \text{nul pour } a = 0 \\ \mathrm{R}^1 j_*(\text{faisceau}) & \text{concentré en } (0,0), \text{ où c'est } \delta \\ \mathrm{R}^2 j_*(\text{faisceau}) & \text{concentré en } (0,0) \end{cases}$$

Ceci se vérifie assez facilement en éclatant 2 fois $(0,0)$, la 2ème fois en éclatant (courbe exceptionnelle) $\cap$ (transformé pur de l’axe des x) : on utilise

$$\begin{tikzcd} & (\text{plan éclaté}) \arrow[dr, "\pi"] & \\ (a \neq 0) \arrow[ur, hook, "\widetilde{j}"] \arrow[rr, hook] & & (\text{plan}) \end{tikzcd} \qquad \mathrm{R}j_* = \mathrm{R}\pi_*\mathrm{R}\widetilde{j}_*$$

On contrôle en projetant sur la droite des a : si p_a est cette projection, on a

$$\mathrm{R}p_{a*}\mathrm{R}j_* = \mathrm{R}j_*\mathrm{R}p_{a*}$$

[où] $\mathrm{R}p_{a*}$ donne $\mathbb{Q}_\ell$ sur la droite a, $-\{0\}$, et $\mathrm{R}j_*$ sur cette droite $(a \neq 0) \hookrightarrow a = 0$, donc

$$\begin{cases} \text{deg. } 0 & \mathbb{Q}_\ell \\ \text{deg. } 1 & \mathbb{Q}_\ell(-1) \text{ en } 0. \end{cases}$$

Ceci nous dit ce que nous devons faire pour construire P:

(a) sur U^+, le noyau s'obtient à partir de

$$\tau_{\leqslant 0}\Big(\mathrm{R}j_*\Big(\Big(\int \mathscr{F}\psi(a^{-1}\frac{x^2}{2})dx\Big)^{-1}\mathscr{F}(\psi(a^{-1}\frac{x^2}{2}))\Big)$$

sur $\mathrm{U}^+ \times \mathbb{G}_a$, comme convolution.

(b) sur $\mathrm{G} \times \mathbb{G}_a \times \mathbb{G}_a$, on prend le noyau déjà construit sur $(\mathrm{G} - \mathrm{B}^-) \times \mathbb{G}_a \times \mathbb{G}_a$, et pour j l'inclusion dans $\mathrm{G} \times \mathbb{G}_a \times \mathbb{G}_a$, on lui applique $\tau_{\leqslant 0}\mathrm{R}j_*$.

Je me suis convaincu que la formule $\mathrm{P}_g \cdot \mathrm{P}_{g'} = \mathrm{P}_{gg'}$ vaut au sens le plus fort possible:

a) sur $\mathrm{G} \times \mathrm{G} \times \mathbb{G}_a \times \mathbb{G}_a$, on prend $\mathrm{P}_{g''}(y,z)\mathrm{P}_{g'}(x,y)$.

b) on intègre par rapport à y: $(\mathrm{P} \cdot \mathrm{P})_{g',g''} = \int dy \cdots$ sur $\mathrm{G} \times \mathrm{G} \times \mathbb{G}_a \times \mathbb{G}_a$

c) si π est $\mathrm{G} \times \mathrm{G} \to \mathrm{G}$: $g', g'' \mapsto g'g''$, on a un isomorphisme

$$(\mathrm{P} \cdot \mathrm{P}) = \pi^*\mathrm{P}$$

d) on a une compatibilité pour un composé triple [en c), on a unicité à une constante près, et on normalise par ce qui se passe à l'origine].

Bien sûr, tout ceci devrait valoir pour un espace vectoriel V, et $\mathrm{Sp}(\mathrm{V} \oplus \mathrm{V}^*)$. Il est facile de se convaincre qu'on a en tout cas un noyau $\mathrm{P}_g(v, v')$ qui est un faisceau virtuel, et que sur la cellule des $g \in \mathrm{Sp}$ où $g\mathrm{V}^* \cap \mathrm{V}^* = 0$, il est donné de façon naturelle par un faisceau de rang 1, localement constant, en degré $-n$. J'espère que le noyau lui-même s'en déduit par une suite d'opérations $\tau_{\leqslant} j_*$, avec un résultat localement constant de rang 1 sur un sous-espace, en degré $-k$, sur la strate $\dim(\mathrm{V}^*/\mathrm{V}^* \cap g\mathrm{V}^*) = k$... (qu'on ait un noyau ainsi stratifié doit pouvoir se vérifier par Fourier).

<u>Question</u>: Le foncteur $\mathrm{K} \mapsto (x \mapsto -x)_*\underline{\mathrm{RHom}}(\mathrm{K}, \mathbb{Q}_\ell)$ commute-t-il à l'action de SL(2) ?

<u>Question bis</u>: pour P_g le noyau, et K sur $\mathbb{G}_a$, a-t-on

$$\mathrm{Rpr}_{2!}(\mathrm{P}_g \otimes \mathrm{pr}_1^*\mathrm{K}) \xrightarrow{\sim} \mathrm{Rpr}_{2*}(\mathrm{P}_g \otimes \mathrm{pr}_1^*\mathrm{K}) \quad ?$$

<u>Question ter</u>: y commute-t-il virtuellement – au moins virtuellement sur $\overline{\mathbb{F}}$?

Bien à toi,

P. Deligne

APPENDIX E

Intuition for analytic number theorists

The goal of this informal appendix is to provide readers who have a background in analytic number theory with some intuition and feeling for objects such as ℓ-adic complexes, perverse sheaves, or tannakian categories, all of which are essential tools in this book.

The focus here concerns trace functions of *more than one variable*. On the other hand, the theory of trace functions in *one* variable is more accessible, as the algebraic objects can be presented more concretely using Galois theory of function fields. Some familiarity with this point of view will certainly also be very helpful in developing intuition. A very concise introduction can be found in the Pisa survey of Fouvry, Kowalski and Michel [40], and a more detailed treatment is contained in the lectures of Michel at the 2016 Arizona Winter School [42].

We fix a finite field k, and denote by k_n the extension of k of degree n inside a fixed algebraic closure $\bar{k}$. For simplicity of notation, we will mostly speak about trace functions on the affine space $\mathbf{A}^m$ for some integer $m \geqslant 0$. However, it will be implicit that most of what we discuss can be done for any algebraic variety Y over k (and this is needed, for instance because we often naturally wish to restrict a trace function to a subvariety, where some particular property holds), for instance for powers of the multiplicative group $\mathbf{G}_m$ (i.e., Y such that $\mathrm{Y}(k_n) = (k_n^\times)^d$ for some $d \geqslant 0$). The reader should keep in mind that for such a subvariety, of dimension $d \leqslant m$, the size of the finite set $\mathrm{Y}(k_n)$ of points of Y with coordinates in k_n is approximately $|k_n|^d$ when n is large.

Throughout, we fix a non-trivial additive character $\psi\colon k \to \mathbf{C}^\times$ and, for $n \geqslant 1$, we define

$$\begin{aligned} \psi_n\colon k_n &\longrightarrow \mathbf{C}^\times \\ x &\longmapsto \psi(\mathrm{Tr}_{k_n/k}(x)). \end{aligned}$$

We finally note that we will completely ignore (here) the distinction between $\overline{\mathbf{Q}}_\ell$ and $\mathbf{C}$.

E.1. Trace functions

The concrete origin for the use of methods of algebraic geometry and étale cohomology in analytic number theory lies in trace functions, and especially in exponential sums. Properly speaking, a trace function on $\mathbf{A}^m$ is the data of a family $(t_n)_{n\geqslant 1}$ of functions $k_n^m \to \mathbf{C}$, and it is associated to some algebraic object M, which we call a "coefficient object". This object is not uniquely determined by (t_n), but we will not worry about this matter in this appendix.

The first examples of trace functions arise from polynomials $f \in k[\mathrm{X}_1, \ldots, \mathrm{X}_m]$ by means of

$$t_n(x_1, \ldots, x_m) = \psi_n(f(x_1, \ldots, x_m)); \tag{E.1}$$

the corresponding coefficient object is denoted by $\mathscr{L}_{\psi(f)}$. Many other examples are then obtained by applying various operations, which are known to preserve the set of trace functions (these are operations on the coefficient objects, which are reflected in a specific operation at the level of trace functions). These operations include the following, where we indicate the algebraic notation for the corresponding coefficient objects:

– The constant function 1 is associated to the coefficient object $\mathrm{M} = \overline{\mathbf{Q}}_\ell$.

– The sum of the trace functions associated to M_1 and M_2 is associated to $\mathrm{M}_1 \oplus \mathrm{M}_2$.

– If (t_n) is a trace function associated to M, then $((-1)^k t_n)$ is a trace function for each integer $k \in \mathbf{Z}$, associated to a coefficient denoted by $\mathrm{M}[k]$ and called a "shift" of M.

– If (t_n) is a trace function associated to M, then $(|k_n|^r t_n)$ is a trace function for each integer $r \in \mathbf{Z}$, associated to a coefficient denoted by $\mathrm{M}(-r)$ and called a "(Tate) twist" of M.

– The product of the trace functions associated to M_1 and M_2 is associated to $\mathrm{M}_1 \otimes \mathrm{M}_2$.

– If $f = (f_1, \ldots, f_d) \colon \mathbf{A}^m \to \mathbf{A}^d$ is a tuple of polynomials in $k[\mathrm{X}_1, \ldots, \mathrm{X}_m]$, and $s = (s_n)$ is a trace function on $\mathbf{A}^d$ associated to a coefficient N, then

$$t_n(x_1, \ldots, x_m) = s_n(f(x_1, \ldots, x_m))$$

defines a trace function (t_n) on $\mathbf{A}^m$, which we also denote by $s \circ f$. The corresponding coefficient is $f^*\mathrm{N}$.

– If $f = (f_1, \ldots, f_d) \colon \mathbf{A}^m \to \mathbf{A}^d$ is a tuple of polynomials in $k[\mathrm{X}_1, \ldots, \mathrm{X}_m]$, and $t = (t_n)$ is a trace function on $\mathbf{A}^m$, associated to a coefficient object M, then

$$s_n(y_1, \ldots, y_d) = \sum_{\substack{x \in k_n^m \\ f(x) = y}} t_n(x) \tag{E.2}$$

defines a trace function on $\mathbf{A}^d$; the associated coefficient object is denoted by $\mathrm{R}f_!\mathrm{M}$.

Example E.1 (Fourier transform). This formalism is already sufficient to explain Deligne's Fourier transform. Let $m \geqslant 1$ be an integer, and consider the projections

$$p_1,\ p_2 \colon \mathbf{A}^{2m} \to \mathbf{A}^m$$

given by

$$p_1(x_1, \ldots, x_m, y_1, \ldots, y_m) = (x_1, \ldots, x_m), \qquad p_2(x_1, \ldots, x_m, y_1, \ldots, y_m) = (y_1, \ldots, y_m).$$

We write

$$\mathrm{X} \cdot \mathrm{Y} = \mathrm{X}_1\mathrm{Y}_1 + \cdots + \mathrm{X}_m\mathrm{Y}_m$$

for variables X_i and Y_j. This is a polynomial with coefficients in k, so the functions

$$\mathrm{F}_n(x, y) = \psi_n(x_1y_1 + \cdots + x_my_m)$$

define a trace function $\mathrm{F} = (\mathrm{F}_n)$ on $\mathbf{A}^{2m}$, associated to the coefficient object $\mathscr{L}_{\psi(\mathrm{X}\cdot\mathrm{Y})}$.

Let $t = (t_n)$ be a trace function on $\mathbf{A}^m$ with coordinates $(x_1, \ldots, x_m)$. Then the discrete Fourier transforms $(\widehat{t}_n)$, which are defined for $n \geqslant 1$ and $y \in k_n^m$ by

$$\widehat{t}_n(y) = \sum_{x \in k_n^m} t_n(x)\mathrm{F}_n(x, y) = \sum_{x \in k_n^m} t_n(x)\psi_n(x \cdot y),$$

also define a trace function $\widehat{t} = (\widehat{t}_n)$. Indeed, for any y, the set of all $x \in k_n^m$ can be identified with the set of $(x, y) \in k_n^{2m}$ such that $p_2(x, y) = y$, and we have $t_n(x) = t_n(p_1(x, y))$, so that if t is associated to the coefficient object M, then the formalism above shows that $\widehat{t}$ is associated to

$$\widehat{\mathrm{M}} = \mathrm{R}p_{2!}(p_1^*\mathrm{M} \otimes \mathscr{L}_{\psi(\mathrm{X}\cdot\mathrm{Y})}).$$

E.2. Weights and purity: lisse sheaves

The formalism of trace functions is useful in analytic number theory *because* of Deligne's Riemann hypothesis over finite fields. This also leads to some understanding of the important qualitative differences between various types of trace functions—corresponding to classes of coefficients which may (for instance) be lisse sheaves, constructible sheaves, complexes of constructible sheaves, or perverse sheaves. We will try in this and the following sections to provide the readers with some intuition of the concrete meaning of these notions.

The key concept (due to Deligne) is that of a coefficient M which is *punctually pure*, or *pure*, of some weight $w \in \mathbf{Z}$. The main conceptual difficulty is that the meaning of this property for the corresponding trace function is not straightforward in general.

The simplest case (from which the others will be derived) is that of M which is a single "lisse sheaf". *In that case*, the concrete meaning[1] of M being *punctually pure of weight* w, in terms of the trace function $t = (t_n)$, is that there exist

- an integer $r \geqslant 0$, the *rank* of M,
- for each $n \geqslant 1$ and $x \in k_n^m$, a unitary matrix $\Theta_{\mathrm{M}}(x; k_n) \in \mathbf{U}_r(\mathbf{C})$, well-defined up to conjugacy,

so that the following equality holds:

$$t_n(x) = |k_n|^{w/2} \operatorname{Tr}(\Theta_{\mathrm{M}}(x; k_n)).$$

In particular, note that this implies the estimate

$$|t_n(x)| \leqslant r|k_n|^{w/2}$$

for all n and $x \in k_n^m$.

In the remainder of this appendix, we will sometimes say that a lisse sheaf, or its trace function, is "pure" instead of the more correct "punctually pure".

Remark E.2. The matrix $\Theta_{\mathrm{M}}(x; k_n)$ is not arbitrary in $\mathbf{U}_r(\mathbf{C})$. For instance, its eigenvalues (which of course determine the trace) are Weil numbers of weight 0, i.e., algebraic numbers in $\mathbf{C}$ for which all Galois conjugates have modulus 1. Moreover, if n' is a multiple of n, then $x \in k_n^m$ can also be viewed as an element of $k_{n'}^m$ through the inclusion $k_n \subset k_{n'}$, and the formula

$$\Theta_{\mathrm{M}}(x; k_{n'}) = \Theta_{\mathrm{M}}(x; k_n)^{n'/n}$$

holds (i.e., the eigenvalues of the matrix $\Theta_{\mathrm{M}}(x; k_{n'})$ are those of $\Theta_{\mathrm{M}}(x; k_n)$ raised to the power n'/n).

As one can expect, the trace functions defined by the formulas (E.1), associated to $\mathscr{L}_{\psi(f)}$, are of this type, with $r = 1$, $w = 0$, and the matrix $\Theta(x; k_n)$ reduced to the single complex number of modulus one $\psi_n(f(x))$. Moreover, it is also intuitively clear (and true) that some of the operations discussed above will respect the special class of trace functions associated to pure lisse sheaves.

For instance:

- If t and t' are trace functions associated to objects M and N which are both lisse sheaves pure of (the same) weight w, then $t + t'$ is also pure of weight w; we have

$$\Theta_{\mathrm{M}\oplus\mathrm{N}}(x; k_n) = \Theta_{\mathrm{M}}(x; k_n) \oplus \Theta_{\mathrm{N}}(x; k_n).$$

[1] But not exactly the precise definition.

– If t and t' are trace functions associated to objects M and N which are both lisse sheaves pure of weights w and w', respectively, then tt' is also pure of weight $w + w'$. In other words, $\mathrm{M} \otimes \mathrm{N}$ is still a lisse sheaf, pure of that weight; in fact, we have

$$\Theta_{\mathrm{M}\otimes\mathrm{N}}(\mathrm{X}; k_n) = \Theta_{\mathrm{M}}(x; k_n) \otimes \Theta_{\mathrm{N}}(x; k_n).$$

– If $f = (f_1, \ldots, f_d) \colon \mathbf{A}^m \to \mathbf{A}^d$ is a tuple of polynomials in $k[\mathrm{X}_1, \ldots, \mathrm{X}_m]$, and s is a trace function on $\mathbf{A}^d$ associated to a lisse sheaf of weight w, then $s \circ f$ is also pure of weight w. In other words, $f^*\mathrm{N}$ is still a lisse sheaf, pure of weight w; in fact, we have

$$\Theta_{f^*\mathrm{N}}(x; k_n) = \Theta_{\mathrm{N}}(f(x); k_n).$$

But elementary examples show that the crucially important operation of "summing over the fiber" (see (E.2)) does not always send a single lisse sheaf to a lisse sheaf, and may also not map a trace function which is pure of some weight to another one.

Example E.3. (1) Let $m = d = 1$ and $f \in k[\mathrm{X}]$ a polynomial of degree 2, viewed as a map from $\mathbf{A}^1$ to itself. We consider the trace function (t_n) with $t_n(x) = \psi_n(x)$, associated to the lisse sheaf $\mathscr{L}_{\psi(\mathrm{X})}$ (of weight 0), and the trace function (s_n) defined by

$$s_n(x) = \sum_{\substack{y \in k_n \\ f(y) = x}} t_n(y) = \sum_{\substack{y \in k_n \\ f(y) = x}} \psi_n(y),$$

for $n \geqslant 1$ and $x \in k_n$, which is associated to the coefficient object $\mathrm{R}f_!\mathscr{L}_{\psi(\mathrm{X})}$. For most x, the value of $s_n(x)$ is either 0 (if $f(y) = x$ has no solutions in k_n) or a sum of two roots of unity, but for the single point $x_0 = f(y_0)$, where y_0 is the unique zero of the derivative of f, the value $s_n(x_0)$ is a single root of unity (note that y_0, and hence x_0, belongs to k, so it also belongs to k_n for all n, but the value of $s_n(x_0)$ does vary with n).

(2) We consider $m = 2$ and the trace function (t_n) defined by $t_n(x, y) = \psi_n(xy^2)$ for $(x, y) \in k_n^2$. It is associated to the coefficient object $\mathscr{L}_{\psi(\mathrm{XY}^2)}$, which is pure of weight 0. Let $d = 1$ and $f = \mathrm{X}$. Then $\mathrm{R}f_!\mathscr{L}_{\psi(\mathrm{XY}^2)}$ has the trace function (s_n) such that

$$s_n(x) = \sum_{y \in k_n} \psi_n(xy^2) = \begin{cases} \text{a quadratic Gauss sum} & \text{if } x \neq 0, \\ |k_n| & \text{if } x = 0. \end{cases}$$

Neither of these examples of trace functions are associated to a single punctually pure lisse sheaf. However, it turns out that the underlying reason is not the same. In Example (1), the issue is that (s_n) is associated to a single constructible sheaf which is "not lisse" at the point x_0. In Example (2), the issue is that (s_n) is associated to a "complex" of constructible sheaves, i.e., not to a single sheaf.

E.3. Weights and purity: constructible sheaves and complexes

In fact, the most general source of trace functions are *(bounded) mixed complexes of constructible sheaves*. We now try to outline the concrete interpretation of these more general conditions.

The first step goes from a single lisse sheaf to a *single constructible sheaf*. Such a sheaf is *(punctually) pure of weight w* if there is a "stratification"

$$\varnothing = \mathrm{X}_0 \subset \mathrm{X}_1 \subset \cdots \subset \mathrm{X}_q = \mathbf{A}^m$$

of $\mathbf{A}^m$, where X_i is a closed subvariety of X_{i+1}, so that the restriction of M to each of the pieces $\mathrm{X}_{i+1} - \mathrm{X}_i$ is a single lisse sheaf, punctually pure of weight w, and of some rank $r_i \geqslant 0$ (which in general depends on i).

Concretely, for a given $x \in k_n^m$, there exists a unique i such that $x \in \mathrm{X}_{i+1} - \mathrm{X}_i$, and then there exists a unitary matrix $\Theta_{\mathrm{M}}(x; k_n)$ of size r_i such that

$$t_n(x) = |k_n|^{w/2} \operatorname{Tr}(\Theta_{\mathrm{M}}(x; k_n)).$$

EXAMPLE E.4. Example (1) above is of this kind, with the stratification

$$\varnothing \subset \{x_0\} \subset \mathbf{A}^1,$$

and with $r_0 = 1$ and $r_1 = 2$. On $\{x_0\}$, the unique eigenvalue is $s_n(x_0) = \psi_n(y_0)$, viewing x_0 as belonging to k_n. On $\mathbf{A}^1 - \{x_0\}$, the two eigenvalues are either opposite (hence the trace is zero) if $x \notin f(k_n)$, or are given by $\psi_n(y)$, for y ranging over the two roots of the quadratic equation $f(y) = x$.

More generally, Deligne defined a *mixed constructible sheaf* of weights $\leqslant w$ by the condition that there is a filtration with associated punctually pure quotients M_j, each of some weight $w_j \leqslant w$. Concretely, this implies that the trace function $t = (t_n)$ is given by

$$t_n(x) = \sum_{j \in \mathrm{J}} t_{n,j}(x)$$

for some finite set J, where each family $(t_{n,j})_{n \geqslant 1}$ is the trace function of a constructible sheaf which is pure of weight $w_j \leqslant w$.

Finally, the most general type of trace functions arises from objects M that are *complexes of constructible sheaves*. Such a complex gives in particular rise to a *sequence* $(\mathscr{H}^i(\mathrm{M}))_{i \in \mathbf{Z}}$ of constructible sheaves, with $\mathscr{H}^i(\mathrm{M}) = 0$ for all but finitely many i, in such a way that

$$t_n(x) = \sum_{i \in \mathbf{Z}} (-1)^i t_{n,i}(x)$$

for all $n \geqslant 1$ and $x \in k_n$, where $(t_{n,i})_{n \geqslant 1}$ is the system of trace functions for the constructible sheaf $\mathscr{H}^i(\mathrm{M})$. (These sheaves are called the *cohomology sheaves* of the complex M.)

EXAMPLE E.5. Example (2) above is obtained from a complex of constructible sheaves M, where there are two non-zero pieces, namely $\mathscr{H}^1(\mathrm{M})$ and $\mathscr{H}^2(\mathrm{M})$.

The sheaf $\mathscr{H}^1(\mathrm{M})$ is constructible for the stratification

$$\varnothing \subset \{0\} \subset \mathbf{A}^1,$$

with the piece on $\{0\}$ of rank 0, and the piece on $\mathbf{A}^1 - \{0\}$ of rank 1, pure of weight 1, with the corresponding unique eigenvalue equal to the quadratic Gauss sum

$$\sum_{y \in k_n} \psi_n(xy^2)$$

for $x \in k_n - \{0\}$.

The sheaf $\mathscr{H}^2(\mathrm{M})$ is also constructible, for the same stratification (but this is not a general feature), with the lisse sheaf of rank 0 on $\mathbf{A}^1 - \{0\}$, and a piece of rank 1 of weight 2 at $\{0\}$, with eigenvalue $|k_n|$.

However, for a complex M, the definition of what it means that M is *pure of weight* w is much more subtle than for a single sheaf. In particular, it does *not* mean that each piece $\mathscr{H}^i(\mathrm{M})$ is itself a punctually pure sheaf of weight w. More precisely, one defines first the *mixed complexes of weights* $\leqslant w$, which are those such that $\mathscr{H}^i(\mathrm{M})$ is a mixed constructible sheaf of weights $\leqslant w + i$ for any $i \in \mathbf{Z}$. There is then furthermore defined another complex D(M), called the *Verdier dual* of M, and M is said to be pure of weight w if M is mixed of weights $\leqslant w$ and D(M) is mixed of weights $\leqslant -w$.

Remark E.6. (1) For a single lisse sheaf M which is punctually pure of weight 0, the corresponding complex has $\mathscr{H}^0(\mathrm{M}) = \mathrm{M}$ and $\mathscr{H}^i(\mathrm{M}) = 0$ for all $i \neq 0$. One can prove that the Verdier dual is a complex $\mathrm{D}(\mathrm{M})$ such that $\mathscr{H}^{-2m}(\mathrm{D}(\mathrm{M}))$ is a lisse sheaf which is pure of weight $-2m$ and all the other cohomology sheaves vanish, so that the two definitions of purity coincide for lisse sheaves. In fact, the trace function of $\mathrm{D}(\mathrm{M})$ is *in this case* the complex conjugate of the trace function of M.

(2) In practice, if an analytic number theorist is interested in a single trace function (e.g., one that represents a concrete family of exponential sums which one is interested in estimating) and one is not applying further operations like $\mathrm{R}f_!$, then one can quite often reduce to the case of a single lisse sheaf. This is for example the case for the hyper-Kloosterman sums in two variables

$$\mathrm{Kl}_3(x; k_n) = \frac{1}{|k_n|} \sum_{\substack{a,b,c \in k_n^\times \\ abc = x}} \psi_n(a+b+c),$$

or the famous sums

$$\mathrm{FI}(x, y; k_n) = \sum_{z \in k_n^\times} \mathrm{Kl}_3(xz; k_n)\, \mathrm{Kl}_3(yz; k_n) \psi_n(z)$$

which arose in the work of Friedlander and Iwaniec on the ternary divisor function [45], and reappeared in the work of Zhang [128].

Indeed, if the exponential sum is mixed, this will often be clear from the definition, or from a preliminary analysis, and one can "isolate" the part of most interest (of highest weight usually), which will be associated to a punctually pure constructible sheaf. Then by restricting the set of definition according to a suitable stratification, one will ensure that one handles a lisse sheaf.

For $m = 1$, this second step means avoiding finitely many values of x where the sheaf has unusual behavior; for $m \geqslant 2$, this means avoiding those that satisfy some non-trivial polynomial equation $g(x_1, \ldots, x_m) = 0$. These special parameters can then be handled separately—giving rise to a kind of inductive process which reflects exactly the algebraic stratification of the corresponding coefficient M.

One good explanation for the focus on mixed objects with bounded weights can be found (a posteriori) from the statement of Deligne's most general form of the Riemann hypothesis. In our context, it can be stated as follows:

Theorem E.7 (Deligne). *Let (t_n) be a trace function on $\mathbf{A}^m$ associated to a complex* M *which is mixed of weights $\leqslant w$. Let $f = (f_1, \ldots, f_d)$ be a tuple of polynomials in $k[\mathrm{X}_1, \ldots, \mathrm{X}_m]$. The complex* $\mathrm{R}f_!\mathrm{M}$ *is* mixed *of weights $\leqslant w$, and so its trace functions*

$$s_n(y) = \sum_{\substack{x \in k_n^m \\ f(x) = y}} t_n(x)$$

are also mixed of weights $\leqslant w$.

Remark E.8. On the other hand, even if M is a single lisse sheaf, punctually pure of weight w, it is *not always the case* that $\mathrm{R}f_!\mathrm{M}$ is pure.

A benefit of introducing these more general definitions is that all operations now respect the property of being mixed for any trace function, with a good understanding of how the weights may change:

- The lisse sheaf $\mathrm{M} = \overline{\mathbf{Q}}_\ell$ is pure of weight 0.
- If M_1 and M_2 have weights $\leqslant w_1$ and $\leqslant w_2$, respectively, then $\mathrm{M}_1 \oplus \mathrm{M}_2$ has weights $\leqslant \max(w_1, w_2)$ and $\mathrm{M}_1 \otimes \mathrm{M}_2$ has weights $\leqslant w_1 + w_2$.

– If M has weights $\leqslant w$, then for any $k \in \mathbf{Z}$, the shifted complex $\mathrm{M}[k]$ has weights $\leqslant w-k$.

– If M has weights $\leqslant w$, then for any $r \in \mathbf{Z}$, the twisted complex $\mathrm{M}(r)$ has weights $\leqslant w-2r$.

– If $f = (f_1, \dots, f_d)\colon \mathbf{A}^m \to \mathbf{A}^d$ is a tuple of polynomials in $k[\mathrm{X}_1, \dots, \mathrm{X}_m]$, and $s = (s_n)$ is a trace function on $\mathbf{A}^d$ associated to a mixed complex N of weights $\leqslant w$, then $f^*\mathrm{N}$ has weights $\leqslant w$.

– If $f = (f_1, \dots, f_d)\colon \mathbf{A}^m \to \mathbf{A}^d$ is a tuple of polynomials in $k[\mathrm{X}_1, \dots, \mathrm{X}_m]$, and if M has weights $\leqslant w$, then $\mathrm{R}f_!\mathrm{M}$ has weights $\leqslant w$ (this is again Deligne's Theorem E.7).

All objects that occur in practice in analytic number theory[2] are mixed complexes. This means that any trace function (t_n) has a decomposition

$$t_n = \sum_{a \leqslant w \leqslant b} t_{n,w}$$

for some a and b (independent of n), where $(t_{n,w})_{n \geqslant 1}$ is a trace function associated to a complex which is pure of weight w.

E.4. Perverse sheaves

There remains the task of attempting to explain a further fundamental subclass of trace functions (hence of complexes), those associated to *perverse sheaves*. This is a distinguished class of complexes with remarkable geometric and arithmetic properties. For analytic purposes, the most important of these is maybe that the *simple* perverse sheaves provide a *canonical basis* of the abelian group of trace functions, and that if we restrict to pure perverse sheaves, then this is in a natural sense a *quasi-orthogonal basis* for the trace functions of pure complexes of weight 0. We will now explain these properties.

The rigorous definition of perverse sheaves is of a similar nature to that of pure complexes: it is the combination for both the complex M and its Verdier dual D(M) of a relatively simple condition, called *semiperversity*.[3] The condition of semiperversity concerns the size of the support of the cohomology sheaves $\mathscr{H}^i(\mathrm{M})$ (which are intuitively the points x where $\mathscr{H}^i(\mathrm{M})$ does not vanish; in the stratification in terms of lisse sheaves, this is where these sheaves have non-zero rank): for any $i \in \mathbf{Z}$, the support of $\mathscr{H}^i(\mathrm{M})$ should be of dimension at most $-i$. (In particular, if $i \geqslant 1$, then the support should be empty, so $\mathscr{H}^i(\mathrm{M})$ should be zero then.)

Remarkably, this condition can be recovered intuitively from basic analytic intuition (which highlights that it is extremely natural).

Thus consider a trace function $t = (t_n)$ associated to a complex M on $\mathbf{A}^m$ and assume that it is mixed of weights $\leqslant 0$. From the analytic point of view, we are often in the situation where the mean-square of the values of the trace function t_n are bounded (after some normalization maybe), and bounded away from zero, i.e., for n large enough, we have

$$\sum_{x \in k_n^m} |t_n(x)|^2 \asymp 1. \tag{E.3}$$

For $i \in \mathbf{Z}$, the cohomology sheaf $\mathscr{H}^i(\mathrm{M})$ should be "essentially" pure of weight i (rigorously, we only know that it is mixed of weights $\leqslant i$). So the contribution to the sum above of the x in the support S_i of $\mathscr{H}^i(\mathrm{M})$ should be expected to be of order of magnitude

$$|k_n|^{2 \cdot i/2} \times |\mathrm{S}_i(k_n)| \approx |k_n|^{i+d_i}$$

[2] And indeed more generally in algebraic geometry.

[3] The complication is that the Verdier dual is often difficult to compute.

if S_i has dimension d_i. Hence the estimate (E.3) only has a chance to hold if $i + d_i \leqslant 0$ for all i, and this is *precisely* the semiperversity condition.

EXAMPLE E.9. Consider a family of exponential sums of type

$$\frac{1}{|k_n|^m} \sum_{y \in k_n^m} \psi_n(f(y) + x_1 y_1 \cdots + x_m y_m)$$

with parameters $(x_1, \ldots, x_m) \in k_n^m$ (these functions of x are the trace functions of a complex M which is a normalized form of Deligne's Fourier transform of the lisse sheaf $\mathscr{L}_{\psi(f)}$).

We expect "generic" square-root cancellation, so as n varies, for "most" choices of $x \in k_n^m$, this sum should be of size about $|k_n|^{-m/2}$. Since $\mathscr{H}^i(\mathrm{M})$ is of weight $\leqslant i$, and hence contributes terms of size typically expected to be $|k_n|^{i/2}$, this expectation corresponds to the fact that $\mathscr{H}^i(\mathrm{M})$ should be "generically" zero unless $i = m$, while $\mathscr{H}^{-m}(\mathrm{M})$ contributes a fixed number of complex numbers of modulus $\leqslant |k_n|^{-m/2}$.

But for special values of x, those satisfying some non-trivial polynomial equation $g(x) = 0$, one may obtain a larger sum than square-root cancellation. Experience teaches that usually this size only jumps by one factor $|k_n|^{1/2}$ (so the sum is about $|k_n|^{-m/2+1/2}$) if only this one condition is imposed; if it is bigger (say of size $|k_n|^{-m/2+1}$), this should mean that a second (independent) equation $h(x) = 0$ holds, and so on.

This "stratification" of bounds getting steadily worse only on smaller subsets corresponds to cohomology sheaves $\mathscr{H}^i(\mathrm{M})$ (contributing terms of size $|k_n|^{i/2}$) vanishing outside of subvarieties of dimension at most $-i$.

In the extreme case, the exponential sum is of size 1 (i.e., there is no cancellation at all) at worse for finitely many values of the parameters, corresponding to $\mathscr{H}^0(\mathrm{M})$ being supported on finitely many points.

This particular example is at the root of the results of Katz, Laumon and Fouvry on stratification for additive exponential sums [**38, 77, 39**]. It should suggest to analytic readers that semiperversity is a relatively easy condition to check, and that it should be natural and ubiquitous in analytic number theory.

The following statement provides a concrete illustration of the advantages of perverse sheaves.

THEOREM E.10. *The* $\mathbf{Z}$*-module of trace functions on* $\mathbf{A}^m$ *over* k *is generated by the trace functions of perverse sheaves, and the trace functions of* simple *perverse sheaves form a basis.*

The first statement is in fact very explicit. Indeed, if $t = (t_n)$ is an arbitrary trace function, associated to a complex M, one can define (in addition to its "usual" cohomology sheaves $\mathscr{H}^i(\mathrm{M})$) its *perverse cohomology sheaves* ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M})$, which are perverse sheaves, zero for $|i| > m$, such that their trace functions $({}^{\mathfrak{p}}t_{i,n})_{n \geqslant 1}$ satisfy the equation

$$t_n = \sum_{i \in \mathbf{Z}} (-1)^i \ {}^{\mathfrak{p}}t_{i,n}$$

for all $n \geqslant 1$. Furthermore, a complex M is mixed of weights $\leqslant w$ if and only if each ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M})$ is also mixed of weights $\leqslant w + i$ (similarly to the cohomology sheaves; see [**8**, Th. 5.4.1]).

REMARK E.11. To say that a complex M is perverse is to say that its perverse cohomology sheaves are $\mathrm{M} = {}^{\mathfrak{p}}\mathscr{H}^0(\mathrm{M})$ and ${}^{\mathfrak{p}}\mathscr{H}^i(\mathrm{M}) = 0$ for all $i \neq 0$.

Up to the terminology and notation, the second statement of Theorem E.10 is proved by Laumon in [**98**, Th. 1.1.2] (it was already mentioned by Deligne in his letter to Kazhdan; see Appendix D).

To understand it, one must explain what are the simple perverse sheaves which are mentioned there. We will content ourselves with stating the quasi-orthonormality property which holds for a simple perverse sheaf that is pure of weight 0. It is another consequence of Deligne's Riemann Hypothesis, proved by Katz, that if $t = (t_n)$ is the trace function of a perverse sheaf M, then

$$\limsup_{n\to+\infty} \sum_{x\in k_n^m} |t_n(x)|^2 = 1 \tag{E.4}$$

if and only if M is *simple*.

Remark E.12. One of the fundamental results of Beilinson, Bernstein, Deligne and Gabber [8, Cor. 5.3.4] is that a simple perverse sheaf which is mixed, as a complex, is in fact *pure* of some weight; since non-mixed complexes do not appear in practice, this means that simple perverse sheaves in analytic number theory are always pure of some weight, and the quasi-orthonormality characterization can be extended to all simple perverse sheaves, up to normalization.

Example E.13. We can illustrate how useful this quasi-orthonormality statement can be to guess or understand some properties of perverse sheaves by noting that it strongly suggests a non-trivial property of simple perverse sheaves. Namely, let M be a simple perverse sheaf, pure of weight 0, and generically non-zero (i.e., the support of M is all of $\mathbf{A}^m$). If we repeat the argument leading to the guess of the semiperversity condition, we see that we expect that the contribution to

$$\sum_{x\in k_n^m} |t_n(x)|^2$$

of each non-zero cohomology sheaf $\mathscr{H}^i(\mathrm{M})$ should be of size

$$\alpha_i|k_n|^{i+d_i}$$

for some integer $\alpha_i \geqslant 1$, and comparison with (E.4) indicates that $i + d_i$ will be < 0 except for one single value of i. Moreover, one knows that the cohomology sheaf $\mathscr{H}^{-m}(\mathrm{M})$ is generically non-zero, so this value must be $i = -m$, so that we expect that

$$d_i \leqslant -i-1 \quad \text{for} \quad i \neq -m,$$

which is stronger than the condition $d_i \leqslant -i$ derived from semiperversity only. This is indeed true (it is the improved support condition of Proposition A.13).

E.5. Tannakian categories

The results of this book also rely in an essential way on another tool that is most likely unfamiliar to analytic number theorists: the formalism of tannakian categories. In very rough terms, this refers to a method to construct or define a *group* (which in our case will be the "symmetry group" that governs the equidistribution properties of a trace function), by recovering it from the way it acts on finite-dimensional K-vector spaces, for some algebraically closed field K of characteristic zero (which can be considered to be $\mathbf{C}$). That this is possible is indicated by the following result:

Theorem E.14 (Tannaka). *Let* $\mathbf{G}$ *be a compact group. Assume that for every finite-dimensional complex vector space* V *on which the group* $\mathbf{G}$ *acts linearly, via a continuous homomorphism* $\varrho\colon \mathbf{G} \to \mathbf{GL}(\mathrm{V})$*, we are given an invertible linear transformation* $\alpha(\varrho)\colon \mathrm{V} \to \mathrm{V}$*, and suppose that these data satisfy the following conditions*:

"Whenever $\mathbf{G}$ *acts by* ϱ *on* V *and by* π *on* W*, we have*

$$\alpha(1) = \mathrm{Id}_{\mathbf{C}}, \qquad \alpha(\varrho\otimes\pi) = \alpha(\varrho)\otimes\alpha(\pi)\,;$$

whenever $\mathbf{G}$ *acts by* ϱ *on* V, *we have*

$$\alpha(\bar{\varrho}) = \overline{\alpha(\varrho)},$$

where $\bar{\varrho}$ *is the same action as* ϱ *but viewed as a representation on the conjugate vector space; and whenever we have a linear map* $u\colon \mathrm{V} \to \mathrm{W}$ *such that*

$$u(\varrho(g)v) = \pi(g)u(v)$$

for all $g \in \mathbf{G}$ *and* $v \in \mathrm{V}$, *then we have*

$$u \circ \alpha(\varrho) = \alpha(\pi) \circ u$$

as linear maps from V *to* W.*"*

Then there exists a unique element $g \in \mathbf{G}$ *such that* $\alpha(\varrho) = \varrho(g)$ *for all actions* ϱ *of* $\mathbf{G}$.

More generally, note that the "set" of all data of all $\alpha(\varrho)$ of the type considered in this theorem can naturally be used to form a group (with $(\alpha\beta)(\varrho) = \alpha(\varrho) \circ \beta(\varrho)$), and then the result identifies the group $\mathbf{G}$ with these data.

In a converse direction, the main theorem of the theory of tannakian categories establishes a list of conditions on a suitable category which guarantees that it is "equivalent" to the category of representations of a group $\mathbf{G}$ (although the context is that of algebraic groups, such as $\mathbf{GL}_n(\mathbf{C})$, instead of compact groups). A key property to apply the "reconstruction theorem" is that one must be able to associate to each object M a finite-dimensional vector space $\omega(\mathrm{M})$ (corresponding to the abstract space on which the group acts), and one needs to have defined a bilinear operation on these objects, say $\mathrm{M} \star \mathrm{N}$, in such a way that $\omega(\mathrm{M} \star \mathrm{N}) = \omega(\mathrm{M}) \otimes \omega(\mathrm{N})$. Such an "assignment" ω is called a *fiber functor*; it is not unique, and its construction may be a delicate matter.

In the applications in this book (following the idea of Katz in [74]), the objects that will correspond in this abstract way to the actions of $\mathbf{G}$ on vector spaces are certain perverse sheaves, and the operation $\star$ is a form of algebraic convolution which respects the corresponding usual convolution operation on trace functions.

For the classical form of Tannaka duality for compact groups, we refer to the presentation by Joyal and Street [67, § 1]. For an accessible treatment of tannakian categories, emphasizing the natural evolution from Galois theory, we refer to the book [124] of Szamuely.

E.6. Frequently asked questions

We conclude by trying to answer some natural questions that an analytically-minded reader of little faith may raise:

– *Is it possible to describe trace functions (or the underlying algebraic objects) "by generators and relations", by listing a number of basic examples and a list of operations preserving trace functions, so that all trace functions are obtained from these basic data in finitely many steps?*

  It is true that in many applications to analytic number theory, the sheaves or trace functions which occur are constructed precisely in such a way (e.g., starting from an additive character, replacing the variable by a polynomials, taking the Fourier transform, etc).

  However, it seems extremely unlikely that one could provide a satisfactory and rigorous version of such an idea, for instance because it is known that there are q geometrically irreducible middle extension sheaves of rank 2 on the projective line over $\mathbf{F}_q$ with 4 singular points and principal tame local monodromy at each point (see for instance [31, Prop. 7.1]; the proof of this fact relies on automorphic methods). All these sheaves have bounded complexity as q varies. However, only six such sheaves are explicitly known (they are associated to certain elliptic curves over $\mathbf{F}_q(t)$ with four singular fibers), as shown by Beauville [4].

Since operations on sheaves tend to increase the complexity in general (although in a controllable manner), it seems very difficult to imagine how one could construct the "other" $q-6$ sheaves in a straighforward way.

– *Why are perverse sheaves essential to the results of this book? Why can one not (even in the simplest cases, such as exponential sums parameterized by multiplicative characters) work around the requirement to use such objects in a way similar to the previous papers of Fouvry, Kowalski and Michel?*

  The simplest reason for this (not the only one) is that the use of tannakian methods (which is the only way we know to produce the symmetry group for arithmetic Fourier transforms) depends on applying many times a number of operations which will have uncontrollable effect on the type of complex we work with, even when starting with a single lisse sheaf.

  More technically, the same tannakian idea requires the construction of an *abelian* category (which will "be" the category of representations of the symmetry group); general complexes do not form an abelian category, whereas perverse sheaves form one—certainly the best known abelian category beyond that of lisse sheaves.

– *Conversely, if perverse sheaves are so natural and have such remarkable properties, and suffice to describe all trace functions, why not dispense with general complexes then?*

  Here the issue is that, although perverse sheaves and their trace functions are individually wonderful things, they are not *in toto* stable by all the operations that one might want to apply. In particular, if M, M_1, M_2 and N are perverse sheaves, then it is *not true* in general that $\mathrm{M}_1 \otimes \mathrm{M}_2$, or $f^*\mathrm{N}$, or $\mathrm{R}f_!\mathrm{M}$, are perverse sheaves (on their respective affine spaces). (A significant and highly non-trivial exception, however, is that if M is perverse on $\mathbf{A}^m$, then its Fourier transform in the sense of Deligne is still perverse.) In the case of our applications, the problem appears in the definition of the algebraic convolution that is used to apply the tannakian formalism—*a priori*, even for M and N perverse, their algebraic convolution is simply a complex of constructible sheaves.

– *Why is there no normalization by the size of the sum in a formula like* (E.4) *?*

  It is a useful property of perverse sheaves, although surprising at first sight, that the definition itself implies a normalization for these sums. If M is a perverse sheaf with support $\mathbf{A}^m$ which is pure of weight 0, then the local eigenvalues at a "generic" point x of k_n^m are of weight $-m$, i.e., they are typically of size $|k|^{-m/2}$. So the sum (E.4) is naturally expected to be of bounded size, *without normalizing*.

Index

Glossary

$\mathbf{1}$	skyscraper sheaf, unit for convolution, 23
$\mathrm{A}(\mathrm{X})$	Albanese variety of $\mathrm{F}(\mathrm{X})$, 169
$\mathrm{D}(\mathrm{M})$	Verdier dual, 14
$\mathbf{D}(\mathrm{G})$	subcategory of $\mathrm{D}^{\mathrm{b}}_c(\mathrm{G}_{\bar{k}})$ of objects defined over a finite field, 54
$\overline{\mathbf{D}}(\mathrm{G})$	convolution category, 57
$\mathrm{D}^{\mathrm{b}}_c(\mathrm{X}) = \mathrm{D}^{\mathrm{b}}_c(\mathrm{X}, \overline{\mathbf{Q}}_\ell)$	category of bounded constructible complexes of $\overline{\mathbf{Q}}_\ell$-sheaves on X, 13
Δ	diagonal embedding $\mathbf{G}_m \to \mathbf{G}_m \times \mathbf{G}_a$, 136
$\mathrm{FM}_!$	Fourier-Mellin transform with compact support, 30
FM_*	Fourier-Mellin transform, 30
$\mathrm{Fr}_{\mathrm{M},k_n}(\chi)$	Frobenius action on $\mathrm{H}^0_c(\mathrm{G}_{\bar{k}}, \chi)$, 66
$\mathrm{Fr}_{\mathrm{M}}(\chi)$	$\mathrm{Fr}_{\mathrm{M},k}(\chi)$, 66
Fr_{k_n}	geometric Frobenius automorphism of k_n, 66
$\mathbf{Fr}_{\mathrm{M},k_n}(\chi)$	Frobenius conjugacy class in $\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$, 66
$\mathbf{Fr}_{\mathrm{M}}(\chi)$	$\mathbf{Fr}_{\mathrm{M},k}(\chi)$, 66
FT_ψ	Fourier transform on a unipotent group, 36
$\mathrm{F}(\mathrm{X})$	Fano variety of lines, 169
$[\widehat{\mathrm{G}}]$	primitive characters modulo Galois action, 95
$\widehat{\mathrm{G}}$	disjoint union of $\widehat{\mathrm{G}}(k_n)$, 23
$\widehat{\mathrm{G}}(k_n)$	characters of $\mathrm{G}(k_n)$, 23
$\widehat{\mathrm{G}}^*$	primitive elements of $\widehat{\mathrm{G}}$, 95
$\widehat{\mathrm{G}}^{(\ell)}$	ℓ-adic characters, 115
$\mathbf{G}^{\mathrm{ari}}_{\mathrm{M}}$	arithmetic tannakian group of M, 64
$\mathbf{G}^{\mathrm{geo}}_{\mathrm{M}}$	geometric tannakian group of M, 60
$\mathbf{H}_{\mathrm{int}}(\mathrm{T})$	hypergeometric group of Gabber and Loeser, 74
$\mathrm{Hyp}(!, \psi, \boldsymbol{\chi}; \boldsymbol{\varrho})$	hypergeometric sheaf, 188
$\mathrm{K}(\mathrm{X}) = \mathrm{K}(\mathrm{X}, \overline{\mathbf{Q}}_\ell)$	Grothendieck group, 14
$\mathrm{K}^\sharp$	space of conjugacy classes in K, 77
$\mathrm{K}_{\mathrm{neg}}(\mathrm{G})$	subgroup of $\mathrm{K}(\mathrm{G})$ generated by negligible objects, 55
Λ_{M}	von Mangoldt function of M, 158
$\widehat{\mathrm{L}}(\mathrm{M}, \mathrm{T})$	$\widehat{\mathrm{L}}$-function of M, 95
$\mathbf{L}(\mathrm{G})$	group of isomorphism classes of objects of rank 1, 73
$\mathscr{L}^\lambda_{\chi,a}$	sheaf $\lambda_*(\mathscr{L}_\chi \otimes j^*\mathscr{L}_{\psi(ay)})$, 136
$\mathscr{L}_{\mathrm{G}}$	Ω_{G}-sheaf associated to $\mathrm{can}_{\mathrm{G}}$, 30
$\mathscr{L}_{\mathrm{U},\mathrm{U}^\vee,\psi}$	lisse sheaf on $\mathrm{U} \times \mathrm{U}^\vee$, 36
$\mathscr{L}_{\mathrm{U},\mathrm{U}^\vee}$	bi-extension, 35
$\mathscr{L}_\chi$	character sheaf, 24
$\mathrm{M} *_* \mathrm{N}$	convolution, 22
$\mathrm{M} *_! \mathrm{N}$	convolution with compact support, 22
$\mathrm{M} *_{\mathrm{int}} \mathrm{N}$	internal convolution, 57
$\mathrm{M} \boxtimes \mathrm{N}$	external tensor product, 14
$\mathrm{M}^\vee$	tannakian dual of M, 23
$\mathrm{M}_{2m}(\mathbf{G})$	$\mathrm{M}_{2m}(\mathbf{G}, \mathrm{Std})$, 122

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