CORRECTIONS AND ADDITIONS TO "AN INTRODUCTION TO THE REPRESENTATION THEORY OF GROUPS" GSM 155

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Corrections

Here are the currently known corrections; those which are mathematically significant are marked with a star.

Thanks to J. Fresán for sending some of these corrections.

(1) Page 3, line 2: U. Schapira should be U. Shapira.

(2) ($\star$) Pages 32, 34, : Zorn's Lemma states that there is a maximal element in any non-empty ordered set where every *non-empty* totally ordered subset $\mathcal{S}$ has an upper bound; page 32, for instance, the assumption that $\mathcal{S}$ is not empty is necessary to check that the subspace $\tilde{F}$ of line -5 is a linear subspace.

(3) Page 115, line 4: x should be in $M_n(k)$ and g in $\mathrm{GL}_n(k)$, instead of the opposite.

(4) ($\star$) Page 145, Lemma 3.3.7: this lemma is *false*; the problem in the proof is that the "reduction" to continuity at $(1, 0)$ does not work. A counter-example is provided by the regular representation ϱ on $E = L^1(G, \mu)$ for an infinite locally-compact group. In this case, the dual space is $L^\infty(G, \mu)$, and one can show that the space of vectors $f \in E'$ such that the map $g \mapsto \check{\varrho}(g)f$ is continuous is the space of uniformly continuous functions on G (see Bourbaki, Intégration, Ch. VIII, p. 191, exercise 3, (d)).

However, the contragredient is continuous if the Banach space E is reflexive (which applies to Hilbert spaces, for instance; see loc. cit., (c)), and it is always continuous if E' is given the topology of uniform convergence on compact subsets (instead of the topology of uniform convergence on bounded subsets, which is the Banach-space topology on the dual); see Bourbaki, Intégration, Ch. VIII, p. 131, prop. 3, (ii).

(5) ($\star$) Page 155, Schur's Lemma and the Spectral Theorem: for the proof of Schur's Lemma to be complete, it is better to strengthen the form of the spectral theorem (Theorem 3.4.18) by asking that the measure space (X, μ) be a locally-compact topological space x with a Radon measure μ; it is then easy to see why, in the last paragraph of Page 156, the conditions $\nu(U) > 0$ and $\nu(V) > 0$ imply that H_1 is a non-zero proper subspace. This stronger version of the spectral theorem is as easy to prove as the version stated, and the result is clear from the statement and the proof in the reference [49].

Date: January 30, 2021.

(6) Page 184, Remark 4.3.24 (2): the expression for the generating series should be
$$g(z) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_\pi(g)}e^{(\chi_\varrho(g)-\dim\varrho)z}.$$
This requires some changes afterwards: the contribution of $g = 1$ is $\dim(\pi)/|G|$, the other terms have modulus bounded by $\dim(\pi)e^{-cz}/|G|$, and the last inequality becomes
$$\Bigl|g(z) - \frac{\dim(\varrho)}{|G|}\Bigr| \leqslant \dim(\pi)e^{-cz},$$
from which the conclusion follows in the same way.

(7) Page 197, line -3: the variable y in the sum should be x.

(8) Page 205, line -2: there is an extra parenthesis in $a \pmod q$).

(9) Page 219, middle of the page, display: χ instead of *chi*.

(10) Page 241, line 1: "Nikolo" should be "Nikolov".

(11) Page 281 and 282, proof of Prop. 5.2.6: all $C(G)$ should be $C_c(G)$ (compactly-supported continuous functions).

(12) Page 338, line 19: $\mathrm{SU}_3(\mathbf{C})$ instead of $\mathrm{U}_3(\mathbf{C})$.

(13) Page 351, line -7: Ψ_1 should be ψ_1 twice in the formula.

(14) ($\star$) Page 402, line 1: $\exp(-X) = \exp(X)^{-1}$ instead of $\exp(-X) = \exp(X)$.

(15) Page 422, reference [31]: "Théorie" instead of "ThÉorie".

(16) Page 423, reference [56]: add space before the title.

Additions

After Exercise 4.5.4, add the following Example 4.5.5.

Example 4.5.5 (Characterization of cyclic Galois extensions) As an application of the representations of finite abelian groups, we explain a proof of an important result from elementary Galois theory: the classification of cyclic Galois extensions when the degree is coprime to the characteristic of the base field and we have enough roots of unity. Readers who are unfamiliar with Galois theory can skip this example.

Proposition 4.5.6 *Let K be a field of characteristic p and L/K a finite Galois extension of degree $n \geqslant 1$ coprime with p. Assume that K contains a primitive n-th root of unity, and that the Galois group of L/K is cyclic, isomorphic to $\mathbf{Z}/n\mathbf{Z}$. Then there exists $b \in L^\times$ such that $L = K(b)$ and $b^n \in K^\times$. In other words, we have $L = K(a^{1/n})$ for some $a \in K^\times$.*

For a standard proof, which does not involve representation theory one can see for instance [40, VIII, Th. 6.2]. The proof below is a bit longer, but it is quite easy to understand conceptually, and it shows another instance of applying representation theory to understand a naturally given representation of a group in order to solve a problem (as we saw in the first proof of Burnside's Irreducibility criterion in Section 2.7.3 and as we will see also in Section 4.7.3).

Sketch of proof. We denote by G the Galois group of L over K, and by $\xi \in K^\times$ a primitive n-th root of unity. The first step is to note that it is enough to find an element $b \in L^\times$ such that $\sigma(b) = \xi b$, where $\sigma \in G$ is a generator of the cyclic group G. Indeed, one then finds that $\sigma(b^n) = b^n$, and since G is generated by σ, that $b^n \in L^G = K$ (by Galois theory). Then we get $K \subset K(b) \subset L$, and it is elementary to show that $[K(b) : K] = n = [L : K]$, and hence deduce that $L = K(b)$, as claimed.

To find this element b, we observe that the condition $\sigma(b) = \xi b$, and the fact that $\xi \in K$, imply in particular that the one-dimensional subspace $Kb \subset L$ is a (one-dimensional) subrepresentation of the K-representation of G on L given by $\sigma \cdot x = \sigma(x)$. It is therefore natural (independently of knowing where this will lead us...) to attempt to understand this representation.

In fact it is very well understood: the *normal basis theorem* (see, e.g., [40, VIII, §13], and note that it applies to any finite Galois extension, not only to cyclic ones) implies that L is isomorphic, as a representation of G over K, to the left-regular representation λ_G on $V = C_K(G)$ (see Exercise 4.5.5). Precisely, there exists $y \in L^\times$ such that $(\tau(y))_{\tau\in G}$ is a K-basis of L, and if we define

$$\Phi : \begin{cases} L & \longrightarrow & V \\ x & \mapsto & \varphi_x \end{cases}$$

by the condition that

$$x = \sum_{\tau\in G} \varphi_x(\tau)\tau(y),$$

one checks that the K-linear isomorphism Φ intertwines the action of G on L and λ_G.

Since $p \nmid n = |G|$, we have a decomposition

$$V = \bigoplus_{\chi} K\chi,$$

where χ runs over all characters of G with values in an algebraic closure $\bar{K}$ of K. Indeed, since $G \simeq \mathbf{Z}/n\mathbf{Z}$, all such characters have values in the group of n-th roots of unity, which is contained in K, and therefore can be seen as elements of $C_K(G)$. Since they are linearly independent, and there are n characters, we obtain this decomposition although K is not algebraically closed.

We observed that to find an element b with $\sigma(b) = \xi b$, we must look among the generators of subrepresentations of L. This means that b must correspond to a character χ by the isomorphism Φ. But for such a χ, we get

$$\lambda_G(\sigma)(\chi) = \chi(\sigma)^{-1}\chi$$

(because χ spans the χ^{-1}-isotypic component of the left-regular representation). So the question is whether we can find χ such that $\chi(\sigma) = \xi^{-1}$. This is certainly the case since G is cyclic of order n generated by σ (see the proof of Theorem 4.5.2 (2) or Remark 4.5.3). $\square$