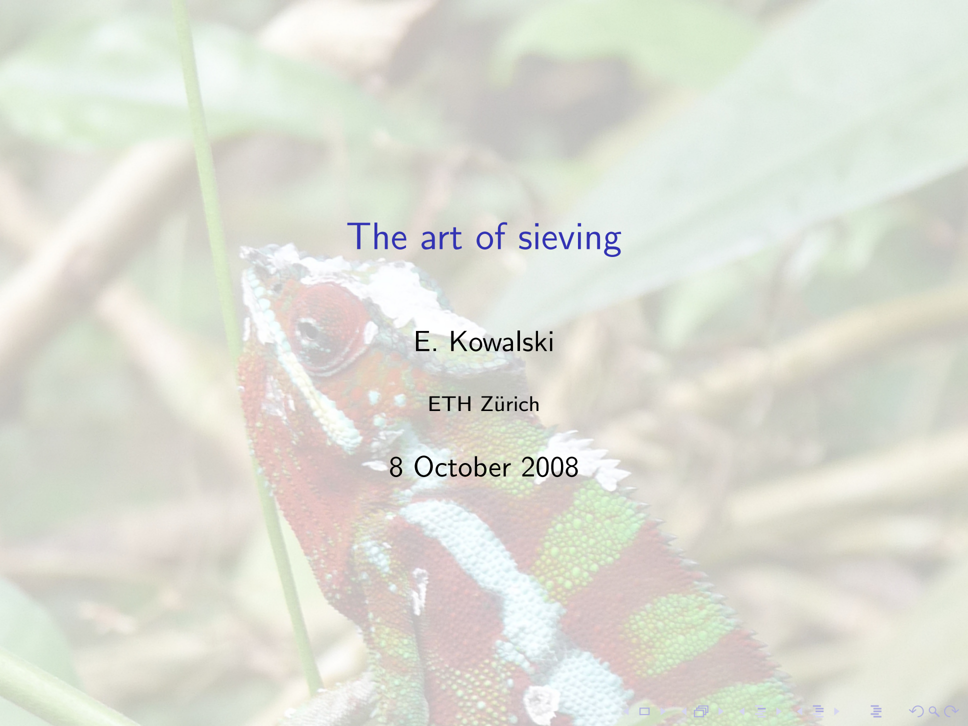

The art of sieving

E. Kowalski

ETH Zürich

8 October 2008

Prime numbers

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but not

$$5007 = 3 \cdot 1669, \qquad 156839 = 2209 \cdot 71, \qquad 8102008 = 8 \cdot 1012751.$$

(The number $2^{43112609} - 1$ is known to be prime since August 2008.)

Factorization

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For example:

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When ordered, the *prime factors* that appear are uniquely determined by n, and so are the number of repetitions of each.

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Un-historical answer. Constructing primes, and checking that numbers are primes is now *easy* (in some sense).

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```
? p1=1634733645809253848443133883865090859841783670033092312181110852
389333100104508151212118167511579;
? isprime(p1)
time = 137 ms.
%1= 1
? p2=1900871281664822113126851573935413975471896789968515493666638539
0880271038021044989571912614655571;
time = 0 ms.
? isprime(p2)
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? isprime(p1)
time = 137 ms.
%1= 1
? p2=19008712816648221131268515739354139754718967899685154936666385
39088027103802104498957191261465571;
time = 0 ms.
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But factoring integers seems to be *extremely hard*.

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? factor(p1*p2)
^C  *** factor: user interrupt after 1hr, 53mn, 39,129 ms.
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RSA challenge: Factoring $p_1 \cdot p_2$ (without knowing p_1 and p_2 in advance!) took the equivalent of 30 years of non-stop computation on a fast personal computer in 2005.

Communicating trust

This (and other similar problems) is the foundation of much of today's computer security protocols.

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If **C** were an impostor, without knowing p_1, she would not be able to factor n and convince **B**.

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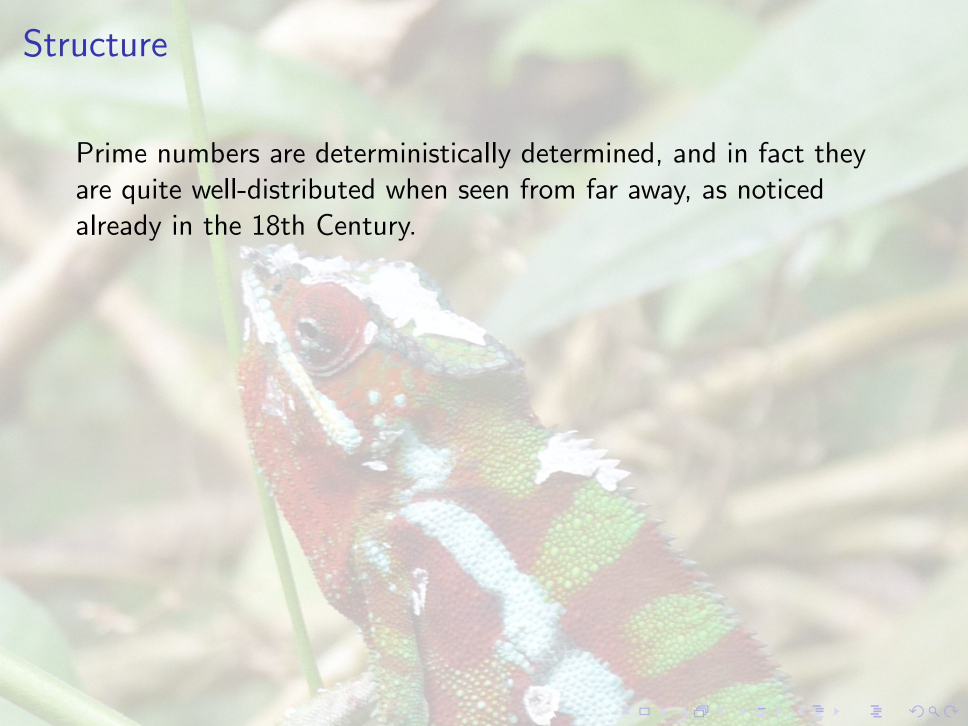
Primes are linked with multiplication; as soon as they interact with *addition*, strange things may happen.

$$2^{30} = 2 \cdot 2 \cdot 2 \cdots 2 \quad \text{(30 times)}$$
$$2^{30} + 1 = 5 \cdot 5 \cdot 13 \cdot 41 \cdot 61 \cdot 1321$$
$$2^{30} + 2 = 2 \cdot 3 \cdot 59 \cdot 3033169$$
$$2^{30} + 3 = 1073741827, \quad 2^{30} + 4 = 2 \cdot 2 \cdot 17 \cdot 15790321$$
$$2^{30} + 5 = 3 \cdot 149 \cdot 2402107$$

Structure

Prime numbers are deterministically determined, and in fact they are quite well-distributed when seen from far away, as noticed already in the 18th Century.



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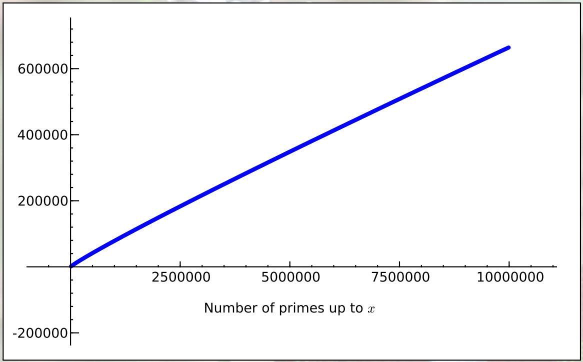
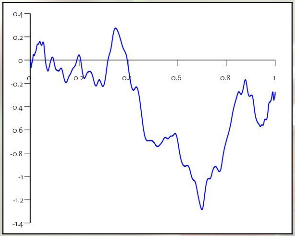


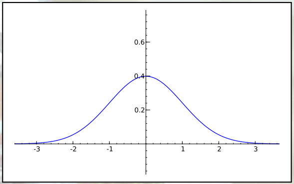
Randomness

But, seen more closely, primes seem to behave chaotically. It seems that any interesting probability distribution may be found naturally within the primes.

The normal gaussian distribution

This is for instance the distribution of the position of a random walker (Brownian motion) at time 1.





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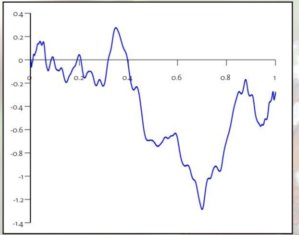
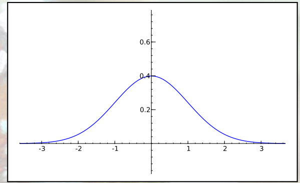




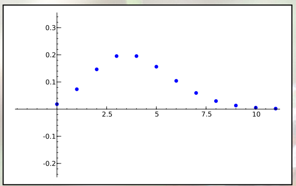
ERDÖS-KÁC theorem:

$$\frac{(\text{nb. of primes dividing } n) - \log\log n}{\sqrt{\log\log n}}$$

is approximately distributed like a normal variable for large n.

Poisson distribution

This is for instance the distribution of the number of atomic disintegrations (of a given substance) observed during a fixed time.



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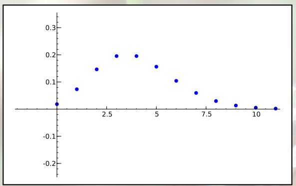

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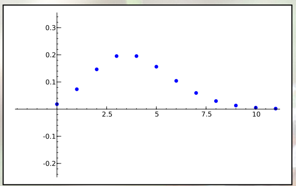

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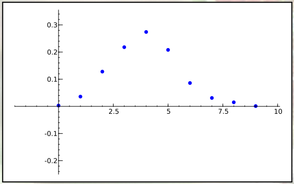

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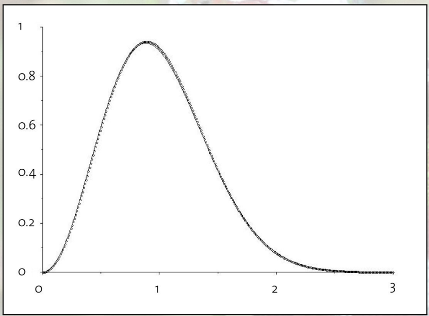

The continuous line is the theoretical distribution of the normalized spacings of GUE matrices, and the small circles are the data from the Riemann zeta function. (Numerical work and graph by X. Gourdon)

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Experimental psychology answer.

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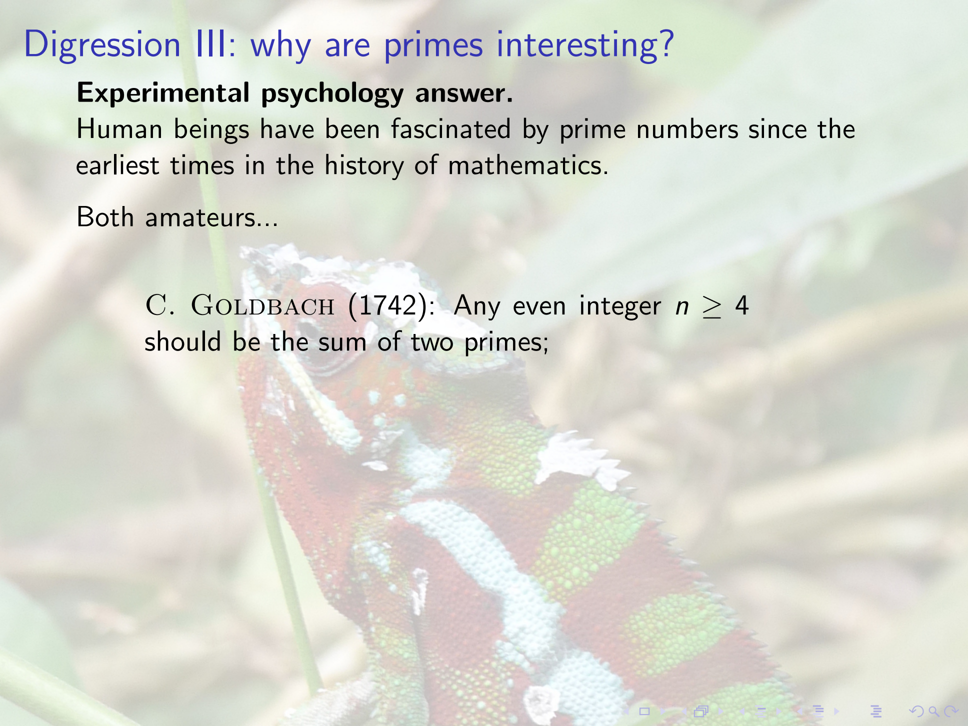

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... and some of the most renowned scientists:

(The two questions above are still open.)

(cont.)

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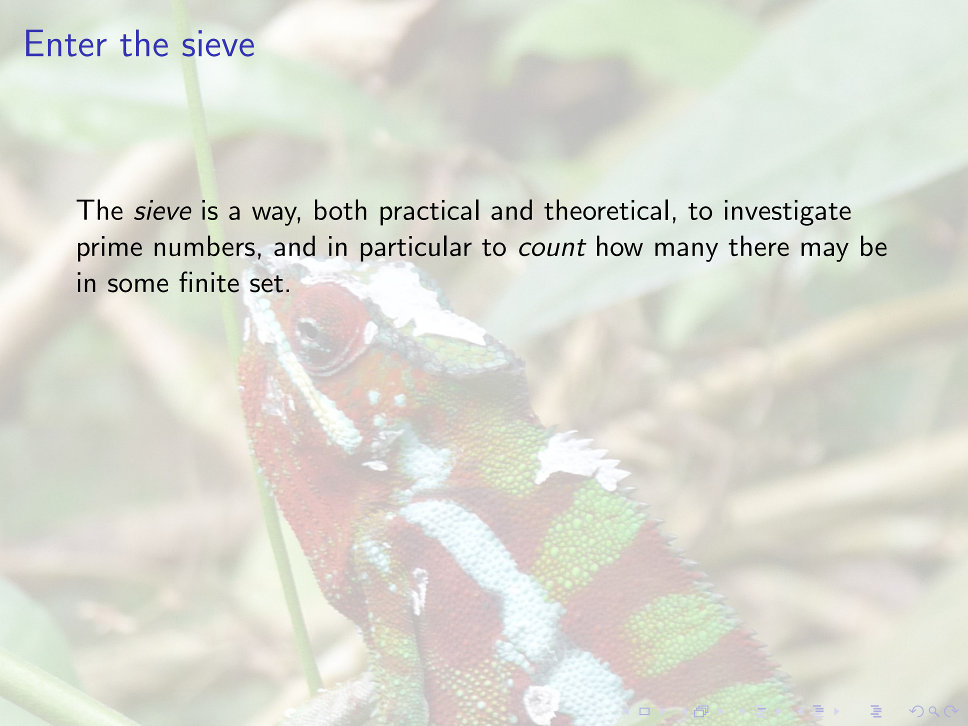
C.F. GAUSS (the probability that a large integer N be prime is about one chance in $\log N$, which is about three times the number of digits of N).

Enter the sieve

The *sieve* is a way, both practical and theoretical, to investigate prime numbers, and in particular to *count* how many there may be in some finite set.



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21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
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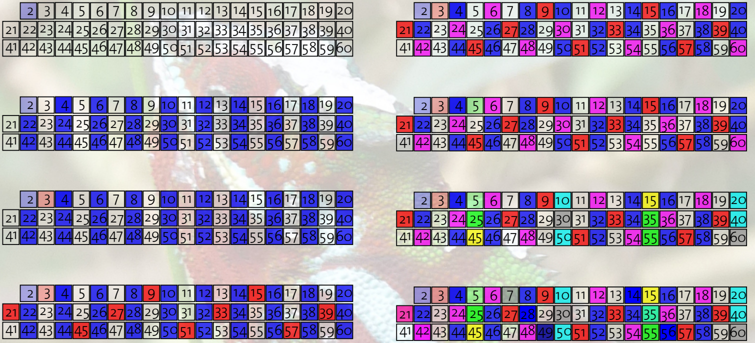

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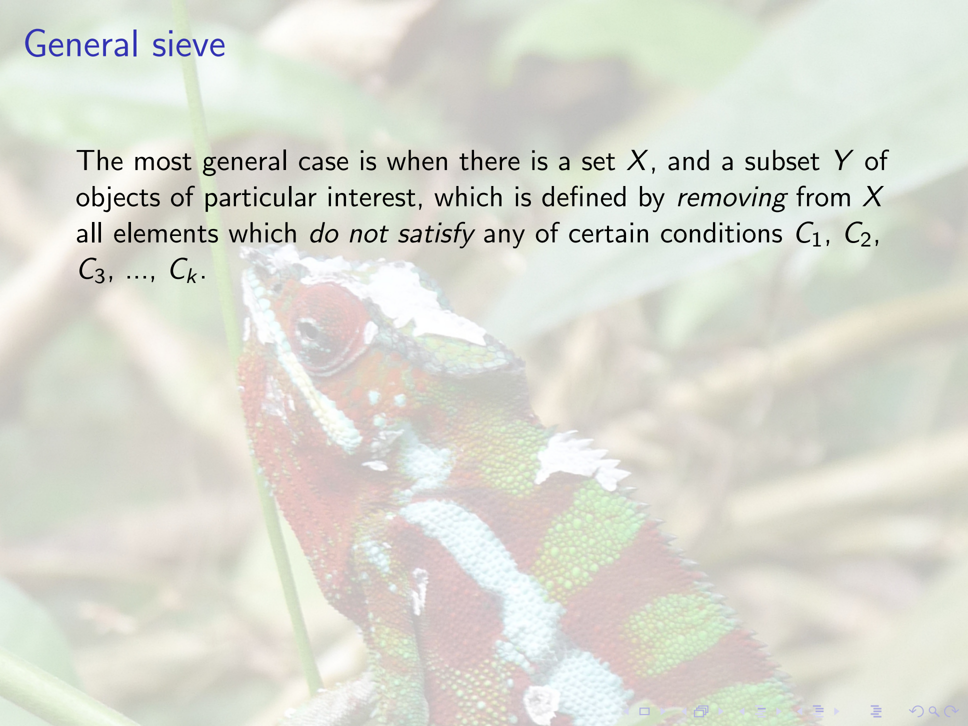
The primes up to 60 are

$$2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, 41, 43, 47, 53, 59.$$

General sieve



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This general "sieve" procedure is based on *inclusion-exclusion*.

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(Cont.)

So the probability that $n \leq N$ be prime should be about

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{p_k}\right)=\prod_{\substack{\ell\leq\sqrt{N}\\ \ell \text{ prime}}}(1-\ell^{-1}).$$

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$$(\text{Number of primes } p \leq N) \sim \frac{N}{\log N}$$

(J. HADAMARD and C. DE LA VALLÉE-POUSSIN: Prime Number Theorem, 1896, confirming the intuition of GAUSS).

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The “sifted set” is now the set of twin primes $\sqrt{N} < p \leq N$ such that $p+2$ is also prime.

(Cont.)

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$$\left(1-\frac{1}{2}\right)\left(1-\frac{2}{3}\right)\cdots\left(1-\frac{2}{p_k}\right)=\frac{1}{2}\prod_{\substack{3\leq\ell\leq\sqrt{N}\\ \ell\text{ prime}}}(1-2\ell^{-1}).$$

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It is expected that

$$(\text{Number of primes } p\leq N \text{ such that } p+2 \text{ is prime}) \sim \frac{s_2 N}{(\log N)^2}$$

with

$$s_2=\frac{1}{2}\prod_{\substack{\ell\geq 3\\ \ell \text{ prime}}}\frac{1-2\ell^{-1}}{(1-\ell^{-1})^2}=1.32032469\ldots$$

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–H. IWANIEC & J. FRIEDLANDER (1998):

There are infinitely many pairs of integers (x, y) such that $x^2 + y^4$ is prime.

The small and large sieves

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This was first developed by LINNIK in 1941, and after many developments, it is now very useful in many (sometimes surprising) applications.

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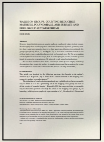
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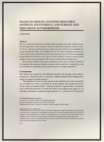
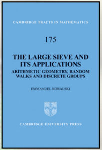


E. Kowalski: *The large sieve and its applications*, Cambridge Univ. Press (2008).



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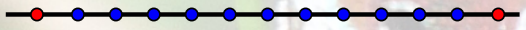

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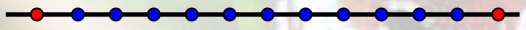

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The boundary of a large disc has length much smaller than the area: πR^2 is much larger than $2\pi R$ if R is large.

Hyperbolic counting

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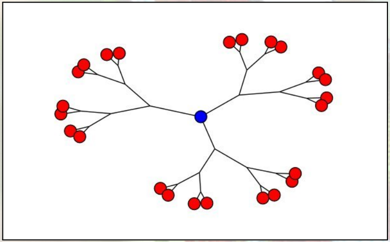

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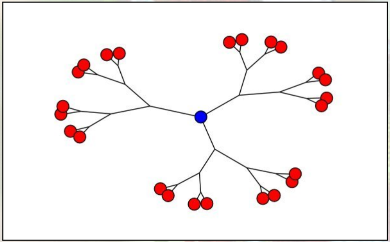

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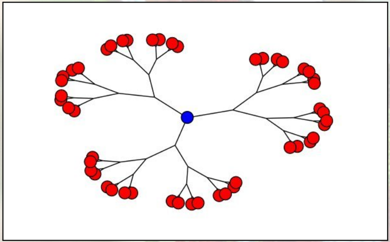

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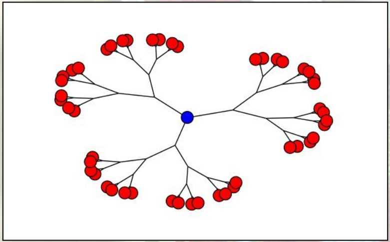

48 vertices among the 94 are on the "boundary".

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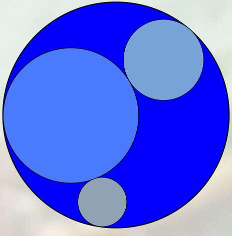

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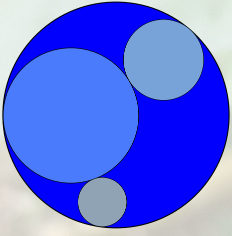

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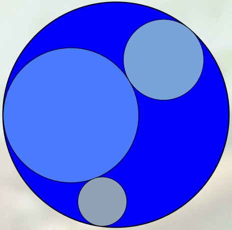

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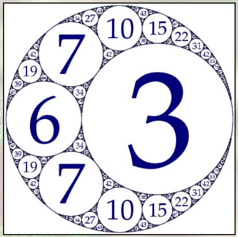

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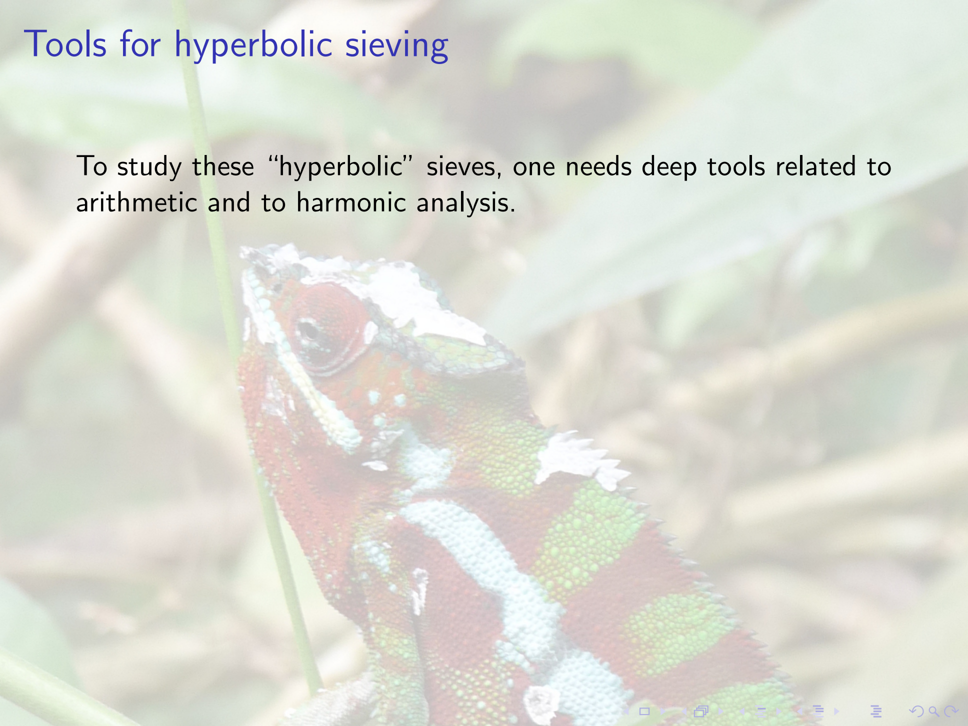
Diophantine properties of apollonian packings

P. SARNAK (2007), following the methods of Bourgain-Gamburd-Sarnak: for some constant C, in any such packing, there exist infinitely many quadruples of tangent circles, all curvatures of which are product of at most C primes.

Tools for hyperbolic sieving

To study these "hyperbolic" sieves, one needs deep tools related to arithmetic and to harmonic analysis.



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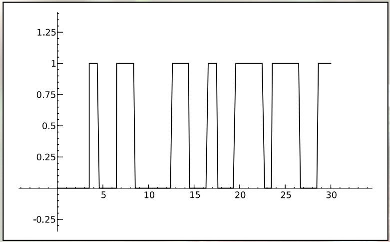

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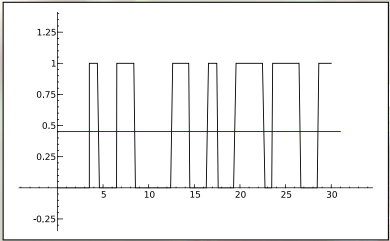

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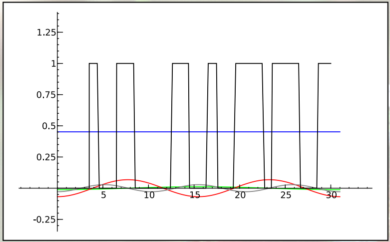

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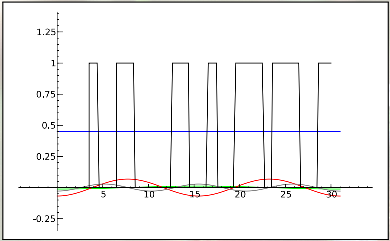
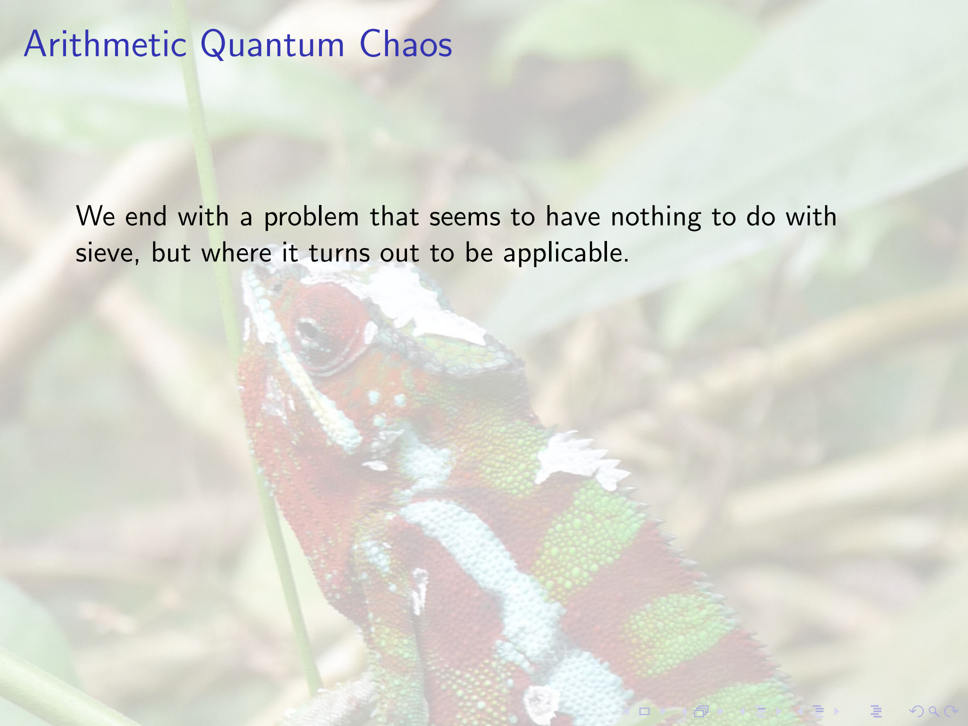


Harmonic analysis is used to decompose the binary signal that says that a condition holds into a steady “main term” (which corresponds to the probability) and a sum of further oscillating contributions of other harmonics.

The oscillating harmonics must be shown to be negligible in some sense. This often involves very deep results, and most of the work lies here.

Arithmetic Quantum Chaos



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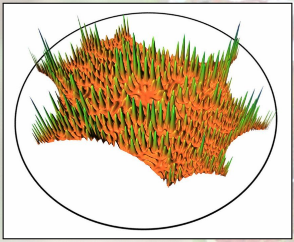

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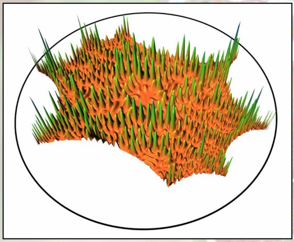
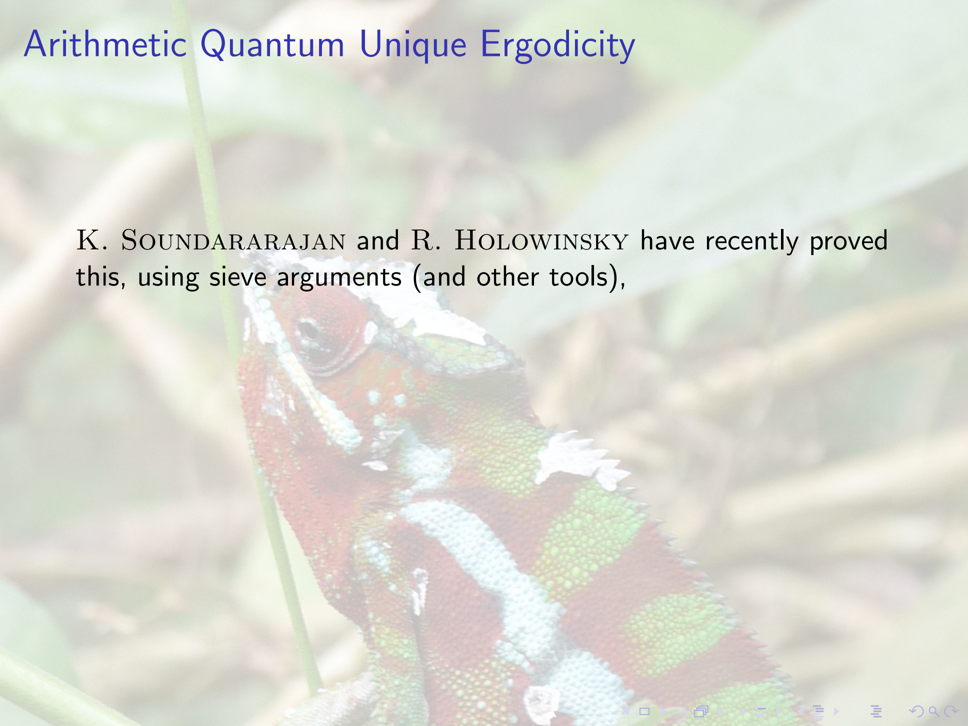


This shows the density of a “wave function” on a hyperbolic surface. It is expected that for certain surfaces, the wave functions with high energy will be uniformly distributed. (Picture by R. Aurich and F. Steiner.)

Arithmetic Quantum Unique Ergodicity



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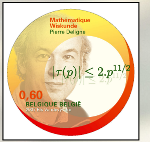
Arithmetic Quantum Unique Ergodicity

K. Soundararajan and R. Holowinsky have recently proved this, using sieve arguments (and other tools), if a certain widely believed conjecture holds.

For a similar but slightly different problem, they are able to prove completely the analogue result, because the corresponding conjecture is known.



Acknowledgements

Most of the pictures in this document were produced using free mathematical and graphics software, in particular:

– **Inkscape**, `www.inkscape.org`

– **Pari/GP**, `pari.math.u-bordeaux.fr`

– **Sage**, `www.sagemath.org`

– **ImageMagick**, `www.imagemagick.org`

and the whole **GNU** and **Linux** desktop environment.