CORRELATION SUMS IN THE WILD

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1. Introduction

We collect in this note, with precise references, examples in the literature where one can find exponential sums over finite fields which turn out to be examples of the general "correlation sums" introduced in [2]. Furthermore, in Section 6, we discuss how the results of [2] imply the bounds for these sums given by the various authors who defined them.

We first recall the notation. For a prime number p and a function $K : \mathbf{Z}/p\mathbf{Z} \to \mathbf{C}$ and $v \in \mathbf{Z}/p\mathbf{Z}$, we denote by

$$\hat{K}(v) = \frac{1}{p^{1/2}} \sum_{x \,(\mathrm{mod}\, p)} K(x) e\Big(\frac{vx}{p}\Big)$$

the (unitarily normalized) Fourier transform modulo p of K. Denoting by $\gamma \cdot z$ the action of PGL_2 (or GL_2) on $\mathbf{P}^1$ by homographies, the correlation sums $\mathcal{C}(K;\gamma)$ of K are defined for $\gamma \in \mathrm{GL}_2(\mathbf{F}_p)$ by

$$\mathcal{C}(K;\gamma) = \sum_{\substack{z \in \mathbf{F}_p \\ z \neq -d/c}} \hat{K}(\gamma \cdot z) \overline{\hat{K}(z)}. \tag{1.1}$$

For upper-triangular matrices, these sums have a direct expression in terms of the original function K, given by

$$\mathcal{C}\Big(K; \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\Big) = \sum_x K(x) \overline{K(a\bar{d}x)} e\Big(\frac{\bar{d}bx}{p}\Big). \tag{1.2}$$

Remark 1.1. In this informal note, we will not always keep track very precisely of what happens at the singularities of the sheaf, so that the identifications of various sums with correlation sums might be off by some $O(1)$ amount, due to the values at the singularities.

2. The Friedlander-Iwaniec sum

In [3, p. 329, 347], Friedlander and Iwaniec consider the sum

$$S_{FI} = \sum_{u \in \mathbf{F}_p^\times - \{1\}} S\Big(\frac{\alpha}{u}, 1; p\Big) S\Big(\frac{\beta}{1-u}, 1; p\Big)$$

where α, $\beta \in \mathbf{F}_p^\times$ and $S(m,m;c)$ denotes the classical Kloosterman sums. For $K(x) = e(\bar{x}/p)$ and $K(0) = 0$, we have

$$\hat{K}(z) = \frac{S(z,1;p)}{\sqrt{p}} = \overline{\hat{K}(z)},$$

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and hence we have

$$S_{FI} = p\ \mathcal{C}\Big(K; \begin{pmatrix} \beta & 0 \\ 1 & -\alpha \end{pmatrix}\Big).$$

These sums are estimated by Birch and Bombieri in the Appendix to [3].

3. The Iwaniec sum

In [4, p. 76], Iwaniec considers the sum

$$S_I = \sum_{x \in \mathbf{F}_p^\times} K(ax)\overline{K(bx)} e\Big(\frac{c\bar{x}}{p}\Big)$$

where [4, p. 68, p. 75] we have

$$K(x) = e\Big(\frac{2x}{p}\Big) S(x, x; p)$$

in terms of Kloosterman sums, and a, $b \in \mathbf{F}_p^\times$, $c \in \mathbf{F}_p$.

Defining $L(x) = K(x^{-1})$, we get

$$S_I = \sum_{z \in \mathbf{F}_p^\times} L(z)\overline{L\Big(\frac{bz}{a}\Big)} e\Big(\frac{acz}{p}\Big)$$

and using (1.2), we recognize that

$$S_I = \mathcal{C}\Big(L; \begin{pmatrix} a & abc \\ 0 & b \end{pmatrix}\Big). \tag{3.1}$$

Iwaniec [4, Lemma 9] proves a bound $S_I \ll p^{3/2}$ when a, b, $c \in \mathbf{F}_p^\times$ using the Riemann Hypothesis and mean-square average methods. Another (more geometric) proof is explained by Bombieri and Sperber in [1, §4, Th. 10].

4. The Pitt sum

In [7, Th. 3], Pitt considers the sum

$$S_P = \sum_{\substack{x,y,z \in \mathbf{F}_p \\ xyz(z+h) \neq 0}} \psi(\bar{x} - rx - \bar{y} + ry + \alpha x\bar{z} - \beta y\overline{z+h})$$

where ψ is any non-trivial additive character modulo p, where α, β, $h \in \mathbf{F}_p^\times$ and $r \in \mathbf{F}_p$.

Summing over z first, we get

$$S_P = \sum_{z \neq 0, -h} S\Big(\frac{\alpha}{z} - r, 1; p\Big) S\Big(r - \frac{\beta}{z+h}, -1; p\Big) = \sum_{z \neq 0, -h} S\Big(\frac{\alpha}{z} - r, 1; p\Big) S\Big(-r + \frac{\beta}{z+h}, 1; p\Big),$$

and hence this becomes

$$S_P = p \sum_{z \neq 0, -h} \overline{\hat{K}\Big(\frac{\alpha}{z} - r\Big)} \hat{K}\Big(-r + \frac{\beta}{z+h}\Big) = p\mathcal{C}(K; \gamma)$$

where

$$\gamma = \begin{pmatrix} \beta - hr & r(\beta - \alpha - hr) \\ h & \alpha + hr \end{pmatrix}.$$

Pitt [7, §1] estimates these sums using results of Adolphson and Sperber and mean-square arguments.

5. The Munshi sum

In [6, §5.2, p. 8, line -6], Munshi considers the sum

$$S_M = p \sum_{x,y,z\in \mathbf{F}_p^\times} \psi\Big(ax + \frac{b}{x} + cy + \frac{d}{y} + \frac{e}{xz} + \frac{f}{y}\frac{1}{gz+h}\Big)$$

where $\psi(x) = e(x/p)$ (using the correspondance

$$(x, y, z) = (\beta, \gamma, \delta), \qquad (a, b, c, d, e, f, g, h) = (\bar{q}_1 h, -\bar{q}_1 m, -\bar{\tilde{q}}_1 h, \bar{\tilde{q}}_1 n, \bar{q}_1, -\bar{\tilde{q}}_1 q_1, \tilde{q}_1, m)$$

between his parameters and variables and the ones in the sum above).

Summing over z first, we obtain

$$\begin{aligned} S_M &= p \sum_{z(gz+h)\neq 0} S\Big(a, b + \frac{e}{z}; p\Big) S\Big(c, d + \frac{f}{gz+h}; p\Big) \\ &= p \sum_{z(gz+h)\neq 0} S\Big(a\Big(b + \frac{e}{z}\Big), 1; p\Big) S\Big(c\Big(d + \frac{f}{gz+h}\Big), 1; p\Big) \\ &= p^2 \sum_{z(gz+h)\neq 0} \overline{\hat{K}\Big(ab + \frac{ae}{z}\Big)} \hat{K}\Big(cd + \frac{cf}{gz+h}\Big) \end{aligned}$$

for (again) $K(x) = e(\bar{x}/p)$, and hence

$$S_M = p^2 \mathcal{C}(K; \gamma)$$

this time with

$$\gamma = \begin{pmatrix} c(dh - f) & acdeg - abcdh - abcf \\ h & aeg - abh \end{pmatrix}.$$

Munshi estimates his sums using results of Bombieri and Sperber [1, §4].

6. Discussion

Example 6.1 (The Friedlander–Iwaniec, Pitt and Munshi sums)**.** For $K(x) = e(\bar{x}/p)$ and $K(0) = 0$, the result of [2, Remark 11.1] gives

$$C(K; \gamma) \ll p^{3/2}$$

uniformly for all p and $\gamma \neq 1$ in $\mathrm{PGL}_2(\mathbf{F}_p)$. This implies the estimates for the exponential sums of Friedlander–Iwaniec, Pitt and Munshi, since these involve the same weight.

The common weight in these three cases is certainly explained by their common $\mathrm{GL}_3 \times \mathrm{GL}_2$ flavor (which is only implicit in the work of Friedlander and Iwaniec, and only partially explicit for Pitt, where GL_3 is represented by $\zeta^3(s)$.)

Example 6.2 (The Iwaniec sum)**.** We analyze here the sum S_I of Iwaniec using the method from [2]. The weight $K(x) = -e(2\bar{x}/p)S(\bar{x}, \bar{x}; p)/\sqrt{p}$ is an irreducible trace weight coming from the sheaf

$$\mathcal{F} = \mathcal{L}_{\psi(2X^{-1})} \otimes i^*\mathcal{K} = i^*(\mathcal{L}_{\psi(2X)} \otimes \mathcal{K}),$$

for some additive ℓ-adic character ψ, where $i : \mathbf{G}_m \longrightarrow \mathbf{G}_m$ is the inverse map and $\mathcal{K} = [x \mapsto x^2]^*\mathcal{K}\ell$ is the Kloosterman sheaf with trace function at $x \in \mathbf{F}_p^\times$ given by

$$-\frac{1}{\sqrt{p}} S(x, x; p) = -\frac{1}{\sqrt{p}} S(x^2, 1; p).$$

By properties of the Kloosterman sheaf, the sheaf $\mathcal{F}$ is tamely ramified at ∞ and wildly ramified at 0. Its conductor is uniformly bounded for all p (since the rank is 2, the number of singularities is 2, and the Swan conductors at 0 are all 1).

Now let $\mathcal{G} = \mathrm{FT}_\psi(\mathcal{F})$ be the Fourier transform of $\mathcal{F}$ (with respect to the same additive character ψ). Since $\mathcal{F}$ is tame at infinity, $\mathcal{G}$ is lisse on $\mathbf{G}_m$ by [5, Cor. 7.4.5 (1)], and $\mathcal{G}$ is tame at 0 by [5, Cor. 7.4.5 (2)]. Applying [5, Th. 7.4.4 (4)], we see that $\mathcal{G}$ is wildly ramified at ∞. Therefore, the Fourier-Möbius group $\mathbf{G}_{\mathcal{F}}$ is a subgroup of the diagonal torus, which fixes the two singularities 0 and ∞. In Iwaniec's work, the difficult case for the sum is when a, b, $c \in \mathbf{F}_p^\times$ are all non-zero, in which case the matrix γ in (3.1) is not diagonal. Hence we obtain immediately the result of [4, Lemma 9], namely

$$|S_I| \ll p^{3/2}$$

uniformly for all p and a, b, $c \in \mathbf{F}_p^\times$. Note that the diagonal case is handled elementarily in [4, Lemma 7], which implies that the same bound holds for all $\gamma \neq 1$ in $\mathrm{PGL}_2(\mathbf{F}_p)$.

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