Expander graphs

E. Kowalski

École Polytechnique Fédérale de Zürich

9 Avril 2015

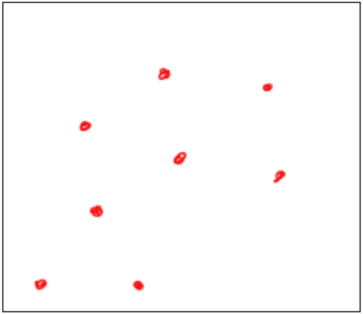
... à l'expansion de mon cœur refoulé s'ouvrirent aussitôt des espaces infinis.
M. Proust, « À l'ombre des jeunes filles en fleurs",
(second part, « Noms de Pays : le Pays")

Outline

- What is a graph?
- What are graphs useful for?
- Expansion in graphs
- Expander graphs
- Applications

What is a graph?

- A set of vertices;



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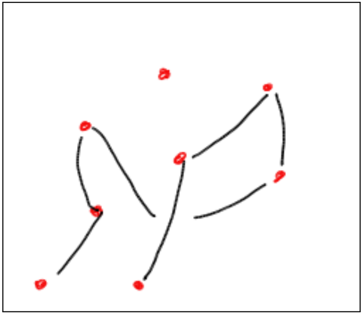

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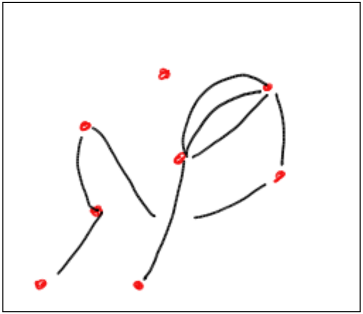

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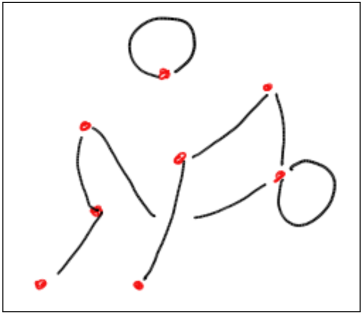

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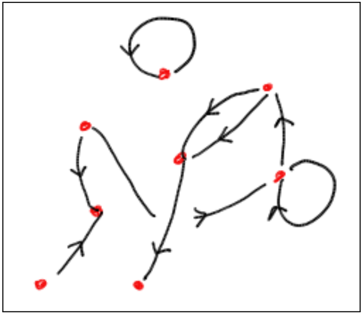

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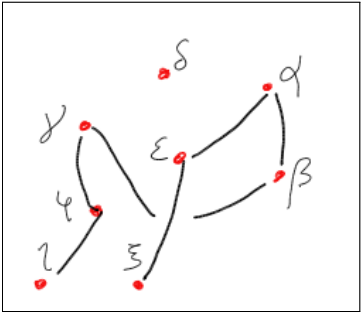
- Maybe with loops;
- Sometimes edges are oriented;
- And sometimes we add data to either vertices or edeges or both.



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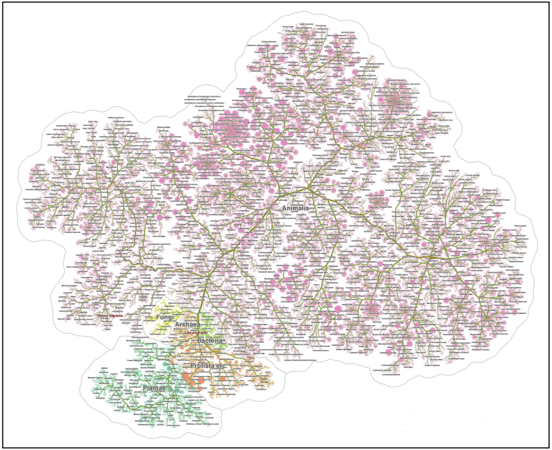

Parts of the tree of life; Source: Yifan Hu

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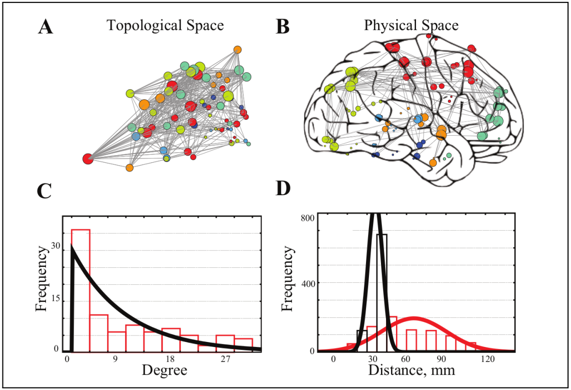

Bullmore and Bassett, Annu. Rev. Clin. Psychol. 2011, 7:113–40

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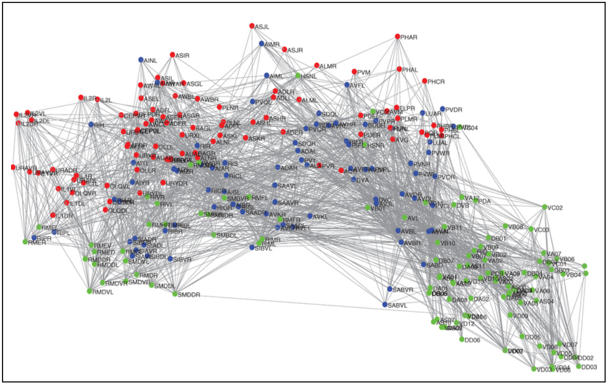

The nervous system of *Caenorhabditis Elegans* (302 neurons, about 8000 synapses), from White, Southgate, Thomson, Brenner (1986), updated and represented by Varshney, Chen, Paniagua, Hall, Chklovskii (2011)

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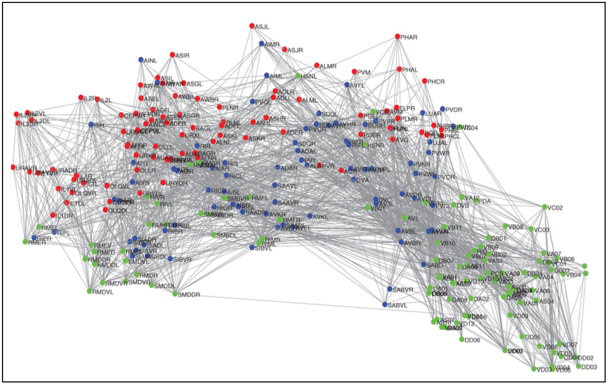

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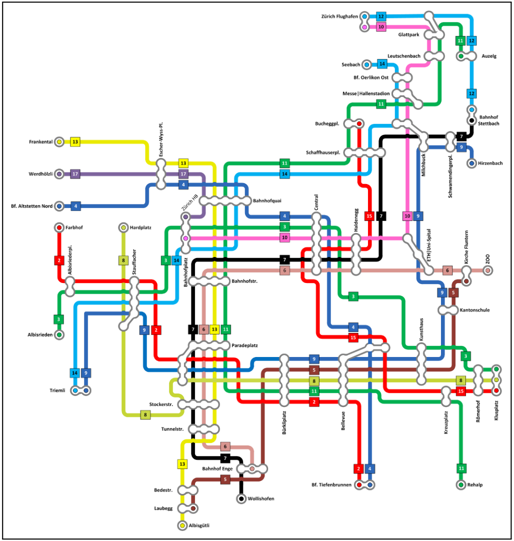

Mateusch, CC Attribution Share-Alike 3.0

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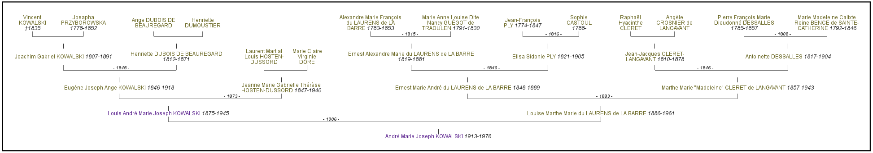

A “normal” family

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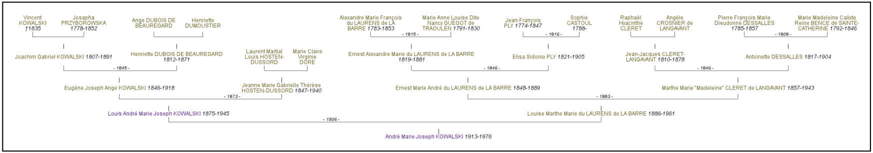

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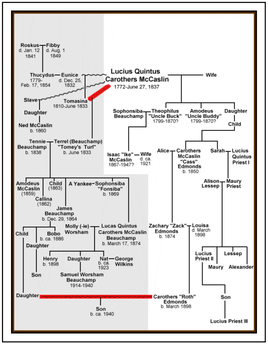

McCaslin family genealogy in “Go Down, Moses” (W. Faulkner); Source: J. Padgett

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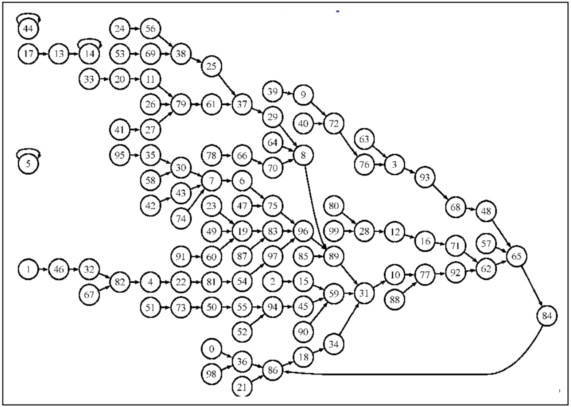

Source: A. Shamir, "Random graphs in cryptography"

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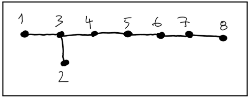
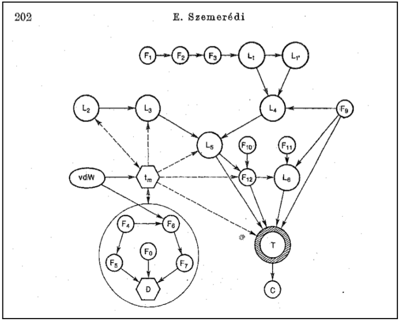


Dynkin diagram of type E_8



Source: E. Szemerédi, "On sets of integers containing no k elements in arithmetic progression", Acta Arith. 1973

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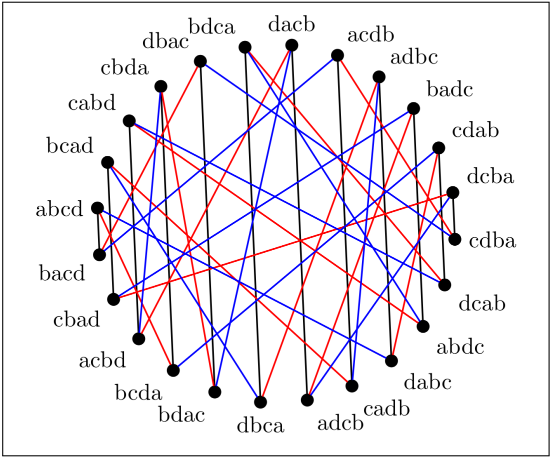

A Cayley graph

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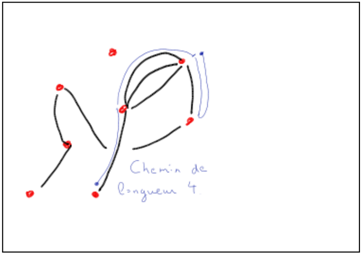

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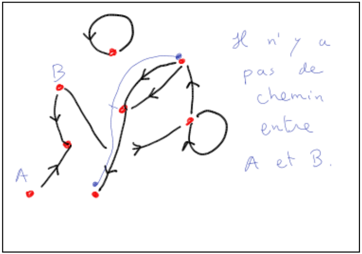

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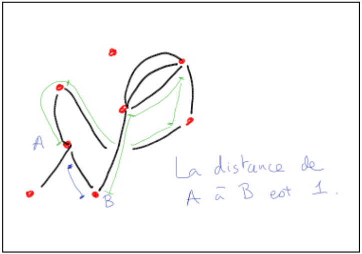

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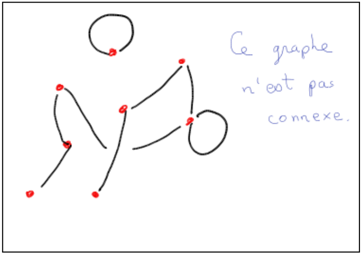

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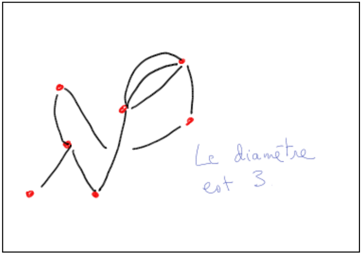

Efficient navigation

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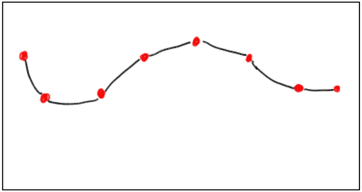

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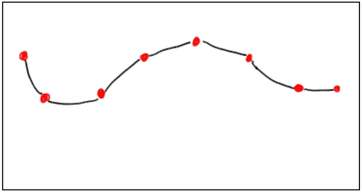
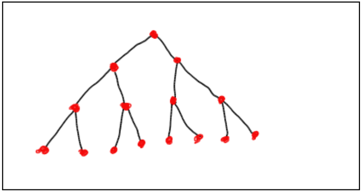
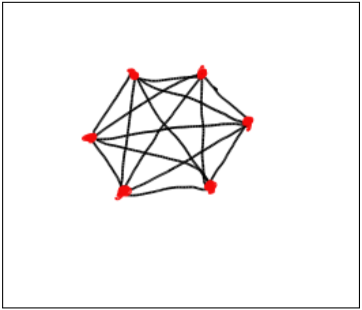


Example 2. If we can put objects in a tree-like configuration, the diameter is much smaller: it is proportional to the logarithm of the number of vertices.



The cost of edges

The diameter may be very small, if one is allowed to put many edges, as in a *complete graph*.



The cost of edges

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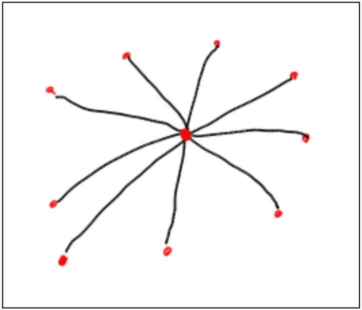

Graphs "not too dense"

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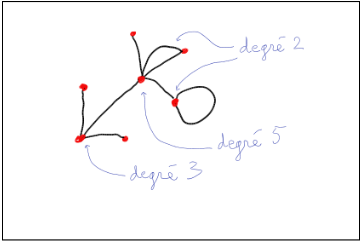

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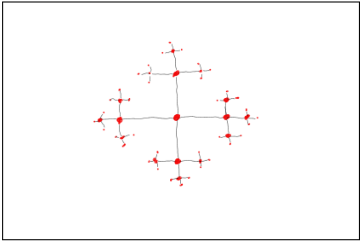

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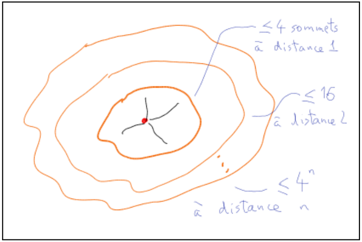

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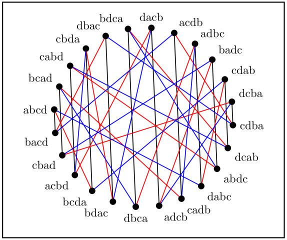

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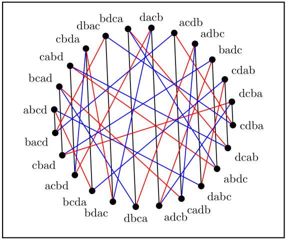

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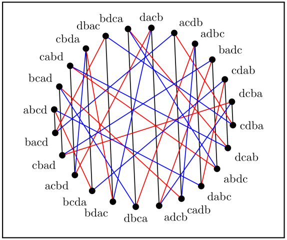

Exercises.
(1) This graph is connected.
(2) Its diameter is of order of magnitude n^2.

Digression: what can diameter be useful for?

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ANALYSIS OF CASINO SHELF SHUFFLING MACHINES

BY PERSI DIACONIS[1], JASON FULMAN[2] AND SUSAN HOLMES[3]

Stanford University, University of Southern California and Stanford University

Many casinos routinely use mechanical card shuffling machines. We were asked to evaluate a new product, a shelf shuffler. This leads to new probability, new combinatorics and to some practical advice which was adopted by the manufacturer. The interplay between theory, computing, and real-world application is developed.

1. Introduction. We were contacted by a manufacturer of casino equipment to evaluate a new design for a casino card-shuffling machine. The machine, already built, was a sophisticated "shelf shuffler" consisting of an opaque box containing ten shelves. A deck of cards is dropped into the top of the box. An internal elevator moves the deck up and down within the box. Cards are sequentially dealt from the bottom of the deck onto the shelves; shelves are chosen uniformly at random at the command of a random number generator. Each card is randomly placed above or below previous cards on the shelf with probability $1/2$. At the end, each shelf contains about $1/10$ of the deck. The ten piles are now assembled into one pile, in random order. The manufacturer wanted to know if one pass through the machine would yield a well-shuffled deck.

Cheeger constant

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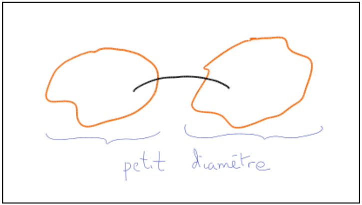

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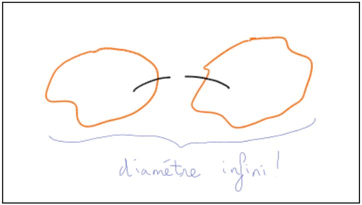

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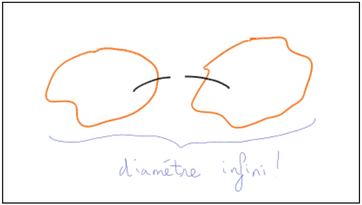

The "robustness" of a non-oriented graph, with vertex set $S \neq \emptyset$, is measured by its *Cheeger constant*:

$$h = \min_{\substack{X \subset S \\ 1 \leq |X| \leq |S|/2}} \frac{|\partial X|}{|X|},$$

where ∂X is the set of edges with one extremity exactly in X (and $|A|$ is the number of elements of a finite set A).

Properties

Proprerties. We defined

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because the number of vertices at distance d from a fixed origin is at least

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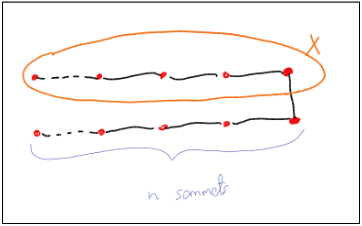

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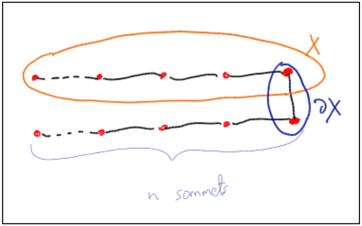

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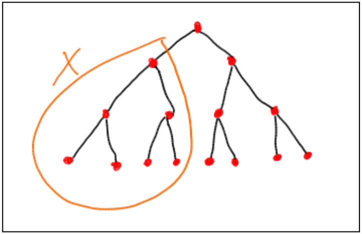

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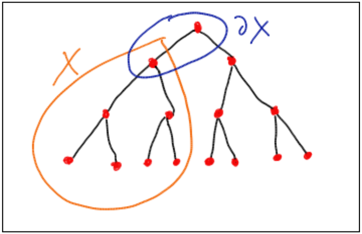

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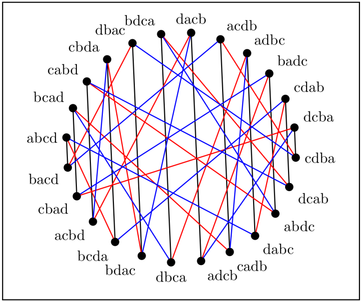

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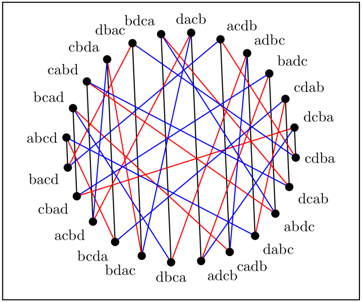

We notice that in each of these cases, h tends to 0 as the number of vertices tends to infinity.

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But do expanders exists? Or is just a castle in the sky?

History

- **1973**, L. A. Bassalygo et M.S. Pinsker, “On complexity of optimal non-blocking system without rearrangement” (Russian; Problemy Peredaci Informacii 9, 84–87).

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- [†]**1967**, Ya. M. Barzdin et A.N. Kolmogorov, “On the realization of networks in three-dimensional space” (Russian; Problemy Kibernetiki 19, 261–268).
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[†]As note by L. Guth (2010); see L. Guth and M. Gromov, “Generalizations of the Kolmogorov-Barzdin embedding estimates”, 2011.

Barzdin–Kolmogorov

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ON THE REALIZATION OF NETWORKS
IN THREE-DIMENSIONAL SPACE*[1]

(*Jointly with Ya. M. Barzdin*)

By a (d, n)-network we shall mean a oriented graph with n numbered vertices $\alpha_1, \alpha_2, \dots, \alpha_n$ and dn marked edges such that precisely d edges are incident to each vertex and one of them is marked by the weight x_1, another by the weight x_2, etc., and finally the last one by the weight x_d.

Examples of such networks are logical networks and neuron networks. It is precisely for this reason that the question of constructing such networks in ordinary three-dimensional space under the condition that the vertices are balls while the edges are tubes of a certain positive diameter is of importance.

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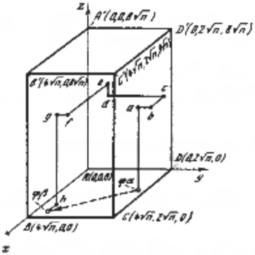
Remark. It is not always possible to represente a graph in the plane, but it is easy to convince oneself that this is possible in space.

The results of Barzdin and Kolmogorov

Theorem 1. It is possible to do this in a sphere of radius $\sqrt{n}$.



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What is the meaning of Theorem 3?

Almost all graphs are expanders

Consider a "large" integer $n \geq 1$. We take the vertex set

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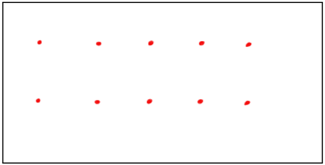

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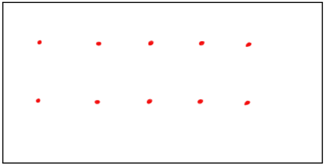

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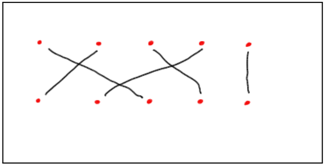

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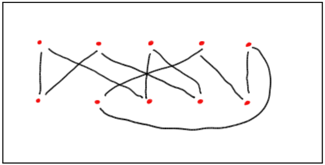

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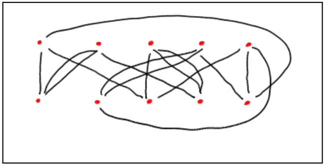

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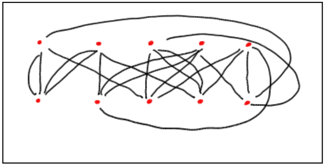

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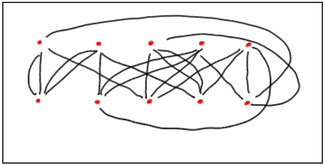

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The Theorem means: there exists $\delta > 0$ such that

$$\lim_{n \to +\infty} \frac{1}{(n!)^4} |\{(\sigma_1, \dots, \sigma_4) \mid h(\Gamma(\sigma_1, \dots, \sigma_4)) \geq \delta\}| = 1.$$

The motivation of Barzdin and Kolmogorov

Barzdin indicates in the notes of the selected works of Kolmogorov:

> *Unfortunately, I do not remember what was the occasion or event at which Andrei Nikolayevich first mentioned these results (I was not present there). I know only that the topic discussed there was the explanation of the fact that the brain (...) is so constituted that the most of its mass is occupied by nerve fibers (axons), while the neurons are only disposed on its surface. The construction of Theorem 1 precisely confirms the optimality (in the sense of volume) of such a disposition of the neuron network.*

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Valiant (1994–2005) has suggested possible algorithms to model realistically certain basic operations that must be performed by the brain; he observes that these methods require some expansion properties of the neuron network:

> *The property of expansion (...) is an archetypal such property. (This property, widely studied in computer science, was apparently first discussed in a neuroscience setting [BK].) The vicinal algorithms for the four tasks considered here need some such connectivity properties. In each case random graphs with appropriate realistic parameters have it, but pure randomness is not necessarily essential.*

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APPENDIX

Proof of Lemma 1.

An auxiliary concept is introduced. An elementary graph f with n inputs and r outputs is called an m expanding graph if any set of $k \leqslant m$ inputs is connected by edges to at least k outputs.

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Remark. There is only one non-constructive part in the proof of the Theorem – the proof of existence of an expanding graph in Lemma 1. Recently, G.A.Margulis [4] has get regular methods for constructing such graphs.

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- Because this might be required in applications (to construct efficient networks, for instance).
- To understand the nature of the expansion property... In particular, to have methods to prove that certain *concretely given* graphs are expanders, or not.

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Margulis graphs. The set of vertices is $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$ (hence there are n^2 vertices); edges connect (a, b) to the vertices

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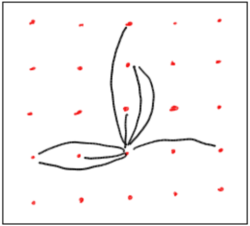
Example: for $n = 5$, the neighbors of $(2, 3)$ are $(0, 3)$, $(4, 3)$, $(2, 0)$, $(2, 1)$, $(1, 3)$, $(0, 3)$, $(2, 1)$, $(2, 2)$.



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- Are there other remarkable properties or applications of expander graphs?

The combinatorial Laplace operator

For a finite non-oriented graph Γ, with vertex set $S \neq \emptyset$, we define $C(\Gamma)$ to be the vector space of functions $f : S \longrightarrow \mathbf{C}$. It is a finite-dimensional space, and its dimension is equal to the number of vertices.

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Assuming the degree at each x is at least 1, we then consider the linear operator ("combinatorial" laplacian)

$$\Delta : C(\Gamma) \longrightarrow C(\Gamma)$$

defined by

$$(\Delta f)(x) = f(x) - \frac{1}{d(x)} \sum_{y \text{ relié à } x} f(y).$$

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We denote

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{|S|-1}$$

the eigenvalues of Δ ("spectrum of the graph").

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In particular, a sequence (Γ_n) of graphs with bounded degree is an expanding family *if and only if* there exists $\delta > 0$ (a "spectral gap") such that

$$\lambda_1(\Gamma_n) \geq \delta$$

for all $n \geq 1$.

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$$\left(\frac{d_-^2}{2d_+}\right)\lambda_1 \leq h \leq d_+\sqrt{2\lambda_1}$$

where $d_- \geq 1$ and d_+ are the minimal and maximal degrees, respectively.

In particular, a sequence (Γ_n) of graphs with bounded degree is an expanding family *if and only if* there exists $\delta > 0$ (a "spectral gap") such that

$$\lambda_1(\Gamma_n) \geq \delta$$

for all $n \geq 1$.

Using (fancy) numerical linear algebra, it is then possible to compute λ_1, hence to estimate h, for very large graphs (up to a billion vertices).

Digression

- Why "laplacian"? If we have a function f of C^3 class on $\mathbf{R}^2$, a Taylor expansion shows that

$$
\begin{aligned}
-\Delta f(a,b) &= -\frac{\partial^2 f}{\partial x^2}(a,b) - \frac{\partial^2 f}{\partial y^2}(a,b) \\
&= \lim_{h\to 0} \frac{1}{h^2}\Big(f(a,b) - \frac{1}{4}(f(a+h,b) + f(a-h,b) + \\
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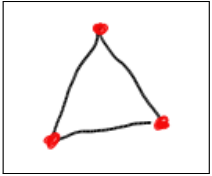
- The spectrum of the Laplace operator has many other applications; for instance, if the graph is regular of degree k, the number of *triangles* in the graph is equal to

$$\frac{k^3}{6}\operatorname{Tr}((\mathrm{Id}-\Delta)^3) = \frac{k^3}{6}\sum_i (1-\lambda_i)^3.$$



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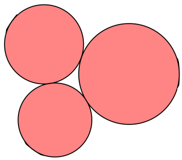
- Geometry;
- Number theory and arithmetic geometry;
- Group theory;
- Theoretical computer science;
- Operator theory;
- Combinatorics;
- And many others...

Application: geometry, arithmetic and sieve

A classical result from plane geometry:
given three circles ($\bigcirc_1, \bigcirc_2, \bigcirc_3$) which are pairwise tangent with disjoint interiors,



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A classical result from plane geometry: given three circles $(\bigcirc_1, \bigcirc_2, \bigcirc_3)$ which are pairwise tangent with disjoint interiors, there exist two other circles $(\bigcirc_4, \bigcirc_4')$ such that

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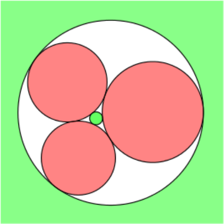

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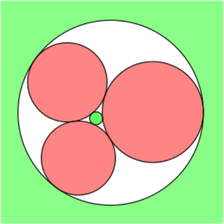

(With the convention that circles with radius < 0 are allowed, in which case the “l’intérieur” is the complement of the disc bounded by the circle.)

Circle packings

Starting with four circles
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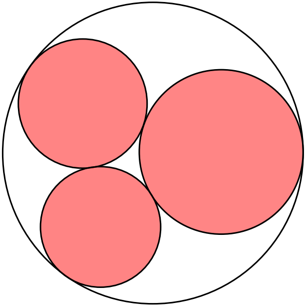

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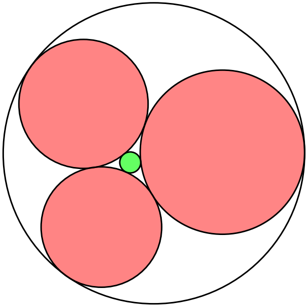

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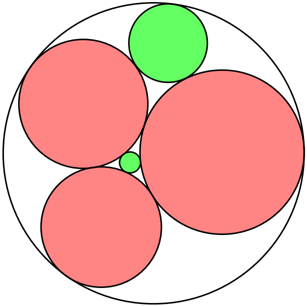

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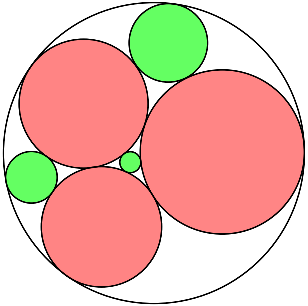

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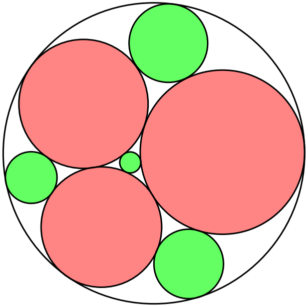

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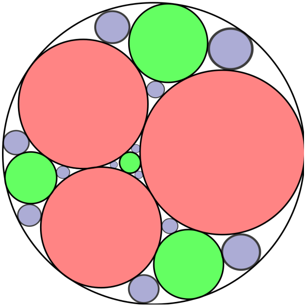

Integral circle packings

Descartes proved that the curvatures $(c_1, c_2, c_3, c_4) = (1/r_1, 1/r_2, 1/r_3, 1/r_4)$ of the four circles $(\bigcirc_1, \bigcirc_2, \bigcirc_3, \bigcirc_4)$ in such a configuration satisfy

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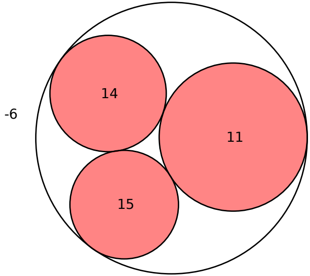

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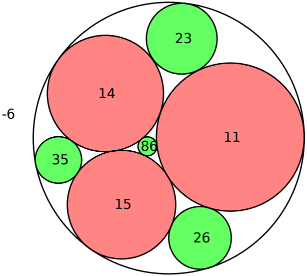

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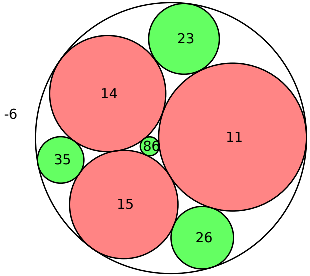

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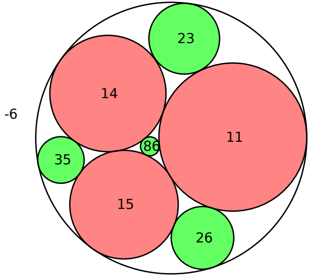

Question. (Graham, Lagarias, Mallows, Wilks, Yan, 2003) What are the properties of the set of integers that appear in such a packing? For instance, does this set contain infinitely many prime numbers?

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- (Bourgain–Kontorovich, 2014) “Almost all” those integers $n \geq 1$ for which it is not the case that $r(n) = 0$ for “obvious reasons” satisfy $r(n) \geq 1$.

Which graphs are involved

A direct computation shows that, at each stage, the curvatures are obtained by formulas like

$$(c_1', c_2, c_3, c_4) = (c_1, c_2, c_3, c_4)^t A_1,$$

and similarly with matrices A_2, A_3, A_4; these 4×4 matrices have integral coefficients. For instance

$$A_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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The set of all curvatures is therefore the set of coordinates of vectors of the form $v_0 B$, where v_0 corresponds to the initial four curvatures, and B runs over the subgroup of the group of 4×4 matrices generated by $\mathcal{S} = (A_1, A_2, A_3, A_4)$: the set $\mathcal{A}$ of all products of matrices A_i and of their inverses (because $A_i^{-1} = A_i$).

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For any integer $q \geq 1$, if we consider the set $\mathcal{A}_q$ of matrices with coefficients in $\mathbf{Z}/q\mathbf{Z}$ that are obtained by replacing each coefficient of A_i and of the elements $B \in \mathcal{A}$ with their values modulo q, we obtain a finite group.

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Theorem (Varjú 2013; Helfgott, Bourgain–Gamburd, Bourgain–Gamburd–Sarnak) The sequence of graphs (Γ_q) is a family of expanders.

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$$\left| \frac{1}{4^N} |\{(B_1, \ldots, B_N) \in \mathcal{S}^N \mid B_1 \cdots B_N \equiv B \pmod{q}\}| - \frac{1}{|\mathcal{A}_q|} \right| \leq (1-\delta)^N.$$

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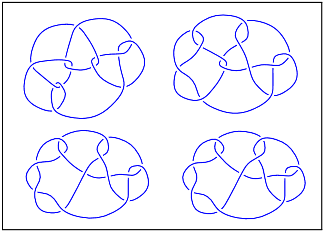
This means that, in the products of matrices defining a complicated element of $\mathcal{A}$, each reduction modulo q has "the same chance" of appearing, as long as q is not too large compared with the length N of the product $B_1 \cdots B_N$. Precisely, since $|\mathcal{A}_q| \leq q^{16}$, the error is negligible as long as

$$(1-\delta)^N \leq 10^{-10} q^{-16}$$

which means that q may grow exponentially with q, *because* δ *is independent of* q.

Application: complicated knots

Around 1980, Gromov asked whether complicated knots exist: it would be impossible to put them in space in such a way that their *distorsion* be small.



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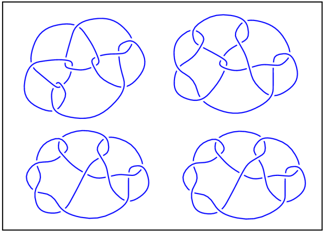

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$$\delta(N) = \min_{\substack{\gamma:[0,1]\to\mathbf{R}^3 \\ \text{realizing } N}} \sup_{\substack{(x,y)\in\gamma \\ x\neq y}} \frac{d_\gamma(x,y)}{\|x-y\|},$$

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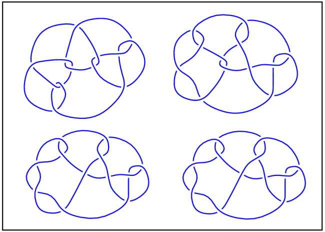

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To say that $\delta(N)$ is “large” means that, whichever way one puts the knot in space, there will be points “close” in space which are “far away” along the knot.

Gromov-Guth (and Pardon)

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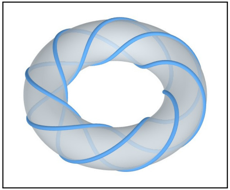
The proof of Pardon is constructive and direct: for toric knots $T_{p,q}$, he gives a lower-bound for the distortion in terms of p and q. (See `http://images.math.cnrs.fr/Des-Noeuds-Indetordables.html`)



$T_{8,3}$; picture B. Klœckner

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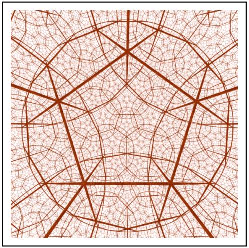

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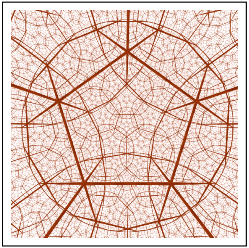

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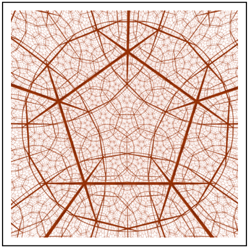

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Gromov and Guth prove that there exists $\alpha > 0$ such that

$$\delta(N_n) \geq \alpha |S_n| h(\Gamma_n)$$

for all n, where S_n is the set of vertices of Γ_n.

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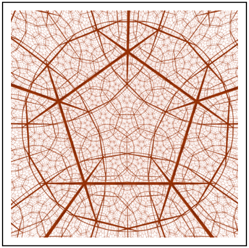

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Finally, they know that (Γ_n) is a sequence of expanding graphs, for subtle reasons close to that used by Margulis ("Property (τ) of Lubotzky"; Selberg, Clozel).

Some some open questions

- Obtain "reasonable" estimates for the Cheeger constant of graphs like the Cayley graphs of the matrix group modulo p generated by

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \pmod{p}, \qquad \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \pmod{p}$$

(known: $\lambda_1 \geq 2^{-2^{80}}$; if 3 is replaced by 1, about $1/100$...)

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- Find efficient *algorithms* to navigate in certain expanders; for instance, find an efficient algorithm to express

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- And find new applications!

Some references

- (Expanders in general) S. Hoory, N. Linial and A. Wigderson: *Expander graphs and their applications*, Bull. AMS 43 (2006), 439–561.
- (Apollonian circle packings) E. Fuchs: *Counting problems in Apollonian packings*, Bull. AMS 50 (2013), 229-266; A. Kontorovich: *From Apollonius to Zaremba: local-global phenomena in thin orbits*, Bull. AMS 50, (2013), No. 2, 187–228.
- (Applications) A. Lubotzky: *Expander graphs in pure and applied mathematics*, AMS Colloquium Lectures 2011, Bull. AMS 49 (2012), 113–162.
- (Arithmetic applications) *Crible en expansion*, Séminaire Bourbaki, exposé 1028; in Astérisque 348 (2012), 17–64.