Families of L-functions and cusp forms

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An individual L-function is, roughly, a Dirichlet series which has two strong "regularity" properties: an Euler product (multiplicativity) and a functional equation (analytic continuation), both of a very special type.

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}, \qquad \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\tfrac{1-s}{2})\zeta(1-s).$$

The notion of a "family" has been important in recent years both as a heuristic guide to understand or guess many important properties of L-functions, and to prove things about them.

(cont.)

However, there is no accepted formal definition of a family of L-functions. This talk will try to make suggestions along these lines.

This will suggest some interesting problems of analytic number theory which haven't been studied yet in any great detail, and also reveal some analogies with general sieve methods. These analogies in particular may help to reach a better understanding of the general situation.

[References. "The large sieve and its applications", CUP 2008; various texts in progress; a letter of Sarnak to Conrey, Iwaniec, Michel, Soundararajan (Nov. 6, 2008; not yet published), and comments.]

Why families?

There are two basic reasons to insert an individual L-function in a family.

- Study *statistic* behavior; e.g., try to determine the asymptotics of

$$\sum_{f\in S_2(q)^*} \mathrm{ord}_{s=1/2} L(f,s) \overset{\text{(supposedly)}}{=} \quad \mathrm{rank}\, J_0^{new}(q)(\mathbf{Q})$$

  (Brumer, K.-Michel, VanderKam).
- Obtain *individual* results:

$$\int_T^{T+T^{2/3}} |\zeta(\tfrac{1}{2}+it)|^4 dt \ll_\varepsilon T^{2/3+\varepsilon}$$

$$\Longrightarrow \quad \zeta(\tfrac{1}{2}+it) \ll_\varepsilon (1+|t|)^{1/6+\varepsilon}$$

  (a proof by Iwaniec of Weyl's estimate)

Examples of empirical families

Many examples have been considered in analytic number theory, particularly in recent years. Here are some of them:

- Primitive Dirichlet character to a given modulus q;
- Primitive Dirichlet character to any modulus $q \leq Q$;
- Values of $\zeta(1/2 + it)$ for t in some interval;
- Primitive holomorphic forms (newforms) of given level and weight and nebentypus;
- Same with weight $k \leq K$, or with eigenvalue $\lambda \leq T$, or with given root number $+1$;
- Rankin-Selberg convolutions $f \times g$ where f is fixed and g varies in one of the earlier families...

General remark

The basic idea is to have "finite" sets of L-functions, which are in some sense "similar", depending on one or more parameters, and to study what happens when the parameter(s) grow.

Finding a good formal framework might be important, first, to formulate general conjectures and then hopefully to make progress on their solution.

We start by describing a few principles that seem reasonable to find the right definition.

First principle:

The L-function is just one invariant (maybe the one of most interest!) of an *automorphic form* (or representation). The latter are more natural objects to put in families. One can also think of L-functions of more algebraic objects (Artin L-functions, Hasse-Weil L-functions) where the object in question displays more information.

Example

If we twist a cusp form of level p^4 with a Dirichlet character of level p^2, getting an L-function of the type

$$L(f \times \chi, s) = \sum_{n \geq 1} \lambda_f(n)\chi(n)n^{-s},$$

getting the "right" conductor and expression is not entirely clear. Similarly, guessing the gamma factor from the coefficients of the L-functions is not obvious (though in principle it is possible...)

Second principle:

Automorphic forms (hence L-functions) have a local-global nature that is exemplified either by the Euler product

$$L(f,s) = \prod_p L_p(f, p^{-s})^{-1}$$

or by the Hecke-eigenvalue property, or more generally (and better) by the automorphic factorization

$$\pi_f = \bigotimes_v \pi_{f,v} \quad \Longrightarrow \quad \Lambda(f,s) = \prod_v L(\pi_{f,v}, s),$$

for the automorphic representation π_f associated to a form of degree n, where $\pi_{f,v}$ are representations of the local groups $GL(n, \mathbf{Q}_v)$.

Third principle:

Many analytic properties of "families" make sense at the level of "finite" conductors. This means, for instance, that one can prove asymptotic formulas like

$$\sum_{0 \leq d \leq X} L(\chi_{8d}, \tfrac{1}{2})^3 \sim cX(\log X)^6 \qquad \text{(Soundararajan)}$$

for a given X without using any property of characters not entering in the sums.

[Here the conductor is the *analytic conductor* of Iwaniec-Sarnak

$$\mathfrak{q}(f) = q(f) \prod_{1 \le j \le n} (3 + |\kappa_j|)$$

where $q(f) \ge 1$ is the arithmetic conductor appearing in the functional equation

$$\Lambda(f, s) = \varepsilon(f) q(f)^{1/2 - s} \Lambda(\bar{f}, 1 - s),$$

and

$$\gamma(f, s) = \prod_{1 \le j \le n} \Gamma\Big(\frac{s + \kappa_j}{2}\Big)$$

is the gamma factor. This is a natural "height" to measure the rough complexity of an automorphic form on $GL(n)/\mathbf{Q}$.]

Fourth principle:

Well-behaved (global) families should also be well-behaved *locally*. Indeed, checking this should be a prerequisite to the study of a new family of global automorphic forms, and should provide valuable information for that purpose. This principle is currently used only implicitly in most works I know. However, I believe it should be considered much more systematically.

Example

The following (conjectural) statement is considered important because until around 2000 and the work of Keating-Snaith, there was no convincing precise conjecture:

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt \overset{(?)}{\sim} \Phi_1(k)\Phi_2(k)(\log T)^{k^2}$$

where

$$\Phi_1(k) = \frac{G(1+k)^2}{G(1+2k)},$$

$$\Phi_2(k) = \prod_p \Big(1-\frac{1}{p}\Big)^{k^2} \sum_{m\geq 0} \Big(\frac{(k+m-1)!}{(k-1)!m!}\Big)^2 p^{-m}.$$

How to understand this better?

The analogy with sieve

In classical sieve method, we have global objects (integers or a subset of them), local information given by reduction modulo primes, and we try to understand averages of interesting arithmetic functions defined on the integers, e.g., characteristic functions of the primes or of "almost" primes, etc.

The basic "sieve axiom" that is used in most treatments asks that the function $a_n \geq 0$ of interest satisfies

$$\sum_{n \equiv 0 \pmod{d}} a_n = \rho(d) X + r_d$$

where ρ is a multiplicative function of d and X a certain positive constant. (Often $X = \sum a_n$).

(Cont.)

From suitable assumptions on the remainder term r_d, one derives explicit inequalities or asymptotic formulas for, e.g.,

$$\sum_{p|n \Rightarrow p>z} a_n$$

with main term given by

$$X \prod_{p \leq z} (1 - \rho(p)).$$

[The interpretation is that $\rho(p)$ measures the *distribution* of a_n modulo p, relative to the equidistribution of integers in residue classes, and the multiplicativity reflects the *independence* of these reductions; assumptions like

$$\sum_{d<D} |r_d| \ll X(\log X)^{-A}$$

reflect *quantitatively* the equidistribution modulo primes.]

Reminder: equidistribution

Let μ_n be a sequence of probability measures on a topological space X; the μ_n *become equidistributed with respect to a measure* μ if, for every bounded continuous function f on X, we have

$$\int_X f(x) d\mu_n(x) \to \int_X f(x) d\mu(x).$$

Example

Let $X = [0, 1]$ and

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{j/n}$$

then μ_n becomes equidistributed with respect to Lebesgue measure.

Small and extended families

For $n \geq 1$, let Aut_n be the automorphic spectrum of $GL(n)/\mathbf{Q}$. If we restrict to cusp forms, it is a countable set.

- A *small family* of automorphic forms of degree n is a (measurable) map $\Pi : (X, \mu) \to \mathrm{Aut}_n$, where (X, μ) is a measure space with $\mu(X) < +\infty$.
- An *extended family* is a collection Π_T of small families parametrized e.g. by $T \geq 1$, where the limit $T \to +\infty$ is of interest.

It is very likely that additional conditions are required!

Sarnak's definition

In an unpublished letter, P. Sarnak suggests a definition which (in the language above) corresponds to specifying precisely a type of *extended families* which are likely to be particularly well-behaved, and that encompass most (but not quite all) examples which have been considered.

In more detail, he looks at an (usually infinite) subset $\mathfrak{F}$ of Aut_n (subject to specific rules) and considers extended families with

$$\mathfrak{F}_T = \{f \in \mathfrak{F} \mid \mathfrak{q}(f) \leq T\}.$$

It is part of the theory that these sets are supposed to be finite, and they are considered with counting measure.

Sarnak's types of families

The building blocks of Sarnak's definition are the following families:

- (Type I) A single form;
- (Type II) Forms with fixed central character, or self-dual forms with a given root numbers;
- (Type III/"CM" forms) Forms with local components equidistributed (in Sato-Tate sense) in some subgroup $H \subset GL(n, \mathbf{C})$;
- (Type IV) Forms with local components in some open set of the spectrum at $v \in S$, S finite, and either unramified or arbitrary outside S;
- (Type V) Geometric families;
- (Type VI) Images of functoriality maps.

He allows imposing more than one such condition, and he then considers possible products of finitely many such families.

Examples

Here are some examples of the families that this definition allows:

- (Type II) Even real Dirichlet characters;
- (Type III) All holomorphic CM modular forms with a fixed CM field;
- (Type IV) Classical modular forms with weight k ($S = \{\infty\}$), or with $\lambda_f(2) > 0$;
- (Type V) Cusp forms associated to a family $y^2 = x^3 + a(t)x + b(t)$ of elliptic curves, $t \in \mathbf{Q}$;
- (Type VI) Symmetric squares of classical modular forms of weight 2.

Why small families?

Allowing (X, μ) arbitrary finite measure space instead of a finite set with counting measure has the following possible advantages:

- Multiplicities might be naturally present and very hard to get rid of analytically (e.g., type V above; type VI also);
- This allows smoothing, e.g., taking $X = $ (Maass forms for $SL(2, \mathbf{Z})$), $\mu(\{f\}) = \Phi(\lambda_f / T)$ for some smooth function Φ on $[1/4, +\infty[$;
- This puts discrete and continuous families on the same footing, e.g., $X = [0, T]$, $\Pi(x) = |\cdot|^{it}$ with $L(\Pi(x), s) = \zeta(s + it)$;
- The nice finite sets $S_2(q)^*$ or Dirichlet characters modulo a fixed q, do not fit well in the previous definition because the extended family is not the union of the small ones.

Local spectrum examples

Let's now consider local families associated with a small family. This requires looking a bit at the nature of the local spectrum where the components $\pi_{f,v}$ lie. We consider only $v = p$ finite for simplicity (there are very unclear issues for $v = \infty$).

Example

The first example is $GL(1)/\mathbf{Q}$. Things are quite explicit here: the local spectrum at p is the dual group of $\mathbf{Q}_p^\times$ and splits as $U_p \times \mathbf{S}^1$, where U_p is the dual of $\mathbf{Z}_p^\times$ and $\mathbf{S}^1$ corresponds to *unramified* characters. A primitive character χ modulo q maps to $(\chi^{(p)}, \theta)$ where we factor (a specific feature of this examples) $\chi = \chi^{(p)}\chi_1$ with $\chi^{(p)}$ of conductor (q, p^∞) and χ_1 unramified at p, and then

$$\chi_p = (\chi^{(p)}, \chi_1(p)),$$

seeing $\chi^{(p)} : \mathbf{Z}_p^\times \to (\mathbf{Z}/p^\nu\mathbf{Z})^\times \xrightarrow{\chi^{(p)}} \mathbf{C}^\times$.

(Cont.)

Example

Consider $GL(2)/\mathbf{Q}$. The local spectrum is again a disjoint union of nice compact spaces. If the nebentypus is trivial, the unramified part of the (tempered) spectrum can be identified with $[-2, 2]$ or with the set of conjugacy classes in $SU(2, \mathbf{C})$.

Global cusp forms correspond to Maass or holomorphic newforms. If $p \nmid q(f)$ (unramified case), and trivial nebentypus, the p-component is given by the p-Hecke eigenvalue $\lambda_f(p) \in [-2, 2]$ (if Ramanujan-Petersson holds).

The ramified components are much more delicate to describe in the ramified case, especially if $p^2 \mid q(f)$ where *supercuspidal representations* can occur; those have $L(\pi, s) = 1$, so the L-function gives no information about their behavior. Yet they contain for instance part of the root numbers.

Local equidistribution question

Here is then an obvious question: given any family (e.g., the extended families from Sarnak's definition, or any other) and a place v, how are the local components $\pi_{f,v}$ distributed?
With each small family Π we can construct an image measure μ_Π on the local spectrum by

$$\mu_\Pi(U) = \mu(\{x \in X \mid \Pi(x)_v \in U\}),$$

and then the best answer one can expect is:

Question

In an extended family $(\Pi_T)_T$, is there equidistribution of $\mu_T = \mu_{\Pi_T}$ as $T \to +\infty$? With respect to which measure? If p is replaced with finitely many primes S, is there independent equidistribution over S?

Guess

The motivation is the following: if this holds for all v, with limiting measures μ_v, and if those and the measures in the extended family are all probability measures $\mu(\Pi_T) = 1$, then the average behavior of the L-function at s_0 should be largely dictated by the expectation that

$$\int_{X_T} L(\Pi(x), s_0)^k d\mu(x) \approx \prod_v \int_{S_v} L(\pi, s_0)^k d\mu_v(\pi),$$

where S_v is the local spectrum at v. If s_0 is on the critical line, one must be much more careful!

An example

Consider $(X_T, \mu_T) = ([0, T], T^{-1}dt)$ and $L(\Pi(t), s) = \zeta(s + it)$. The local p-component is unramified and is given by $\Pi(t)_p = p^{it} \in \mathbf{S}^1$. Take $S = \{2, 3, \ldots, p_k\}$; then there is S-equidistribution by Kronecker's Theorem: for reasonable f, we have

$$\frac{1}{T}\int_0^T f(2^{it}, \ldots, p_k^{it})dt \to \int_{(\mathbf{S}^1)^k} f(\theta_1, \ldots, \theta_k)d\theta_1 \cdots d\theta_k.$$

Guess: the average of $|\zeta(\frac{1}{2} + it)|^{2k}$ should be related to

$$\prod_p \int_0^1 |1 - e(\theta)p^{-\frac{1}{2}}|^{2k} d\theta = \prod_p \sum_{m \geq 0} \Big(\frac{(k+m-1)!}{(k-1)!m!}\Big)^2 p^{-m}.$$

Refined local equidistribution

For further work it is absolutely essential to have not only local equidistribution everywhere, but also independence, and a quantitative form over a range of S as large as possible. Even a proper formulation is not so easy since, in contrast to sieve, the local target S_v is typically an infinite compact set (e.g., $[-2, 2]$) for which infinitely many test functions are required in the Weyl Criterion. This is also crucial to detect the *symmetry type* which is supposed to be the group-theoretical cause explaining the factors like $\Phi_1(k)$ in the moment conjecture for $\zeta(s)$. Indeed, the standard "explicit formula" argument leads to a guess if one knows local equidistribution modulo all primes, *and* the average behavior over primes.

Examples of local equidistribution

A number of existing results can be interpreted to cases of local spectral equidistribution, usually at a fixed prime or finite set of primes in the unramified case. The $GL(1)$ situation is also very simple to look at to have a guess of what may happen. In general the most striking aspects may be the following: even when there is local equidistribution, the limiting measures may be quite varied! This can be contrasted with the two fields of number theory where equidistribution statements are most important: Deligne's Equidistribution Theorem, and ergodic-theory areas (Ratner's Theorems), where the equidistribution is always with respect to some canonical measure.

(Cont.)

The unfixedness of limit is of interest partly because there *is* a fairly natural measure on the local spectrum S_v, namely the *Plancherel measure* μ_v^P, which is defined (from a choice of Haar measure on $GL(n, \mathbf{Q}_v)$) by a "Fourier inversion formula"

$$f(1) = \int_{S_v} \hat{f}(\pi) d\mu_v^P(\pi),$$

for reasonable functions f and some Fourier transform $\hat{f}$ on the local spectrum. For $GL(1)$, where S_v is the dual group, the Plancherel measure is the Haar measure on the dual group for which the Fourier inversion formula holds with $\hat{f}(\chi) = \int f(x)\chi(x^{-1})dx$.

The $GL(1)$ case

This is of course very simple and explicit. Let $\mathcal{F}_q = \{\chi \,(\mathrm{mod}\, q) \text{ primitive}\}$, and $\mathcal{F}_{\leq Q}$ the union of $\mathcal{F}_q$ over $q \leq Q$, with probability counting measure. Fix a prime p and recall that $S_p = U_p \times \mathbf{S}^1$.

- As $q \to +\infty$ with $(q, p) = 1$, $\{\chi_p \mid \chi \in \mathcal{F}_q\}$ becomes equidistributed with respect to the Plancherel measure normalized so that the unramified part has measure 1. *Proof*: check that

$$\lim_{(q,p)=1} \frac{1}{|\mathcal{F}_q|} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \chi(p)^m = 0, \qquad \text{if } m \in \mathbf{Z} - \{0\}.$$

(Cont.)

- Let $\nu \geq 0$ fixed. As $q \to +\infty$ with $(q, p^\infty) = p^\nu$, the χ_p for $\chi \in \mathcal{F}_q$ become equidistributed with respect to the normalized Plancherel measure on $\{\psi \pmod{p^\nu}\} \times \mathbf{S}^1$.
- As $Q \to +\infty$, the χ_p with $\chi \in \mathcal{F}_{\leq Q}$ become equidistributed with respect to a measure given by $c_\nu d\theta$ on each component $\{\psi\} \times \mathbf{S}^1$ with ψ of conductor p^ν, for some (explicit) $c_\nu > 0$. In particular this is not a multiple of the Plancherel measure!

The $GL(2)$ case

Historically, the first example of local spectral equidistribution is implicit in the 1978 paper *Fourier coefficients of cusp forms* of Bruggeman where it is shown that the Ramanujan-Petersson conjecture at p holds for most Maass forms. In the language above, let $\mathcal{M}_T$ be the set of primitive Maass forms for $SL(2, \mathbf{Z})$ with $\lambda_f \leq T$. Fix p and put

$$\mu(\{f\}) = \frac{1}{\|f\|^2}.$$

Then as $T \to +\infty$, the p-component (given by the p-Hecke eigenvalue) becomes equidistributed with respect to

$$\mu_{ST} = \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}\mathbf{1}_{|x|\leq 2}dx$$

on $\mathbf{R}$.

(Cont.)

Other results:

- Sarnak (1984) considers the problem for unramified primes in some generality; for Maass forms *with counting measure*, he finds that the limit is

$$\mu_p = \frac{p+1}{(p^{1/2}+p^{-1/2})^2 - x^2} d\mu_{ST},$$

  and he shows independence for multiple primes. This measure is the Plancherel measure, normalized with the tempered spectrum $[-2,2]$ having measure 1.

Corollary

Fix S finite set of primes. The values $(\lambda_f(p))_{p\in S}$ are dense in $[-2,2]^{|S|}$.

(Cont.)

- Serre (1997) obtains the same result for holomorphic cusp forms of weight k, q with $k \geq 2$, $k + q \to +\infty$ and $p \nmid q$. Royer (2000) gave quantitative versions.
- Expanding a bit these considerations one gets proper *sieve results*: for certain sets $\Omega_p \subset [-2, 2]$, we can estimate the number of $f \in S_k(q)^*$ with $\lambda_f(p) \notin \Omega_p$ for p in some set of primes.

Question

What about equidistribution at ramified places? E.g., for $S_2(p^3 q)$, $(q, p) = 1$, $q \to +\infty$*?*

Algebraic families

For an algebraic family $E_t : y^2 = x^3 + a(t)x + b(t)$, non-constant, let $\mathcal{A}_T = \{E_t\}_{|t| \leq T}$. For $p \nmid \Delta(t)$, the local representation at p is given by

$$a_p(t) = -\frac{1}{\sqrt{p}} \sum_{x \pmod{p}} \left(\frac{x^3 + a(t)x + b(t)}{p}\right).$$

This is periodic in t modulo p; so local equidistribution holds with limit a weighted sum of Dirac masses at $j/\sqrt{p} \in [-2, 2]$ with $|j| \leq 2\sqrt{p}$.

Whither universality?

In view of this diversity of local behavior, why should one expect almost universal asymptotics, say, for moments of special values in families? The answer seems to be that μ_p, as $p \to +\infty$, does *itself* converge to some very universal measures! For instance, the μ_p on $[-2, 2]$ do converge to μ_{ST} as p grows. For algebraic families, the corresponding fact is also true (in great generality) by the "vertical Sato-Tate laws" of Deligne, Katz, Michel, based on Deligne's Equidistribution Theorem.