# Counting the $\zeta$ zeroes in distinct intervals

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Random matrices, L-functions and primes

Gaussian fluctuations for the number of eigenvalues in distinct arcs of the circle (Wieand, Diaconis-Evans). Analogous result for $\zeta$ ?

Conjecture for the $\limsup$ of $|\zeta|$.

The Keating-Snaith central limit theorem

$$u_n \sim \mu_{U(n)}$$

$$\frac{\log \det(\mathrm{Id} - u_n)}{\sqrt{\frac{1}{2}\log n}} \xrightarrow{\text{law}} \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2$$

as $n \longrightarrow \infty$.

Selberg's central limit theorem

$\omega$ uniform on $(0, 1)$

$$\frac{\log \zeta\left(\frac{1}{2} + \mathrm{i}\omega t\right)}{\sqrt{\frac{1}{2}\log \log t}} \xrightarrow{\text{law}} \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2$$

as $t \longrightarrow \infty$.

$\mathcal{N}_1$ and $\mathcal{N}_2$ independent standard real normal variables

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$\epsilon_n \to 0$, $\epsilon_n \gg 1/n$, $u_n \sim \mu_{U(n)}$

$$\frac{\log \det(\mathrm{Id} - e^{-\epsilon_n} u_n)}{\sqrt{-\frac{1}{2}\log \epsilon_n}} \xrightarrow{\text{law}} \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2$$

Normalization $\sqrt{\frac{1}{2}\log n}$ if $\epsilon_n \ll 1/n$

$\epsilon_t \to 0$, $\epsilon_t \gg 1/\log t$, $\omega$ uniform on $(0,1)$

$$\frac{\log \zeta\left(\frac{1}{2} + \mathrm{i}\epsilon_t + \mathrm{i}\omega T\right)}{\sqrt{-\frac{1}{2}\log \epsilon_t}} \xrightarrow{\text{law}} \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2$$

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$N_n(\alpha,\beta)$ : number of eigenvalues $e^{\mathrm{i}\theta}$ of $u_n$ with $\alpha<\theta<\beta$,
$\delta_n(\alpha,\beta)=N_n(\alpha,\beta)-\mathbb{E}_{\mu_{U(n)}}(N_n(\alpha,\beta))$,

$$\delta_n(\alpha,\beta)=\frac{1}{\pi}(\mathfrak{Im}\log Z(u_n,e^{\mathrm{i}\beta})-\mathfrak{Im}\log Z(u_n,e^{\mathrm{i}\alpha}))$$

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$N(t)$ for the number of zeroes $z$ of $\zeta$ with $0<\mathfrak{Im} z\leq t$

$$N(t)=\frac{t}{2\pi}\log\frac{t}{2\pi e}+\frac{1}{\pi}\mathfrak{Im}\log\zeta(1/2+\mathrm{i}t)+\frac{7}{8}+O\left(\frac{1}{t}\right)$$

$$\Delta(t_1,t_2)=(N(t_2)-N(t_1))-\left(\frac{t_2}{2\pi}\log\frac{t_2}{2\pi e}-\frac{t_1}{2\pi}\log\frac{t_1}{2\pi e}\right),$$

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$N(t)$ for the number of zeroes $z$ of $\zeta$ with $0 < \mathfrak{Im} z \le t$

$$N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi e} + \frac{1}{\pi}\mathfrak{Im}\log\zeta(1/2 + it) + \frac{7}{8} + O\left(\frac{1}{t}\right)$$

$$\Delta(t_1,t_2) = (N(t_2) - N(t_1)) - \left(\frac{t_2}{2\pi}\log\frac{t_2}{2\pi e} - \frac{t_1}{2\pi}\log\frac{t_1}{2\pi e}\right),$$

**Wieand, 1998** As $n \to \infty$, the finite-dimensional distributions of the process

$$\frac{\delta_n(\alpha, \beta)}{\frac{1}{\pi}\sqrt{\log n}}, \ 0 \leq \alpha < \beta < 2\pi,$$

converge to those of a Gaussian process $\{\delta(\alpha, \beta), 0 \leq \alpha < \beta < 2\pi\}$ with covariance function

$$\mathbb{E}\left(\delta(\alpha, \beta)\delta(\alpha', \beta')\right) = \begin{cases} 1 & \text{if } \alpha = \alpha' \text{ and } \beta = \beta' \\ 1/2 & \text{if } \alpha = \alpha' \text{ and } \beta \neq \beta' \\ 1/2 & \text{if } \alpha \neq \alpha' \text{ and } \beta = \beta' \\ -1/2 & \text{if } \beta = \alpha' \\ 0 & \text{elsewhere} \end{cases} .$$

**Analogue on $\zeta$.** $\omega$ uniform on $(0,1)$. As $t \to \infty$, the finite-dimensional distributions of the process

$$\frac{\Delta(\omega t+\alpha, \omega t+\beta)}{\frac{1}{\pi}\sqrt{\log\log t}},\ 0 \leq \alpha < \beta < 2\pi,$$

converge to those of a Gaussian process $\{\delta(\alpha,\beta), 0 \leq \alpha < \beta < 2\pi\}$ with covariance function

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For $\alpha_1 < \cdots < \alpha_\ell$, convergence of

$$\frac{1}{\sqrt{\log\log t}}\left(\log\zeta(1/2+\mathrm{i}\omega t+\mathrm{i}\alpha_1),\dots,\log\zeta(1/2+\mathrm{i}\omega t+\mathrm{i}\alpha_\ell)\right)$$

to a vector of independent Gaussian random variables. Finite Dirichlet series are sufficient thanks to Selberg :

$$\frac{1}{t}\int_0^t \left|\log\zeta(1/2+\mathrm{i}s)-\sum_{p\leq t}\frac{p^{-\mathrm{i}s}}{\sqrt{p}}\right|^2 \mathrm{d}s < c.$$

## Landau

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n\le x}\frac{\Lambda(n)}{n^s} + \frac{x^{1-s}}{1-s} - \sum_{\rho}\frac{x^{\rho-s}}{\rho-s} + \sum_{n=1}^{\infty}\frac{x^{-2n-s}}{2n+s},$$

## Selberg

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n\le x^2}\frac{\Lambda_x(n)}{n^s} + \frac{x^{2(1-s)}-x^{1-s}}{(1-s)^2\log x} + \frac{1}{\log x}\sum_{\rho}\frac{x^{\rho-s}-x^{2(\rho-s)}}{(\rho-s)^2} + \frac{1}{\log x}\sum_{n=1}^{\infty}\frac{x^{-2n-s}-x^{-2(2n+s)}}{(2n+s)^2}$$

where

$$\Lambda_x(n) = \begin{cases} \Lambda(n) & \text{for } 1\le n\le x, \\ \Lambda(n)\frac{\log\frac{x^2}{n}}{\log n} & \text{for } x\le n\le x^2, \end{cases}$$

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Complex numbers $a_{pt}$ $(p \in \mathcal{P}, t \in \mathbb{R}^+)$, $\sup_p |a_{pt}| \to 0$, $\sum_p |a_{pt}|^2 \to \sigma^2$ as $t \to \infty$.
Existence of $(m_t)$ with $\log m_t / \log t \to 0$ and

$$\sum_{p>m_t} |a_{pt}|^2 \left(1 + \frac{p}{t}\right) \underset{t\to\infty}{\longrightarrow} 0.$$

Then, if $\omega$ is a uniform random variable on $(0,1)$,

$$\sum_{p\in\mathcal{P}} a_{pt} p^{-\mathrm{i}\omega t} \xrightarrow{\text{law}} \sigma Y$$

as $t \to \infty$, $Y$ being a standard complex normal variable.

Analogue of this result, which implies the fluctuations observed by Wieand.

**Diaconis-Evans, 2001.** Complex constants $\{a_{nj} \mid n \in \mathbb{N}, j \in \mathbb{N}\}$, $\lim_{n\to\infty} \sum_{j=1}^{\infty} |a_{nj}|^2 (j \wedge n) = \sigma^2$. Existence of positive integers $\{m_n \mid n \in \mathbb{N}\}$ such that $\lim_{n\to\infty} m_n/n = 0$ and

$$\lim_{n\to\infty} \sum_{j=m_n+1}^{\infty} |a_{nj}|^2 (j \wedge n) = 0,$$

Then, if $u_n \sim \mu_{U(n)}$, as $n \to \infty$

$$\sum_{j=1}^{\infty} a_{nj} \operatorname{Tr}(u_n^j) \xrightarrow{\text{law}} \sigma Y.$$

$$\limsup_{t\to\infty} \frac{\log|\zeta(1/2+\mathrm{i}t)|}{\sqrt{\log t/\log\log t}} \geq c$$

Montgomery, 1977 : on RH, $c = 1/20$
Soundararajan, 2007 : $c = 1$

Farmer, Gonek, Hughes, 2007. Analogy with a convergence in probability for matrices of increasing size :

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$$\frac{1}{\sqrt{\log\log n}}\left(\log\zeta(1/2+\mathrm{i}\omega n+\mathrm{i}),\dots,\log\zeta(1/2+\mathrm{i}\omega n+n\mathrm{i})\right)$$

**Resnick-Tomkins.** $(X_i)$ i.i.d. random variables and $F(x)=\mathbb{P}(X_1<x)$. Let $\mu_n=F^{-1}(1-1/n)$. Then

$$\max(X_1,\dots,X_n)/\mu_n\to 1$$

almost surely when $n\to\infty$ iff, for all $\delta>1$, $\int_1^\infty \frac{\mathrm{d}F(x)}{1-F(\delta x)}<\infty$.
For i.i.d. standard Gaussians,

$$\frac{\max(X_1,\dots,X_n)}{\sqrt{2\log n}}\to 1$$

almost surely. This supports the conjecture

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Shrinking intervals $(\alpha/K_t, \beta/K_t)$ :

- $K_t/\log t \to 0$ : Gaussian fluctuations.
- $K_t = \lambda \log t$ : Montgomery-Odlyzko law, the number of zeroes may converge to a discrete random variable (Gram law).

Correlations in the mesoscopic regime : $\delta \in (0,1)$,

$$\frac{1}{\sqrt{\frac{1}{2}\log\log t}}\left(\log\left|\zeta\left(\frac{1}{2}+\mathrm{i}\omega t\right)\right|, \log\left|\zeta\left(\frac{1}{2}+\mathrm{i}\omega t+\frac{\mathrm{i}}{(\log t)^{\delta}}\right)\right|\right)$$

converges in law to

$$(\mathcal{N}_1, \delta\mathcal{N}_1 + \sqrt{1-\delta^2}\mathcal{N}_2).$$