Distribution of gaps between zeros of the derivative of the Riemann ξ-function

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Riemann ζ-function

The Riemann ζ-function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (\sigma > 1).$$

- Riemann hypothesis: all the nontrivial zeros satisfy $\sigma = 1/2$. Denote the zeros by $1/2 + i\gamma$.

- Number of zeros up to height T

$$N(T) := \#\{0 < \gamma \le T : \zeta(1/2 + i\gamma) = 0\} \sim \frac{T}{2\pi} \log T.$$

- For $0 < \gamma \le \gamma'$ two consecutive ordinates of zeros, define

$$\delta(\gamma) = (\gamma' - \gamma)\frac{\log \gamma}{2\pi}.$$

  On average $\delta(\gamma) = 1$.

Define the distribution function

$$D(\alpha,T)=\left(\frac{T}{2\pi}\log T\right)^{-1}\sum_{\substack{0<\gamma\le T\\ \delta(\gamma)\le\alpha}}1.$$

$$D^+(\alpha)=\limsup_{T\to\infty}D(\alpha,T)\text{ and }D^-(\alpha)=\liminf_{T\to\infty}D(\alpha,T).$$

Expected: $D^+(\alpha)=D^-(\alpha)=D(\alpha)$, and

$$D(0)=0\qquad\text{and}\qquad D(\alpha)<1\text{ for all }\alpha.$$

Selberg (1940's)

$$D^-(\alpha_1)>0\qquad\text{and}\qquad D^+(\alpha_2)<1$$

for some $\alpha_1<1<\alpha_2$.

Conrey, Ghosh, Gonek, Goldston, Heath-Brown (1985)

$$D^-(0.77)>0\qquad\text{and}\qquad D^+(1.33)<1.$$

Riemann ξ-function

The Riemann ξ-function is defined by

$$\xi(s) = \frac{s(s-1)}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

- Riemann hypothesis $\Rightarrow$ all the zeros of $\xi'(s)$ satisfy $\sigma = 1/2$. Denote the zeros by $1/2 + i\gamma_1$.
- Number of zeros up to height T

$$N_1(T) := \#\{0 < \gamma_1 \leq T : \xi'(1/2 + i\gamma_1) = 0\} \sim \frac{T}{2\pi}\log T.$$

- For $0 < \gamma_1 \leq \gamma_1'$ two consecutive ordinates of zeros, define

$$\delta(\gamma_1) = (\gamma_1' - \gamma_1)\frac{\log \gamma_1}{2\pi}.$$

  On average $\delta(\gamma_1) = 1$.

Distribution function

$$\tilde{D}(\alpha, T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\substack{0<\gamma_1\leq T \\ \delta(\gamma_1)\leq\alpha}} 1.$$

$$\tilde{D}^{+}(\alpha) = \limsup_{T\to\infty} \tilde{D}(\alpha, T) \text{ and } \tilde{D}^{-}(\alpha) = \liminf_{T\to\infty} \tilde{D}(\alpha, T).$$

Farmer, Gonek (2008)

$$\tilde{D}^{-}(0.91) > 0 \qquad \text{and} \qquad \tilde{D}^{-}(1) > 0.035,$$

i.e a positive proportion of gaps between zeros of $\xi'(s)$ are less than 0.91 times the average spacing, and more than 3.5% of the normalized neighbour gaps are smaller than average.

Theorem

$$\tilde{D}^{-}(0.881) > 0 \qquad \text{and} \qquad \tilde{D}^{+}(1.149) < 1.$$

Sketch

- Define

$$h_k(\alpha, M) = \frac{\int_{-\alpha/2L}^{\alpha/2L} \sum_{T<\gamma_1\le 2T} |M(\frac{1}{2}+i(\gamma_1+t))|^{2k} dt}{\int_T^{2T} |M(\frac{1}{2}+it)|^{2k} dt},$$

where

$$M(s) = \sum_{n\le y} \frac{a(n)P(\frac{\log n}{\log y})}{n^s}.$$

  - $L = \frac{1}{2\pi}\log T$; $y = T^\theta$; $0 < \theta < 1/2$.
  - $a(n)$: arithmetic function; $P(x)$: weight.

- For $\gamma_1^\dagger \le \gamma_1 \le \gamma_1'$ three consecutive ordinates of zeros of $\xi'(s)$, let

$$\delta^+(\gamma_1) = (\gamma_1' - \gamma_1)L, \text{ and } \delta^-(\gamma_1) = (\gamma_1 - \gamma_1^\dagger)L;$$

$$\delta_0(\gamma_1) = \min\{\delta^+(\gamma_1), \delta^-(\gamma_1)\}, \text{ and } \delta_1(\gamma_1) = \max\{\delta^+(\gamma_1), \delta^-(\gamma_1)\}.$$

Then

$$\int_T^{2T} |M(\tfrac12+it)|^2 dt = \sum_{T<\gamma_1\le 2T} \int_{\gamma_1-\delta^-(\gamma_1)/2L}^{\gamma_1+\delta^+(\gamma_1)/2L} |M(\tfrac12+it)|^2 dt$$

$$\ge \sum_{\substack{T<\gamma_1\le 2T\\ \delta_0(\gamma_1)<\mu}} \int_{-\delta^-(\gamma_1)/2L}^{\delta^+(\gamma_1)/2L} |M(\tfrac12+i(\gamma_1+t))|^2 dt$$

$$+ \sum_{\substack{T<\gamma_1\le 2T\\ \delta_0(\gamma_1)\ge\mu}} \int_{-\mu/2L}^{\mu/2L} |M(\tfrac12+i(\gamma_1+t))|^2 dt$$

$$\ge h_1(\mu, M) \int_T^{2T} |M(\tfrac12+it)|^2 dt$$

$$- \sum_{\substack{T<\gamma_1\le 2T\\ \delta_0(\gamma_1)<\mu}} \int_{\delta_0(\gamma_1)/2L}^{\mu/2L} \big(|M(\tfrac12+i(\gamma_1+t))|^2 + |M(\tfrac12+i(\gamma_1-t))|^2\big) dt.$$

That leads to

$$(h_1(\mu, M) - 1)\int_T^{2T} |M(\tfrac{1}{2} + it)|^2 dt + O(T^{1-\varepsilon}) \quad \leq$$

$$\sum_{\substack{T<\gamma_1\leq 2T\\ \delta_0(\gamma_1)<\mu}} \int_{\delta_0(\gamma_1)/2L}^{\mu/2L} \big(|M(\tfrac{1}{2} + i(\gamma_1 + t))|^2 + |M(\tfrac{1}{2} + i(\gamma_1 - t))|^2\big) dt.$$

Cauchy's inequality

$$\text{RHS} \leq \left(\frac{\mu}{L}\right)^{\frac{1}{2}} \Bigg(\sum_{\substack{T<\gamma_1\leq 2T\\ \delta_0(\gamma_1)<\mu}} 1\Bigg)^{\frac{1}{2}} \left(h_2(\mu, M)\int_T^{2T} |M(\tfrac{1}{2} + it)|^4 dt\right)^{\frac{1}{2}}.$$

Thus, if $h_1(\mu, M) > 1$,

$$\sum_{\substack{T<\gamma_1\le 2T \\ \delta_0(\gamma_1)<\mu}} 1 \ge \frac{(h_1(\mu,M)-1)^2\left(\int_T^{2T}|M(\frac{1}{2}+it)|^2dt\right)^2 L}{h_2(\mu,M)\int_T^{2T}|M(\frac{1}{2}+it)|^4dt} + o(1).$$

- The condition $h_2(\mu, M) \ll 1$ is easy to be satisfied.
- We need

$$T\int_T^{2T}|M(\tfrac{1}{2}+it)|^4dt \asymp \left(\int_T^{2T}|M(\tfrac{1}{2}+it)|^2dt\right)^2 !!! \qquad (1)$$

- Then if $h_1(\mu, M) > 1$ for some μ then we have $\tilde{D}^-(\mu) > 0$.

- What form of $M(s)$ should we take?
- Condition (1) is satisfied with, for example,

$$a(n) = \Lambda(n).$$

In this case

$$\int_T^{2T} |M(\tfrac{1}{2} + it)|^2 dt \asymp TL^2,$$

and

$$\int_T^{2T} |M(\tfrac{1}{2} + it)|^4 dt \asymp TL^4.$$

The simplest choice

$$a(n) = \begin{cases} 1 & \text{if } n = 1 \\ -1 & \text{if } n = p \text{ prime} \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the optimal weight is

$$P(x) = \begin{cases} 1 & \text{if } x = 0 \\ C(1-x)\sin(\frac{\pi\mu x}{2}) & \text{otherwise,} \end{cases}$$

where

$$C = \left(\int_0^1 \frac{(1-t)^2 \sin(\frac{\pi\mu t}{2})^2 dt}{t} \right)^{-1/2}.$$

- Then

$$h_1(\mu) = \mu + \frac{1}{\pi}\left(\int_0^1 \frac{(1-t)^2 \sin(\frac{\pi\mu t}{2})^2 dt}{t}\right)^{1/2}.$$

$h_1(0.881) = 1.0005.$

- Similarly

$$h_1(\lambda) = \lambda - \frac{1}{\pi}\left(\int_0^1 \frac{(1-t)^2 \sin(\frac{\pi\lambda t}{2})^2 dt}{t}\right)^{1/2}.$$

$h_1(1.149) = 0.9994.$

- The improvement could be better if $a(n)$ is supported on primes and products of two distinct primes (in progress).