Characteristic Polynomials of Unitary Matrices

We survey repr'n theoretic ideas underlying work of:

- Diaconis and Shahshahani
- Keating and Snaith (following Gamburd)
- Conrey, Farmer, Zirnbauer (Bump-Gamburd)
- Szegö LT: Bump-Diaconis, Tracy-Widom, Dehaye
- Derivatives: Dehaye

The last two topics are not covered but would fit in. Also omitted are symplectic and orthogonal groups (see Bump-Gamburd for example).

In A Nutshell

- Basic idea is using a **correspondence** to move computation from one group to another.

These slides

`http://sporadic.stanford.edu/bump/zurich.pdf`

Motivation: From CUE to ζ

GUE (Random Hermitian Matrices)

- Physicists (Wigner, Gaudin, Dyson, Mehta) investigated random Hermitian matrices (GUE).
- Interest is in local statistics: eigenvalue correlations. (Eigenvalues repel).
- From Montgomery and Dyson GUE also models zeros of ζ.

CUE (Random Unitary Matrices)

Dyson: the exponential map

$$X \longmapsto e^{iX}\colon \left\{ \begin{array}{l} \text{Hermitian} \\ \text{matrices (GUE)} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Haar Unitary} \\ \text{matrices (CUE)} \end{array} \right\}$$

- Maps eigenvalues from $\mathbb{R} \longrightarrow \{|z| = 1\}$.
- Preserves **local statistics** (eigenval correlations)
- CUE (Circular Unitary Ens.) is $(U(n), d\mu_{\text{Haar}})$.
- CUE is easier to work with since compact.

The Idea of a Correspondence

Howe (commenting on Weyl, Weil) observed naturally occurring representations of groups $G \times H$ are multiplicity free in a strong sense.

- (ω, V_ω) a unitary rep'n of $G \times H$
- **For simplicity** G, H compact
- $\omega = \bigoplus \pi_i^G \otimes \pi_i^H$ with π_i^G and π_i^H irreducible
- **Peter-Weyl:** π_i^G and π_i^H are finite-dim'l
- **Assume no repetitions in π_i^G or π_i^H.**

Then call ω a **correspondence**.

- $\pi_i^G \Leftrightarrow \pi_i^H$ is a **bijection** $\{\pi_i^G\} \cong \{\pi_i^H\}$.

Examples:

- $G \times G$ acting on $L^2(G)$
- $S_k \times U(n)$ acting on $\otimes^k V$ $(V = \mathbb{C}^n)$
- Dual reductive pairs in $\mathrm{Sp}(2n)$ – Weil rep'n

Frobenius-Schur Correspondence

Let $G = S_k$, $H = U(n)$. H acts on $V = \mathbb{C}^n$ and both act on $V_\omega = \otimes^k V$.

- $\omega(\sigma) \in S_k$: $v_1 \otimes \cdots \otimes v_k \longrightarrow v_{\sigma^{-1}1} \otimes \ldots \otimes v_{\sigma^{-1}k}$
- $\omega(g) \in U(n)$: $v_1 \otimes \cdots \otimes v_k \longrightarrow gv_1 \otimes \cdots \otimes gv_k$
- Actions commute so ω is a rep'n of $S_k \times U(n)$

Theorem. *This is a correspondence.*

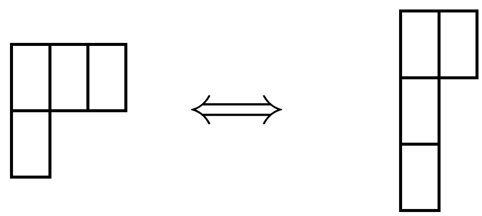
- So there is a bijection between certain rep's of S_k and certain rep's of $U(n)$.
- Explains Frobenius' use of symmetric functions (related to $U(n)$) to compute characters of S_k.
- It is useful to study S_k and $U(n)$ together.

Reps of S_k

- Let $\lambda = (\lambda_1, \cdots, \lambda_m)$ be a partition of k.
- $\lambda_1 \geqslant \ldots \geqslant \lambda_m$ and $\sum \lambda_i = k$.
- $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_m} \subset S_k$.

Let $\mu = \lambda'$ be conjugate partition.

$$\lambda = (3, 1) \quad \Longleftrightarrow \quad \mu = (2, 1, 1)$$



Theorem. $\mathrm{Hom}_{\mathbb{C}[S_k]}(\mathrm{Ind}_{S_\lambda}^{S_k}(1), \quad \mathrm{Ind}_{S_\mu}^{S_k}(\mathrm{sgn}))$ *is one-dimensional.*

Proof. Mackey Theory. □

- Mackey Theory computes intertwinings of induced rep's by double coset computations.
- More on Mackey Theory for S_k later.

Let $\pi_\lambda^{S_k}$ be the unique irreducible constituent of $\mathrm{Ind}_{S_\lambda}^{S_k}$ that can be mapped into $\mathrm{Ind}_{S_\mu}^{S_k}(\mathrm{sgn})$.

Reps of $U(n)$

- If $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ associate the character $\mathrm{diag}(t_1, \cdots, t_n) \longmapsto \prod t_i^{\lambda_i}$, called a **weight**.
- Call λ a **dominant weight** if $\lambda_1 \geqslant \ldots \geqslant \lambda_n$.
- Call λ **effective** if $\lambda_n \geqslant 0$.
- An effective dominant weight is a partition.

Let λ be dominant. Define **Schur polynomial**

$$s_\lambda(t_1, \cdots, t_n) = \frac{\det(t_i^{\lambda_j + n - j})}{\det(t_i^{n-j})}.$$

- It is a symmetric polynomial of degree $\sum \lambda_j$.

Theorem. (Schur, Weyl) *Given a dominant weight* λ *there is a rep'n* $\pi_\lambda^{U(n)}$ *with character*

$$\chi_\lambda^{U(n)}(g) = s_\lambda(t_1, \cdots, t_n), \qquad t_i = \textit{eigenval's of } g.$$

Frobenius-Schur duality

In the Frobenius-Schur correspondence partitions λ of k of length $\leqslant n$ are effective dominant weights.

$$\pi_\lambda^{S_k} \Longleftrightarrow \pi_\lambda^{U(n)}.$$

Diaconis-Shahshahani

As a first application, the distribution of traces $\mathrm{tr}(g)$ for $g \in U(n)$ is approximately Gaussian. More precisely, let ϕ be any smooth test function on $\mathbb{C}$. Then

$$\lim_{n \longrightarrow \infty} \int_{U(n)} \phi(\mathrm{tr}(g))\, dg = \int_{\mathbb{C}} \phi(z) \left[\frac{e^{-\pi(x^2+y^2)}}{\pi} \right] dx\, dy,$$

$z = x + iy$. This is surprising since the right-hand side is independent of n. We might expect the traces to spread out as $n \longrightarrow \infty$ since they are sums of many eigenvalues.

- If $\phi(x) = |x|^k$ and $k < n$ then **exactly**

$$\int_{U(n)} \phi(\mathrm{tr}(g))\, dg = \int_{\mathbb{C}} \phi(x+iy) \left[\frac{e^{-\pi(x^2+y^2)}}{\pi} \right] dx\, dy.$$

- **Method of moments:** This is sufficient.
- Assume ϕ homogeneous of degree k and **transfer the computation to S_k**.
- If $\phi(x) = |x|^k$ then RHS $= k!$

Transferring The Computation

- Let $\omega : G \times H \longrightarrow \mathrm{End}(V_\omega)$ be correspondence.
- Remember $\omega = \bigoplus \pi_i^G \otimes \pi_i^H$.
- Let f be a class function on G. We construct a class function f' on H.
- If $f = \chi_i^G$ is a character of a π_i^G let $f' = \chi_i^H$.
- If f is orthogonal to the χ_i^G let $f' = 0$.
- $f \longmapsto f'$ is an isometry on the span of the χ_i^G by Schur orthogonality.

So if f is in the span of the χ_i^G and we can compute $\|f'\|_{L_2}$ we can compute $\|f\|_{L_2}$.

Transferring Diaconis-Shahshani

For example, let $G = S_k$ and $H = U(n)$. Let

$$f(\sigma) = \begin{cases} \frac{1}{|C|} & \text{if } \sigma \in C \\ 0 & \text{otherwise} \end{cases}$$

where C is the conjugacy class of k-cycles. Then f' is the function $\mathrm{tr}(g)^k$ on $U(n)$, and if $n \geqslant k$

$$\|f'\|^2 = \|f\|^2 = k!$$

or

$$\int_{U(n)} |\mathrm{tr}(g)|^{2k} dg = k!$$

which is the Diaconis-Shahshahani result. Indeed

$$\int_{\mathbb{C}} |x + iy|^{2k} \left[\frac{e^{-\pi(x^2+y^2)}}{\pi} \right] dx\, dy = k!$$

- This is their method.
- They proved more: the distributions on $U(n)$ of $\mathrm{tr}(g), \mathrm{tr}(g^2), \cdots, \mathrm{tr}(g^m)$ converge in measure to independent Gaussians as $n \longrightarrow \infty$.
- Same trick, other conjugacy classes of S_k.

$G \times G$ Correspondences

- $\mathrm{GL}_n \times \mathrm{GL}_m$ is a dual reductive pair and so there is a Howe correspondence.
- We do not need the Weil representation to discuss it but it is in the background.

1. Very general. Let $G = H$ be any compact group. Then $G \times H$ acts on $L^2(G)$ by

$$\omega(g, h) f(x) = f(g^{-1} x h).$$

This is a correspondence. All irreducible reps appear:

$$\{\pi_i^G\} = \{\pi_i^H\} = \text{all irreducibles}$$

and $\pi_i^H = \hat{\pi}_i^G$ is the **contragredient representation**.

2. $G = U(n)$. If $G = H = U(n)$ we can modify this construction as follows.

- G has an involution $g \longmapsto {}^t g^{-1}$ that interchanges π and $\hat{\pi}$.
- So let $G \times H$ act on $L^2(G)$ by

$$\omega(g, h) f(x) = f({}^t g x h).$$

Then $\pi_i^G = \pi_i^H$.

3. $G = \mathrm{GL}(n, \mathbb{C})$.

- $K = U(n)$ is maximal compact in $G = \mathrm{GL}(n, \mathbb{C})$.
- From last example $K \times K$ acts on $L^2(K)$ by
$$\omega(g, h) f(x) = f({}^t g x h).$$
- Polynomial functions are dense in $L^2(K)$ and closed under this action (finite functions).
- Action on on polynomials extends to $\mathrm{GL}(n, \mathbb{C})$ by same formula.
- Polynomials $= \mathbb{C}[g_{ij}, \det^{-1}]$ ($g_{ij} =$ coordinates).
- Every irreducible rep'n $\pi_\lambda^{U(n)}$ of $U(n)$ extends uniquely to an analytic rep'n $\pi_\lambda^{\mathrm{GL}(n)}$ of $\mathrm{GL}(n, \mathbb{C})$.

Conclusion:

$$\mathbb{C}[g_{ij}, \det^{-1}] \cong \bigoplus_{\text{dominant weight } \lambda} \pi_\lambda^{\mathrm{GL}(n)} \otimes \pi_\lambda^{\mathrm{GL}(n)}$$

as $\mathrm{GL}(n, \mathbb{C})$-modules. Sum is over dominant weights λ.

4. $G = \mathrm{GL}(n, \mathbb{C})$: regular on $\det = 0$

Question: Which elements of $\mathbb{C}[g_{ij}, \det^{-1}]$ are regular on the determinant locus in $\mathbb{C}^{n^2} = \mathrm{Mat}_n(\mathbb{C})$?
Answer: λ must be effective ($\lambda_1 \geqslant \ldots \geqslant \lambda_n \geqslant \mathbf{0}$).

$$\mathbb{C}[g_{ij}] \cong \bigoplus_{\substack{\text{effective} \\ \text{dominant weight } \lambda}} \pi_\lambda^{\mathrm{GL}(n)} \otimes \pi_\lambda^{\mathrm{GL}(n)}$$

as $\mathrm{GL}(n, \mathbb{C})$-modules.

- Restrict to $T \times T$ ($T =$ diagonal subgroup)

$$\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}, \begin{pmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_n \end{pmatrix} \in T \times T$$

- Assume $|\alpha_i|, |\beta_j| \leqslant 1$
- Take traces to obtain **Cauchy identity**

$$\prod_{i,j} (1 - \alpha_i \beta_j)^{-1} = \sum_{\text{effective dominant } \lambda} s_\lambda(\alpha) s_\lambda(\beta).$$

The Ring Λ

- We return to Frobenius-Schur duality.
- **Recall:** $S_k \times U(n)$ acts on $\bigotimes^k V$ $(V = \mathbb{C}^n)$ and so there is a Frobenius-Schur correspondence

$$\pi_\lambda^{S_k} \Longleftrightarrow \pi_\lambda^{U(n)} \qquad \lambda = (\lambda_1, \cdots, \lambda_n), \quad \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0, \quad \Sigma_i \lambda_i = k.$$

- Let $\Lambda^{(n)} =$ ring of symmetric poly in n variables
- **Recall:** The character $\chi_\lambda^{U(n)}$ of $\pi_\lambda^{U(n)}$ is

  $$\chi_\lambda^{U(n)}(g) = s_\lambda(\alpha_1, \cdots, \alpha_n), \qquad \alpha_i = \text{eigenvalues of } g$$

  where $s_\lambda = s_\lambda^{(n)} =$ Schur polynomial in $\Lambda^{(n)}$.
- We have homomorphisms $\Lambda^{(n+1)} \longrightarrow \Lambda^{(n)}$ setting the last variable to zero.
- Let $\Lambda = \lim\limits_{\leftarrow} \Lambda^{(n)}$.
- Λ is the ring of symmetric poly's in ∞ variables.
- We have $s_\lambda^{(n)}(\alpha_1, \cdots, \alpha_n) = s_\lambda^{(n+1)}(\alpha_1, \cdots, \alpha_n, 0)$.
- So $s_\lambda \in \Lambda$.
- The s_λ are a VS basis of the $\mathbb{C}$-algebra Λ.

The Involution

We have an involution $\iota\colon \Lambda \longrightarrow \Lambda$ such that $s_\lambda \longrightarrow s_{\lambda'}$ where λ' is the conjugate partition.

- In terms S_k, the involution tensors a representation of S_k with the sign charactor.
- In terms of $U(n)$ it turns symmetric tensors (bosons) into skew-symmetric ones (fermions).

Let $e_k = s_{(1^k)}$ the k-th elementary symmetric poly.
Let $h_k = s_{(k)}$ the k-th complete symmetric poly.

$$\iota\colon e_k \longleftrightarrow h_k$$

The Dual Cauchy Identity

In $\Lambda^{(n)}$

$$\sum e_k x^k = \prod_{i=1}^{n} (1 + \alpha_i x), \qquad \sum h_k x^k = \prod_{i=1}^{n} (1 - \alpha_i x)^{-1}$$

so (roughly) ι interchanges these two expressions.

- ι acts on Λ not $\Lambda^{(n)}$ so this needs interpretation.

Applying ι to one set of variables it transforms the Cauchy identity

$$\prod_{i,j} (1 - \alpha_i \beta_j)^{-1} = \sum_{\lambda} s_\lambda(\alpha) s_\lambda(\beta)$$

into the **dual Cauchy identity**:

$$\prod_{i,j} (1 + \alpha_i \beta_j) = \sum_{\lambda} s_\lambda(\alpha) s_{\lambda'}(\beta).$$

The $\mathrm{GL}_n \times \mathrm{GL}_m$ Correspondences

We proved the Cauchy identity

$$\prod_{i=1}^{n} \prod_{j=1}^{m} (1-\alpha_i\beta_j)^{-1} = \sum_{\lambda} s_\lambda^{(n)}(\alpha)^{(m)} s_\lambda(\beta)$$

when $n = m$ but we can specialize some parameters to zero and hence obtain the same formula for $n \neq m$. Similarly in the dual Cauchy identity we do not need $n = m$.

$$\prod_{i=1}^{n} \prod_{j=1}^{m} (1+\alpha_i\beta_j) = \sum_{\lambda} s_\lambda^{(n)}(\alpha) s_{\lambda'}^{(m)}(\beta)$$

- The Cauchy identity describes the decomposition of the symmetric algebra over $\mathrm{Mat}_{n\times m}(\mathbb{C})$ under the natural action of $\mathrm{GL}_n \times \mathrm{GL}_m$ by left and right multiplication.
- Dual Cauchy identity describes the decomposition of the exterior algebra on $\mathrm{Mat}_{n\times m}(\mathbb{C})$.
- In Cauchy identity we need $|\alpha_i|$, $|\beta_j| < 1$ for convergence.
- In dual identity we do not. The sum is essentially finite since only finitely many λ have length $\leqslant n$ (so $s_\lambda^{(n)} \neq 0$) with λ' of length $\leqslant m$.

Keating and Snaith

Theorem. (Keating and Snaith) *We have*

$$\int_{U(n)} |\det(I-g)|^{2k} dg = \prod_{j=0}^{n-1} \frac{j!(j+2k)!}{((j+k)!)^2}$$

- This was proved by K&S using Selberg integral.
- Gamburd found another proof that we describe.
- This is in the CMP paper of Bump-Gamburd.
- The proofs give different information, viz:
- In KS the argument k could be real.
- In BG dg could be replaced by $\overline{\chi_\nu(g)}\, dg$.

Idea is to use $U(n)$–$U(2k)$ correspondence to transfer the calculation to $U(2k)$.

Proof (Gamburd)

Suppose $\alpha_1, \cdots, \alpha_n$ are the eigenvalues of $g \in U(n)$ so $|\alpha_i| = 1$. Let $m = 2k$ and take $\beta_1 = \cdots = \beta_{2k} = 1$. Then

$$
\begin{aligned}
|\det(I+g)|^{2k} &= \prod_{i=1}^{n} \prod_{j=1}^{k} (1+\alpha_i)(1+\alpha_i^{-1}) \\
&= \det(g)^{-k} \prod_{i=1}^{n} \prod_{j=1}^{k} (1+\alpha_i)(\alpha_i+1).
\end{aligned}
$$

Apply dual Cauchy with $\beta_1 = \cdots = \beta_{2k} = 1$. Then

$$
|\det(I+g)|^{2k} = \det(g)^{-k} \sum_{\lambda} s_{\lambda}^{(n)}(\alpha_1, \cdots, \alpha_n) s_{\lambda'}^{(2k)}(1, \cdots, 1).
$$

Now (k^n) being the partition $(k, \cdots, k)$:

$$
\det(g)^k = \chi_{(k^n)}^{U(n)}(g) = s_{(k^n)}^{n}(\alpha_1, \cdots, \alpha_n)
$$

$$
|\det(I+g)|^{2k} = \chi_{(k^n)}^{U(n)}(g)^{-1} \sum_{\lambda} \chi_{\lambda}^{U(n)}(g)^{-1} s_{\lambda'}^{(2k)}(1, \cdots, 1).
$$

Integrating picks off a single term $\lambda = (k^n)$, $\lambda' = (n^k)$.

$$
\int_{U(n)} |\det(I+g)|^{2k} dg = s_{(n^k)}^{(2k)}(1, \cdots, 1) = \prod_{j=0}^{n-1} \frac{j!(j+2k)!}{((j+k)!)^2}
$$

by the Weyl dimension (=hook length) formula.

What Happened?

Just as in the proof of the Diaconis-Shahshahani result, the computation was moved to another group. It was moved from $U(n)$ to $U(2k)$, where it is easer to do.

Ratios

$\Xi_{L,K}$ consist of all permutations $\sigma \in S_{K+L}$ such that

$$\sigma(1) < \cdots < \sigma(L), \qquad \sigma(L+1) < \cdots < \sigma(L+K).$$

Theorem. (Conrey, Farmer and Zirnbauer) *Assume $n \geqslant Q, R$ and that $|\gamma_q|, |\delta_r| < 1$.*

$$\int_{U(n)} \frac{\prod_{l=1}^{L} \det(I + \alpha_l^{-1} \cdot g^{-1}) \cdot \prod_{k=1}^{K} \det(I + \alpha_{L+k} \cdot g)}{\prod_{q=1}^{Q} \det(I - \gamma_q \cdot g) \prod_{r=1}^{R} \det(I - \delta_r \cdot g^{-1})} dg =$$
$$\sum_{\sigma \in \Xi_{L,K}} \prod_{k=1}^{K} (\alpha_{\sigma(L+k)}^{-1} \alpha_{L+k})^n \times$$
$$\frac{\prod_{q=1}^{Q} \prod_{l=1}^{L} (1 + \gamma_q \alpha_{\sigma(l)}^{-1}) \prod_{r=1}^{R} \prod_{k=1}^{K} (1 + \delta_r \alpha_{\sigma(L+k)})}{\prod_{k=1}^{K} \prod_{l=1}^{L} (1 - \alpha_{\sigma(l)}^{-1} \alpha_{\sigma(K+k)}) \prod_{r=1}^{R} \prod_{q=1}^{Q} (1 - \gamma_q \delta_r)}.$$

After the initial proof by CFZ other proofs were given by Conrey, Forrester, Snaith and by Bump Gamburd.

We will not discuss the proof in detail but we isolate a couple of important ingredients.

Laplace-Levi expansions

- Let $G =$ complex reductive group, $\Phi =$ roots
- $P = MU$ parabolic with Levi M and radical U.
- W and W_M the Weyl groups of G and M.
- $\mathcal{C}$ and $\mathcal{C}_M =$ positive Weyl chambers.
- $\Xi =$ coset reps for $W_M \backslash W$ such that $w\mathcal{C} \subset \mathcal{C}_M$.
- $\rho_U = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

Then

$$\chi_\lambda^G = \frac{1}{e^{-\rho_U} \prod_{\alpha \in \Phi^+ - \Phi_U^+} (1 - e^\alpha)} \sum_{w \in \Xi} (-1)^{l(w)} \chi_{\lambda_w}^M.$$

This follows from the Weyl character formula.

- With $G = \mathrm{GL}_{L+K}$ and $M = \mathrm{GL}_L \times \mathrm{GL}_K$,
- $\lambda = (\lambda_1, \ldots, \lambda_{L+K})$,
- $\tau = (\lambda_1, \ldots, \lambda_L), \qquad \rho = (\lambda_{L+1}, \ldots, \lambda_{L+K})$.

$$s_\lambda(\alpha_1, \cdots, \alpha_{L+K}) = \sum_{\sigma \in \Xi_{L,K}} \prod_{\substack{1 \leqslant l \leqslant L \\ 1 \leqslant k \leqslant K}} (\alpha_{\sigma(l)} - \alpha_{\sigma(L+k)})^{-1}$$

$$s_{\tau + \langle K^L \rangle}(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(L)}) s_\rho(\alpha_{\sigma(L+1)}, \cdots, \alpha_{\sigma(L+K)}) \ .$$

This accounts for sum over $\Xi_{L,K}$ in the ratios formula.

Littlewood-Schur sym polynomials

Due to Littlewood, rediscovered by Berele and Regev. Let $c^{\lambda}_{\mu\nu}$ be Littlewood-Richardson coefficients:

$$s_{\mu}s_{\nu} = \sum_{\lambda} c^{\lambda}_{\mu\nu} s_{\lambda}.$$

Define (for two sets of variables)

$$\mathrm{LS}_{\lambda}(x_1, \cdots, x_k; y_1, \cdots, y_l) = \sum_{\mu,\nu} c^{\lambda}_{\mu\nu} s_{\mu}(x_1, \cdots, x_k) s_{\nu'}(y_1, \cdots, y_l).$$

The **generalized Cauchy identity** (Berele, Remmel)

$$\sum \mathrm{LS}_{\lambda}(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n) \mathrm{LS}_{\lambda}(\gamma_1, \cdots, \gamma_s; \delta_1, \cdots, \delta_t) = \prod_{i,k} (1-\alpha_i\gamma_k)^{-1} \prod_{i,l} (1+\alpha_i\delta_l) \prod_{j,k} (1+\beta_j\gamma_k) \prod_{j,l} (1-\beta_j\delta_l)^{-1}.$$

We will discuss the significance of this momentarily. First we outline the proof of the ratios theorem.

- Cauchy and dual Cauchy identities are applied to LHS producing a sum of Schur functions.
- Some of these are multiplied producing Littlewood-Richardson coefficients
- Regrouped into Littlewood-Schur polynomials.
- Generalized Cauchy identity is applied.
- Laplace-Levi expansion is applied.
- It all works out.

Sketch

Left-Hand-Side:

$$\int_{U(n)} \frac{\prod_{l=1}^{L} \det\left(I + \alpha_l^{-1} \cdot g^{-1}\right) \cdot \prod_{k=1}^{K} \det\left(I + \alpha_{L+k} \cdot g\right)}{\prod_{q=1}^{Q} \det\left(I - \gamma_q \cdot g\right) \prod_{r=1}^{R} \det\left(I - \delta_r \cdot g^{-1}\right)} dg$$

Expand Π with dual Cauchy (up), Cauchy (down)

$$\sum_{\lambda,\mu,\nu} \left\langle \chi_{\lambda'} \chi_\mu, \det{}^L \otimes \chi_\nu \right\rangle$$

$$\prod_{l=1}^{L} \alpha_l^{-N} s_\lambda(\alpha_1, \cdots, \alpha_{L+K}) s_\mu(\gamma_1, \cdots, \gamma_Q) s_\nu(\delta_1, \cdots, \delta_R).$$

inner product is $c_{\lambda'\mu}^{\tilde{\nu}}$ with $\tilde{\nu} = \nu + (L^n)$, $\tilde{\nu}' = N^L \cup \nu'$

$$\prod_{l=1}^{L} \alpha_l^{-N} \sum_{\nu} \mathrm{LS}_{N^L \cup \nu'}(\alpha_1, \ldots, \alpha_{L+K}; \gamma_1, \ldots, \gamma_Q) s_\nu(\delta_1, \cdots, \delta_R)$$

Use Laplace-Levi: $\mathrm{LS}_{\tau \cup \rho}(\alpha_1, \cdots, \alpha_{L+K}; \gamma_1, \ldots, \gamma_Q) =$

$$\sum_{\sigma \in \Xi_{L,K}} \prod_{\substack{1 \leqslant l \leqslant L \\ 1 \leqslant k \leqslant K}} \left(\alpha_{\sigma(l)} - \alpha_{\sigma(L+k)}\right)^{-1}$$

$$\mathrm{LS}_{\tau + \langle K^L \rangle}(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(L)}; \gamma_1, \ldots, \gamma_Q)$$

$$\mathrm{LS}_{\rho}(\alpha_{\sigma(L+1)}, \cdots, \alpha_{\sigma(L+K)}; \gamma_1, \ldots, \gamma_Q)$$

Sketch (continued)

Generalized Cauchy:

$$\sum \mathrm{LS}_\lambda(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n)\mathrm{LS}_\lambda(\gamma_1, \cdots, \gamma_s; \delta_1, \cdots, \delta_t) = \prod_{i,k} (1-\alpha_i\gamma_k)^{-1} \prod_{i,l} (1+\alpha_i\delta_l) \prod_{j,k} (1+\beta_j\gamma_k) \prod_{j,l} (1-\beta_j\delta_l)^{-1}.$$

Right-Hand Side:

$$\sum_{\sigma \in \Xi_{L,K}} \prod_{k=1}^{K} (\alpha_{\sigma(L+k)}^{-1}\alpha_{L+k})^n \times \frac{\prod_{q=1}^{Q} \prod_{l=1}^{L} (1+\gamma_q\alpha_{\sigma(l)}^{-1}) \prod_{r=1}^{R} \prod_{k=1}^{K} (1+\delta_r\alpha_{\sigma(L+k)})}{\prod_{k=1}^{K} \prod_{l=1}^{L} (1-\alpha_{\sigma(l)}^{-1}\alpha_{\sigma(K+k)}) \prod_{r=1}^{R} \prod_{q=1}^{Q} (1-\gamma_q\delta_r)}.$$

Hopf Algebra (Geissinger)

The multiplication in Λ induces a map $m\colon \Lambda \otimes \Lambda \longrightarrow \Lambda$, whose adjoint with respect to the basis for which the s_λ are orthonormal is a map $m^*\colon \Lambda \longrightarrow \Lambda \otimes \Lambda$. Thus

$$m(s_\mu \otimes s_\nu) = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda, \qquad m^*(s_\lambda) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu \otimes s_\nu.$$

The map m^* is a comultiplication making Λ a coalgebra. The Hopf axiom is the commutativity of:

$$\begin{array}{ccccc} \Lambda\otimes\Lambda & \xrightarrow{m^*\otimes m^*} & \Lambda\otimes\Lambda\otimes\Lambda\otimes\Lambda & \xrightarrow{1\otimes\tau\otimes 1} & \Lambda\otimes\Lambda\otimes\Lambda\otimes\Lambda \\ \big\downarrow m & & & & \big\downarrow m\otimes m \\ \Lambda & & \xrightarrow{m^*} & & \Lambda\otimes\Lambda \end{array},$$

where $\tau\colon R\otimes R \longrightarrow R\otimes R$ is the map $\tau(u\otimes v) = v\otimes u$.

- Interpreting the multiplication in Λ as induction (from $S_k \times S_l$ to S_{k+l}) and the comultiplication as restriction (from S_{k+l} to $S_k \times S_l$), the Hopf property boils down to Mackey theory.

Mackey Theory

If G is a finite group and H, K are subgroups, Mackey theory is schematically a commutative diagram

$$\begin{array}{ccc} \text{Reps of } H & \xrightarrow{\text{induce}} & \text{Reps of } G \\ \Big\downarrow \text{restrict} & & \Big\downarrow \text{restrict} \\ \text{Reps of } \ldots\ldots & \xrightarrow{\text{induce}} & \text{Reps of } K \end{array}$$

Here ...... means that we intersect H with K in all possible ways. That is, let γ run through a set of coset reps of $H\backslash G/K$ and let $H \cap_\gamma K$ mean $H \cap \gamma K \gamma^{-1}$. If π is a repn' of H then as K-modules

$$\underset{G \to K}{\text{Res}} \circ \underset{H \to G}{\text{Ind}} (\pi) \cong \bigoplus_{\Gamma} \underset{H \cap_\gamma K \to K}{\text{Ind}} \circ \underset{G \to H \cap_\gamma K}{\text{Res}} (\pi).$$

Here $H \cap_\gamma K$ is not a subgroup of K but is conjugate to one which is enough.

- Nutshell: "induction and restriction commute"

Hopf = Mackey

- Λ is a graded ring. In symmetric group optic $\Lambda_k =$ (virtual) representations of S_k.
- $(\Lambda \otimes \Lambda)_k =$ Reps of $S_m \times S_n$ $(m+n=k)$.
- Multiplication $\Lambda_k \otimes \Lambda_l \longrightarrow \Lambda_{k+l}$ is induction from $S_k \times S_\lambda$ to S_{k+l}.
- Coultiplication $\Lambda_{k+l} \longrightarrow \Lambda_k \otimes \Lambda_l$ is restriction from S_{k+l} to $S_k \times S_\lambda$.
- $\Lambda \otimes \Lambda \xrightarrow{m} \Lambda \xrightarrow{m^*} \Lambda \otimes \Lambda$ is Ind $\circ$ Res
- $\Lambda \otimes \Lambda \xrightarrow{m^*} \Lambda \otimes \Lambda \otimes \Lambda \otimes \Lambda \xrightarrow{m \circ 1 \otimes t \otimes 1} \Lambda \otimes \Lambda$: Res $\circ$ Ind
- To verify equivalence consider one component

$$\Lambda_m \otimes \Lambda_n \longrightarrow \Lambda_p \otimes \Lambda_q \qquad (m+n=p+q)$$

- $H = S_m \times S_n$, $K = S_p \times S_q$
- $H \cap_\gamma K = S_x \times S_y \times S_z \times S_w$
- $x+y=m$, $z+w=n$, $x+z=p$, $y+w=q$
- Grading is a bookkeeping device.
- **Hopf = Mackey**.

Generalized Schur identity

Theorem. *The Generalized Cauchy formula is equivalent to the Hopf property of* Λ.

Proof. The Hopf axiom reduces to the formula

$$\sum_{\lambda} c^{\lambda}_{\mu\nu} c^{\lambda}_{\sigma\tau} = \sum_{\varphi,\eta} c^{\sigma}_{\varphi\eta} c^{\tau}_{\psi\xi} c^{\mu}_{\varphi\xi} c^{\nu}_{\psi\eta}. \tag{1}$$

- Apply $m^* \circ m$ to $s_\mu \otimes s_\nu$, then extract the coefficient of $s_\sigma \otimes s_\tau$ the left-hand side in (1).
- Same with $(m \otimes m) \circ (1 \otimes \tau \otimes 1) \circ (m^* \otimes m^*)$ gives the right-hand side.

$$\sum \mathrm{LS}_\lambda(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n) \mathrm{LS}_\lambda(\gamma_1, \cdots, \gamma_s; \delta_1, \cdots, \delta_t) \; = \; \prod_{i,k} (1 - \alpha_i \gamma_k)^{-1} \prod_{i,l} (1 + \alpha_i \delta_l) \prod_{j,k} (1 + \beta_j \gamma_k) \prod_{j,l} (1 - \beta_j \delta_l)^{-1}.$$

The LHS is

$$\sum c^{\lambda}_{\mu\nu} s_\mu(\alpha) s_{\nu'}(\beta) c^{\lambda}_{\sigma\tau} s_\sigma(\gamma) s_{\tau'}(\delta)$$

while the right-hand side is (using Cauchy & dual)

$$\begin{aligned} & \sum s_\varphi(\alpha) s_\varphi(\gamma) s_{\psi'}(\beta) s_{\psi'}(\delta) s_\xi(\alpha) s_{\xi'}(\delta) s_{\eta'}(b) s_\eta(\gamma) \\ = & \sum c^{\sigma}_{\varphi\eta} c^{\tau}_{\psi\xi} s_\varphi(\alpha) s_\xi(\alpha) s_{\psi'}(\beta) s_{\eta'}(\beta) s_\sigma(\gamma) s_{\tau'}(\delta) \\ = & \sum c^{\sigma}_{\varphi\eta} c^{\tau}_{\psi\xi} c^{\mu}_{\varphi\xi} c^{\nu}_{\psi\eta} s_\mu(\alpha) s_{\nu'}(\beta) c^{\lambda}_{\sigma\tau} s_\sigma(\gamma) s_{\tau'}(\delta). \end{aligned}$$

Comparing, the equivalence amounts to (1). $\square$