Primes, partitions and permutations

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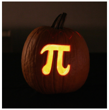
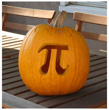
   

Outline

- Review of Bump & Gamburd's method
- A theorem of Okounkov-Olshanski
- Moments of derivatives of characteristic polynomials
  - Hypergeometric functions
  - Integral representations
  - Markov chain on a Young graph
- Comments
- Explain the title

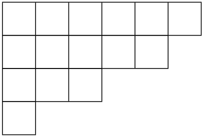
Definitions

Let

$$Z_U(\theta) = \prod_{j=1}^{N} \left(1 - e^{\mathrm{i}(\theta_j - \theta)}\right)$$

be the char. pol. of $U \in \mathrm{U}(N)$, which has eigenvalues $\left\{e^{\mathrm{i}\theta_j}\right\}$.

A partition λ is a non-increasing sequence $(\lambda_1, \cdots, \lambda_{l(\lambda)}, 0, 0, \cdots)$ of integers trailing to 0s. We define its length $l(\lambda)$ as the number of non-zero entries, and its weight $|\lambda|$ as the total value $\sum \lambda_i$ of the entries in the sequence.

We always identify a partition with its Young diagram. The diagram for $(6, 5, 3, 1)$, for instance, is  .

Definitions (II)

Schur polynomials are symmetric polynomials. They form compatible families, indexed by partitions:

$$\mathfrak{s}_\lambda(x_1, \cdots, x_N) = \mathfrak{s}_\lambda(x_1, \cdots, x_N, 0).$$

There is one reduction property:

$$\mathfrak{s}_\lambda(x_1, \cdots, x_N) = 0 \text{ if } l(\lambda) > N.$$

Excluding those, we have irreducible characters of $\mathrm{U}(N)$, and

$$\left\langle \mathfrak{s}_\lambda(U)\overline{\mathfrak{s}_\mu(U)} \right\rangle_{\mathrm{U}(N)} = \begin{cases} \delta_{\lambda\mu} & \text{if } N \geq l(\lambda) \\ 0 & \text{if } l(\lambda) > N \end{cases}$$

with $\mathfrak{s}_\lambda(U) := \mathfrak{s}_\lambda(e^{\mathrm{i}\theta_1}, \cdots, e^{\mathrm{i}\theta_N})$.

"For large N, the $\mathfrak{s}_\lambda$ are orthonormal over $\mathrm{U}(N)$."

The method of Bump and Gamburd

Bump and Gamburd have a way to compute

$$\left\langle \left|Z_U(0)\right|^{2k} \right\rangle_{\mathrm{U}(N)}$$

which uses the dual Cauchy identity

$$\sum_{\lambda \text{ partitions}} \mathfrak{s}_\lambda(x_1, x_2, \cdots, x_M)\mathfrak{s}_{\lambda^{\mathrm{t}}}(y_1, y_2, \cdots, y_N) = \prod_{m,n}^{M,N}(1+x_m y_n).$$

$$\begin{aligned}
\sum_\lambda \mathfrak{s}_\lambda\left(\{1\}^{2k}\right)\overline{\mathfrak{s}_{\lambda^t}(U)} &= \det(\mathrm{Id}+\overline{U})^{2k} \\
&= \overline{\det(U)}^k \left|\det(\mathrm{Id}+U)\right|^{2k} \\
&= \overline{\mathfrak{s}_{\langle k^N \rangle}(U)} \left|\det(\mathrm{Id}+U)\right|^{2k}
\end{aligned}$$

or (replacing U by $-U$)

$$\left|Z_U(0)\right|^{2k} = (-1)^{kN}\mathfrak{s}_{\langle k^N \rangle}(U)\sum_\lambda (-1)^{|\lambda|}\mathfrak{s}_\lambda\left(\{1\}^{2k}\right)\overline{\mathfrak{s}_{\lambda^t}(U)}.$$

The method of Bump and Gamburd (II)

$$|Z_U(0)|^{2k} = (-1)^{kN} \mathfrak{s}_{\langle k^N \rangle}(U) \sum_{\lambda} (-1)^{|\lambda|} \mathfrak{s}_\lambda\left(\{1\}^{2k}\right) \overline{\mathfrak{s}_{\lambda^t}(U)}$$

So

$$\left\langle |Z_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)} = \mathfrak{s}_{\langle N^k \rangle}\left(\{1\}^{2k}\right),$$

which can be evaluated combinatorially, provides many alternate expressions and gives analytic continuations.

This gives an interpreation of the Keating-Snaith "geometric" term as a dimension.

Evaluation

There are many ways to evaluate a Schur function, based on

- (ratios of Vandermonde determinants)
- determinants (valid for any values of the variables)
  - Jacobi-Trudi identity (2 expressions)
  - Giambelli formula
- the hook-content formula (a dimension formula, only 1s)

$$\mathfrak{s}_\lambda\left(\{1\}^K\right) = \prod_{\square\in\lambda} \frac{K + c(\square)}{\mathrm{H}(\square)} = \frac{K \uparrow \lambda}{H(\lambda)}$$

The $K \uparrow \lambda$ notation is a generalization of the Pochhammer symbol, which provides analytic continuations in K.

Evaluation (II)

$$
\begin{aligned}
\mathfrak{s}_\lambda\left(\{1\}^K\right) &= \prod_{\square\in\lambda} \frac{K + c(\square)}{\mathrm{H}(\square)} \\
&= \frac{K \uparrow \lambda}{H(\lambda)}
\end{aligned}
$$

- The way these expressions vanish when $l(\lambda) > K$ is very important.
- Boxes can be grouped in many ways: along rows, columns or "half-hooks" [BHNY: columns]
- The numerators $K \uparrow \lambda$ can be obtained as exponentials of integrals against Russian diagrams of the logarithmic derivative of the Barnes G-function

Moments of derivatives

We now consider the question of moments of derivatives of characteristic polynomials, i.e. we look at

$$\left\langle |Z_U(0)|^{2k} |Z'_U(0)|^{2h} \right\rangle_{\mathrm{U}(N)}$$

(remember: $Z_U(\theta) = \prod_{j=1}^{N} \left(1 - e^{\mathrm{i}(\theta_j - \theta)}\right)$).

We also look at

$$\left\langle |V_U(0)|^{2k} |V'_U(0)|^{2h} \right\rangle_{\mathrm{U}(N)}$$

with $V_U(\theta) = e^{\mathrm{i}N(\theta+\pi)/2} e^{-\mathrm{i}\sum_{j=1}^{N} \theta_j/2} Z_U(\theta)$

The two are simple linear combinations obtained from the moments

$$\left\langle |Z_U(0)|^{2k} \left(\frac{Z'_U(0)}{Z_U(0)} \right)^r \right\rangle_{\mathrm{U}(N)}$$

A theorem of Okounkov and Olshanski

Generalized Binomial Theorem: If $\mathfrak{s}^*_\mu$ stand for "shifted Schur functions", then

$$\frac{\mathfrak{s}_\lambda(1+a_1,\cdots,1+a_n)}{\mathfrak{s}_\lambda(\{1\}^n)} = \sum_{\substack{\mu \\ l(\mu)\le n}} \frac{\mathfrak{s}^*_\mu(\lambda_1,\cdots,\lambda_n)\mathfrak{s}_\mu(a_1,\cdots,a_n)}{n\uparrow\mu}.$$

(The case $n=1$ is the binomial theorem).

Also,

$$\mathfrak{s}^*_\mu(\{N\}^k) \quad = \quad (-1)^{|\mu|}\frac{(-N\uparrow\mu)(k\uparrow\mu)}{H(\mu)},$$

so $\mathfrak{s}_{\langle N^k\rangle}(1+a_1,\cdots,1+a_n)$ has a nice expression as a sum over partitions.

"The Taylor expansion of a Schur function near the identity."

$$|Z_U(0)|^{2k}\left(\frac{Z_U'(0)}{Z_U(0)}\right)^r = (-1)^{kN}\overline{\mathfrak{s}_{\langle k^N\rangle}(U)}\cdot\sum_\lambda (-1)^{|\lambda|}\mathfrak{s}_{\lambda^t}(U)\times$$
$$\partial_1\cdots\partial_r\mathfrak{s}_\lambda\left(\{1\}^{2k-r}\cup\{1-\mathrm{i}a_1,\cdots,1-\mathrm{i}a_r\}\right)\big|_{a_j=0}$$

and hence

$$\left\langle |Z_U(0)|^{2k}\left(\frac{Z_U'(0)}{Z_U(0)}\right)^r\right\rangle_{\mathrm{U}(N)} =$$
$$\partial_1\cdots\partial_r\mathfrak{s}_{\langle N^k\rangle}\left(\{1\}^{2k-r}\cup\{1-\mathrm{i}a_1,\cdots,1-\mathrm{i}a_r\}\right)\big|_{a_j=0}.$$

Using Okounkov-Olshanski, we get

Proposition: When $0 \leq r \leq 2k$,

$$\left\langle |Z_U(0)|^{2k} \left(\frac{Z_U'(0)}{Z_U(0)} \right)^r \right\rangle_{\mathrm{U}(N)} =$$

$$\mathrm{i}^r r! \left\langle |Z_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)} \times \sum_{\mu \vdash r} \frac{1}{H(\mu)^2} \frac{(-N \uparrow \mu)(k \uparrow \mu)}{2k \uparrow \mu}$$

Remarks:

- r has to be an integer
- At first k is an integer, and $r \leq 2k$
- This can be extentded to a rational function of k
- The sum yields a polynomial in N of degree r, with coefficients even rational functions in k (Hughes, unpublished).

Generating series (I)

$$\sum_r \left\langle |Z_U(0)|^{2k} \left(\frac{Z_U'(0)}{Z_U(0)} \right)^r \right\rangle_{\mathrm{U}(N)} \frac{(\mathrm{i}z)^r}{r!}$$

$$= \left\langle |Z_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)} \times \sum_\mu \frac{(-k \uparrow \mu)}{-2k \uparrow \mu} \frac{(N \uparrow \mu) z^{|\mu|}}{H(\mu)^2}$$

$$= \left\langle |Z_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)} \times \sum_\mu \frac{-k \uparrow \mu}{-2k \uparrow \mu} \frac{\mathfrak{s}_\mu \left(z \, \mathrm{Id}_{N \times N} \right)}{H(\mu)}$$

Generating series (II)

Proposition (Borodin, D.):

$$\sum_r \left\langle |Z_U(0)|^{2k} \left(\frac{Z_U'(0)}{Z_U(0)}\right)^r \right\rangle_{\mathrm{U}(N)} \frac{(\mathrm{i}z)^r}{r!}$$

$$= \left\langle |Z_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)} \times \sum_\mu \frac{(-k \uparrow \mu)}{(-2k \uparrow \mu)} \frac{1}{H(\mu)} \frac{(N \uparrow \mu) z^{|\mu|}}{H(\mu)}$$

$$= \left\langle |Z_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)} \, {}_1F_1\left(-k; -2k; z\,\mathrm{Id}_{N\times N}\right)$$

See work of Richards, Gross, Yan on hypergeometric functions of matrix arguments.

At finite N, any problem of analytic continuation (in k, h) is not much harder than classical hypergeometric functions (cf. integral representations, differential equations, recurrence relations).

Integral representations

Hypergeometric functions have integral representations. When they have matrix arguments, Selberg integrals appear.

As meromorphic functions of k,

$$\sum_r \frac{\left\langle |Z_U(0)|^{2k} \left(\frac{Z_U(0)'}{Z_U(0)}\right)^r \right\rangle_{\mathrm{U}(N)}}{\left\langle |Z_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)}} \frac{(\mathrm{i}z)^r}{r!} =$$

$$\frac{\int_0^1\!\int_0^1 e^{z\sum t_i} \prod_i^N t_i^{-k-N}(1-t_i)^{-k-N}\Delta(t_i)^2 \ \mathrm{d}t_1\cdots\mathrm{d}t_N}{\int_0^1\!\int_0^1 \prod_i^N t_i^{-k-N}(1-t_i)^{-k-N}\Delta(t_i)^2 \ \mathrm{d}t_1\cdots\mathrm{d}t_N}$$

$$\sum_h \frac{\left\langle |V_U(0)|^{2k} \left| \frac{V_U(0)'}{V_U(0)} \right|^{2h} \right\rangle_{\mathrm{U}(N)}}{\left\langle |V_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)}} \frac{z^h}{h!} =$$

$$\frac{\int_0^1 \int_0^1 e^{-\frac{1}{4}(2\sum_i t_i - N)^2 z} \prod_i^N t_i^{-k-N}(1-t_i)^{-k-N} \Delta(t_i)^2 \; \mathrm{d}t_1 \cdots \mathrm{d}t_N}{\int_0^1 \int_0^1 \prod_i^N t_i^{-k-N}(1-t_i)^{-k-N} \Delta(t_i)^2 \; \mathrm{d}t_1 \cdots \mathrm{d}t_N}$$

$$= \frac{z^{N(N+2k)}}{\prod_{j=1}^N \Gamma(2k+j)\Gamma(j+1)} \times$$

$$\int_0^\infty \int_0^\infty e^{-\frac{1}{4}(2\sum_i t_i + N)^2 z} \prod_i^N t_i^k (1+t_i)^k \prod_{1 \le i < j \le N} |t_i - t_j|^2 \; \mathrm{d}t_1 \cdots \mathrm{d}$$

In the limit in N, I don't know. The existence of an analytic continuation in h in the limit in N is unproved and very possibly false.

Probabilistic reformulation

Fact:

$$\dim \lambda := \dim \chi_\lambda = \frac{|\lambda|!}{H(\lambda)} = \#\text{paths to } \lambda \text{ in the Young graph.}$$

Set (Poissonized Plancherel measure)

$$m(\lambda) = \frac{\dim(\lambda)^2}{|\lambda|!},$$

which satisfies

$$\sum_{\lambda \text{ s.t. } |\lambda| = \text{cst}} m(\lambda) = 1.$$

The formula obtained before then takes the form

$$\sum_r g(r) \frac{z^r}{r!} = \sum_\lambda f(\lambda) m(\lambda) \frac{z^{|\lambda|}}{|\lambda|!},$$

for $f(\lambda)$ given by a product over the boxes of the partition λ.

Probabilistic reformulation (II)

$$\sum_r \left\langle |Z_U(0)|^{2k} \left(\frac{Z_U'(0)}{Z_U(0)} \right)^r \right\rangle_{\mathrm{U}(N)} \frac{(\mathrm{i}z)^r}{r!}$$

$$= \left\langle |Z_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)} \times \sum_\mu \frac{(-k \uparrow \mu)(N \uparrow \mu)}{(-2k \uparrow \mu)} m(\mu) \frac{z^{|\mu|}}{|\mu|!}$$

$$\sum_r g(r) \frac{z^r}{r!} \quad = \quad \sum_\lambda f(\lambda) m(\lambda) \frac{z^{|\lambda|}}{|\lambda|!}$$

- If the RMT conjectures are to be believed, the partitions are necessary to express the joint moments of the derivatives of ζ.
- In the limit in N, all partitions λ appear, not just those with $l(\lambda) \leq N$.
- Hypergeometric functions of operator arguments have not been studied, but the probabilistic interpretation is still present, and points to the study of the whole Young graph, not just restrictions of it.

Cauchy identities

$$\sum_{\lambda} \mathfrak{s}_\lambda(x_1, x_2, \cdots, x_M)\mathfrak{s}_\lambda(y_1, y_2, \cdots, y_N) = \prod_{m,n}^{M,N} \frac{1}{1 - x_m y_n}$$

$$\sum_{\lambda} \mathfrak{s}_\lambda(x_1, x_2, \cdots, x_M)\mathfrak{s}_{\lambda^{\mathfrak{t}}}(y_1, y_2, \cdots, y_N) = \prod_{m,n}^{M,N} 1 + x_m y_n$$

Theorem (Bourgade, D., Nikeghbali) For $k \in \mathbb{N}$, when $\Re s \geq 1$,

$$\zeta(s)^k = \sum_{\lambda} \mathfrak{s}_\lambda\left(\{1\}^k\right)\mathfrak{s}_\lambda(p^{-s})$$

and one can continue analytically $\mathfrak{s}_\lambda(p^{-s})$ to $\Re s > 0$.

This is merely a reorganization of an absolutely convergent sum. Note that when $k \to \infty$ or $k \notin \mathbb{N}$, this is not trivial. This preserves a symmetry of the Riemann zeta function (invariance under permutation of the primes).

$$\sum_{\lambda} \mathfrak{s}_{\lambda}(x_1, x_2, \cdots, x_M)\mathfrak{s}_{\lambda}(y_1, y_2, \cdots, y_N) = \prod_{m,n}^{M,N} \frac{1}{1 - x_m y_n}$$

$$\sum_{\lambda} \mathfrak{s}_{\lambda}(x_1, x_2, \cdots, x_M)\mathfrak{s}_{\lambda^t}(y_1, y_2, \cdots, y_N) = \prod_{m,n}^{M,N} 1 + x_m y_n$$

Conjecture (D.) There exist functions $f_\lambda(z, s)$ such that $\mathfrak{s}_\lambda(f_\lambda(\gamma_i, s))$ admits an analytic continuation to $\Re s > 0$ and

$$\mathfrak{s}_\lambda(f_\lambda(\gamma_i, s)) = \mathfrak{s}_{\lambda^t}(p^{-s})$$

Selecting N zeroes would then correspond to looking at partitions λ with $\lambda_1 \leq N$ (cf. Keating-Snaith).

The involution on partitions is crucial (cf. graded Hopf algebra of symmetric functions has an antipode). By using partitions we can index characters in a way that is more natural for the symmetric group, and less for the unitary group.

We then have 3 types of objects:

- primes
- zeroes
- eigenvalues of random matrices

Question

We know that

$$\lim_{N\to\infty} \left\langle \mathfrak{s}_\lambda(U), \overline{\mathfrak{s}_\mu(U)} \right\rangle_{\mathrm{U}(N)} = \delta_{\lambda\mu},$$

can we conjecture

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathfrak{s}_{\lambda^t}\left(p^{1/2+\mathrm{i}t}\right) \overline{\mathfrak{s}_{\mu^t}\left(p^{1/2+\mathrm{i}t}\right)} \,\mathrm{d}t = C_{\lambda^t}\delta_{\lambda\mu}$$

and pass from proofs using orthonormality of characters of $\mathrm{U}(N)$ directly to conjectures using orthogonality?

Main computations

Just as before,

$$Z_U(a_1)\cdots Z_U(a_r) = \sum_{\lambda}(-1)^{|\lambda|}\mathfrak{s}_{\lambda^t}(U)\mathfrak{s}_{\lambda}\left(e^{-\mathrm{i}a_1},\cdots,e^{-\mathrm{i}a_r}\right).$$

To the first order in small a, we have

$$e^{-\mathrm{i}a} \approx 1-\mathrm{i}a,$$

so

$$Z_U'(0)^r = \sum_{\lambda}(-1)^{|\lambda|}\mathfrak{s}_{\lambda^t}(U)\times \partial_1\cdots\partial_r\mathfrak{s}_{\lambda}\left(1-\mathrm{i}a_1,\cdots,1-\mathrm{i}a_r\right)\Big|_{a_1=\cdots=a_r=0},$$

where $\partial_j := \partial_{a_j}$.
Also,

$$\overline{Z_U(0)}^k = (-1)^{kN}\overline{\det U}^k Z_U(0)^k = (-1)^{kN}\overline{\mathfrak{s}_{\langle k^N\rangle}(U)}Z_U(0)^k.$$

$$Z'_U(0)^r = \sum_{\lambda} (-1)^{|\lambda|} \mathfrak{s}_{\lambda^t}(U) \times \partial_1 \cdots \partial_r \mathfrak{s}_{\lambda}\left(1 - \mathrm{i}a_1, \cdots, 1 - \mathrm{i}a_r\right)\Big|_{a_1 = \cdots = a_r = 0}$$

and

$$\overline{Z_U(0)}^k = (-1)^{kN} \overline{\mathfrak{s}_{\langle k^N \rangle}(U)} Z_U(0)^k$$

imply

$$|Z_U(0)|^{2k} \left(\frac{Z'_U(0)}{Z_U(0)}\right)^r = (-1)^{kN} \overline{\mathfrak{s}_{\langle k^N \rangle}(U)} \cdot \sum_{\lambda} (-1)^{|\lambda|} \mathfrak{s}_{\lambda^t}(U) \times \partial_1 \cdots \partial_r \mathfrak{s}_{\lambda}\left(\{1\}^{2k-r} \cup \{1 - \mathrm{i}a_1, \cdots, 1 - \mathrm{i}a_r\}\right)\Big|_{a_j = 0}$$

and hence

$$\left\langle |Z_U(0)|^{2k} \left(\frac{Z'_U(0)}{Z_U(0)}\right)^r \right\rangle_{\mathrm{U}(N)} = \partial_1 \cdots \partial_r \mathfrak{s}_{\langle N^k \rangle}\left(\{1\}^{2k-r} \cup \{1 - \mathrm{i}a_1, \cdots, 1 - \mathrm{i}a_r\}\right)\Big|_{a_j = 0}.$$

Using Okounkov-Olshanski,

$$\frac{\left\langle |Z_U(0)|^{2k}\left(\frac{Z_U'(0)}{Z_U(0)}\right)^r\right\rangle_{\mathrm{U}(N)}}{\left\langle |Z_U(0)|^{2k}\right\rangle_{\mathrm{U}(N)}}$$

$$= (-\mathfrak{i})^r \sum_{\mu\vdash r} \frac{\mathfrak{s}^*_\mu\left(\{N\}^k\right)\ \partial_1\cdots\partial_r \mathfrak{s}_\mu(a_1,\cdots,a_r)\big|_{a_j=0}}{2k\uparrow\mu}$$

$$= (-\mathfrak{i})^r \sum_{\mu\vdash r} \frac{\mathfrak{s}^*_\mu\left(\{N\}^k\right)\left\langle \mathfrak{s}_\mu, \mathfrak{p}_{\langle 1^r\rangle}\right\rangle_{\mathcal{S}_r}}{2k\uparrow\mu}$$

$$= (-\mathfrak{i})^r \sum_{\mu\vdash r} \frac{\mathfrak{s}^*_\mu\left(\{N\}^k\right)\dim\mu}{2k\uparrow\mu}$$

$$= \mathfrak{i}^r r! \sum_{\mu\vdash r} \frac{1}{H(\mu)^2}\frac{(-N\uparrow\mu)(k\uparrow\mu)}{2k\uparrow\mu},$$

with a condition that $0\le r\le 2k$.

transpose

$\mathfrak{s}_{k^N}$

$s(U) = 0$ when length

rectangle $< N^k >$

Frobenius

vect

sort

ones

character symmetric group