Random Matrix Theory and Zero Statistics of Elliptic Curve L-functions

Duc-Khiem Huynh, Zürich, October 2008

Joint work with Jon Keating, Eduardo Dueñez,
Steven Miller and Nina Snaith

Random Matrix Theory $\leftrightarrow$ Number Theory

Random Matrix = square matrix where the entries are independent and identically-distributed

Random Matrices of interests are

1. $U(N)$ – unitary group, $XX^* = I_N$ (entries normally distributed)
2. $SO(2N)$ and $SO(2N+1)$ – orthogonal group, $XX^T = I_N$
3. $USp(2N)$ – symplectic group, $XZX^T = Z$, where
$$Z = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$$

- We are in particular interested in the *eigenvalues* of a random matrix. They all lie on the unit circle.
- Each of these compact Lie groups have a Haar measure and integration formulas. This allows us to do analysis.

- An *L-function* is formally a complex valued function of the form

$$L(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where $a_n \in O(n^\varepsilon) \quad \forall \varepsilon > 0$. It has an meromorphic continuation in the whole complex plane, a functional equation and an Euler product.

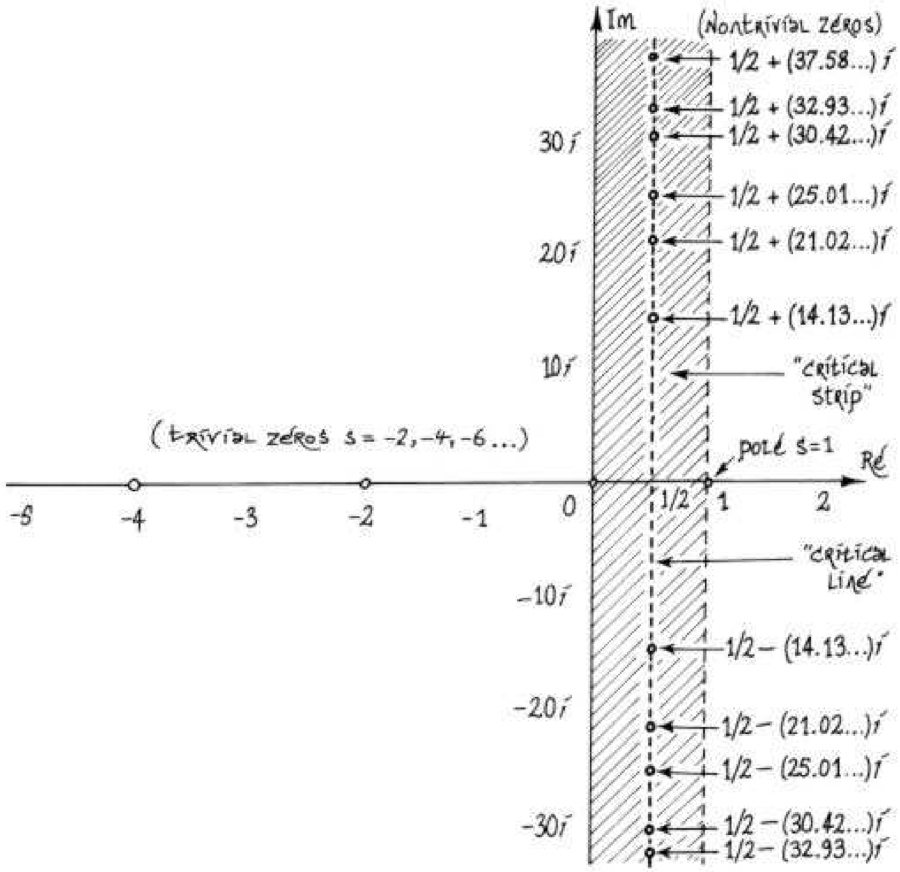
- We can form a family of L-functions by 'twisting' a given L-function:

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \frac{a_n \chi_d(n)}{n^s}.$$

Here d is an integer [fundamental discriminant] and χ_d is the Legendre symbol, it is either -1, 0 or $+1$.

- The *conductor* is a ordering quantity within our family of L-functions. In analogy to RMT is via the matrix size.

The L-functions considered have a *'Generalized Riemann Hypothesis'* (GRH): Their non-trivial zeros all lie on the critical line 1/2.



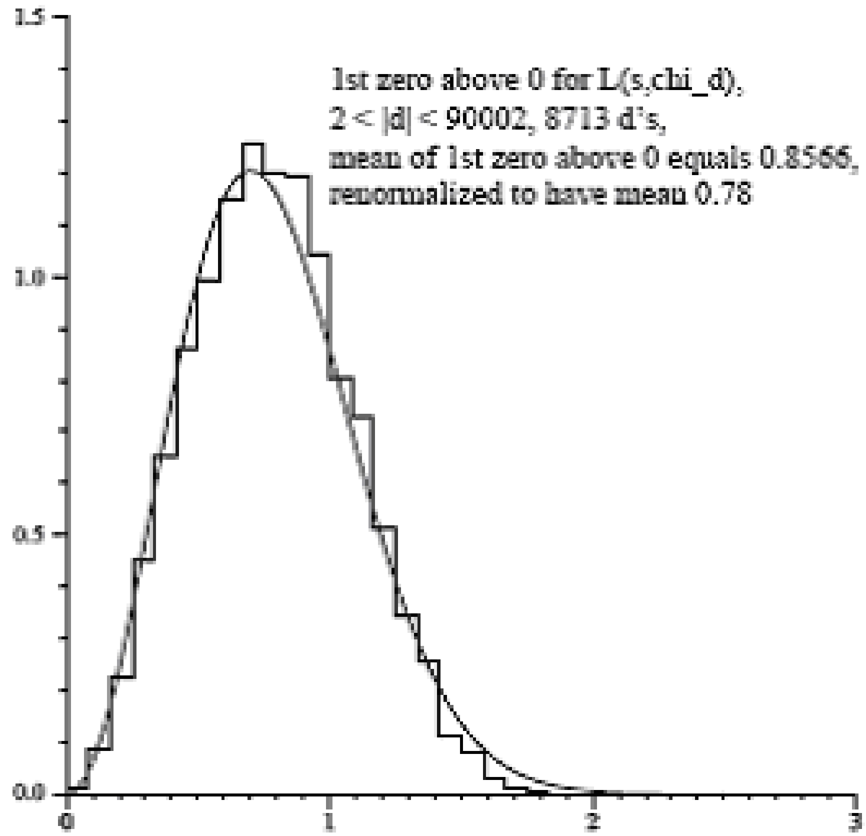
Low lying zeros of $\zeta(s)$

Assuming GRH we can consider various zero statistics of

- one individual L-function
- a family of L-functions

E.g. in the latter case: How likely it is to find the first zero of a L-function from a given family above the critical point with a certain height?

This was investigated by M Rubinstein in his thesis: Restricting $d < X$ and twisting the Riemann ζ-function gives a finite set of L-functions. Now we count how many of them have 1st zeros at a certain height. We increase the height stepwise and obtain an histogram:



Graphic by M Rubinstein

The step-function is the result of this discrete zero statistic of this finite set of L-functions. The smooth curve is the distribution of 1st eigenvalues of USp. Observe that the plots match very nicely.

- In this concrete example we can model the zeros of these L-functions as the eigenvalues of random matrices of the group USp.
- In general: The zeros of a family of L-functions show the same statistics as the eigenvalues of matrices of one of the classical compact groups. [Katz-Sarnak philosophy]

That's a *big mystery* because noone knows why this relation holds.

There are plenty of other examples giving evidence that zeros of L-functions have spectral interpretation.

- Via this relation/link we can use RMT to model NT-objects.
- This approach is because we can *do* concrete calculations in RMT while in NT this is sometimes hardly possible.
- Once we have found the right RMT-model for an NT-problem we can do RMT-calculations to make a prediction in NT.
- We will demonstrate this for a family of L-functions coming from an elliptic curve.

- An *elliptic curve* over the field $\mathbb{Q}$ of rational numbers is a curve defined by
$$E : y^2 = x^3 + ax + b$$
where $a, b \in \mathbb{Z}$ with discriminant $\Delta := -16(4a^3 + 27b^2) \neq 0$.

- Like in the case of the L-functions we can twist E by varying some integer d:
$$E_d : dy^2 = x^3 + ax + b$$
where d satisfies some conditions.

- *Basic problem:* Given an elliptic curve E, how many solutions in rational numbers are there?

- The number of rational points is related to the *rank of* E, which is an integer. In general the rank of E is hard to determine.

- To an elliptic curve we can associate an L-function

$$L_E(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where $a_p = p + 1 - \#E(\mathbb{F}_p)$ and $\#E(\mathbb{F}_p)$ denotes the number of points on E regarded over $\mathbb{F}_p$, p prime.

- A beautiful connection between the arithmetic of E and analytic properties of its L-function is spelled out by the *Birch/Swinnerton-Dyer conjecture* (Millennium Prize Problems): The order of vanishing of $L_E(s)$ at $s = 1/2$ is equal to the rank of E.

- This connection enables us to use RMT to investigate the rank of elliptic curves by considering their associated L-functions.

- Goal: We want a RMT-model for a given order of vanishing at the critical point of L-functions.

- For simplicity consider a family of elliptic curve L-functions with all having even functional equation. From heuristics their zeros should show the same statistics as the eigenvalues of random matrices of $SO(2N)$.
- Hence the subset of even L-functions having order of $2r$ vanishing at the critical point should correspond to the subset of $SO(2N)$ of having $2r$ zeros at 1.
- Do the multiple eigenvalues at 1 effect near by eigenvalues?
  - ⋆ No $\rightarrow$ *independent model*
  - ⋆ Yes $\rightarrow$ *interaction model*

- The independent model has the statistics of $SO(2N-2r)$ with $2r$ eigenvalues located at 1. Note $SO(2N)$ has Haar measure

$$c_n \times \prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 \prod_{1\le j\le N} d\theta_j,$$

where c_N is some constant.

- RMT-calculation (Snaith & Miller/Dueñez) gives for the interaction model the alternative measure

$$\tilde{c}_N \times \prod_{1\le j<k\le N-r} (\cos\theta_k - \cos\theta_j)^2 \prod_{1\le j\le N-r} (1-\cos\theta_j)^{2r} d\theta_j$$

- Observe that this differs from Haar measure of $SO(2N)$ by

$$\prod_{1\le j\le N-r} (1-\cos\theta_j)^{2r} d\theta_j.$$

- With the interaction model we expect to see on the NT side that
  - ⋆ first zero is repelled by zeros at the critical point
  - ⋆ the more central point zeros the greater the repulsion: zeros not likely to be close to the critical point.
- With the independent model we expect to see on the NT side zeros do *not* repell near by zeros.
- Which model is correct?
  - ⋆ Young and Miller showed that for restricted test functions that some zero statistics of families of elliptic curve L-functions in the large conductor limit have orthogonal symmetry $\longrightarrow$ independent model
  - ⋆ However for finite conductor Miller's experimental data shows repulsion of the 1st zero for his one-parameter family of elliptic curve L-functions $\longrightarrow$ interaction model

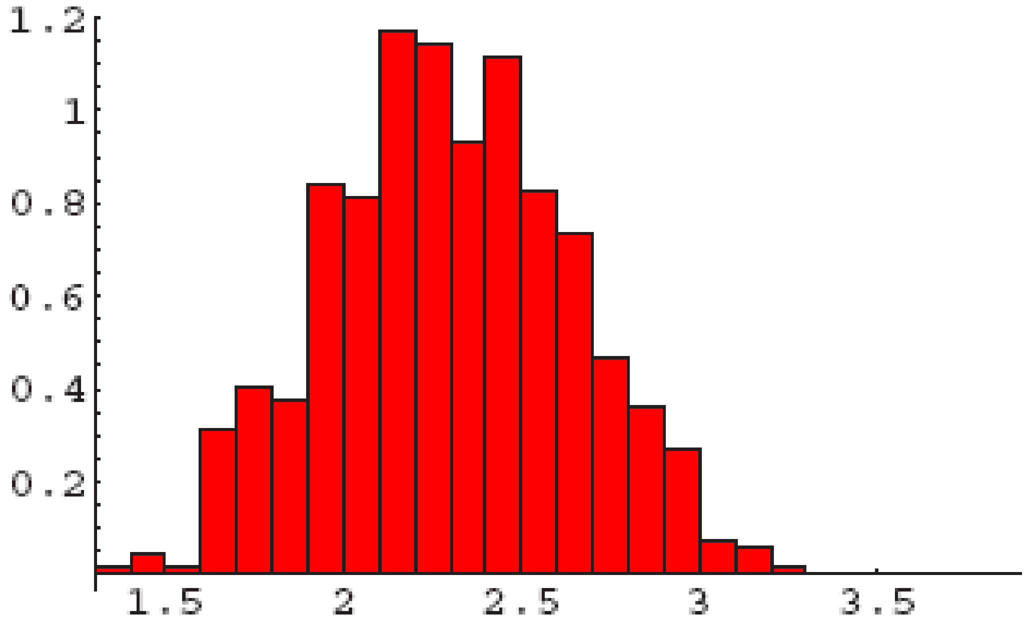


Figure 5: First normalized zero above the central point:
665 rank 2 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [10, 10.3125]$, median $= 2.29$, mean $= 2.30$

Graphic by S Miller

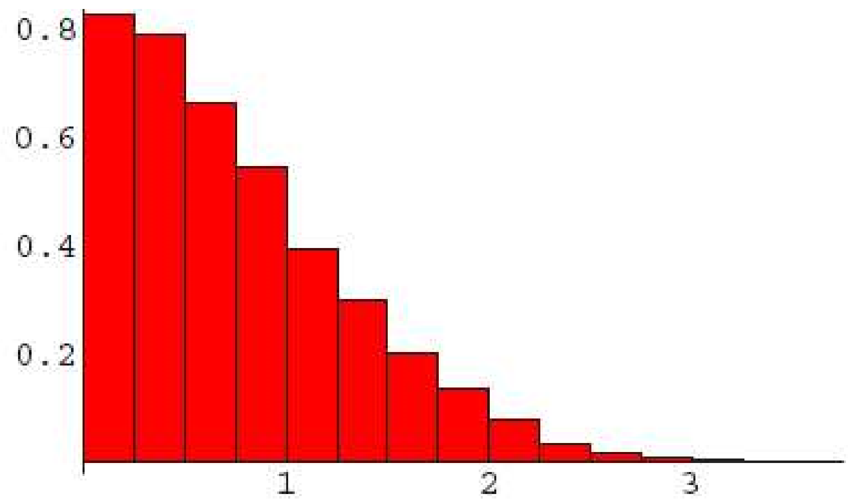


Figure 1b: First normalized eigenangle above 1: 23,040 SO(6) matrices
Mean = .635, Standard Deviation about the Mean = .574, Median = .635

Graphic by S Miller

How can we explain the phenomenom of having interaction for finite conductor and also independency in the large conductor limit?

$\rightarrow$ A study of lower order terms might help.

We investigate the family quadratic twists coming from an elliptic curve L-function with even functional equation because

- Experimental data from Rubinstein's lcalc can be obtained
- Lower order terms for this family from the ratios conjectures

So we can compare theory and data.

- For the rest of the talk we focus on the 1-level-density:

$$\tilde{S}_1(\varphi) = \frac{1}{X^*} \sum_{d \le X} \sum_{\gamma_d} \varphi(\gamma_d)$$

where φ is a even Schwartz test function and γ_d the ordinate of a generic zero of $L_E(s, \chi_d)$ on the critical line.

- By the argument principle we can write

$$\tilde{S}_1(\varphi) = \frac{1}{X^*} \sum_{d \le X} \frac{1}{2\pi i} \left(\int_{(c)} - \int_{(1-c)} \right) \frac{L'(s, \chi_d)}{L(s, \chi_d)} \varphi(-i(s - 1/2)) ds$$

where $3/4 > c > 1/2 + 1/\log X$.

- Hence: If we have a conjecture for

$$\sum_{d \le X} \frac{L_E'(s,\chi_d)}{L_E(s,\chi_d)} \qquad (1)$$

we can also give a conjectural answer for $\tilde{S}_1(\varphi)$.

Using the ratios conjectures we get an estimate for (1).

- The *ratios conjectures* (Conrey, Farmer, Zirnbauer) give precise formulas for quantities like

$$\sum_{0<d\le X} \frac{\prod_{k=1}^{K} L(1/2+\alpha_k,\chi_d)}{\prod_{q=1}^{Q} L(1/2+\gamma_q,\chi_d)}.$$

Simpliest case when $K = Q = 1$.

- For our family we are interested in the following ratio with $\Re(\alpha), \Re(\gamma) > 0$

$$\sum_{d \le X} \frac{L_E(1/2+\alpha, \chi_d)}{L_E(1/2+\gamma, \chi_d)} =: R_E(\alpha, \gamma)$$

and observe that

$$\sum_{d \le X} \frac{L'_E(1/2+r, \chi_d)}{L_E(1/2+r, \chi_d)} = \frac{d}{d\alpha} R_E(\alpha, \gamma)\big|_{\alpha=\gamma=r}.$$

From the ratios conjecture we get for the 1-level-density

$$\tilde{S}_1(g) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)\frac{1}{X^*}\sum_{d\le X}\left(2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right)\right.$$

$$+\frac{\Gamma'}{\Gamma}(1+it)+\frac{\Gamma'}{\Gamma}(1-it)$$

$$+2\Big[-\frac{\zeta'(1+2it)}{\zeta(1+2it)}+\frac{L_E'(\mathrm{sym}^2,1+2it)}{L_E(\mathrm{sym}^2,1+2it)}+A_f'(it,it)$$

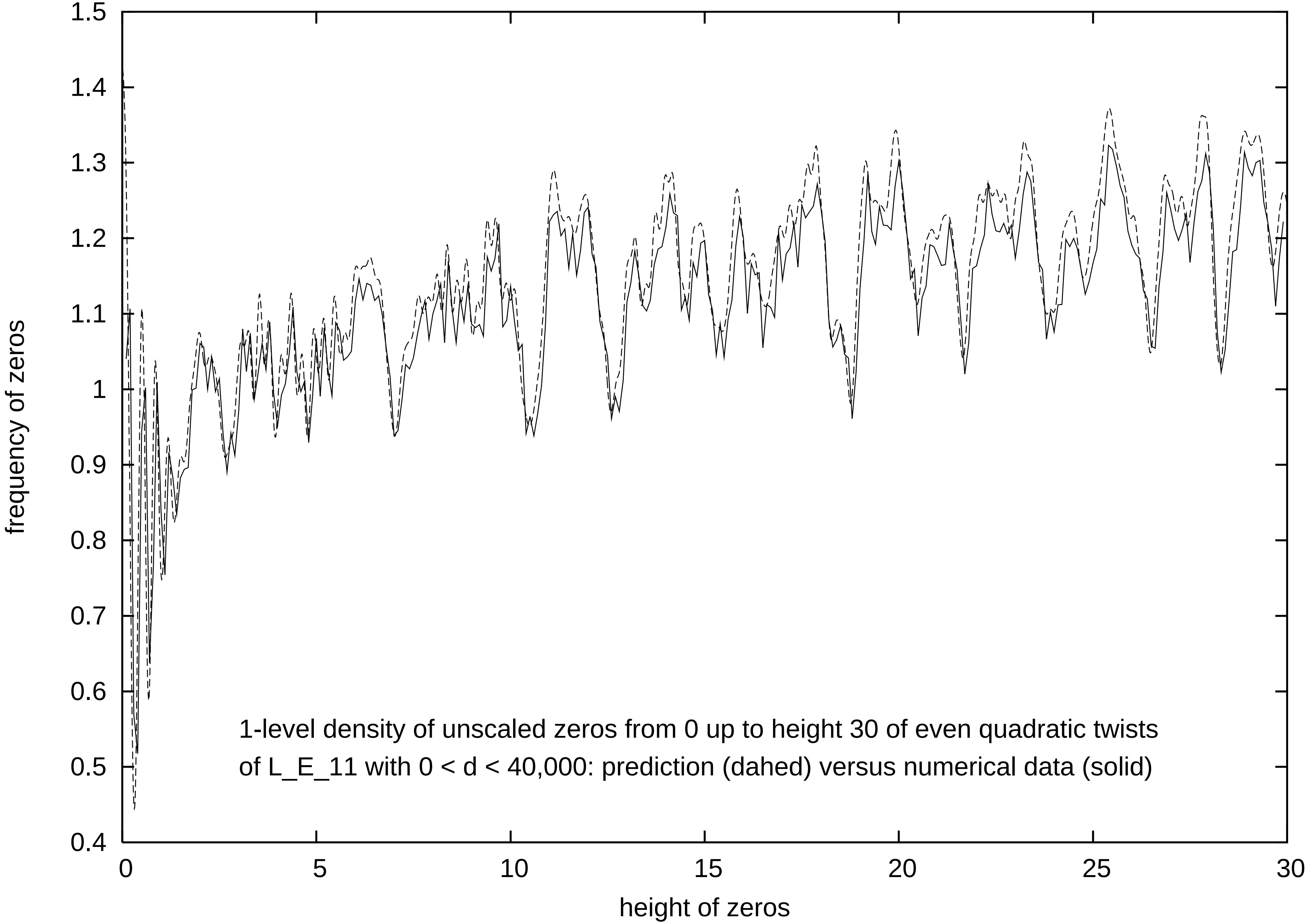
$$\left.-\left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2it}\frac{\Gamma(1-it)}{\Gamma(1+it)}\frac{\zeta(1+2it)L_E(\mathrm{sym}^2,1-2it)}{L_E(\mathrm{sym}^2,1)}A_f(-it,it)\Big]\right)dt$$

$$+O(X^{-1/2+\varepsilon})$$

where M is the conductor of the elliptic curve E and A_f is a product over primes. We note that the ratios conjecture give all terms down to $O(X^{-1/2+\varepsilon})$ which is a very precise prediction.

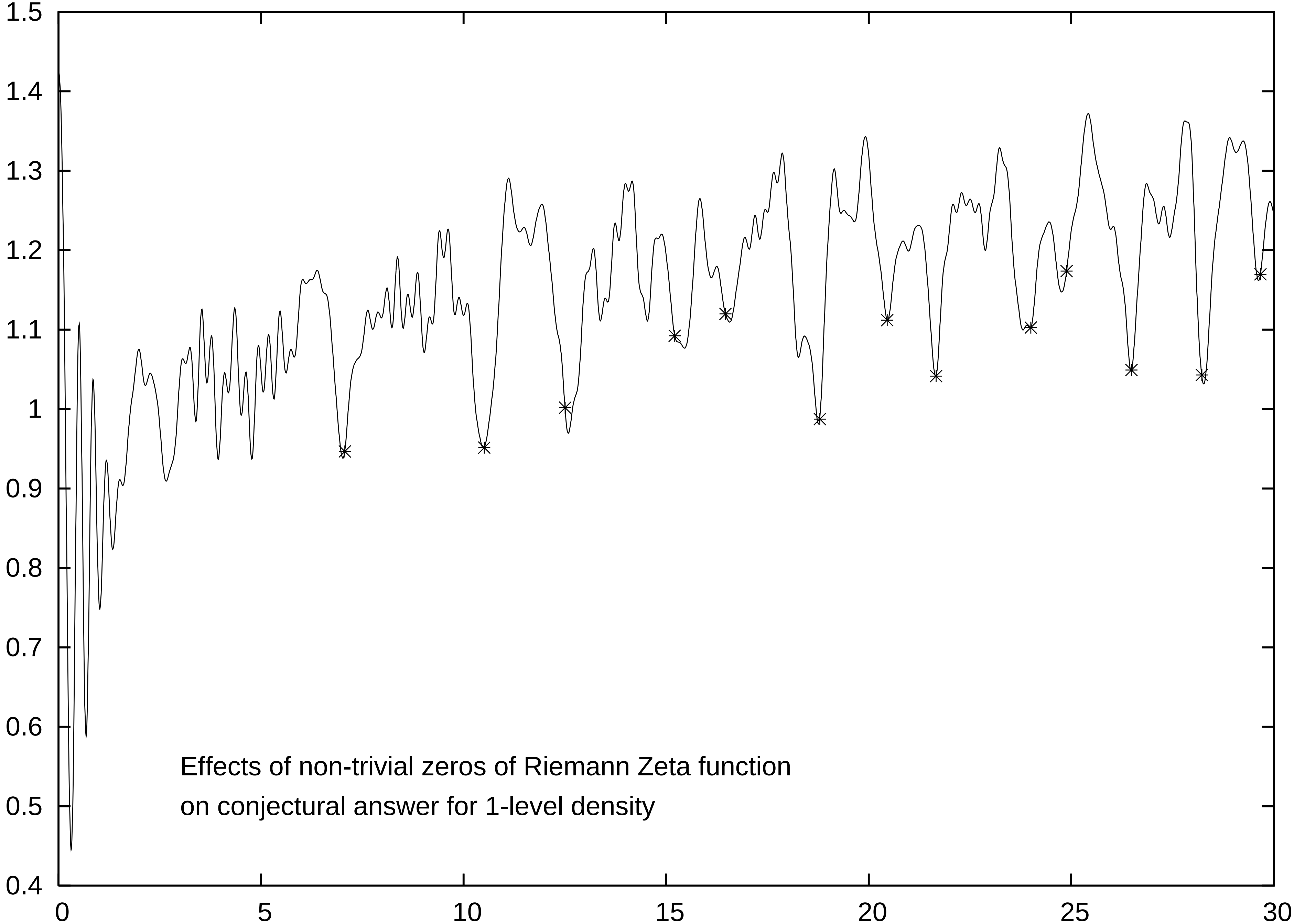
Next we test our prediction:

- We fix the elliptic curve E_{11} and consider its even quadratic twists between 0 and 40,000.
- We use Rubinstein's program to calculate the zeros for each twist up to height 30.
- With this data we obtain the 1-level-density.
- Then we compare the data with our prediction for finite conductor.


1-level density of unscaled zeros from 0 up to height 30 of even quadratic twists
of L_E_11 with 0 < d < 40,000: prediction (dahed) versus numerical data (solid)
frequency of zeros
height of zeros
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0
5
10
15
20
25
30

Our prediction for the 1-level-density of the family of L-functions with even functional equation coming from an elliptic curve L-function is

$$\tilde{S}_1(g) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)\frac{1}{X^*}\sum_{d\le X}\left(2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right)\right.$$
$$+\frac{\Gamma'}{\Gamma}(1+it)+\frac{\Gamma'}{\Gamma}(1-it)$$
$$+2\Big[-\frac{\zeta'(1+2it)}{\zeta(1+2it)}+\frac{L_E'(\mathrm{sym}^2,1+2it)}{L_E(\mathrm{sym}^2,1+2it)}+A_f'(it,it)$$
$$\left.-\left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2it}\frac{\Gamma(1-it)}{\Gamma(1+it)}\frac{\zeta(1+2it)L_E(\mathrm{sym}^2,1-2it)}{L_E(\mathrm{sym}^2,1)}A_f(-it,it)\Big]\right)dt$$
$$+O(X^{-1/2+\varepsilon})$$


1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0
5
10
15
20
25
30
Effects of non-trivial zeros of Riemann Zeta function
on conjectural answer for 1-level density

Our prediction for the 1-level-density of the family of L-functions with even functional equation coming from an elliptic curve L-function is

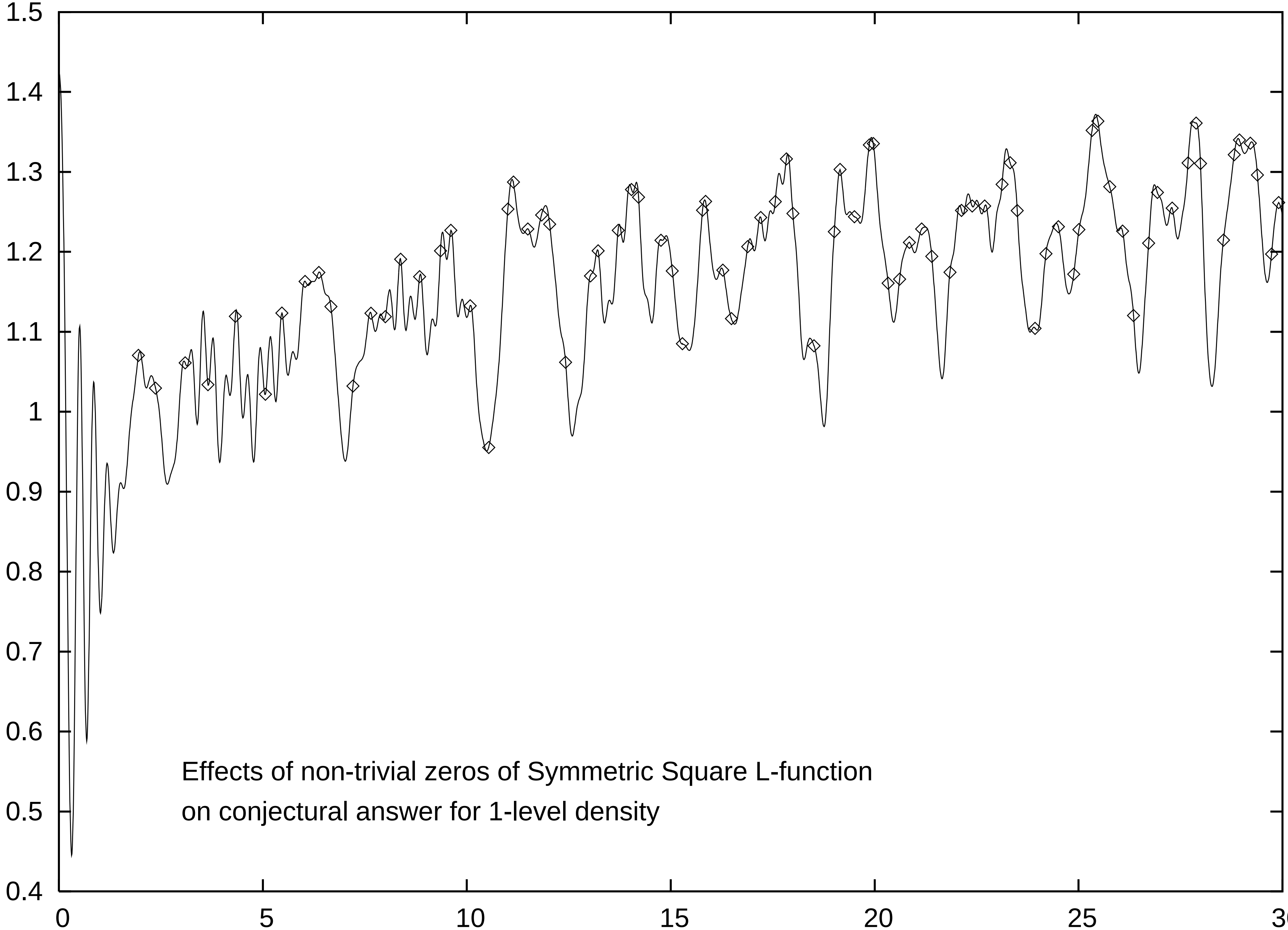
$$\tilde{S}_1(g) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)\frac{1}{X^*}\sum_{d\le X}\Bigg(2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right)$$

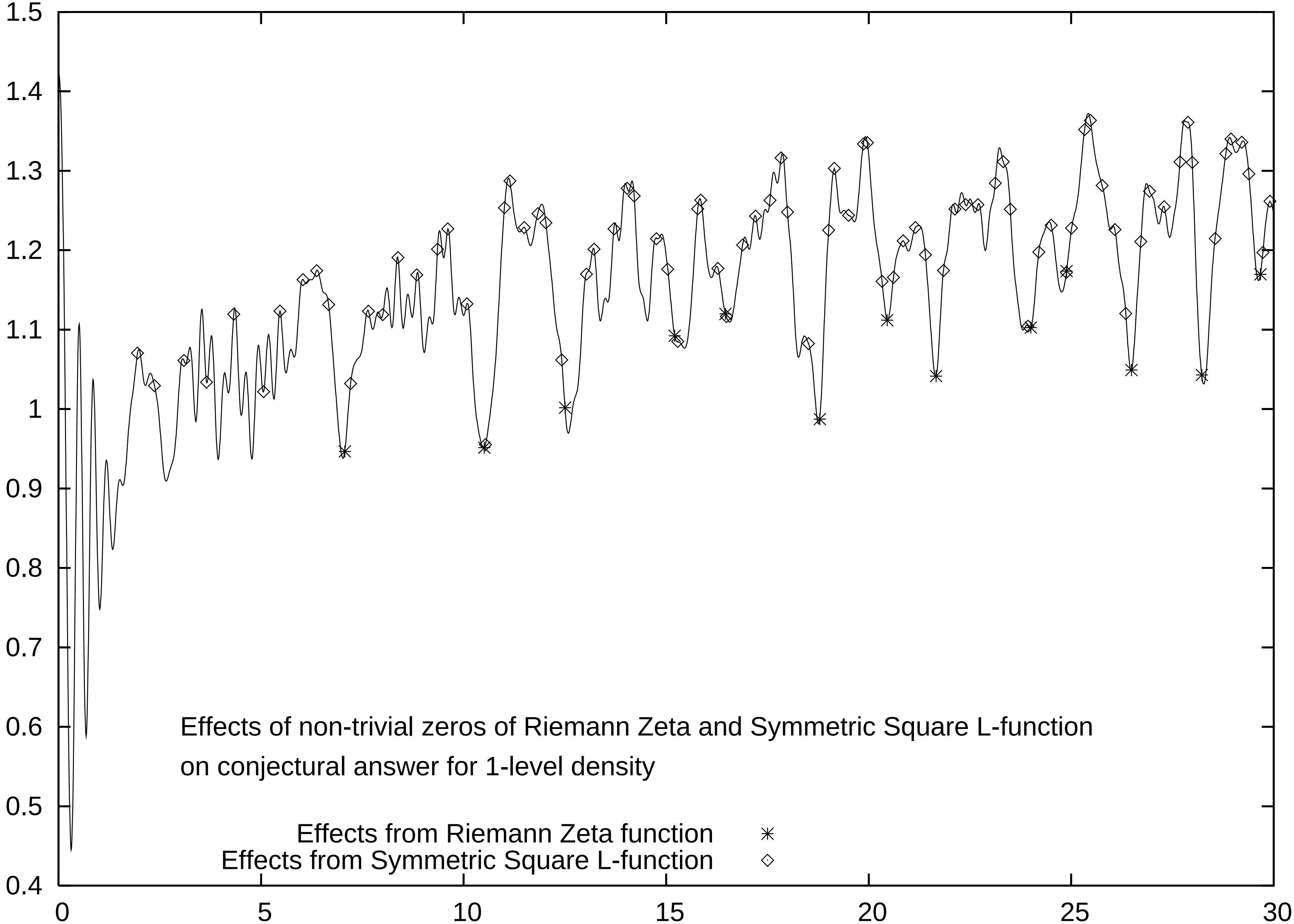
$$+\frac{\Gamma'}{\Gamma}(1+it)+\frac{\Gamma'}{\Gamma}(1-it)$$

$$+2\Big[-\frac{\zeta'(1+2it)}{\zeta(1+2it)}+\frac{L_E'(\mathrm{sym}^2,1+2it)}{L_E(\mathrm{sym}^2,1+2it)}+A_f'(it,it)$$

$$-\left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2it}\frac{\Gamma(1-it)}{\Gamma(1+it)}\frac{\zeta(1+2it)L_E(\mathrm{sym}^2,1-2it)}{L_E(\mathrm{sym}^2,1)}A_f(-it,it)\Big]\Bigg)dt$$

$$+O(X^{-1/2+\varepsilon})$$

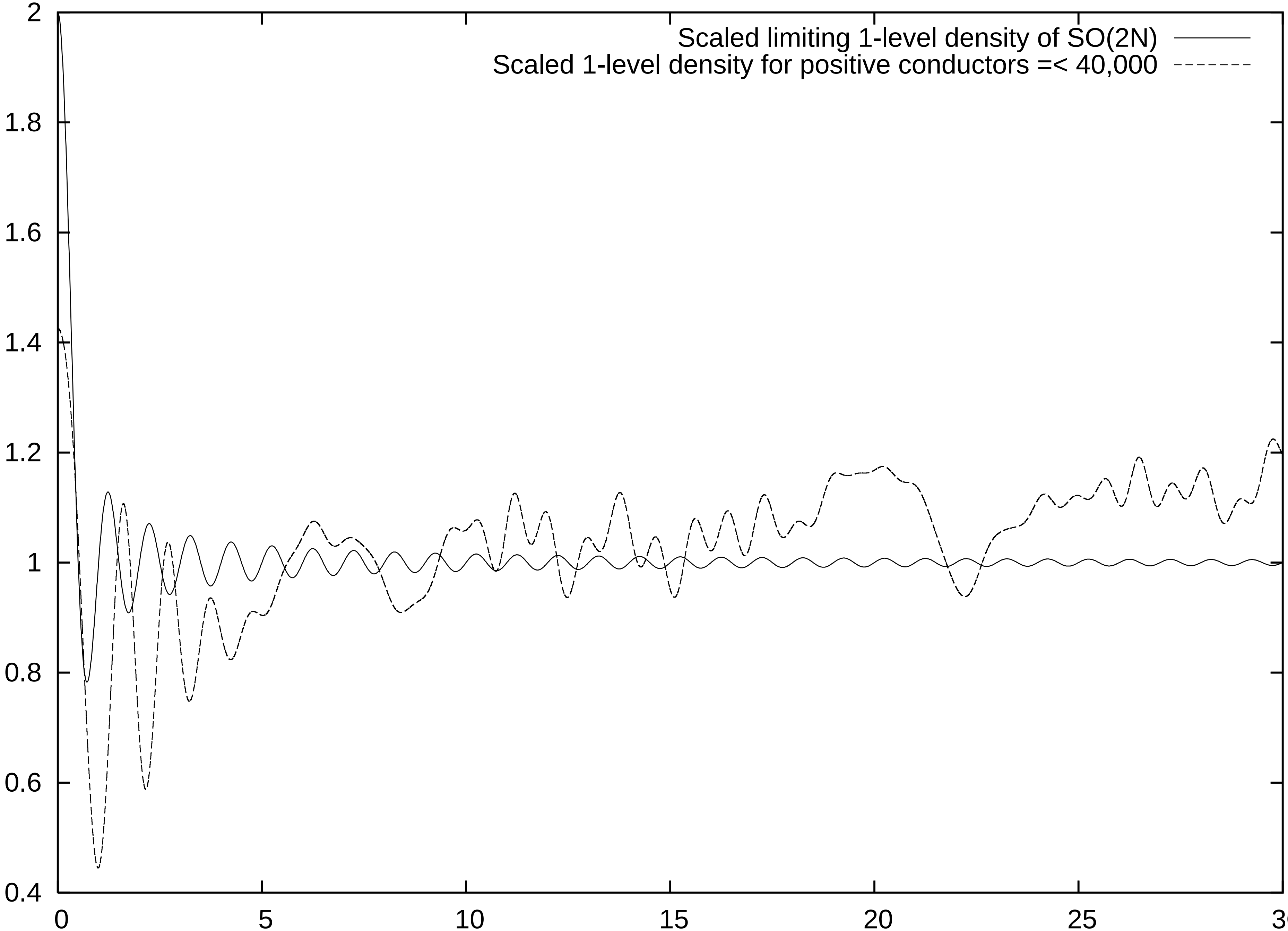

1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0
5
10
15
20
25
30
Effects of non-trivial zeros of Symmetric Square L-function
on conjectural answer for 1-level density

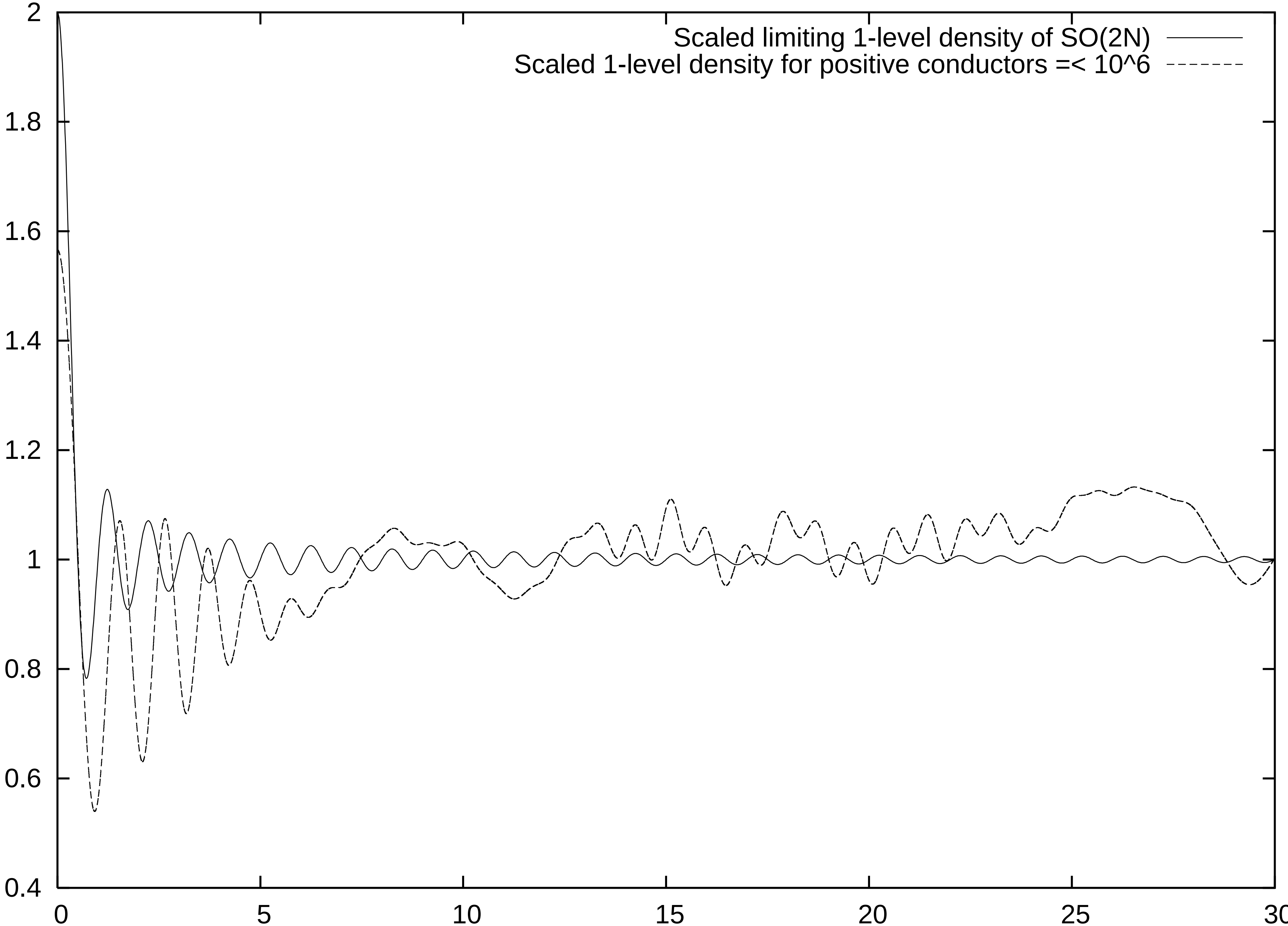

1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0
5
10
15
20
25
30
Effects of non-trivial zeros of Riemann Zeta and Symmetric Square L-function
on conjectural answer for 1-level density
Effects from Riemann Zeta function
Effects from Symmetric Square L-function

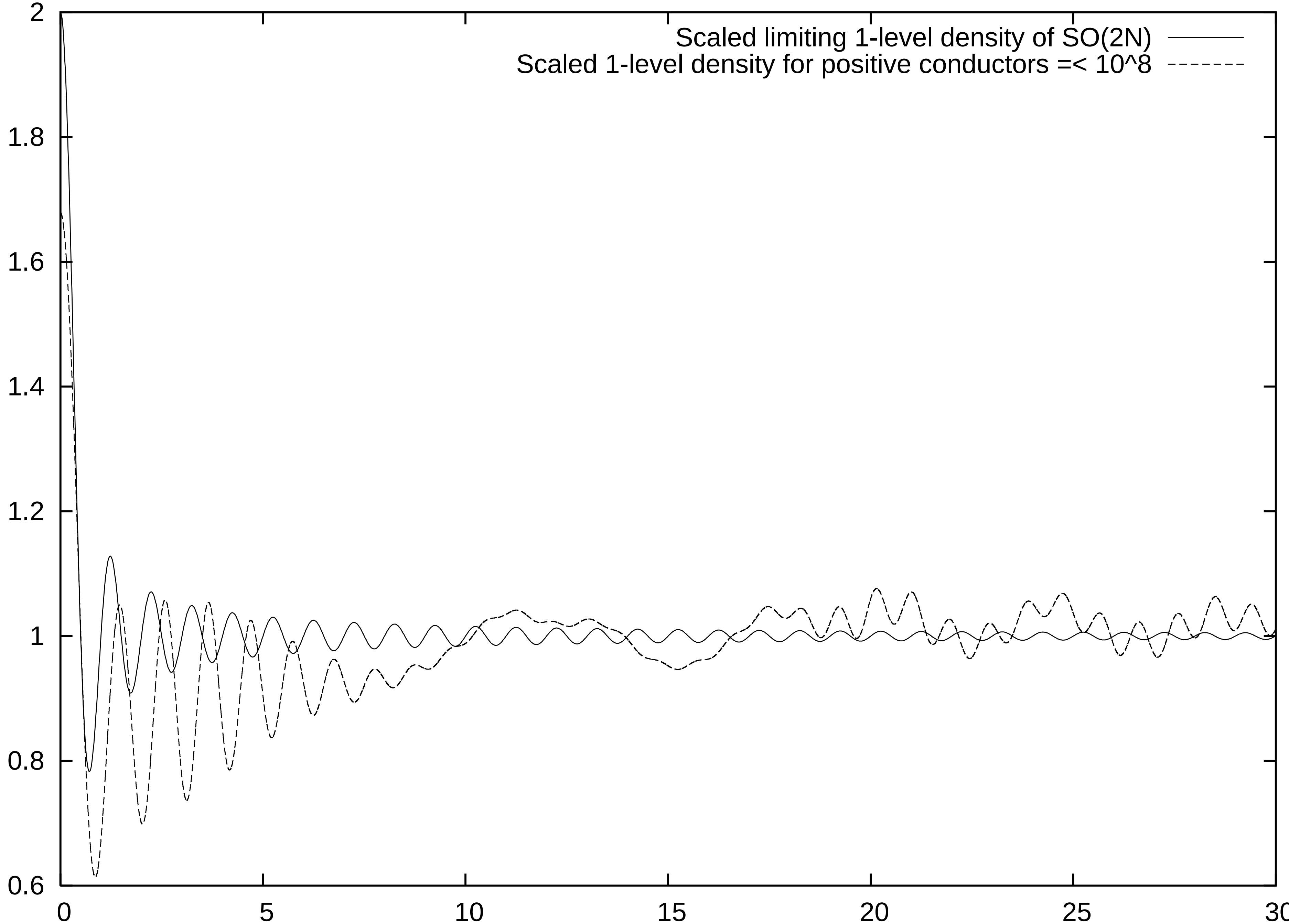
- The scaled 1-level-density in the large N limit of $SO(2N)$ is

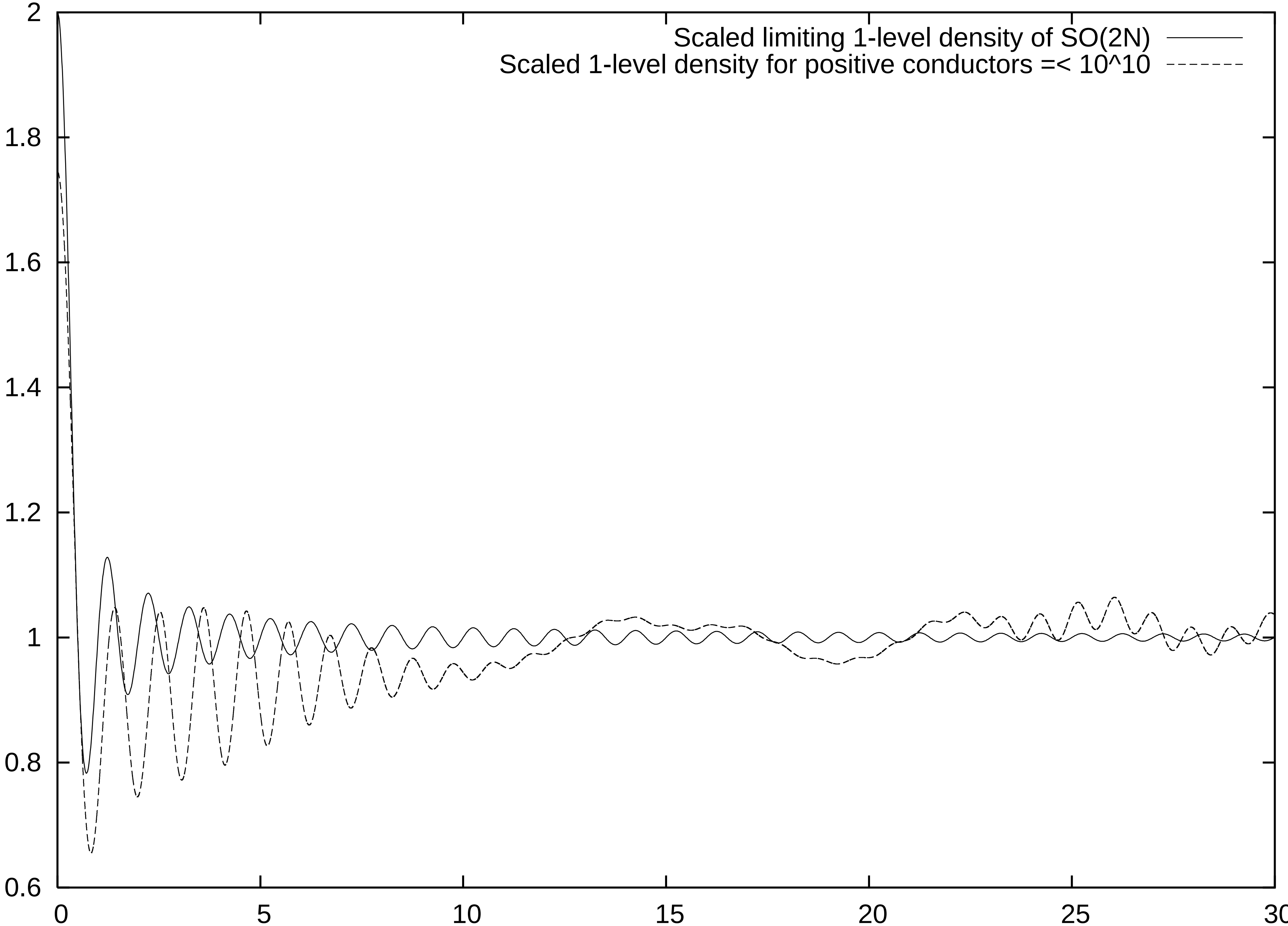
$$1 + \frac{\sin(2\pi x)}{2\pi x}.$$

- Next we compare this with our conjectural answer for the scaled 1-level-density for the family of quadratic twists for finite conductor.
- For this we increase the conductor.

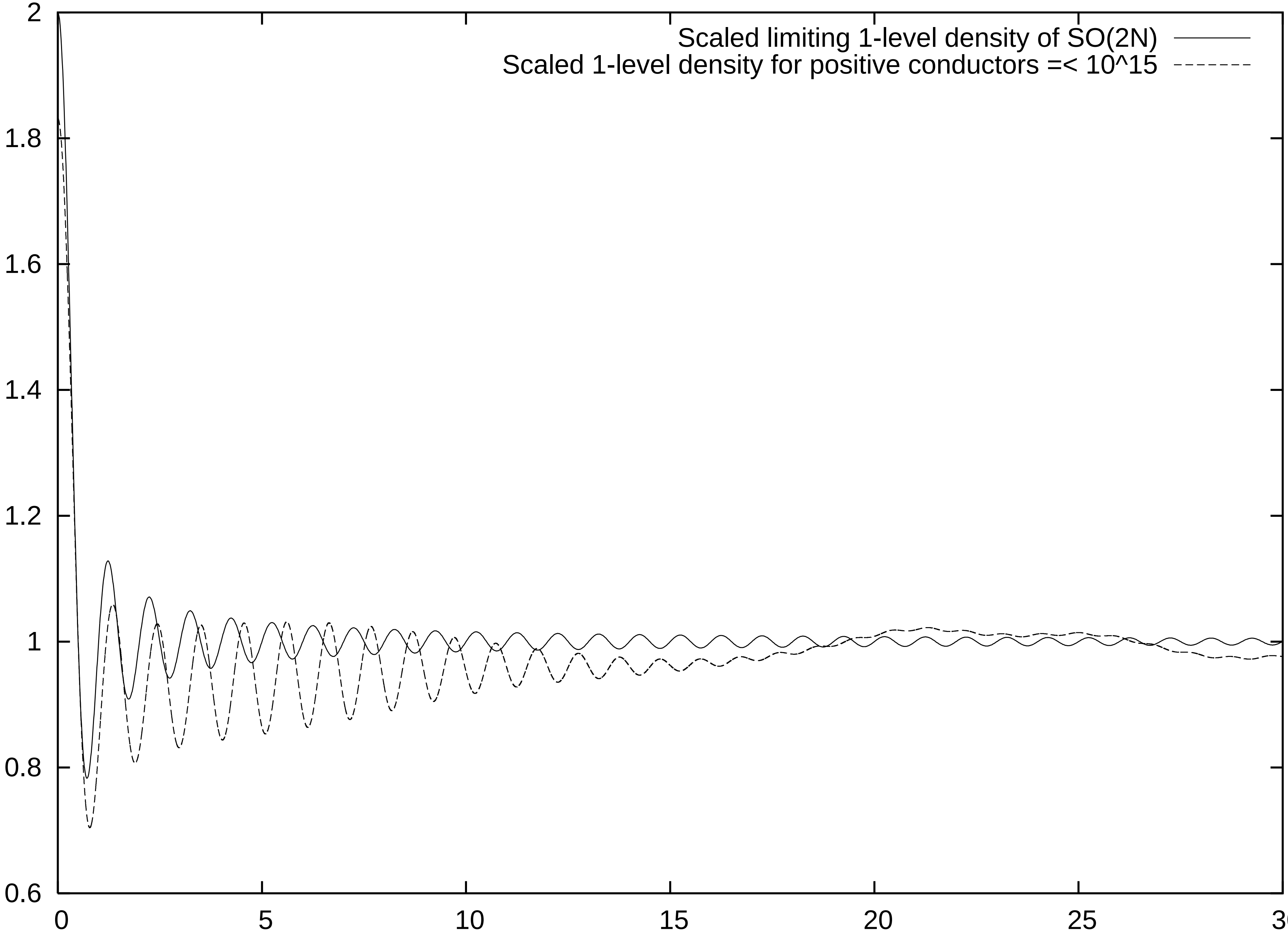

Scaled limiting 1-level density of SO(2N)
Scaled 1-level density for positive conductors =< 40,000
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0
5
10
15
20
25
30

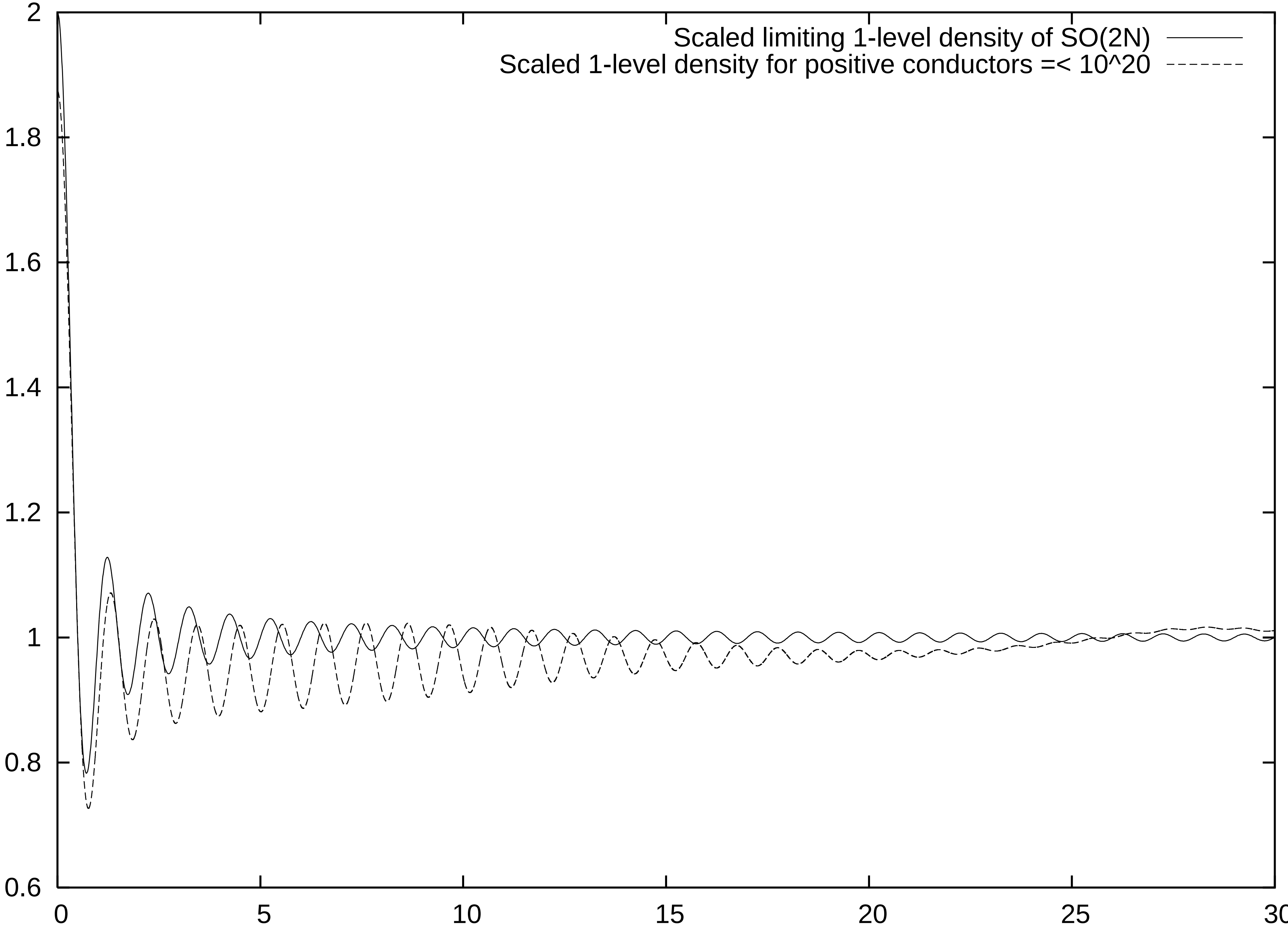

Scaled limiting 1-level density of SO(2N)
Scaled 1-level density for positive conductors =< 10^6
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0
5
10
15
20
25
30

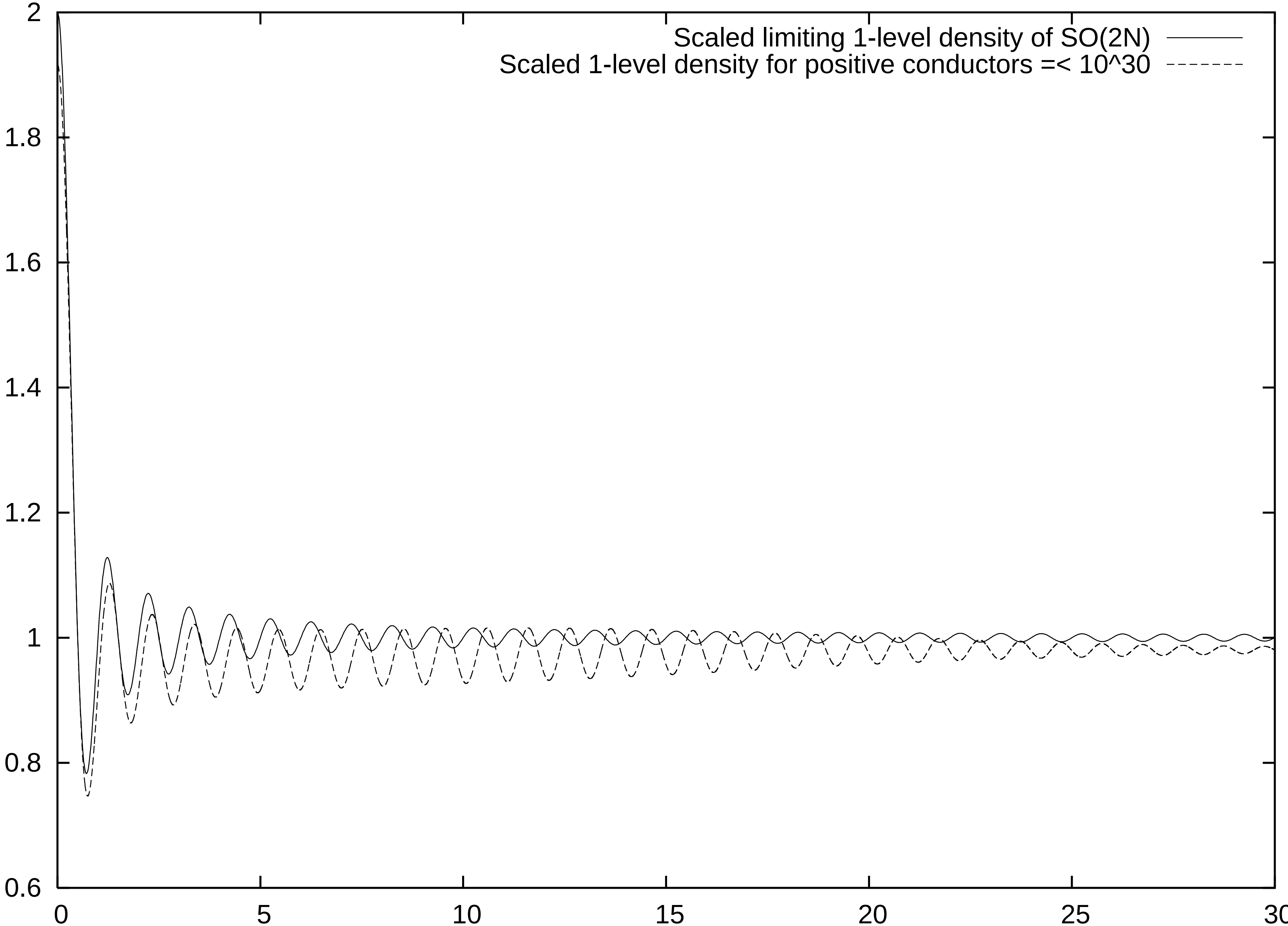

Scaled limiting 1-level density of SO(2N)
Scaled 1-level density for positive conductors =< 10^8
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0
5
10
15
20
25
30

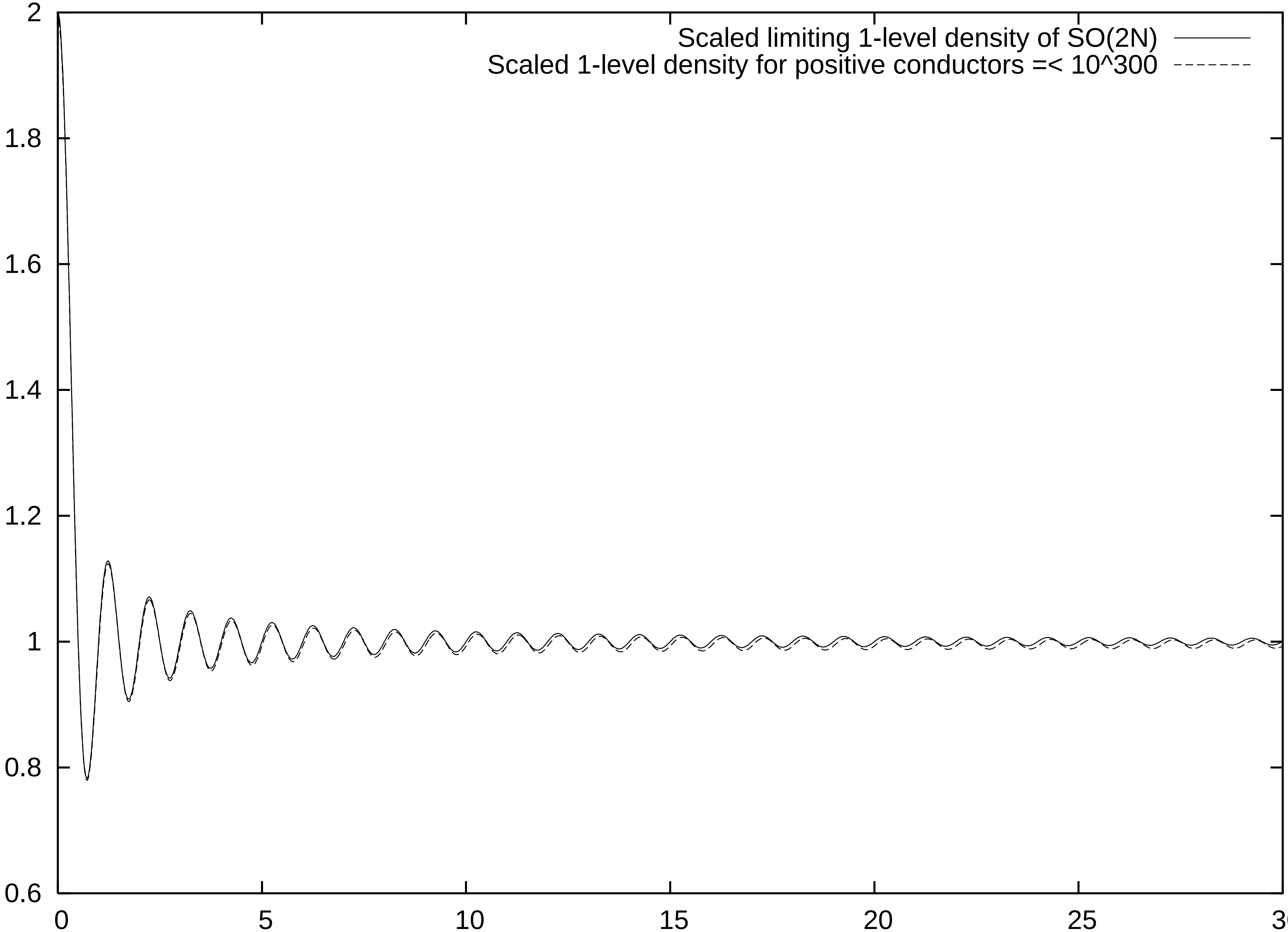


Scaled limiting 1-level density of SO(2N)
Scaled 1-level density for positive conductors =< 10^10
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0
5
10
15
20
25
30


Scaled limiting 1-level density of SO(2N)
Scaled 1-level density for positive conductors =< 10^15
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0
5
10
15
20
25
30


Scaled limiting 1-level density of SO(2N)
Scaled 1-level density for positive conductors =< 10^20
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0
5
10
15
20
25
30


Scaled limiting 1-level density of SO(2N)
Scaled 1-level density for positive conductors =< 10^30
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0
5
10
15
20
25
30



Scaled limiting 1-level density of SO(2N)
Scaled 1-level density for positive conductors =< 10^300
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0
5
10
15
20
25
30

- Using the ratios conjectures we are able to determine all lower order terms. We see that: lower order terms can be large and they dominate strongly the behaviour of the zeros for relative small conductor. Thus we can
  - ⋆ explain many features of the 1-level density for relatively small conductor
  - ⋆ model the slow convergence to the infinite conductor limit
- Are there limitations of the ratios conjectures?

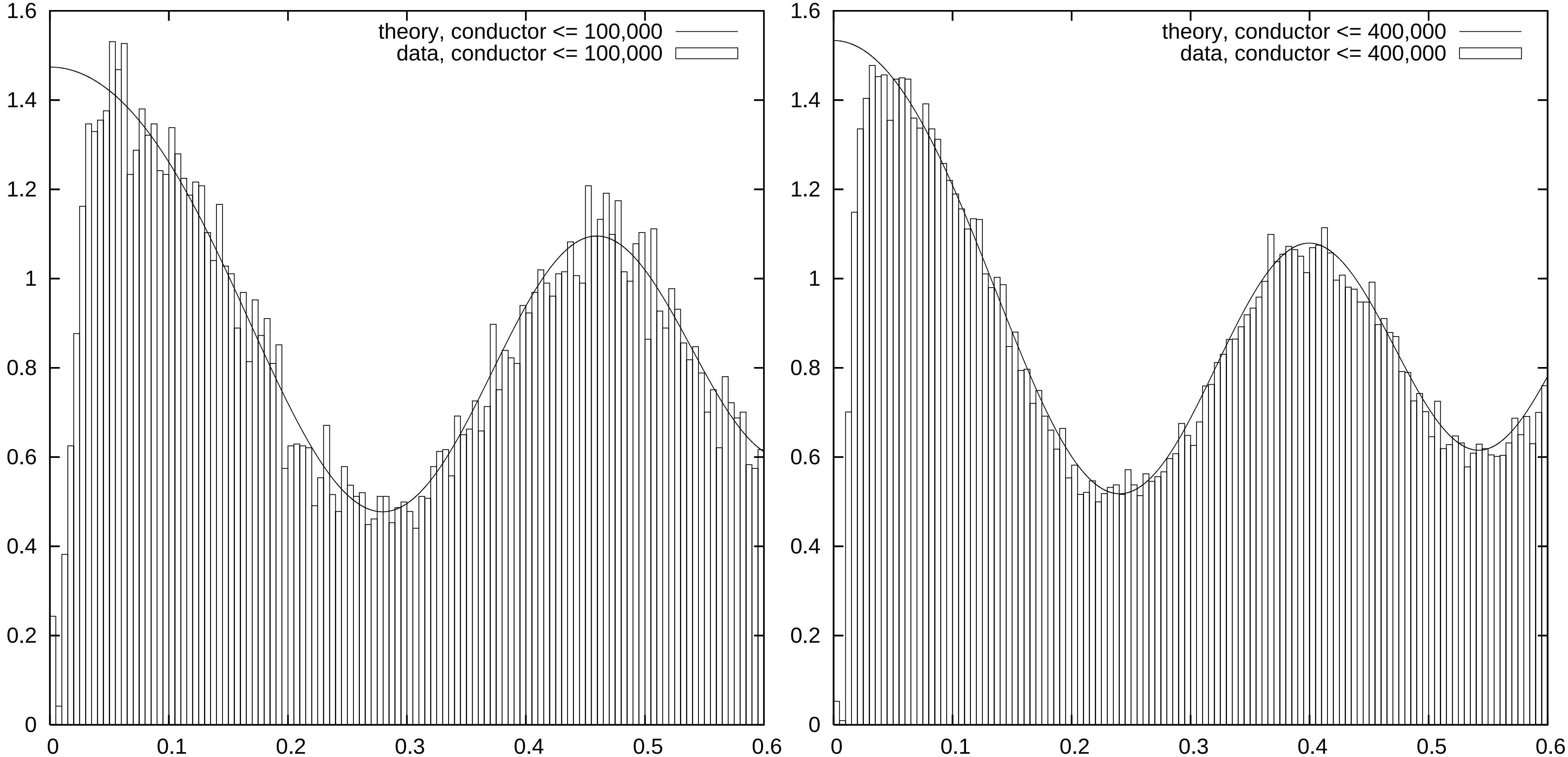


Figure 1: 1-level density of unscaled zeros from 0 up to height 0.6 of even quadratic twists of $L_{E_{11}}$ with $0 < d < 100,000$ for *left* and $0 < d < 400,000$ for *right* hand side, prediction (solid) versus data (bar chart)

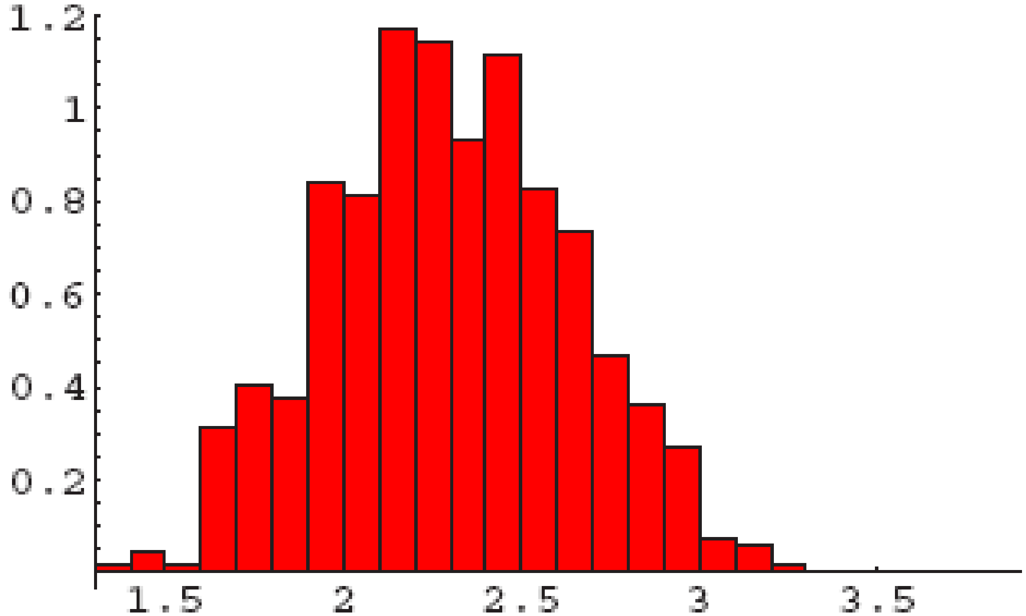
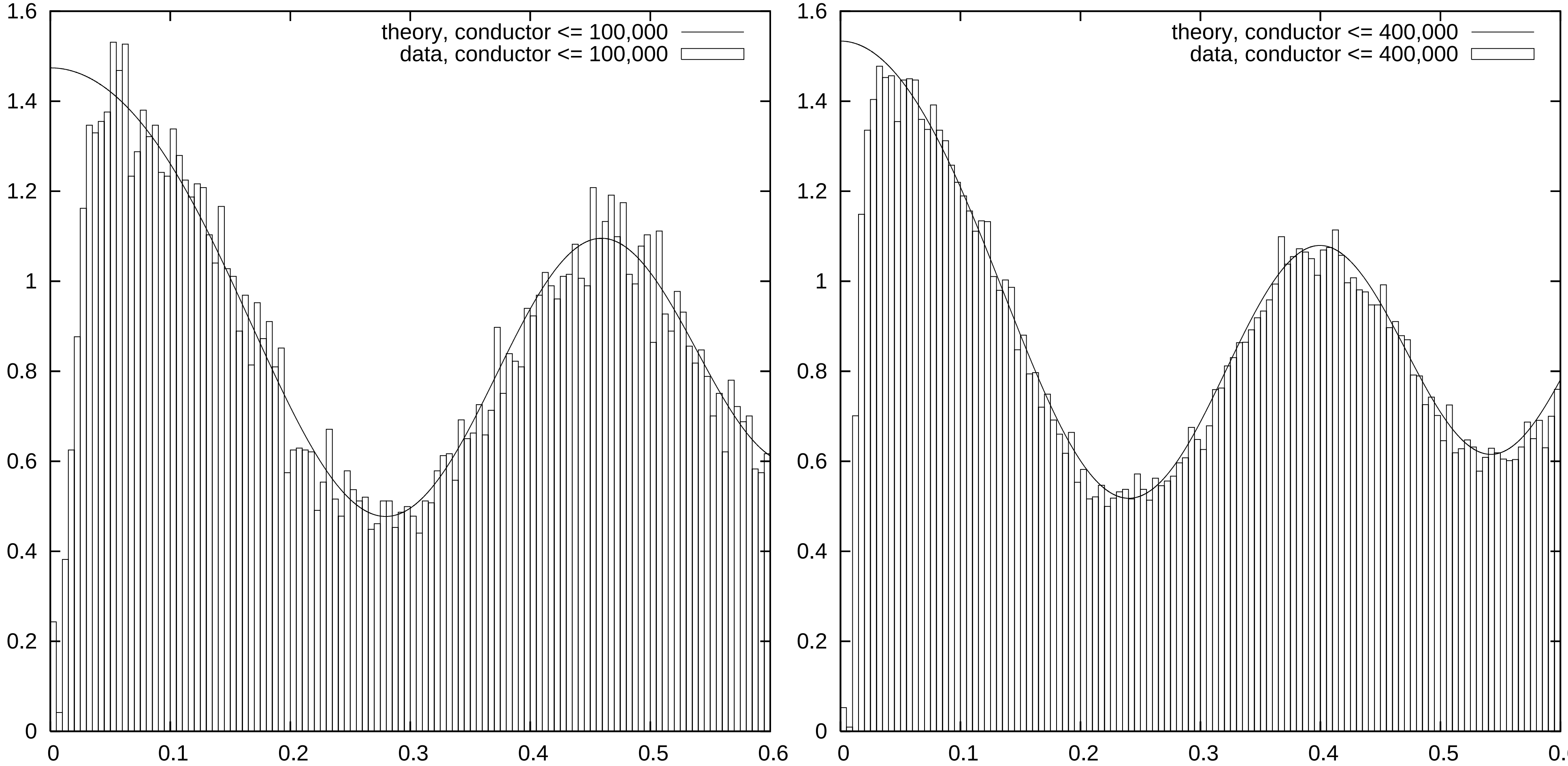


Figure 5: First normalized zero above the central point:
665 rank 2 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [10, 10.3125]$, median $= 2.29$, mean $= 2.30$

Graphic by S Miller

- It seems that our prediction coming from the ratios conjectures does not capture the observed repulsion in the data from the critical point.
- Natural question: can the discrepancy can by accounted for by the error term? Let's do a test!



theory, conductor <= 100,000
data, conductor <= 100,000
theory, conductor <= 400,000
data, conductor <= 400,000
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
0
0.1
0.2
0.3
0.4
0.5
0.6

- For each cut-off parameter X_0 we obtain a prediction "$theory(X_0)$" from our formula and corresponding data, call it "$data(X_0)$".

- We fix a specific height t_0 to compare "$theory(X_0)$" at t_0 and "$data(X_0)$" at t_0.

- Now we vary $0 < X < 400,000$:
  how big is

$$|\Delta(t_0, X)| := |theory(t_0, X) - data(t_0, X)|?$$

  In other words: $|\Delta(t_0, X)| = O(X^{b+\varepsilon})$, what is b?

- Now we vary $0 < X < 400,000$:
  how big is

$$|\Delta(t_0, X)| := |theory(t_0, X) - data(t_0, X)|?$$

  In other words: $|\Delta(t_0, X)| = O(X^{b+\varepsilon})$, what is b?

- If the error term from the ratios conjectures is correct we have

$$|\Delta(t_0, X)| = O(X^{-1/2+\varepsilon}),$$

  thus $b = -1/2$.

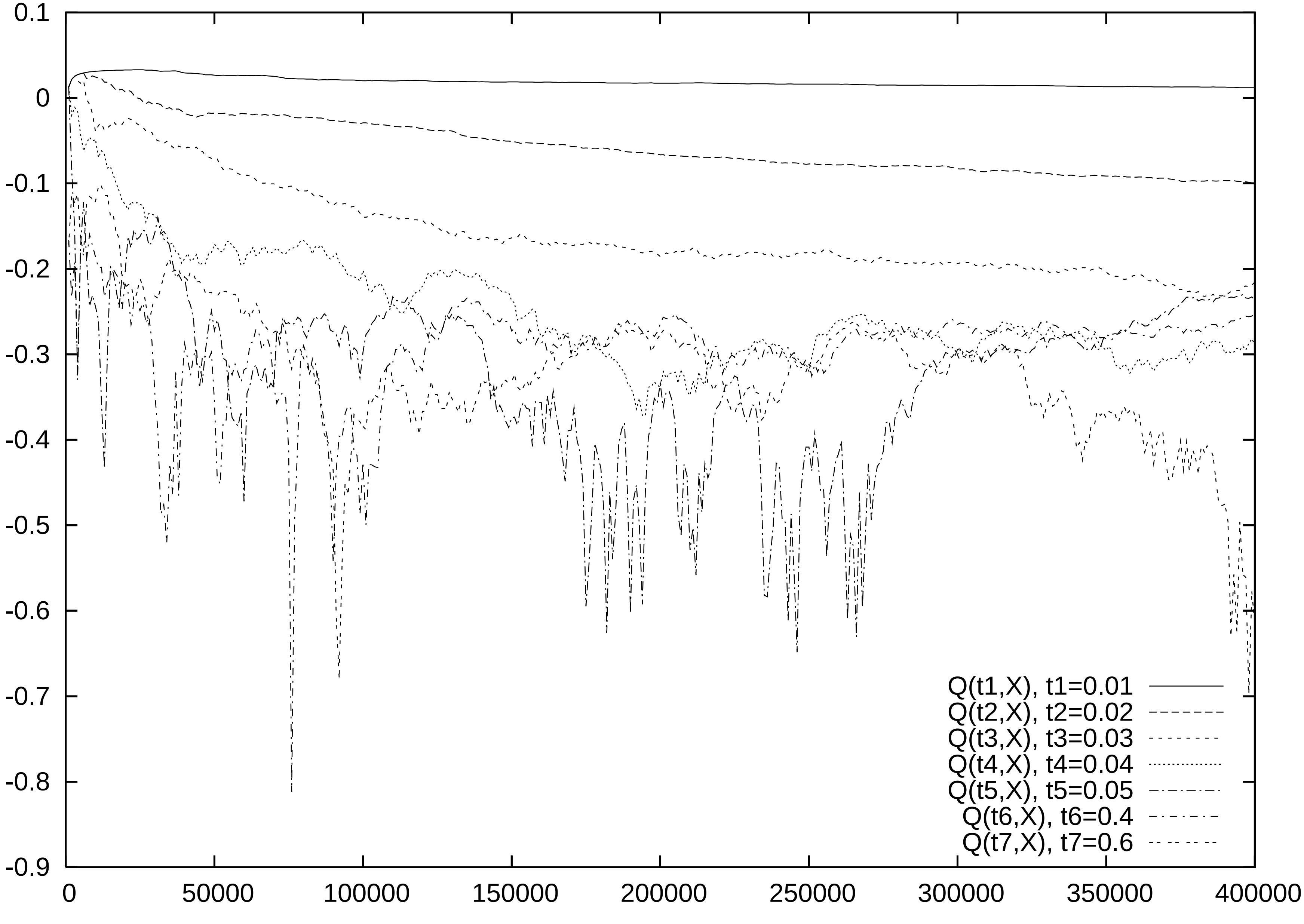
- We plot

$$Q_\Delta(t_0, X) = \frac{\log(|\Delta(t_0, X)|)}{\log X}$$

  for various fixed points t_0.

- We plot
$$Q_\Delta(t_0, X) = \frac{\log(|\Delta(t_0, X)|)}{\log X}$$
for various fixed points t_0.

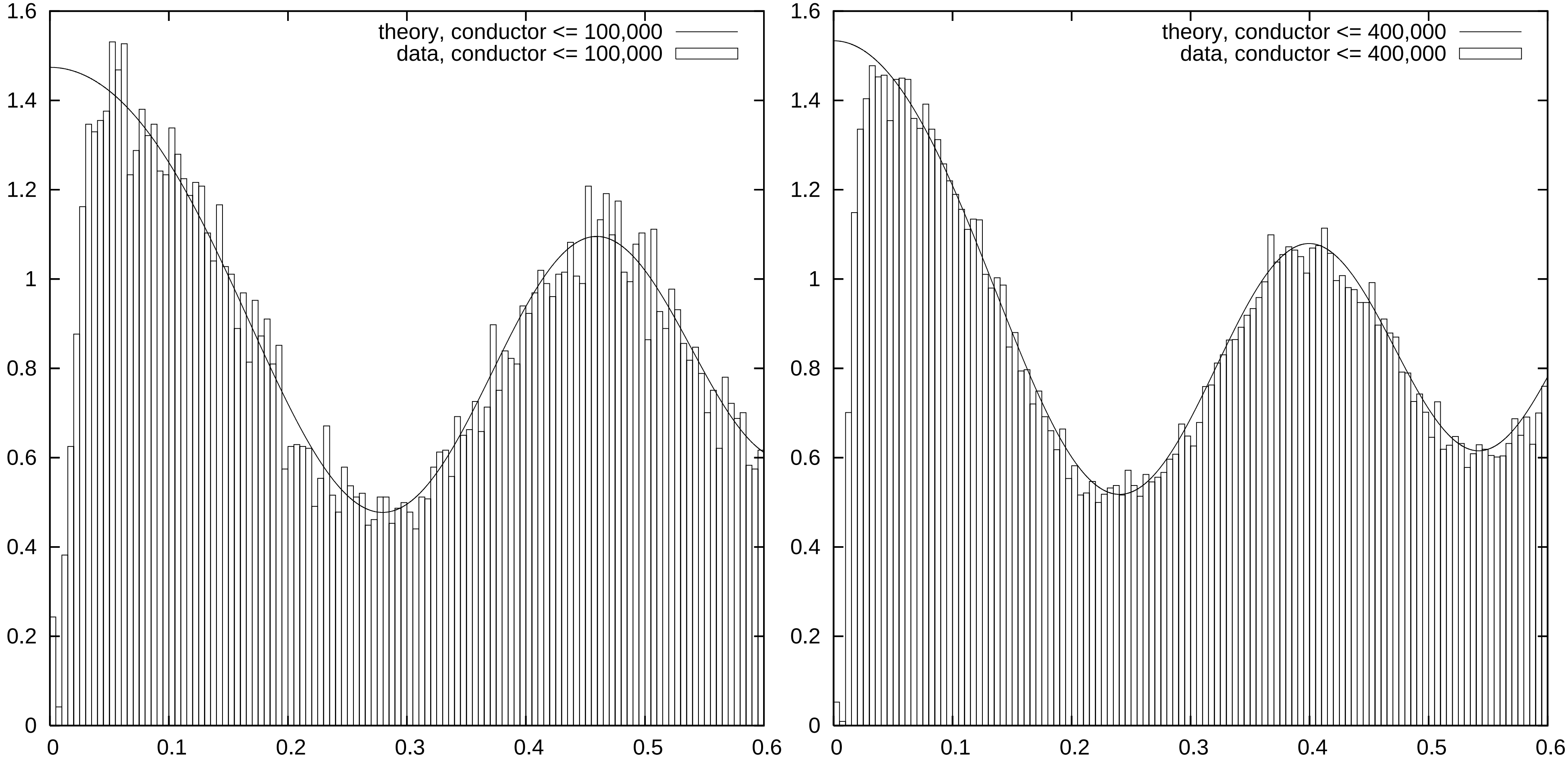
- The question is: do we get
$$Q_\Delta(t_0, X) = b + O\left(\frac{\log\log X}{\log X}\right)$$
for $X \to \infty$ and with $b = -\frac{1}{2}$?


0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
0
50000
100000
150000
200000
250000
300000
350000
400000
Q(t1,X), t1=0.01
Q(t2,X), t2=0.02
Q(t3,X), t3=0.03
Q(t4,X), t4=0.04
Q(t5,X), t5=0.05
Q(t6,X), t6=0.4
Q(t7,X), t7=0.6

We make the following observations:

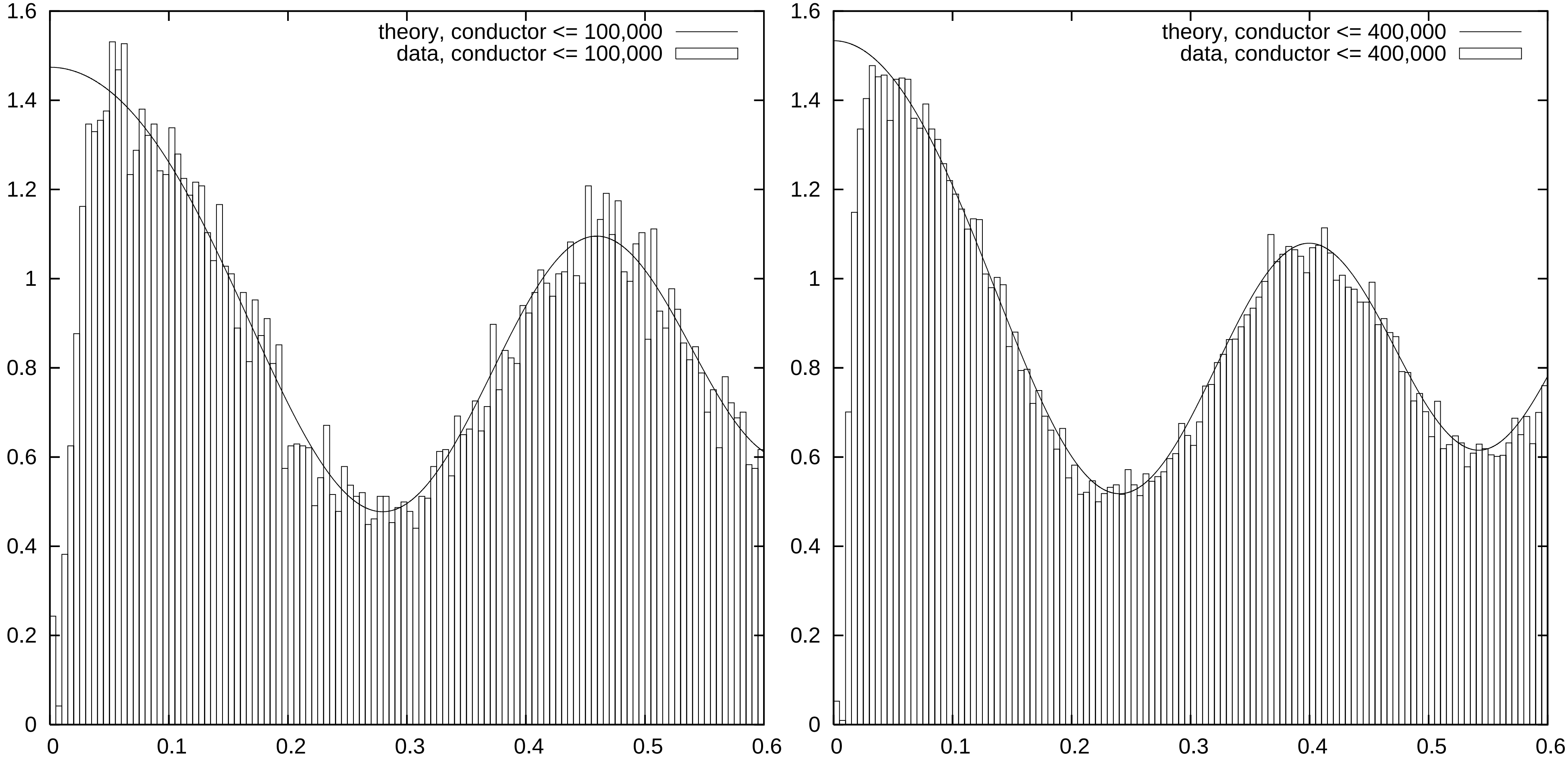
- Curves for sample points near the critical point are smoother $\rightarrow$ no sign changes. Those sample points t_1, t_2 and, t_3 are in the region of repulsion. It appears for $t_1 = 0.01$ in the restricted range that $b > 0$. Too few data to decide whether $b < 0$ if X was large enough.

- Other sample points suggest that $b < 0$. $\rightarrow$ It appears that the lower order terms give a power savings over the main term.



theory, conductor <= 100,000
data, conductor <= 100,000
theory, conductor <= 400,000
data, conductor <= 400,000
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
0
0.1
0.2
0.3
0.4
0.5
0.6

- Let us call $\delta > 0$ the width of the band where we observe repulsion.
- For $t_j > \delta = 0.03$ the curves are jagged and look similar $\rightarrow$ many sign changes.
- It seems that $\delta \rightarrow 0$ for $X \rightarrow \infty$.


theory, conductor <= 100,000
data, conductor <= 100,000
theory, conductor <= 400,000
data, conductor <= 400,000
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
0.1
0.2
0.3
0.4
0.5
0.6

- Natural questions:
  - ⋆ Where is the repulsion coming from?
  - ⋆ Can we model the repulsion using random matrix theory?

- By formulas of Waldspurger, Shimura, Kohnen-Zaiger the values of $L_E(1/2, \chi_d)$ are *discretized*, i.e.,

$$L_E(1/2, \chi_d) = \frac{\kappa_E c_E(|d|)^2}{\sqrt{d}}$$

where κ_E only depends on E and $c_E(|d|)$ are the Fourier coefficients of a half-integral weight form and only take integer values. One way of thinking of this is

$$L_E(1/2, \chi_d) < \frac{\kappa_E}{\sqrt{d}} \Longrightarrow L_E(1/2, \chi_d) = 0.$$

- A working hypothesis: the discretization of $L_E(1/2, \chi_d)$ causes the observed repulsion.

- Here is an ad hoc test for our working hypothesis: we consider "discretized random matrices" at 1.
- More specific, we generate many random matrices from $SO(2N)$ and consider only those with characteristic polynomial

$$|Z(1)| > v > 0$$

for suitable v.

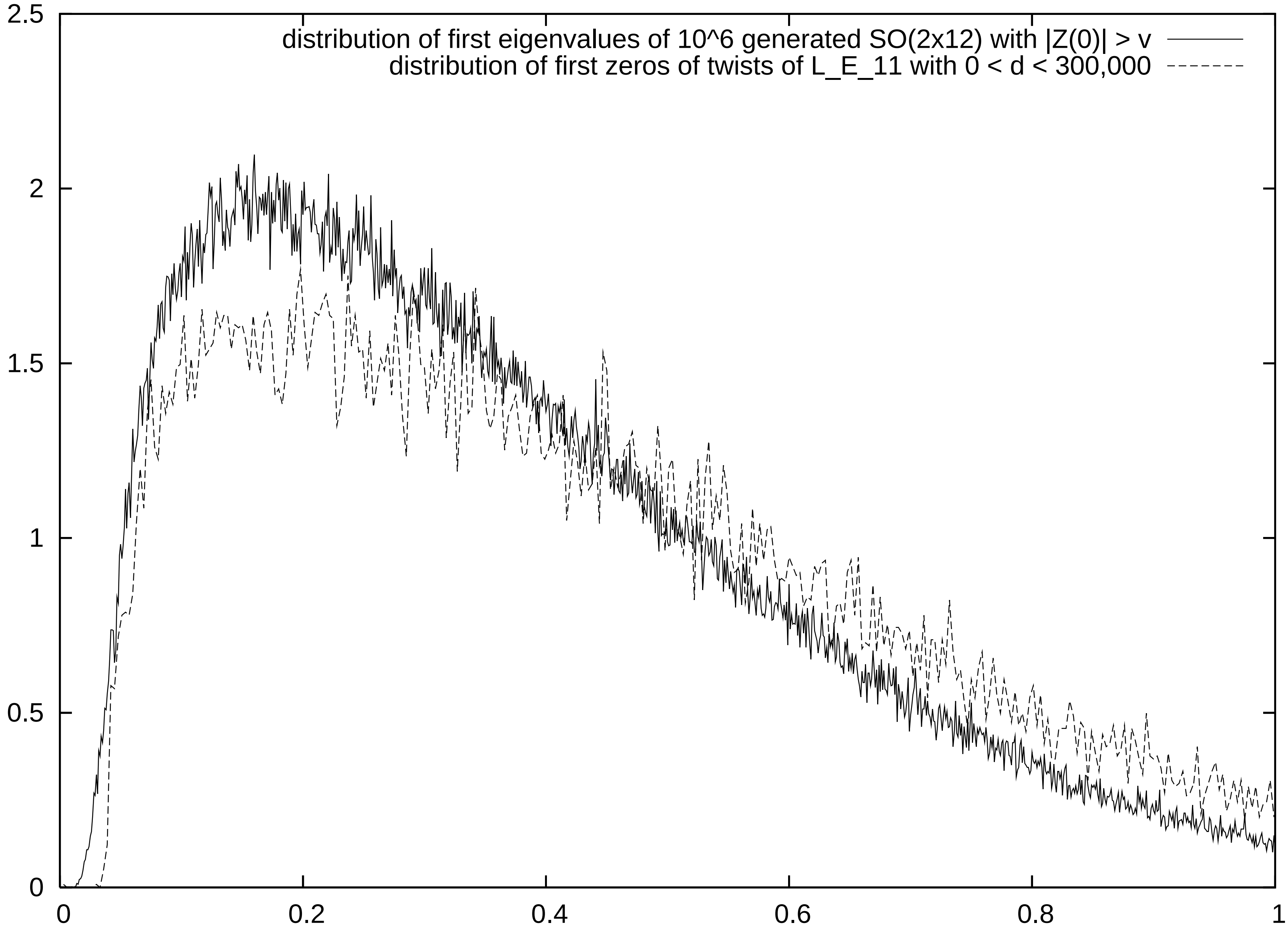
- Then we compare the distribution of the first eigenvalues with the distribution of the first zeros for our family of elliptic curve L-functions.

- On the RMT-side the matrix size N/π plays the role of the mean density of zeros. Hence we set $N/\pi = \log(\frac{\sqrt{M}X}{2\pi})$ and the choice

$$|Z(1)| > v := \kappa_E \times \sqrt{\frac{2\pi}{\sqrt{M}}} \times e^{-N/(2\pi)}$$

corresponds for $0 < d < X$ to

$$L_E(1/2, \chi_d) > \frac{\kappa_E}{\sqrt{d}}.$$


2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
distribution of first eigenvalues of 10^6 generated SO(2x12) with |Z(0)| > v
distribution of first zeros of twists of L_E_11 with 0 < d < 300,000

- The repulsion of zeros away from the central point of the family of quadratic twists of $L_{E_{11}}$ is qualitively captured in terms of eigenvalues of random matrices from $SO(2N)$ with $|Z(1)| > v$. $\rightarrow$ our working hypothesis is pointing into the right direction.
- Further work needs to be done to explain the observed repulsion and how to model it.
- Our work so far give evidence that $SO(2N)$ is the correct limit for zero statistics for the family of quadratic twists of an elliptic curve L-function.

Summary

- For relatively small conductor and away from the critical point the lower order terms dominate strongly the behaviour of the 1-level density.
- Our work so far give evidence that $SO(2N)$ is the correct limit for zero statistics for the family of quadratic twists of an elliptic curve L-function.
- From our data it appears that the lower order terms give a power savings over the main term.
- Data suggests that the discretization causes the repulsion.