The derivative of the Riemann-zeta function, Toeplitz determinants and the Riemann-Hilbert problem

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Random Matrices, L-functions and primes
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Outline

The problem

- The j.p.d.f. for the eigenvalues for matrices in the CUE:

$$P_{\mathrm{CUE}}(\theta_1, \ldots, \theta_N) = \frac{1}{(2\pi)^N N!} \prod_{1 \leq j < k \leq N} \left| e^{i\theta_j} - e^{i\theta_k} \right|^2 .$$

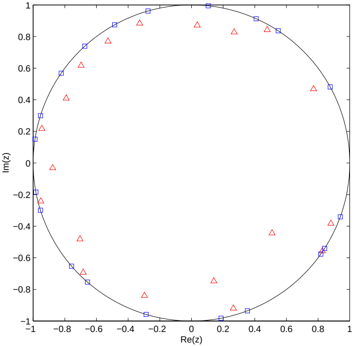
- Characteristic polynomial of a matrix in the CUE:

$$\Lambda(z) = \det(Iz - U) = \sum_{k=0}^{N} a_k z^{N-k}.$$

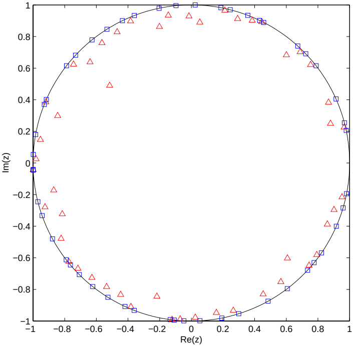
What can we say about the distributions of the roots of

$$\Lambda'(z) = \frac{d\Lambda(z)}{dz} \quad ?$$

The problem



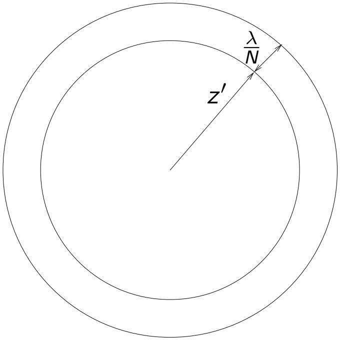
$N = 20$



$N = 50$

The problem

Define $\lambda := (1 - |z'|)N$.



1. Does the limit distribution $Q(\lambda) = \lim_{N\to\infty} Q(\lambda; N)$ exist?
2. What does $Q(\lambda)$ look like?

Where we are

Theorem (FM 2003)

$$Q(\lambda) \sim \frac{1}{\lambda^2}, \quad \lambda \to \infty.$$

Conjecture (FM 2003)

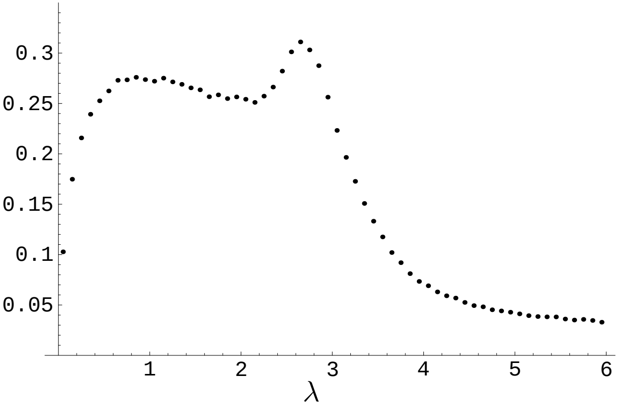
$$Q(\lambda) \sim \frac{4}{3\pi}\lambda^{1/2}, \quad \lambda \to 0.$$

Theorem (Dueñez, Farmer, Froehlich, Hughes, FM, and Phan 2008)

$$Q(\lambda) = \frac{4}{3\pi}\lambda^{1/2} - \frac{14}{15\pi}\lambda^{3/2} + O(\lambda^{5/2}).$$

Question to answer

What happens in between?



Zeros of $\Lambda'(z)$, $N = 100$

Motivations

- The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1.$$

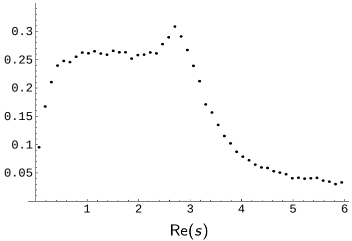
- The local correlations the zeros of $\zeta(1/2 + it)$ for large t are the same as those of eigenvalues of matrices in the CUE.
- The local statistical properties of $\zeta(1/2 + it)$ as $t \to \infty$ are accurately modelled by $\Lambda(z)$.

Theorem (Speiser 1934)

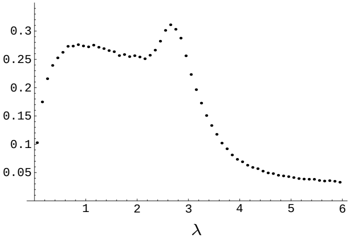
The Riemann hypothesis is equivalent to the statement that $\zeta'(s)$ has no zeros to the left of the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

Motivations

$\zeta'(s)$	$\Lambda'(z)$
$\frac{1}{2\pi}\log\frac{t}{2\pi}$	$N/(2\pi)$
Right of the line $\mathrm{Re}(s) = 1/2$	Interior of the unit circle



Re. part of zeros of $\zeta'(s)$.
10^5 zeros,
$t \in [10^6, 10^6 + 60,000]$



Zeros of $\Lambda'(z)$, $N = 100$.

The Riemann-Hilbert approach

$Q(\lambda; N)$ can be written as

$$Q(\lambda; N) = -\frac{N^2}{4\pi(N-1)} \times \frac{\partial^2}{\partial \alpha^2}\left[\iint_{\mathbb{C}} \Big(D_{N-1}[\exp(i\varphi^1)](w,\alpha,z) + D_{N-1}[\exp(i\varphi^2)](w,\alpha,z)\Big)\, d^2 w\right]_{\alpha=0},$$

where $\varphi^1(\theta, w, \alpha, z)$ and $\varphi^2(\theta, w, \alpha, z)$ are

$$\varphi^1(\theta, w, \alpha, z) = \mathrm{Re}\left(\frac{\overline{w}}{N(z-e^{i\theta})} - \frac{\alpha}{(N(z-e^{i\theta}))^2}\right)$$

$$\varphi^2(\theta, w, \alpha, z) = \mathrm{Re}\left(\frac{\overline{w}}{N(z-e^{i\theta})}\right) - \mathrm{Im}\left(\frac{\alpha}{(N(z-e^{i\theta}))^2}\right)$$

with $|z| = 1 - \lambda/N$.

The Riemann-Hilbert approach

- Why the Riemann-Hilbert Problem (RHP)?
- There is a close connection between the RHP and *integrable operators.*
- This is the kernel of an integrable operator:

$$K(x,y) = \frac{\sin\bigl(\pi(x-y)\bigr)}{\pi(x-y)}.$$

- The Christoffel-Darboux formula is another one:

$$\sum_{j=0}^{N-1} \frac{p_j(x)p_j(y)}{(p_j,p_j)} = C_N \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x-y}.$$

The Riemann-Hilbert approach

- Toeplitz operators are integrable.
- Let $\varphi(z)$ be analytic in some annulus

$$\Gamma_\rho = \left\{ z : \rho < |z| < \rho^{-1}, \quad 0 < \rho < 1 \right\}$$

and consider

$$T_{N-1} = \{\varphi_{j-k}\}_{0 \le j,k \le N-1} \, .$$

- T_{N-1} induces a map on the space of trigonometric polynomials $\tau_{N-1} : P_{N-1} \to P_{N-1}$ defined by

$$\tau_{N-1} z^k = \sum_{j=0}^{N-1} \varphi_{j-k} z^j, \quad z \in \mathbb{S}^1, \quad 0 \le k \le N-1.$$

The Riemann-Hilbert approach

- If $p(z) = \sum_{j=0}^{N-1} a_j z^j$ then

$$\left[\tau_{N-1} p\right](z) = \left[\left(1 - K_{N-1}\right) p\right](z)$$
$$= p(z) - \int_{\mathbb{S}^1} K_{N-1}(z, z') p(z') dz'$$

- K_{N-1} is an integrable operator:

$$K_{N-1}(z, z') = \frac{1}{2\pi i} \frac{\left(z^N \left(z'\right)^{-N} - 1\right)\left(1 - \varphi(z')\right)}{z - z'}$$

- Then

$$D_{N-1} = \det\left(T_{N-1}\right) = \det\left(1 - K_{N-1}\right).$$

The Riemann-Hilbert approach

- Let Σ be an oriented contour in $\mathbb{C}$. An operator on $L^2(\Sigma, |dz|)$ is called *integrable* if its kernel is of the form

$$K(z, z') = \frac{\sum_{j=1}^{k} f_j(z) g_j(z')}{z - z'}.$$

- The action of K on $L^2(\Sigma, |dz|)$ is

$$\left[Kh\right](z) = \pi i \sum_{j=1}^{k} f_j(z) \left[Hhg_j\right](z), \quad z \in \Sigma,$$

where H is the Cauchy Principal Value Operator

$$\left[Hf\right](z) = \lim_{\epsilon \to 0} \frac{1}{\pi i} \int_{\{z' \in \Sigma,\ |z - z'| > \epsilon\}} \frac{f(z')}{z - z'} dz'.$$

The Riemann-Hilbert approach

- The resolvent R is an *integrable operator* too:

$$R(z,z') = \frac{\sum_{j=1}^{k} F_j(z) G_j(z')}{z - z'},$$

where

$$F_j = (1-K)^{-1} f_j = (1+R) f_j$$
$$G_j = (1-K)^{-1} g_j = (1+R) g_j$$

It is a remarkable fact that the functions F_J and G_j can be computed in terms of the solution of a *Riemann-Hilbert Matrix Factorization Problem*.

The Riemann-Hilbert approach

- An $k \times k$ matrix function $M(z)$ is the unique solution (if it exists) of the RHP (Σ, v) if
  - $M(z)$ is analytic in $\mathbb{C} \setminus \Sigma$,
  - $M_+(z) = M_-(z)v(z), \quad z \in \Sigma$,
  - $M(z) \to I$ as $z \to \infty$.
- $v(z)$ is called jump matrix.

Recipe to compute the resolvent R:

1. Let $f = (f_1, \dots, f_k)^t$ and $g = (g_1, \dots, g_k)^t$. Construct the following jump matrix:

$$v(z) = I - \left(\frac{2\pi i}{1 + i\pi \langle g, f \rangle} \right) fg^t.$$

2. Find the solution (if it exists) of the RHP (Σ, v).

The Riemann-Hilbert approach

3. We have

$$F = (F_1, \ldots, F_k)^t = (1 \mp \langle g, f \rangle)^{-1} M_{\pm} f.$$

$$G = (G_1, \ldots, G_k)^t = (1 \mp \langle g, f \rangle)^{-1} \left(M_{\pm}^t\right)^{-1} g.$$

4. Insert F and G into the formula

$$R(z, z') = \frac{\sum_{j=1}^k F_j(z) G_j(z')}{z - z'},$$

The Riemann-Hilbert approach is particularly powerful when the integrable operator K depends on one or more asymptotic parameters and singularities need to be handled.

The Riemann-Hilbert approach

Theorem (Deift 1999)

Let φ_t be the symbol of a Toeplitz determinant D_{N-1}, then

$$\frac{d \log D_{N-1}}{dt} = \int_{\mathbb{S}^1} (1-\varphi_t)^{-1} \frac{d\varphi_t}{dt} \sum_{j=1}^{2} F'_{t,j}(\zeta) G_{t,j}(\zeta) d\zeta,$$

$$(F_{t,1}, F_{t,2})^T = M_+^t(\zeta)(\zeta^N, 1)^T$$

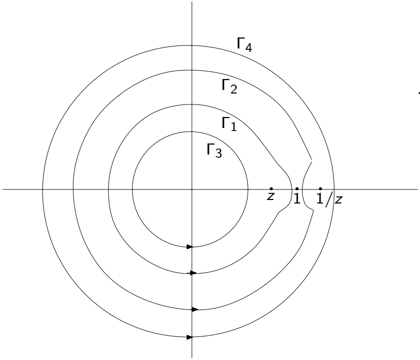
$$(G_{t,1}, G_{t,2})^T = \frac{1-\varphi_t}{2\pi i} \left((M_+^t)^T\right)^{-1}(\zeta)\left(\zeta^{-N}, -1\right)^T,$$

where $M_{t,+}(\zeta)$ is the boundary value from the left of the solution of the RHP

$$M_+^t(\zeta) = M_-^t(\zeta) \begin{pmatrix} \varphi_t & -(\varphi_t^{-1} - 1)\zeta^N \\ \zeta^{-N}(\varphi_t - 1) & 2 - \varphi_t \end{pmatrix}, \quad \zeta \in \mathbb{S}^1$$

$$M^t(\zeta) = I + O(\zeta^{-1}), \quad \zeta \to \infty$$

The Riemann-Hilbert approach



$$z = 1 - \frac{\lambda}{N}$$

$$\Gamma_1 = \left\{ |\zeta| = 1 - N^{-\frac{1}{2}} \right\}, \qquad \Gamma_2 = \Gamma_1^{-1}$$

$$\Gamma_3 = \{ |\zeta| = \rho, \rho < 1, \rho = O(1) \}, \qquad \Gamma_4 = \Gamma_3^{-1}$$

The Riemann-Hilbert approach

The previous RHP becomes

$$M^{(1,t)}(\zeta) = M^t(\zeta)\varphi_t^{-\frac{\sigma_3}{2}} \qquad |\zeta| < \rho$$

$$M^{(1,t)}(\zeta) = M^t(\zeta)\begin{pmatrix} 1 & \zeta^N \\ 0 & 1 \end{pmatrix}\varphi_t^{-\frac{\sigma_3}{2}} \qquad \text{between } \Gamma_3 \text{ and } \Gamma_1$$

$$M^{(1,t)}(\zeta) = M^t(\zeta)\begin{pmatrix} 1 & (1-\varphi_t^{-1})\zeta^N \\ 0 & 1 \end{pmatrix}\varphi_t^{-\frac{\sigma_3}{2}} \qquad \text{between } \Gamma_1 \text{ and } \mathbb{S}^1$$

$$M^{(1,t)}(\zeta) = M^t(\zeta)\begin{pmatrix} 1 & 0 \\ \zeta^{-N}(1-\varphi_t^{-1}) & 1 \end{pmatrix}\varphi_t^{\frac{\sigma_3}{2}} \qquad \text{between } \mathbb{S}^1 \text{ and } \Gamma_2$$

$$M^{(1,t)}(\zeta) = M^t(\zeta)\begin{pmatrix} 1 & 0 \\ \zeta^{-N} & 1 \end{pmatrix}\varphi_t^{\frac{\sigma_3}{2}} \qquad \text{between } \Gamma_2 \text{ and } \Gamma_4$$

$$M^{(1,t)}(\zeta) = M^t(\zeta)\varphi_t^{\frac{\sigma_3}{2}} \qquad |\zeta| > \rho^{-1}$$

The Riemann-Hilbert approach

Then $M^{(1,t)}(\zeta)$ satisfies the jump conditions

$$\nu^{(1)}(\zeta) = \begin{pmatrix} 1 & -\varphi_t \zeta^N \\ 0 & 1 \end{pmatrix} \quad |\zeta| = \rho$$

$$\nu^{(1)}(\zeta) = \begin{pmatrix} 1 & \zeta^N \\ 0 & 1 \end{pmatrix} \quad \zeta \in \Gamma_1$$

$$\nu^{(1)}(\zeta) = \begin{pmatrix} 1 & 0 \\ -\zeta^{-N} & 1 \end{pmatrix} \quad \zeta \in \Gamma_2$$

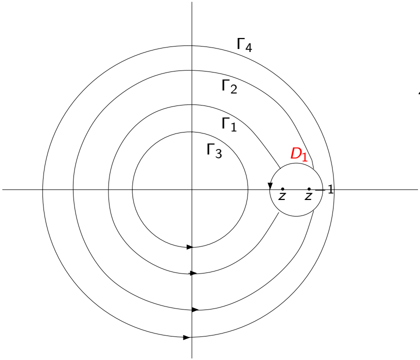
$$\nu^{(1)}(\zeta) = \begin{pmatrix} 1 & 0 \\ \varphi_t^{-1} \zeta^{-N} & 1 \end{pmatrix} \quad |\zeta| = \rho^{-1}$$

$M^{(1,t)}$ has singularities at $z = \left(1 - \frac{\lambda}{N}\right)^{\pm 1}$:

$$M^{(1,t)}(\zeta) = \left(M_{\lambda}^{(1,t)} + O(\zeta - z)\right)\varphi_t^{-\frac{\sigma_3}{2}} \quad \zeta \to z$$

$$M^{(1,t)}(\zeta) = \left(M_{-\lambda}^{(1,t)} + O(\zeta - z^{-1})\right)\varphi_t^{\frac{\sigma_3}{2}} \quad \zeta \to z^{-1}$$

The Riemann-Hilbert approach



$$z = 1 - \frac{\lambda}{N}$$

$$D_1 = \left\{ \zeta : |\zeta - 1| \leq N^{-\frac{1}{2}} \right\}$$

The Riemann-Hilbert approach

- Outside D_1, $\nu^{(1)} \to I$ as $N \to \infty$. Thus, $M^{(1,t)}(\zeta) \to I$.
- We need to solve the RHP exactly in D_1, then match it with the approximate solution outside.
- The substitution

$$\xi = N \log \zeta$$

  maps D_1 to $\mathbb{C}$.
- The RHP inside D_1 is mapped into

The Riemann-Hilbert approach

$$M_+^{(2,t)}(\xi) = M_-^{(2,t)}(\xi) \begin{pmatrix} 1 & e^{\xi} \\ 0 & 1 \end{pmatrix} \quad \xi \in \Gamma_1$$

$$M_+^{(2,t)}(\xi) = M_-^{(2,t)}(\xi) \begin{pmatrix} 1 & 0 \\ -e^{-\xi} & 1 \end{pmatrix} \quad \xi \in \Gamma_2$$

$$M^{(2,t)}(\xi) = (C_+ + O(\xi - \gamma_N(\lambda)))e^{-i\left(\frac{\beta}{(\xi-\gamma_N(\lambda))^2} - \frac{u}{(\xi-\gamma_N(\lambda))}\right)\sigma_3}$$
$$\xi \to \gamma_N(\lambda)$$

$$M^{(2,t)}(\xi) = (C_- + O(\xi + \gamma_N(\lambda)))e^{i\left(\frac{\beta}{(\xi+\gamma_N(\lambda))^2} + \frac{\overline{u}}{(\xi+\gamma_N(\lambda))}\right)\sigma_3}$$
$$\xi \to -\gamma_N(\lambda)$$

$$M^{(2,t)}(\xi) = \left(I + O(\xi^{-1})\right) \quad \xi \to \infty,$$

where $\gamma_N(\lambda) = N \log\left(1 - \frac{\lambda}{N}\right)$ and β, u are related to the parameters in φ^1 and φ^2.

The Riemann-Hilbert approach

- The solution of the original RHP is

$$M^{(1,t)}(\zeta) = \begin{cases} I + O(N^{-\frac{1}{2}}) & \zeta \in \mathbb{C} \setminus D_1; \\ \left(I + O(N^{-\frac{1}{2}})\right) M^{(2,t)}(\zeta) & \zeta \in D_1. \end{cases}$$

- The Toeplitz determinant D_{N-1} in terms of $M^{(2,t)}$ becomes

$$\log D_{N-1} = N\hat{\eta}_0 + \sum_{k=1}^{\infty} k\hat{\eta}_k\hat{\eta}_{-k} + \int_0^1 \int_{i\mathbb{R}} \frac{d\log\varphi_t}{dt}(\xi) \times \operatorname{tr}\left(\left(M^{(2,t)}(\xi)\right)^{-1}\left(M^{(2,t)}(\xi)\right)^t \begin{pmatrix} 1 & -e^{\xi} \\ e^{-\xi} & 1 \end{pmatrix}\right) d\xi dt + O(N^{-\frac{1}{2}})$$

The Riemann-Hilbert approach

- If we define $Y(\xi) = M^{(2,t)} e^{\frac{\xi}{2}\sigma_3}$, then

  $$\partial_\xi Y(\xi) = A(\xi) Y(\xi)$$

  $$A(\xi) = \sum_{i=1}^{3} \frac{A_i^+}{(\xi - \gamma_N(x))^i} + \sum_{i=1}^{3} \frac{A_i^-}{(\xi + \gamma_N(x))^i} + \frac{\sigma_3}{2},$$

  where the A_i^+s and A_i^-s are complicated functions of the parameters in the symbols φ^1 and φ^2.
- This equation has three multiple poles at $\pm\gamma_N(x)$ and ∞.

Comments

- Szegő's theorem does not apply because of essential singularities in the symbols.
- The RHP problem needs to be solved exactly in a neighbourhood of these points and matched with the approximate solution outside.
- Similar techniques as in the study of the double-scaling limit of RMT but with a more complicated singularity structure (three essential singularities)
- Isomonodromic problem with three multiple poles