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- joint work with Eduardo Dueñez, Jon Keating and Steven J. Miller

Zero statistics of elliptic curve L-functions

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University of BRISTOL

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MOTIVATION:

To use random matrix theory to study the distribution of rank amongst families of elliptic curves

Elliptic curve L-functions:

eg.

$$E_{11}:\ y^2 = 4x^3 - 4x^2 - 40x - 79$$

L-function:

$$L_{E_{11}}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

a_n determined by E_{11}

Conjecture (Conrey, Keating, Rubinstein, Snaith):

Let E be an elliptic curve defined over $\mathbb{Q}$. Then there is a constant $c_E > 0$ such that

$$\sum_{\substack{p \le T \\ L_E(1/2,\chi_p)=0 \\ L_E(s,\chi_p)\in\mathcal{F}_{E+}}} 1 \sim c_E T^{3/4} (\log T)^{-5/8}$$

Conjecture (Birch and Swinnerton-Dyer):

$L_E(1/2, \chi_d) = 0$ if and only if E_d has infinitely many rational points (ie. rank greater than zero)

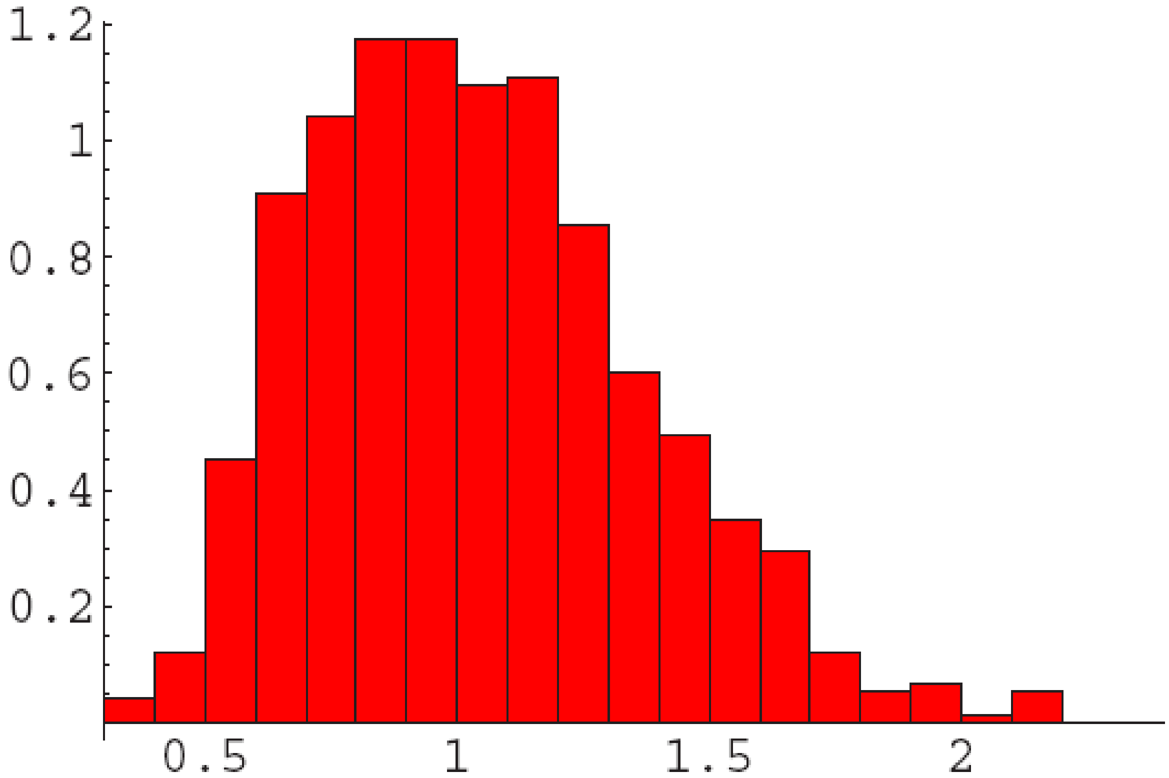

Figure 3: First normalized zero above the central point: 750 rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, $\log(\text{cond}) \in [3.2, 12.6]$, median $= 1.00$ mean $= 1.04$, standard deviation about the mean $= .32$

From: **Miller SJ,** Investigations of zeros near the central point of elliptic curve L-functions EXPERIMENTAL MATHEMATICS 15(3):257-279 2006

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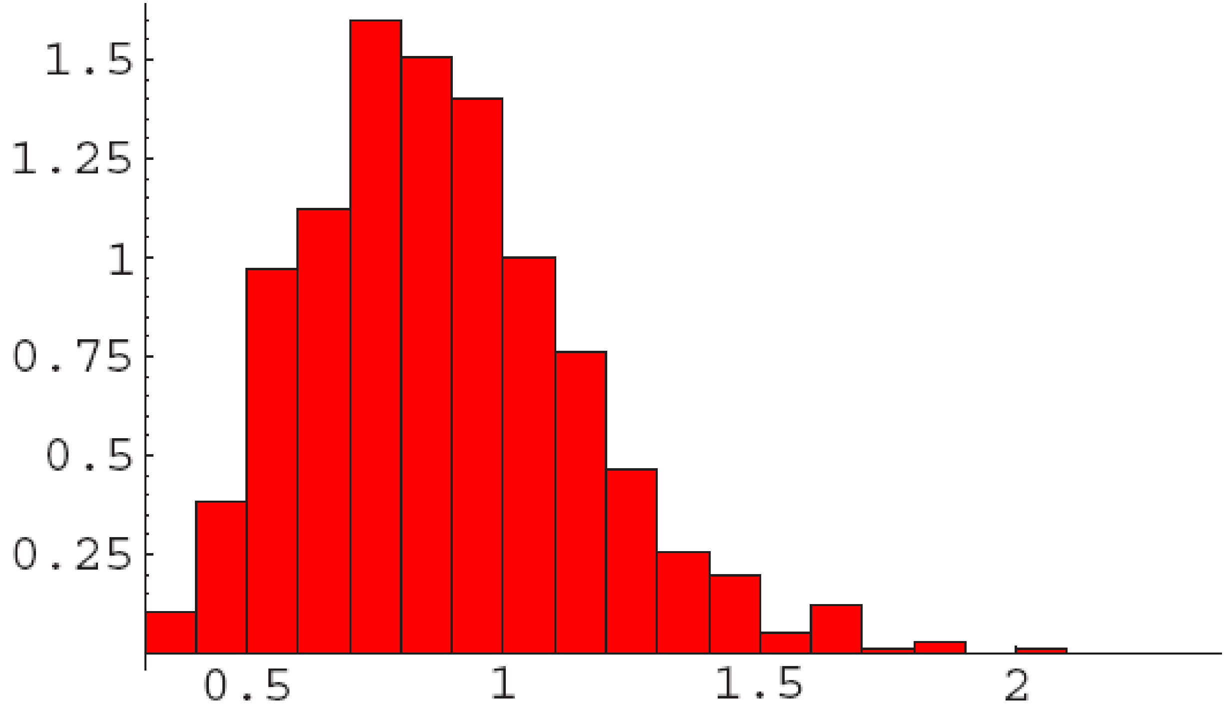

Figure 4: First normalized zero above the central point: 750 rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, $\log(\text{cond}) \in [12.6, 14.9]$, median $= .85$, mean $= .88$, standard deviation about the mean $= .27$

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$SO(2N)$

Orthogonal $2N \times 2N$ matrices with determinant $+1$:

The eigenvalues come in complex conjugate pairs $e^{i\theta_1}$, $e^{-i\theta_1}$, $e^{i\theta_2}$, $e^{-i\theta_2}, \ldots e^{i\theta_N}, e^{-i\theta_N}$.

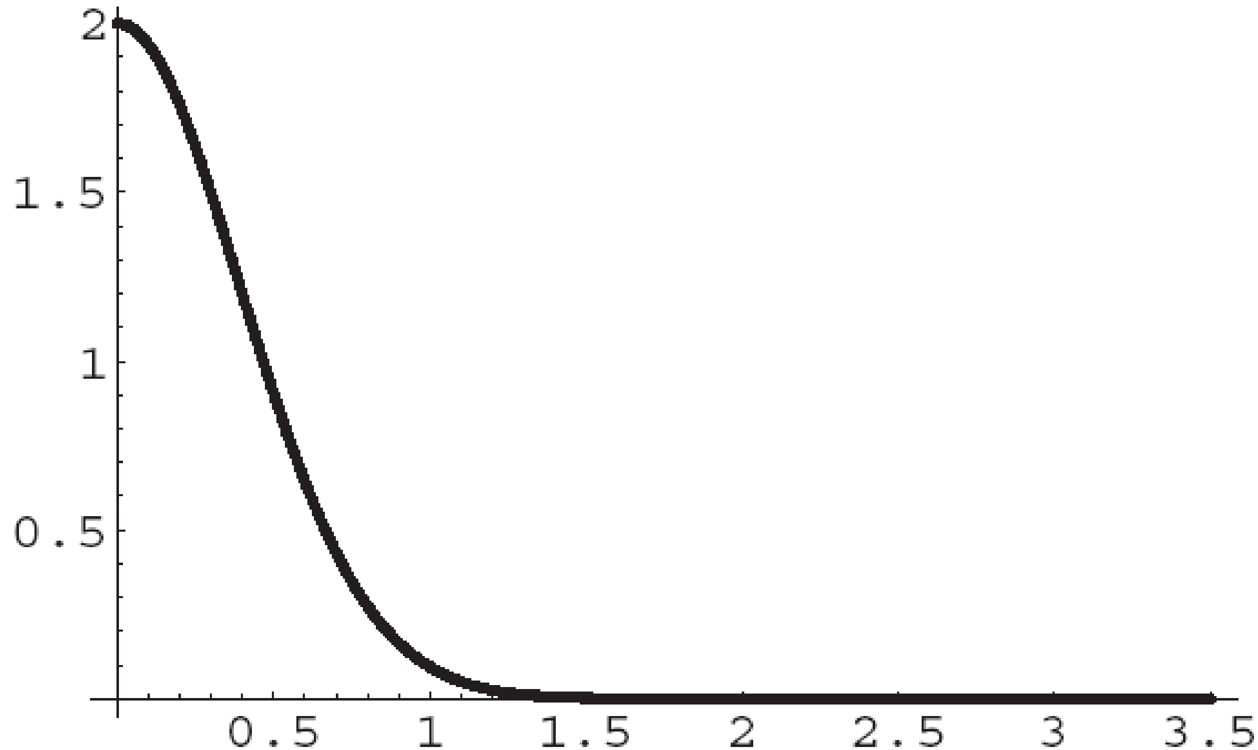

Figure 1c: First normalized eigenangle above 1: $N \to \infty$ scaling limit of $SO(2N)$: Mean = .321.

From: **Miller SJ,** Investigations of zeros near the central point of elliptic curve L-functions

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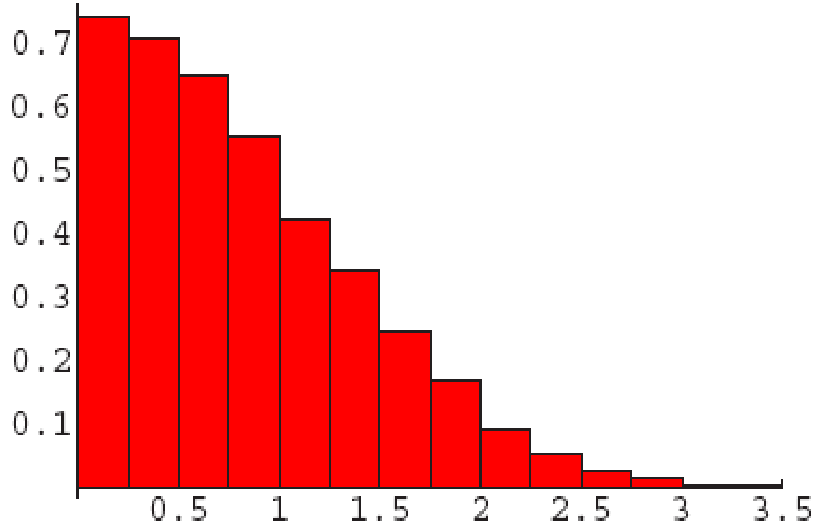

Figure 1a: First normalized eigenangle above 1: 23,040 SO(4) matrices
Mean = .709, Standard Deviation about the Mean = .601, Median = .709

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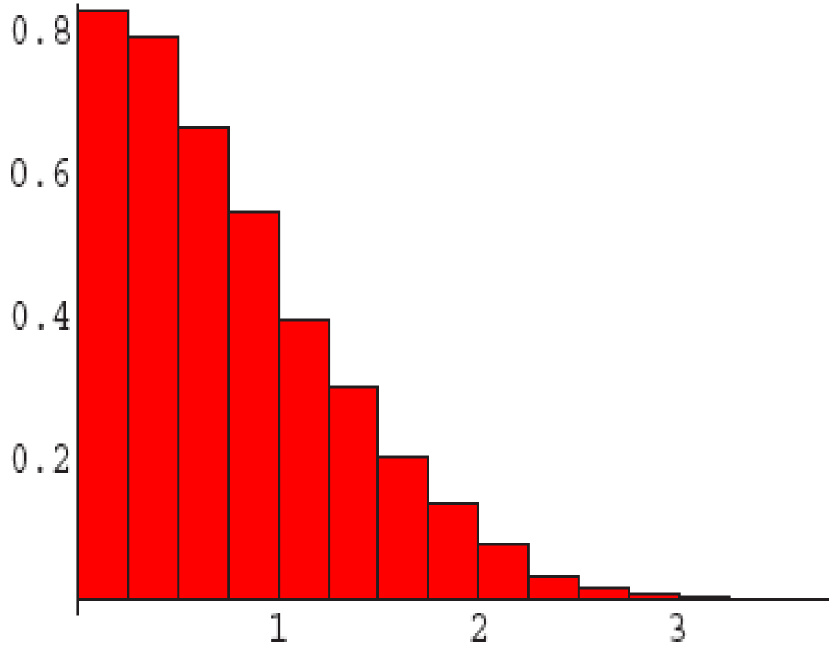

Figure 1b: First normalized eigenangle above 1: 23,040 SO(6) matrices
Mean = .635, Standard Deviation about the Mean = .574, Median = .635

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Inspired by random matrix theory:

Ratios Conjecture: Conrey, Farmer, Zirnbauer

For the Riemann zeta function:

$$\frac{1}{T}\int_0^T \frac{\prod_{k=1}^{K}\zeta(1/2+it+\alpha_k)\ \prod_{\ell=K+1}^{K+L}\zeta(1/2-it-\alpha_\ell)}{\prod_{q=1}^{Q}\zeta(1/2+it+\gamma_q)\ \prod_{r=1}^{R}\zeta(1/2-it+\delta_r)}dt$$

For families of L-functions

$$\sum_{f\in\mathcal{F}} \frac{L_f(1/2+\alpha_1)\cdots L_f(1/2+\alpha_k)}{L_f(1/2+\gamma_1)\cdots L_f(1/2+\gamma_k)}$$

One-level density:

$$S_1(g) = \sum_{f \in \mathcal{F}} \sum_{\rho_f} g(\rho_f)$$

$$= \sum_{f \in \mathcal{F}} \frac{1}{2\pi i} \oint \frac{L'_f(s)}{L_f(s)} g(s) ds$$

$\mathcal{F}$ a family of L-functions

L_f an L-function from the family

ρ_f a zero of L_f

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$$S_1(g) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t) \sum_{|d|\le X} \left(2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right) \right.$$

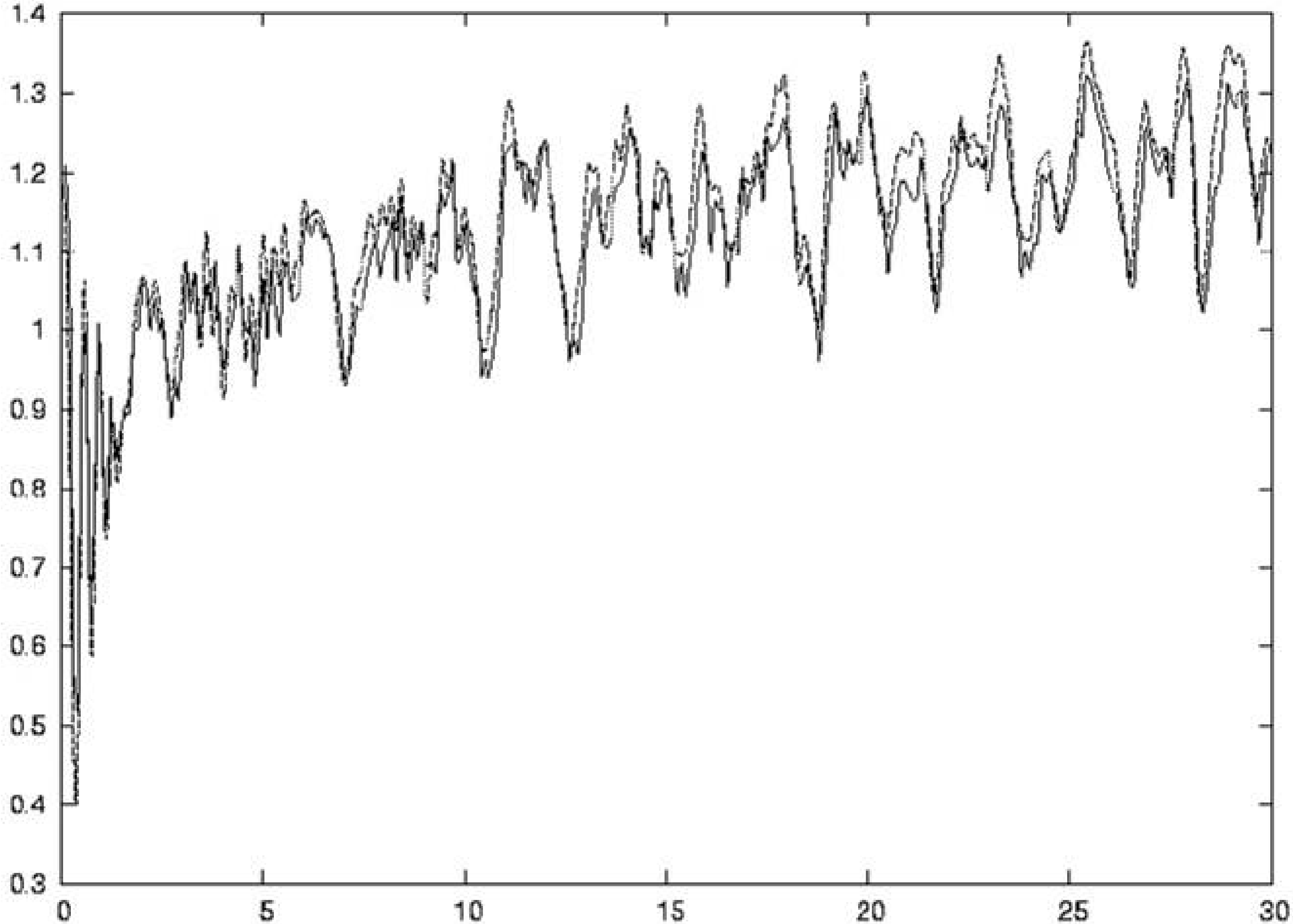
$$+\frac{\Gamma'}{\Gamma}(1/2+it) + \frac{\Gamma'}{\Gamma}(1/2-it)$$

$$+2\Bigg[-\frac{\zeta'}{\zeta}(1+2it) + \frac{L_E'}{L_E}(\mathrm{sym}^2, 1+2it) + A_f'(it,it)$$

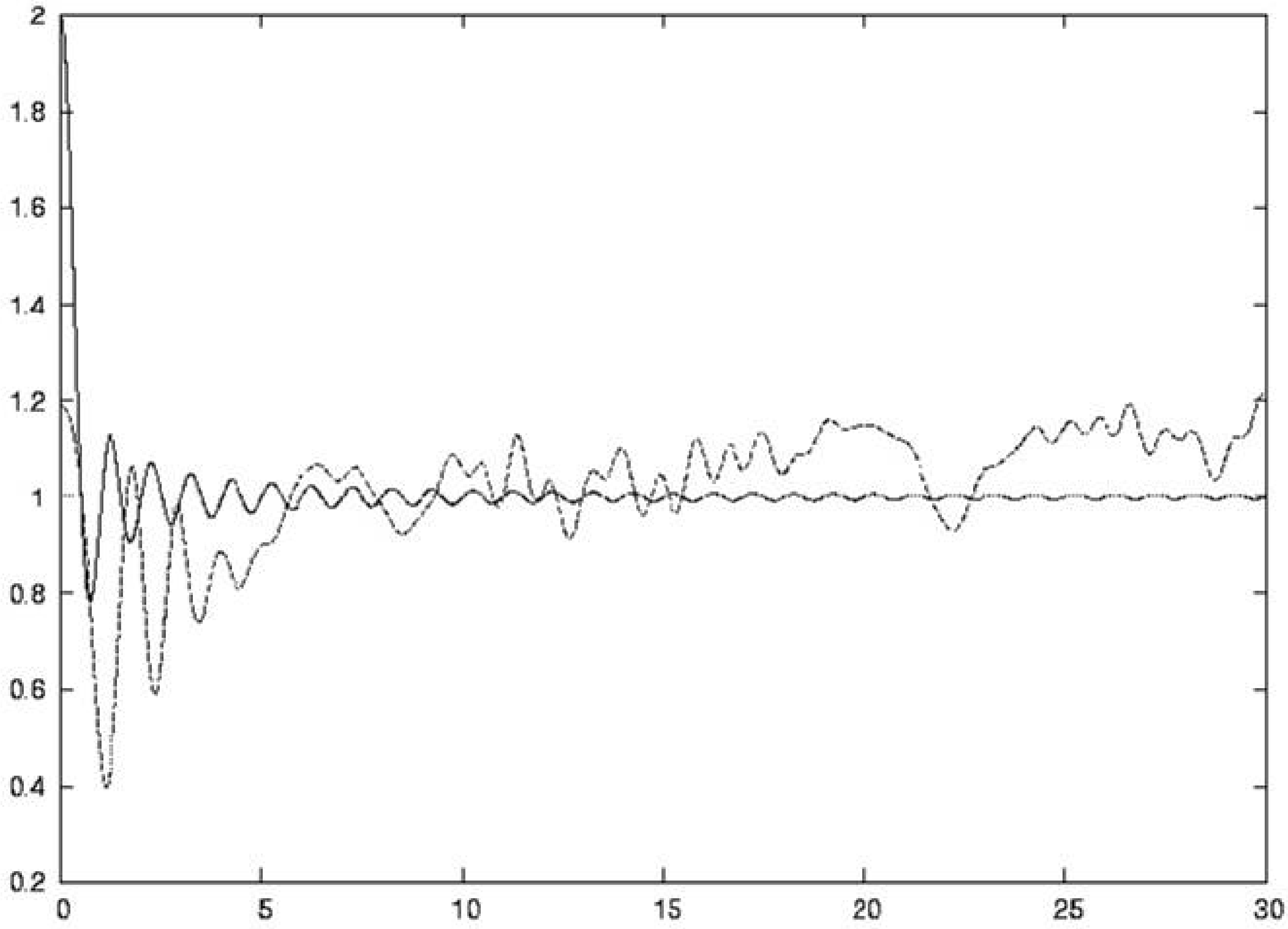
$$-\left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2it} A_f(-it,it)$$

$$\left.\times\frac{\Gamma(1/2-it)}{\Gamma(1/2+it)}\frac{\zeta(1+2it)L_E(\mathrm{sym}^2,1-2it)}{L_E(\mathrm{sym}^2,1)}\Bigg]\right) dt$$

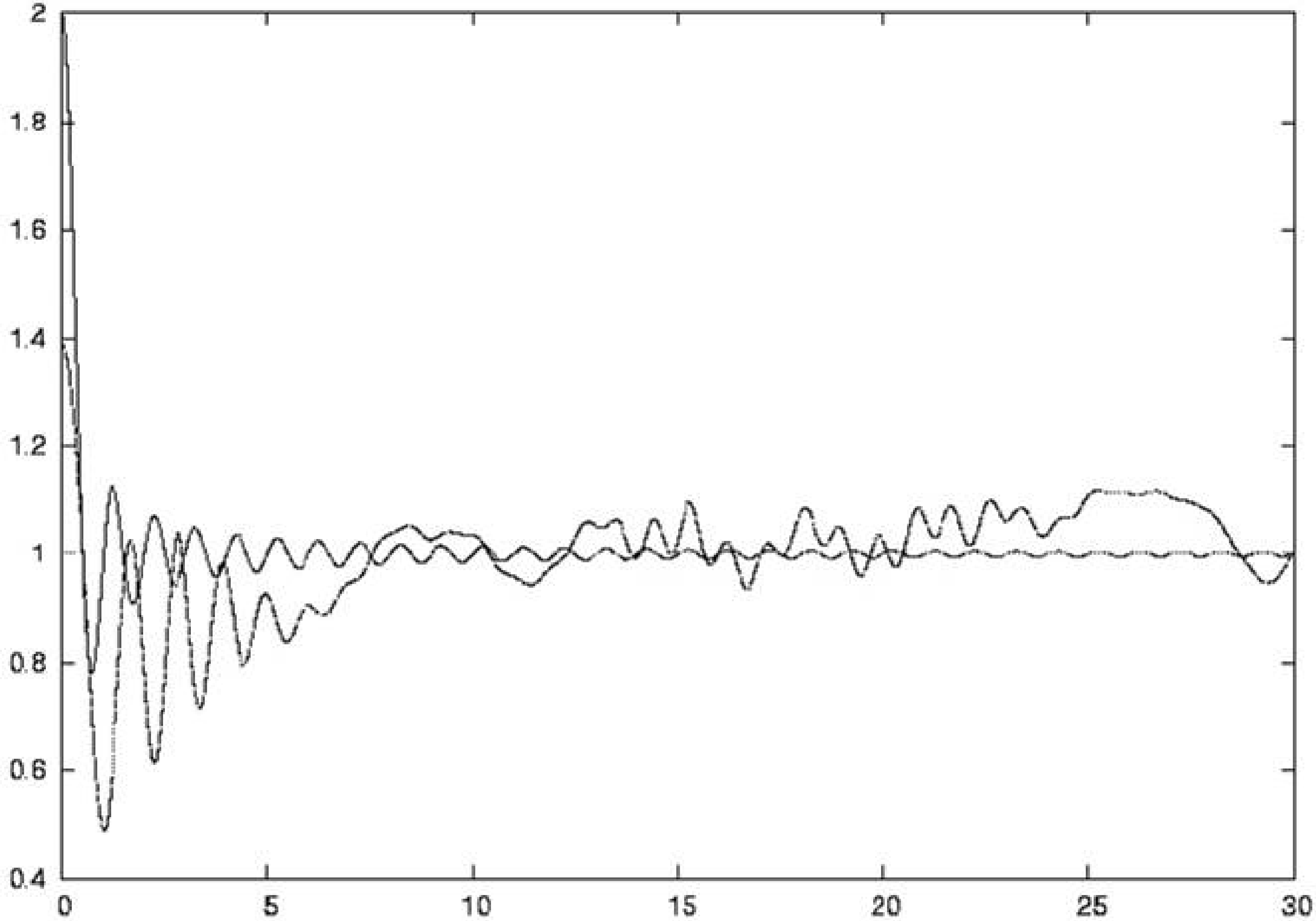
$$+O(X^{1/2+\epsilon})$$



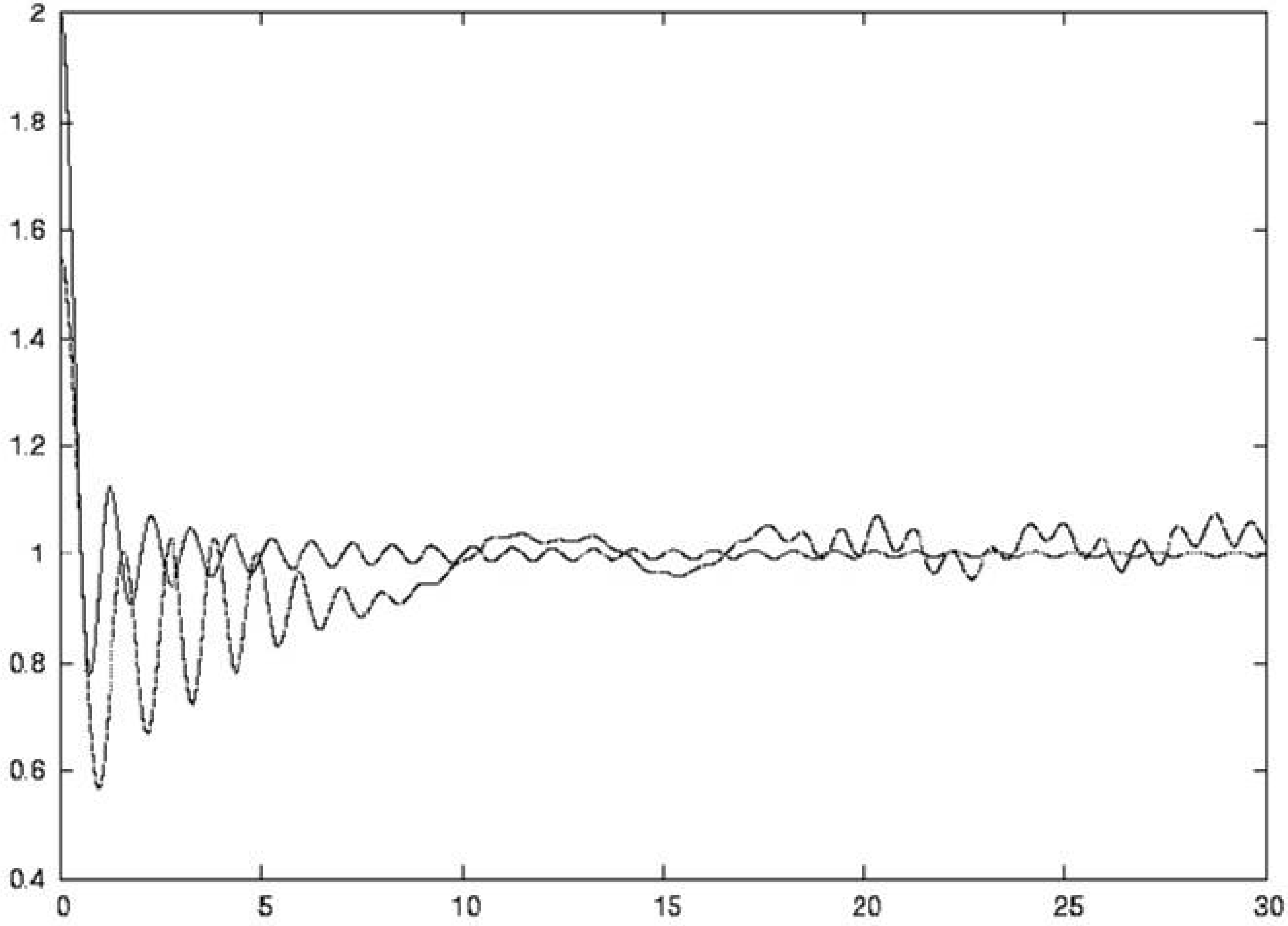
One level density of even twists of $L_{E_{11}}$ with 0<d<40 000 (dashed) versus numerical data (solid)



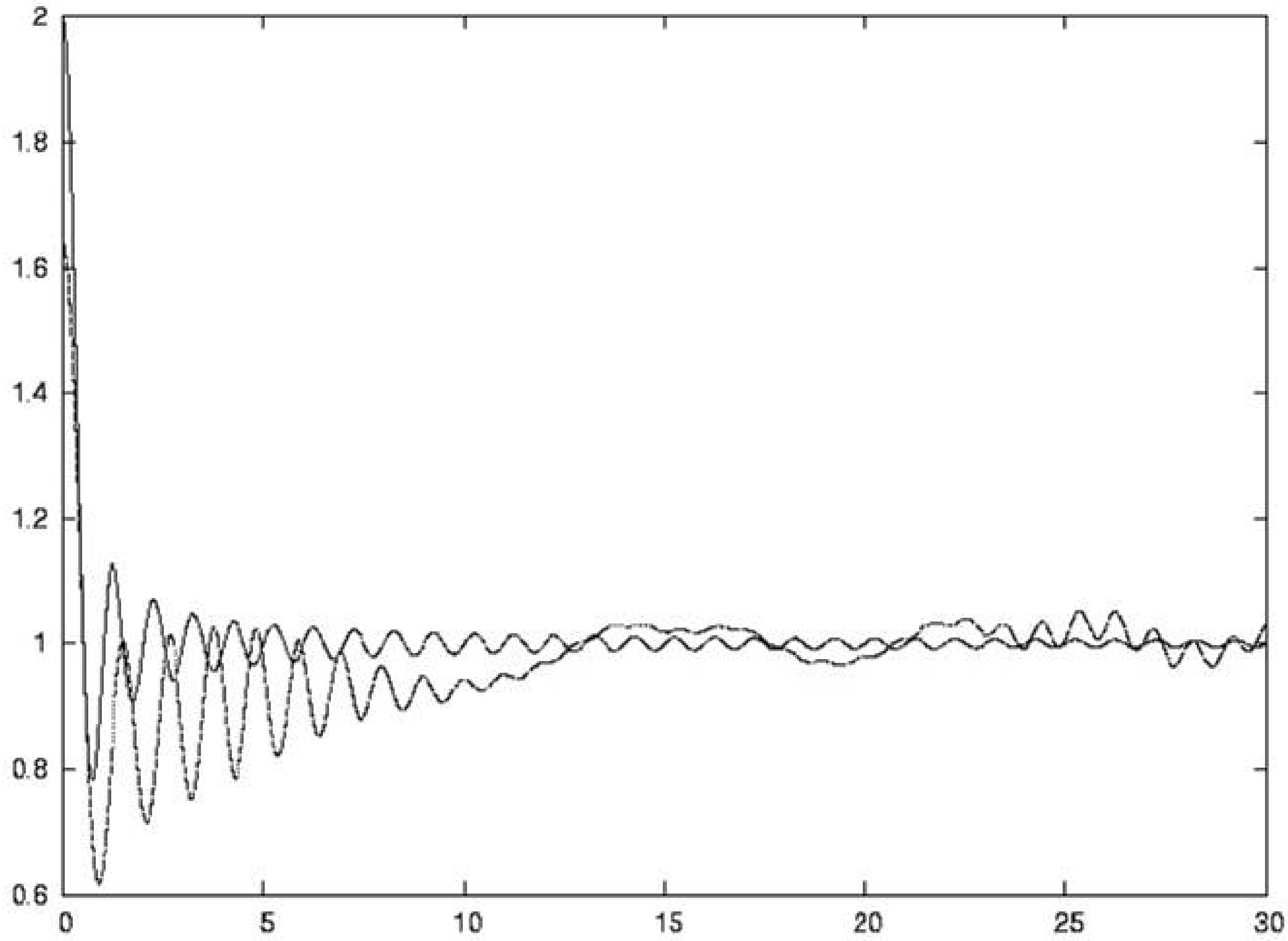
One level density for conductors up to 40 000 compared with the scaling limit



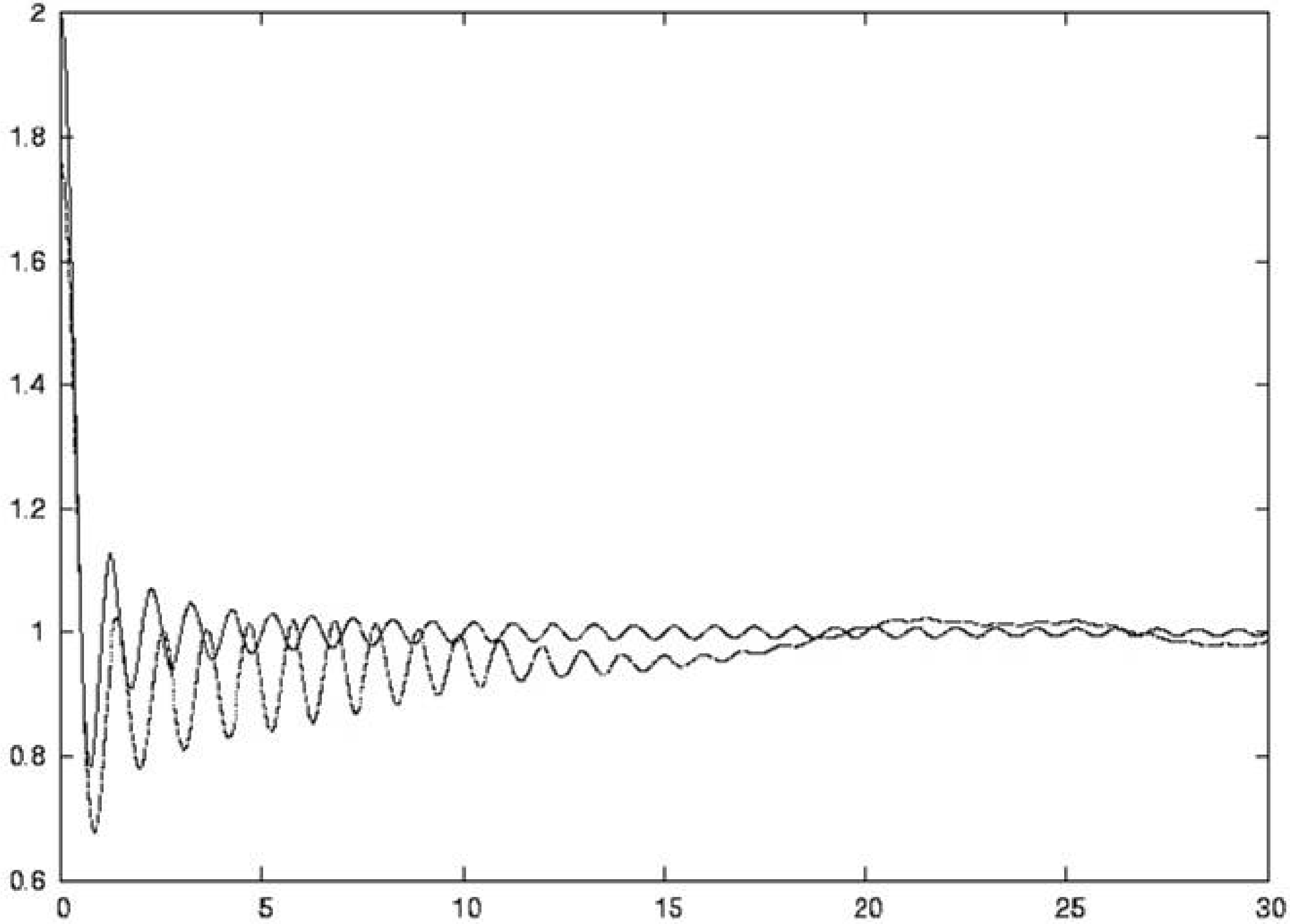
One level density for conductors up to 10^6 compared with the scaling limit



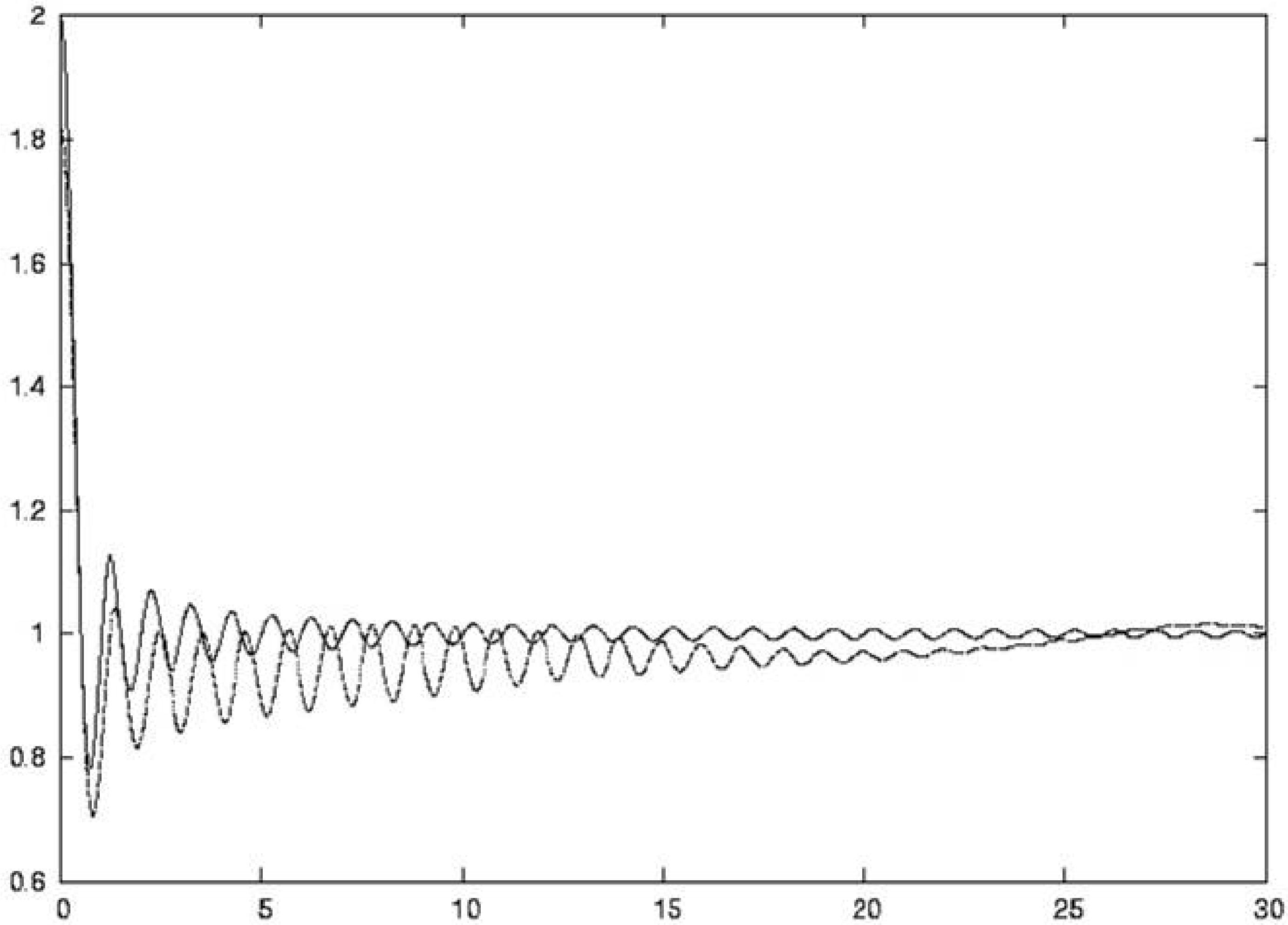
One level density for conductors up to 10^8 compared with the scaling limit



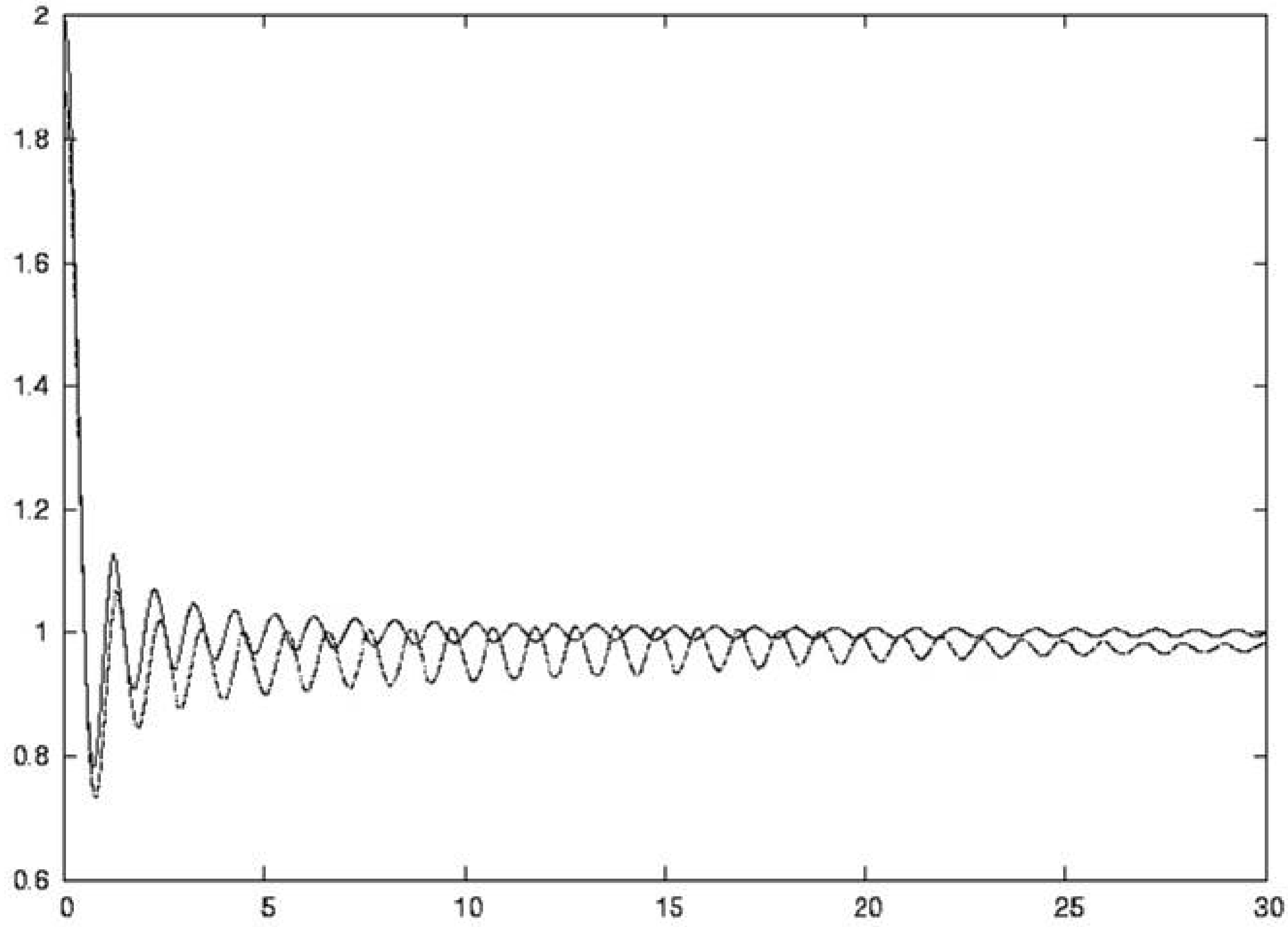
One level density for conductors up to 10^10 compared with the scaling limit



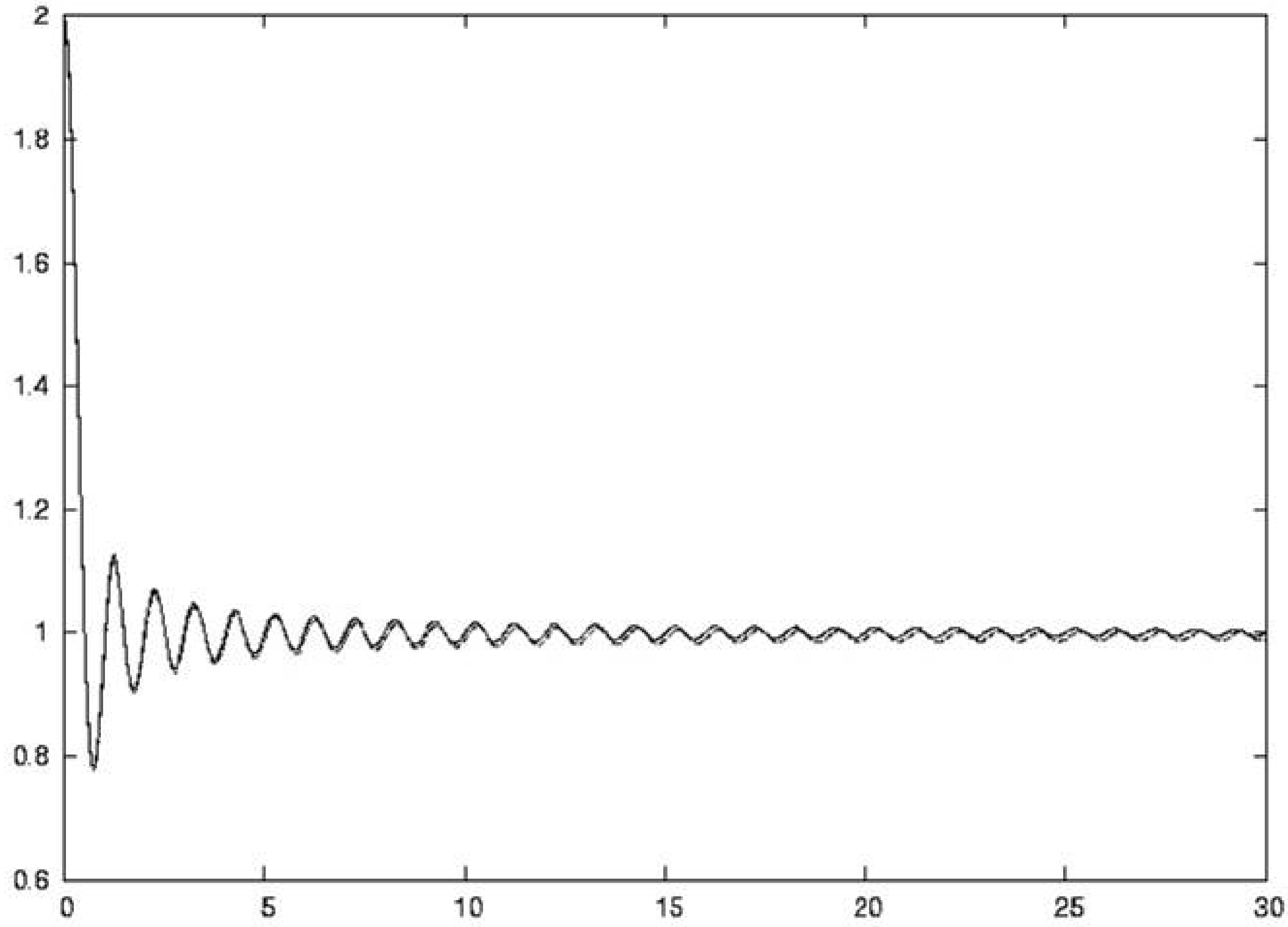
One level density for conductors up to 10^15 compared with the scaling limit



One level density for conductors up to 10^20 compared with the scaling limit



One level density for conductors up to 10^30 compared with the scaling limit



One level density for conductors up to 10^300 compared with the scaling limit

$$
\begin{aligned}
S_1(g) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t) \sum_{|d|\le X} \Bigg(& 2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right) \\
& + \frac{\Gamma'}{\Gamma}(1/2+it) + \frac{\Gamma'}{\Gamma}(1/2-it) \\
& + 2\Bigg[-\frac{\zeta'}{\zeta}(1+2it) + \frac{L_E'}{L_E}(\text{sym}^2, 1+2it) + A_f'(it,it) \\
& - \left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2it} A_f(-it,it) \\
& \qquad \times \frac{\Gamma(1/2-it)}{\Gamma(1/2+it)} \frac{\zeta(1+2it)L_E(\text{sym}^2, 1-2it)}{L_E(\text{sym}^2,1)} \Bigg] \Bigg) dt \\
& + O(X^{1/2+\epsilon})
\end{aligned}
$$

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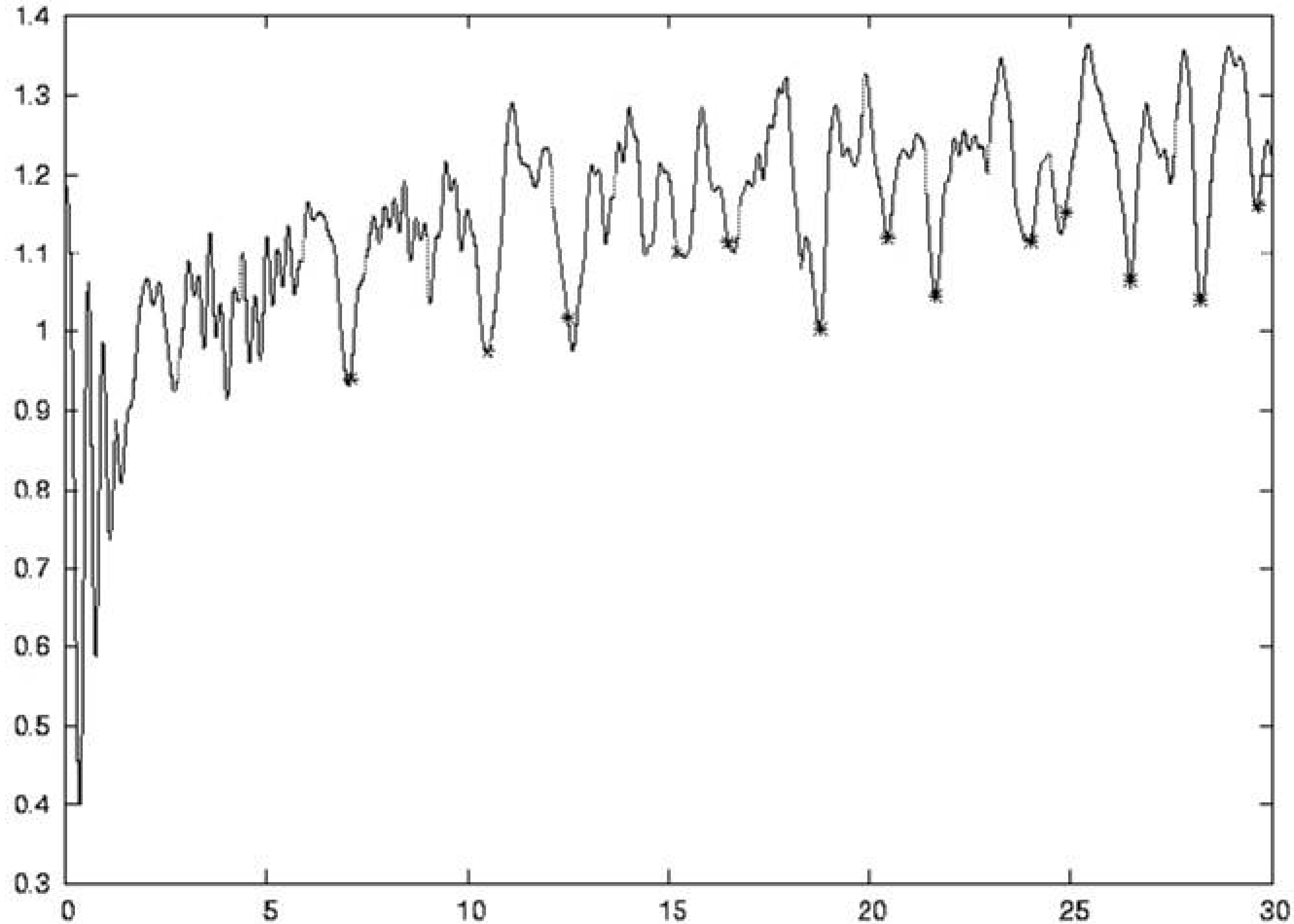

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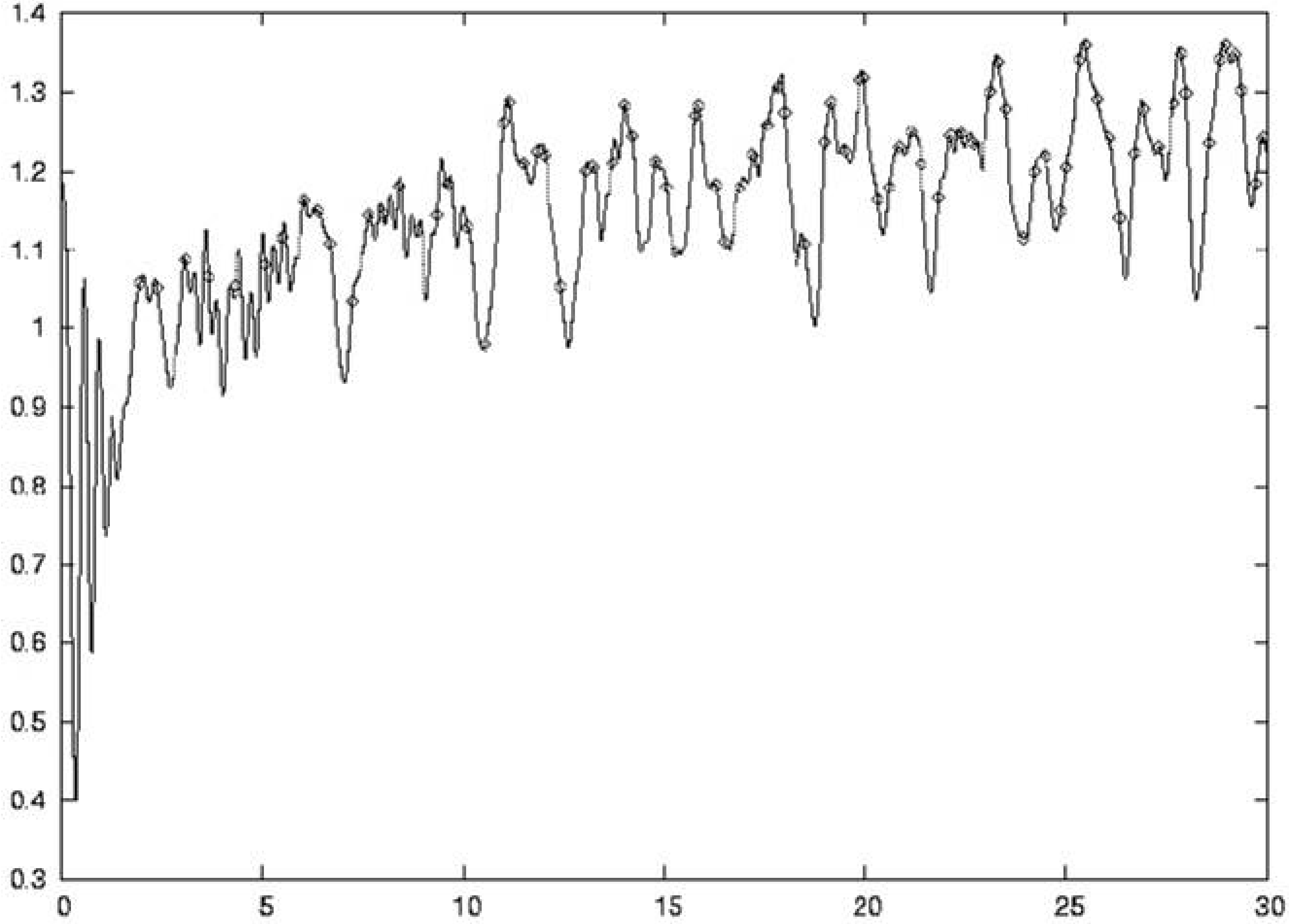
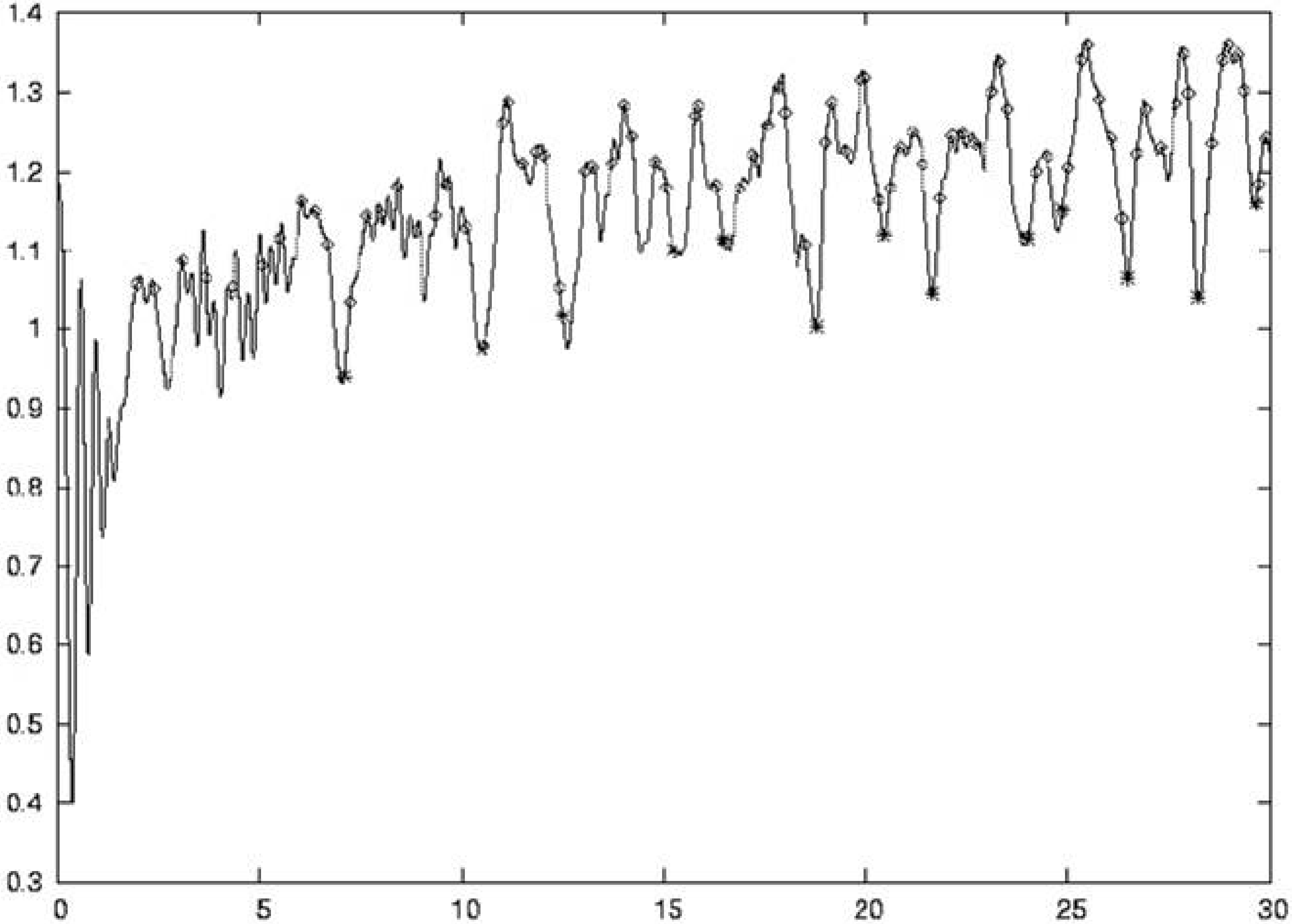
$$
\begin{aligned}
S_1(g) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t) \sum_{|d|\le X} \Bigg(& 2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right) \\
& + \frac{\Gamma'}{\Gamma}(1/2+it) + \frac{\Gamma'}{\Gamma}(1/2-it) \\
& + 2\Bigg[-\frac{\zeta'}{\zeta}(1+2it) + \frac{L_E'}{L_E}(\mathrm{sym}^2, 1+2it) + A_f'(it,it) \\
& - \left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2it} A_f(-it,it) \\
& \qquad \times \frac{\Gamma(1/2-it)}{\Gamma(1/2+it)} \frac{\zeta(1+2it) L_E(\mathrm{sym}^2, 1-2it)}{L_E(\mathrm{sym}^2,1)} \Bigg] \Bigg) dt \\
& \qquad + O(X^{1/2+\epsilon})
\end{aligned}
$$


1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0
5
10
15
20
25
30



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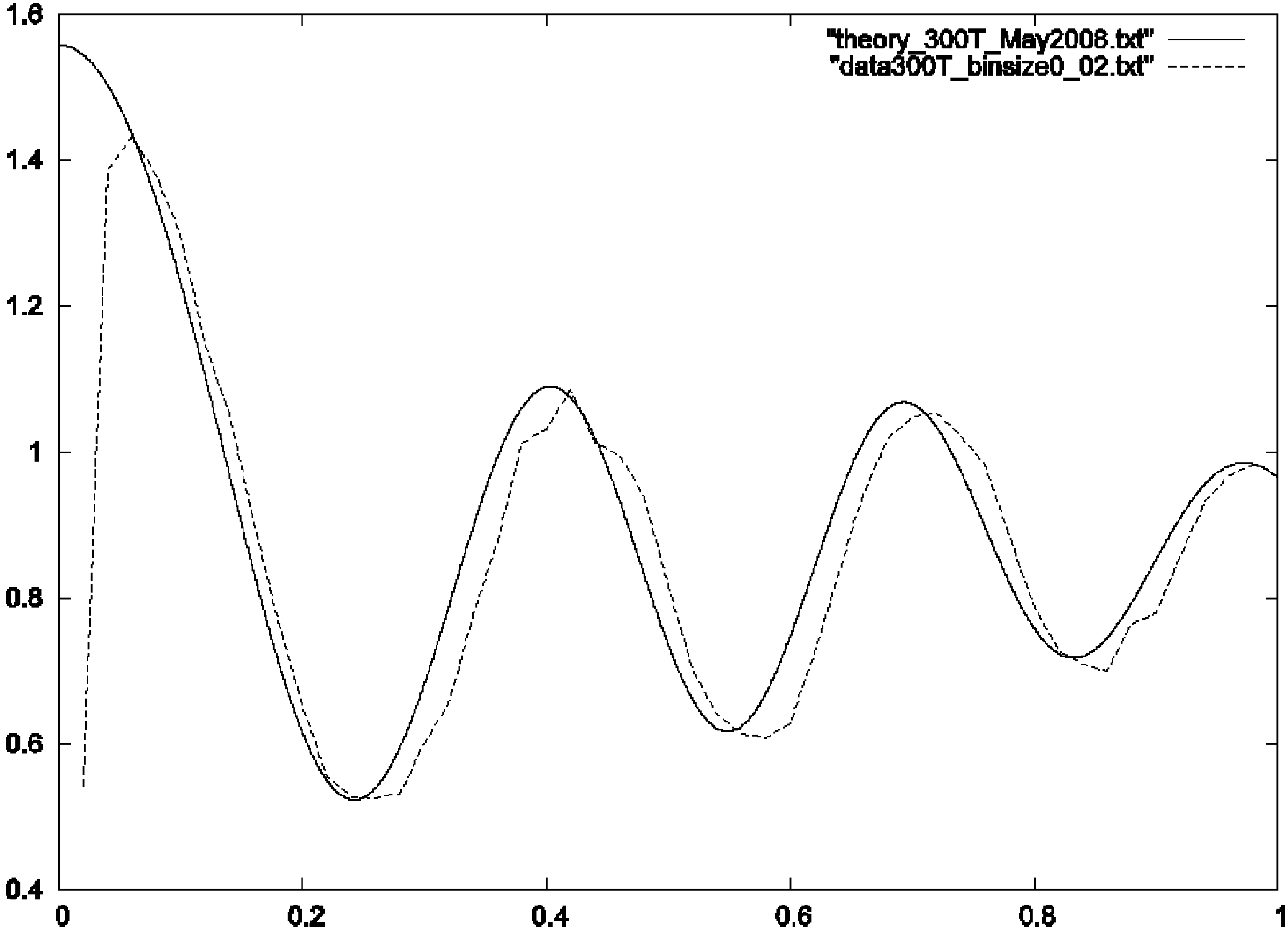
$$
\begin{aligned}
S_1(g) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t) \sum_{|d|\le X} \Bigg(& 2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right) \\
& + \frac{\Gamma'}{\Gamma}(1/2+it) + \frac{\Gamma'}{\Gamma}(1/2-it) \\
& + 2\Bigg[-\frac{\zeta'}{\zeta}(1+2it) + \frac{L_E'}{L_E}(\mathrm{sym}^2, 1+2it) + A_f'(it,it) \\
& - \left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2it} A_f(-it,it) \\
& \qquad \times \frac{\Gamma(1/2-it)}{\Gamma(1/2+it)} \frac{\zeta(1+2it)L_E(\mathrm{sym}^2, 1-2it)}{L_E(\mathrm{sym}^2,1)}\Bigg]\Bigg) dt \\
& \qquad + O(X^{1/2+\epsilon})
\end{aligned}
$$



One level density for quadratic twists of E11for

0<d<300000 (solid:ratios conjecture, dashed:numerical)

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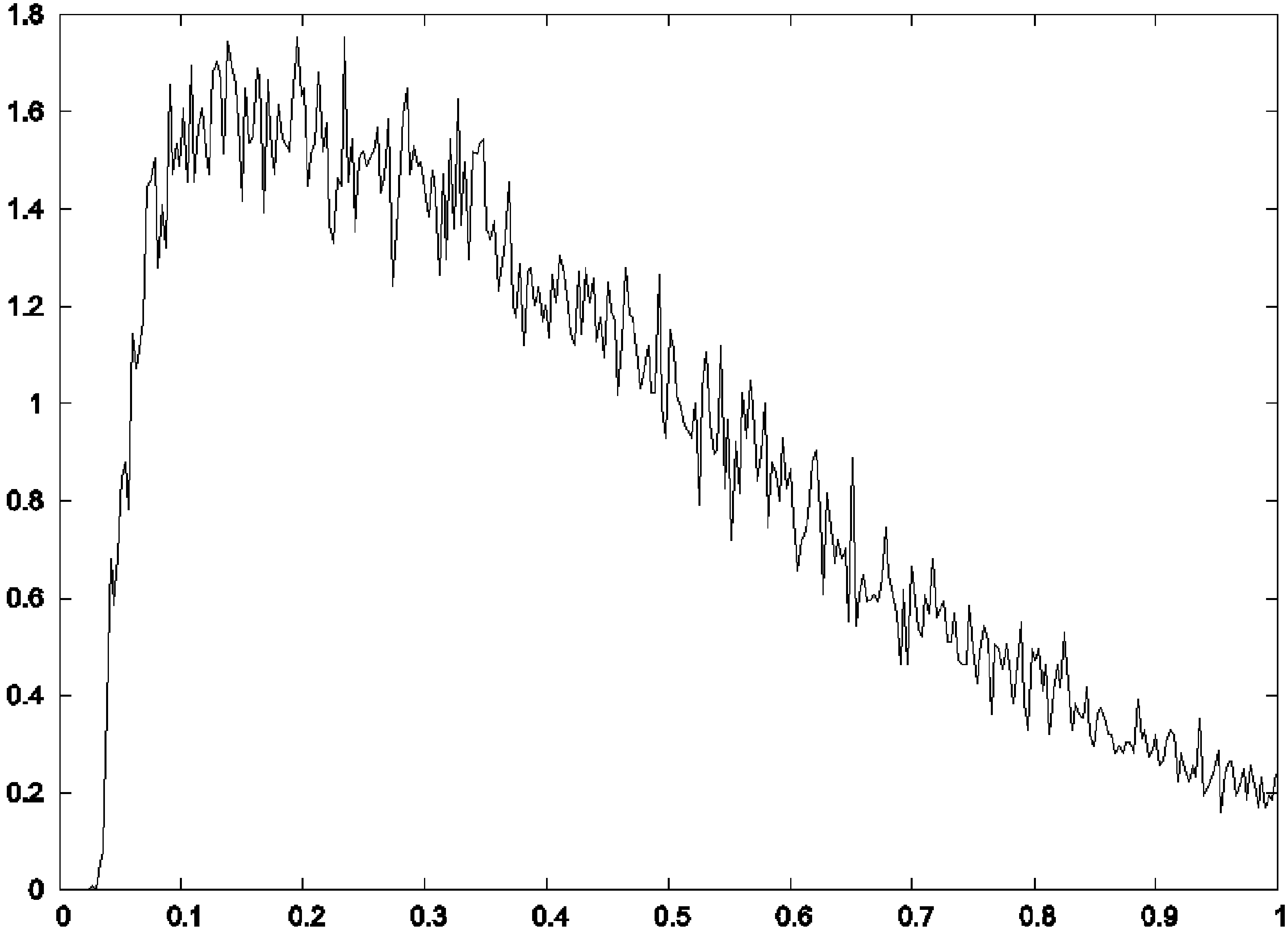

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Density of the first zero for quadratic twists of E11for

0<d<300000 (41660 zeros)

L-values discretised

$$L_E(1/2, \chi_d) = \kappa_E \frac{c_E(|d|)^2}{\sqrt{d}},$$

(Waldspurger,Shimura,Kohnen-Zagier)

where the $c_E(|d|)$ are integers; the fourier coefficients of a half-integral weight form.

So, if

$$L_E(1/2, \chi_d) < \frac{\kappa_E}{\sqrt{d}}$$

then

$$L_E(1/2, \chi_d) = 0.$$

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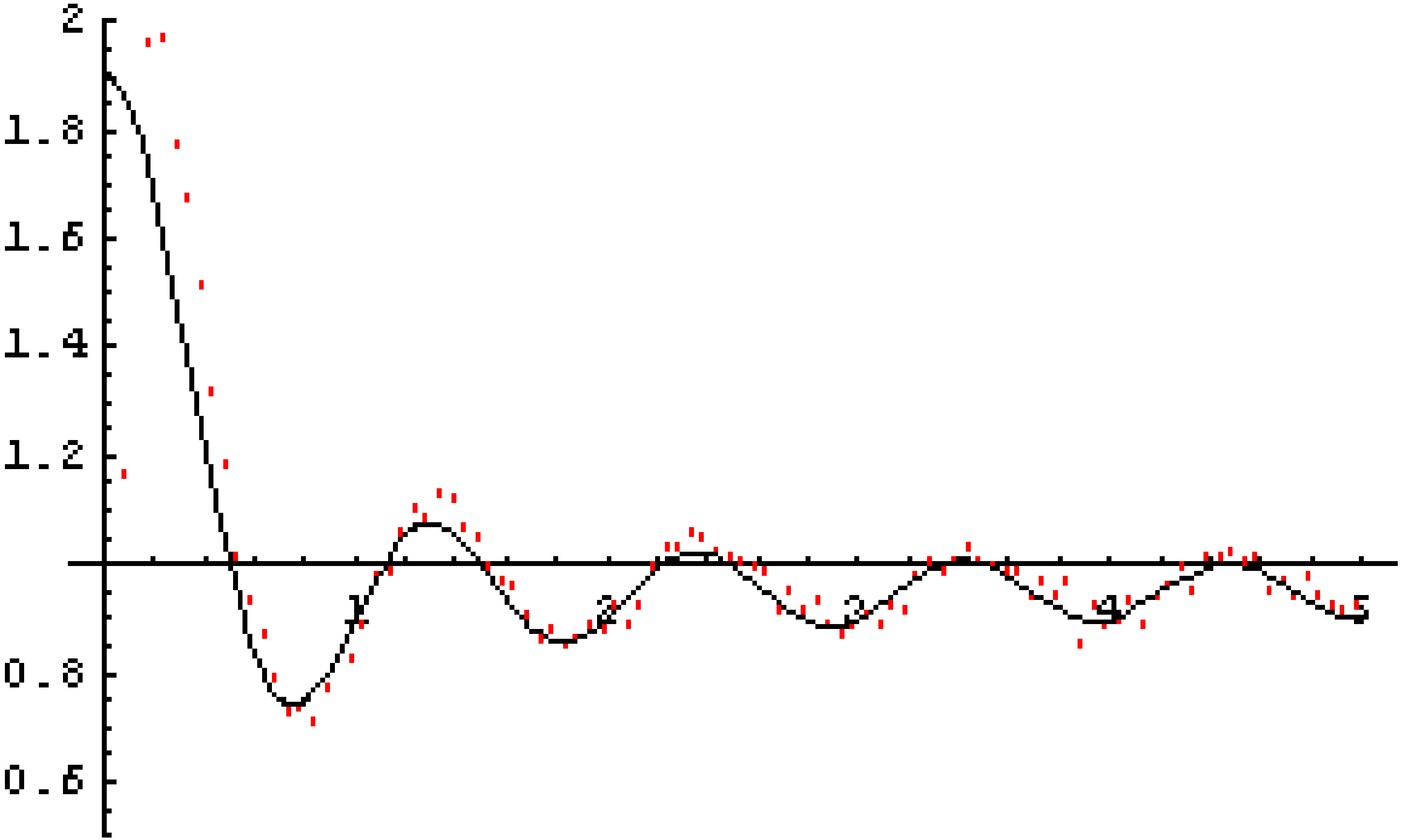
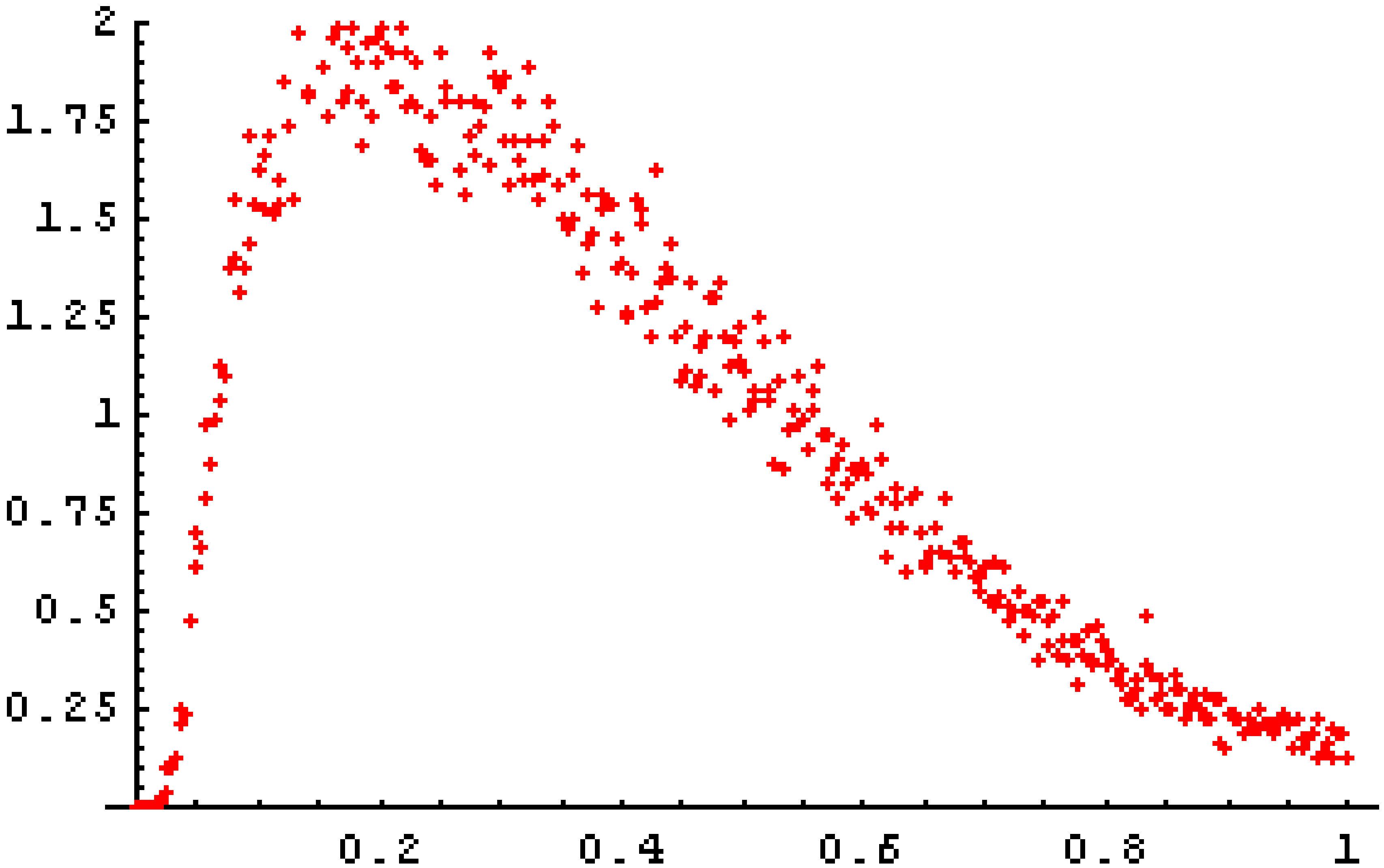
Hypothesis:

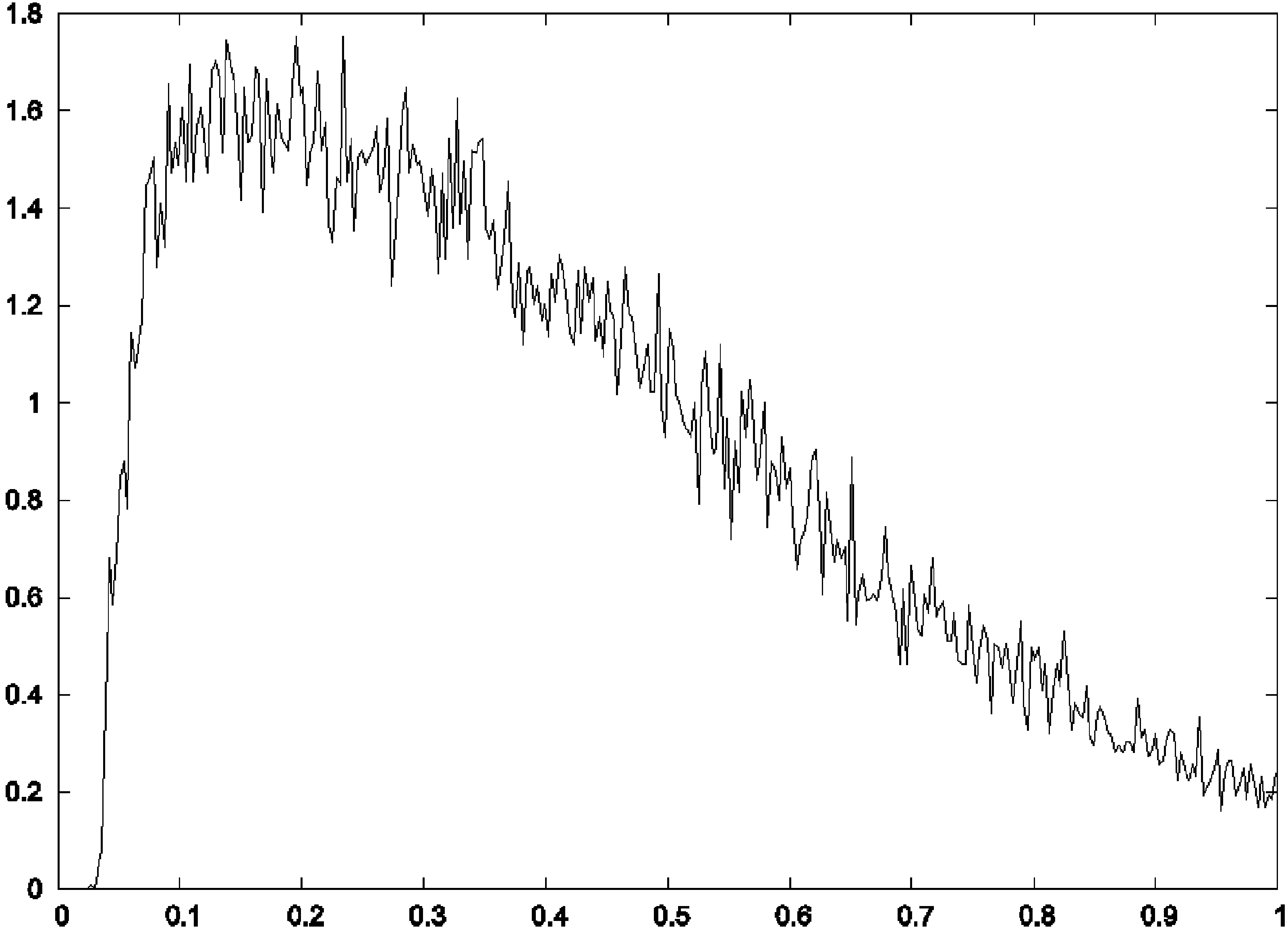
Discretisation causes apparent repulsion from the symmetry point of the zeros?



One level density for random SO(2N) matrices, N=10 (solid:exact curve for full group, red: numerics from 100000 randomly generated matrices with characteristic polynomial > 3 exp(-N/2)



Numerical density of the first eigenvalue from 100000 randomly generated SO(20) matrices with characteristic polynomial > 3 exp(-N/2)



Density of the first zero for quadratic twists of E11for

0<d<300000

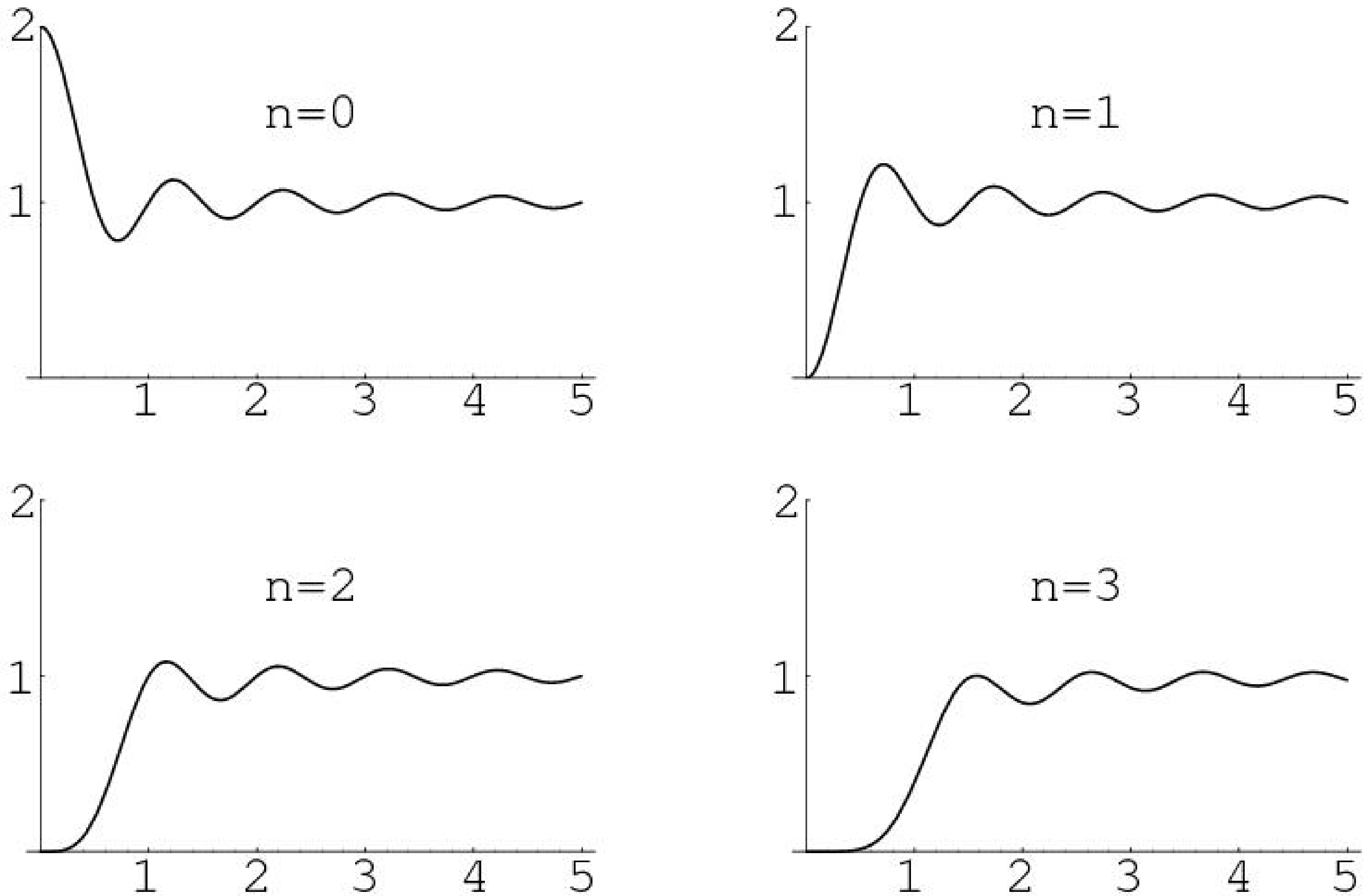
Joint Probability Density $SO(2N)$ eigenvalues

$$\text{const} \prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 \, d\theta_1 \cdots d\theta_N$$

Joint Probability Density for matrices in $SO(2N+n)$

constrained to have n eigenvalues at 1 and
$2N$ eigenvalues elsewhere

$$\text{const} \prod_{j=1}^{N} (1-\cos\theta_j)^n \prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 \, d\theta_1 \cdots d\theta_N$$



One level densities with n eigenvalues at the symmetry point

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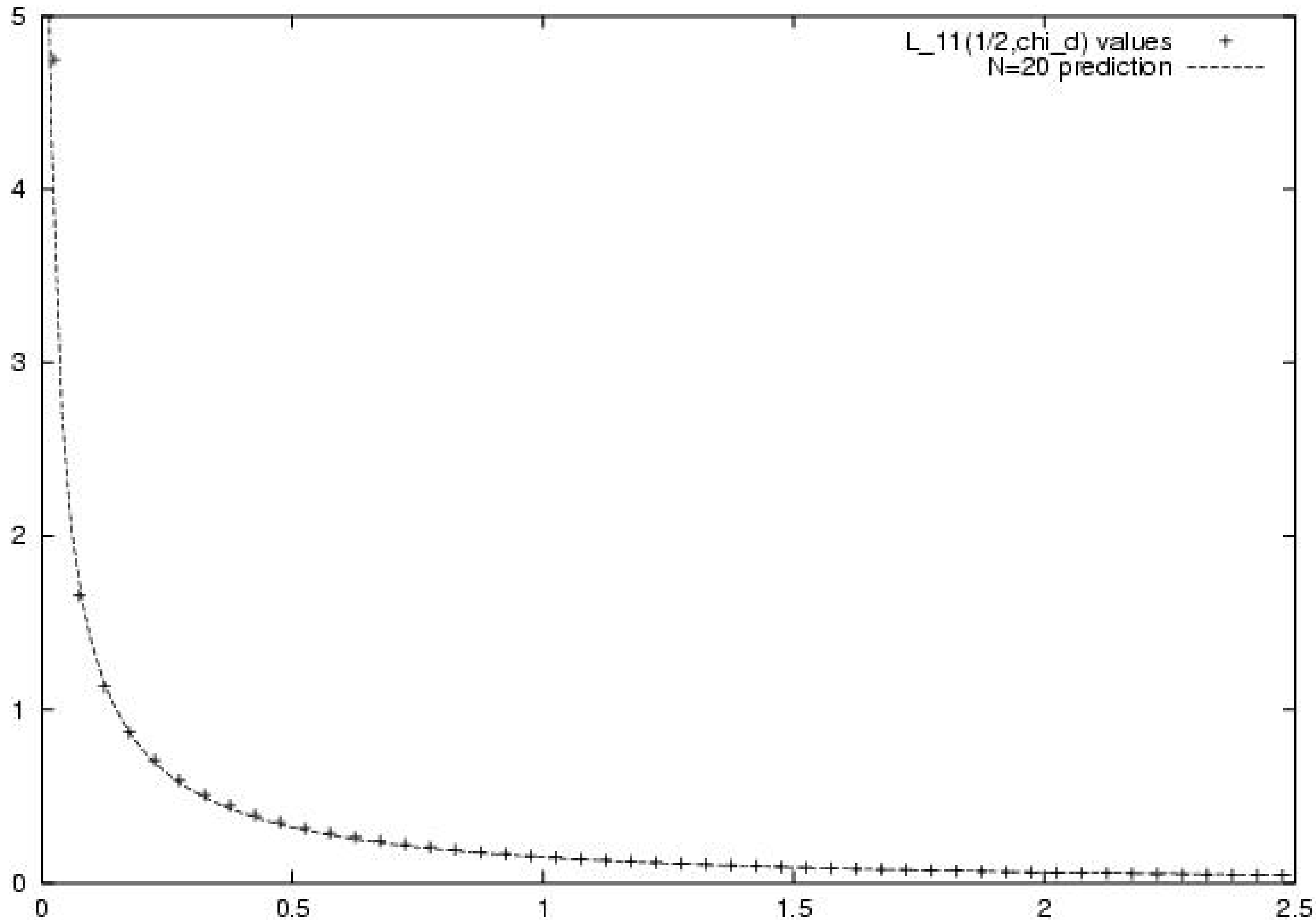
To model repulsion from the symmetry point:

$$\text{const} \prod_{j=1}^{N} (1-\cos\theta_j)^{a(N)} \prod_{1\leq j<k\leq N} (\cos\theta_k - \cos\theta_j)^2 \, d\theta_1 \cdots d\theta_N$$

where $a(N) \to 0$ as $N \to \infty$



The value distribution of $L_{E_{11}}(1/2, \chi_d)$ for prime |d|, -788299808<d<0, even functional equation, compared to the value distribution of characteristic polynomials from SO(40)