A FOURIER-THEORETIC CRITERION FOR CONVERGENCE IN LAW

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A fairly standard approach to prove the convergence in law towards X of a sequence of random variables (X_n) with values in the Banach space $C([0,1])$ of continuous functions on $[0,1]$ is to prove convergence of finite distributions (i.e., for any integer $m \geqslant 1$ and any

$$0 \leqslant t_1 < t_2 < \cdots < t_{m-1} < t_m = 1$$

the vectors

$$(X_n(t_1), \ldots, X_n(t_m))$$

converge in law to $(X(t_1), \ldots, X(t_m)))$ with tightness. This is used, for instance, in many proofs of Donsker's Invariance Principle for random walks, or in certain constructions of Brownian motion.

When dealing with random walks or similar objects with suitable markovian properties, convergence in finite distributions is relatively accessible. However, in other circumstances (such as random Fourier series or exponential sums, as in the paper [3] of Kowalski and Sawin), one may have to rely on the method of moments, which is sometimes awkward. Although the complexity is partly purely a matter of notation, the requirement of existence of moments (or the use of truncation to avoid this condition) may be problematic.

In this note, we describe an elementary Fourier-theoretic criterion for convergence of finite distributions, which may be useful in certain circumstances to shortcut the notational complexity involved in moment computations. For an example, see the proof in [2, Ch. 4] of the functional limit theorem of [3].

Intuitively, the statement is quite simple: a sequence (X_n) as above converges in finite distributions if and only if the Fourier coefficients converge in finite distribution. This is well-adapted to applications where the limit X has independent Fourier coefficients.

To state the result precisely, we must be a bit careful because we wish to deal with the whole space $C([0,1])$, whereas Fourier series are only suitable for periodic functions satisfying $f(0) = f(1)$. We handle this by adding the identity function to the usual periodic exponentials for our Fourier coefficients, observing that for $f \in C([0,1])$, the function $t \mapsto f(t) - t(f(1) - f(0))$ is continuous and periodic.

Let $\bullet$ be a symbol and let $\widetilde{\mathbf{Z}} = \mathbf{Z} \cup \{\bullet\}$, with the obvious topology where $\bullet$ is isolated. We denote $e_\bullet(t) = t$ for $t \in [0,1]$ and for $h \in \mathbf{Z}$, we put $e_h(t) = e(ht)$.

We denote by $C_0(\widetilde{\mathbf{Z}})$ the Banach space of complex-valued functions on $\widetilde{Z}$ converging to 0 at infinity, with the sup norm. We then have a continuous linear map $\mathrm{FT}\colon C([0,1]) \to C_0(\widetilde{\mathbf{Z}})$ that maps a function f to the function determined by

$$\widetilde{f}(\bullet) = f(1) - f(0),$$

and

$$\widetilde{f}(h) = \int_0^1 (f(t) - t(f(1) - f(0)))e(-ht)dt = \int_0^1 (f - \widetilde{f}(\bullet)e_\bullet)e_{-h}$$

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for $h \in \mathbf{Z}$.

We will relate convergence in law in $C([0,1])$ with convergence in law of these (generalized) Fourier coefficients in $C_0(\widetilde{\mathbf{Z}})$.

First, since FT is a continuous map, we have one obvious implication:

Lemma 1. *If $(X_n)_n$ is a sequence of $C([0,1])$-valued random variables that converges in law to a random variable X, then $\mathrm{FT}(X_n)$ converges in law to $\mathrm{FT}(X)$ in $C(\widetilde{\mathbf{Z}})$.*

Next, we prove that the Fourier coefficients also determine the law of a $C([0,1])$-valued random variable.

Lemma 2. *If X and Y are $C([0,1])$-valued random variables and if $\mathrm{FT}(X)$ and $\mathrm{FT}(Y)$ have the same finite distributions, then X and Y have the same law.*

Proof. Let $f \in C([0,1])$. The function $g = f - \widetilde{f}(\bullet)e_\bullet$ extends to a 1-periodic continuous function on $\mathbf{R}$. By Féjer's Theorem on the uniform convergence of Cesàro means of Fourier series of continuous periodic functions (see, e.g, [4, III, Th. 3.4]), we have

$$g(t) = \lim_{H \to +\infty} \sum_{|h| \leqslant H} \Big(1 - \frac{|h|}{H}\Big) \widetilde{f}(h) e(ht)$$

uniformly for $t \in [0,1]$, hence we deduce that $f = \lim_{H \to +\infty} C_H(f)$ where

$$C_H(f) = \widetilde{f}(\bullet)e_\bullet + \sum_{\substack{|h| \leqslant H \\ h \neq 0}} \Big(1 - \frac{|h|}{H}\Big) \widetilde{f}(h) e_h.$$

Note that $C_H(f) \in C([0,1])$ for all $H \geqslant 1$.

We now claim that the $C([0,1])$-valued random variable $C_H(X)$ converges to X in $C([0,1])$. Indeed, let φ be a continuous and bounded function on $C([0,1])$, say $|\varphi(f)| \leqslant M$ for all $f \in C([0,1])$. By the above, we have $\varphi(C_H(X)) \to \varphi(X)$ as $H \to +\infty$ pointwise on $C([0,1])$. Since $|\varphi(C_H(X))| \leqslant M$, which is integrable on the underlying probability space, Lebesgue's dominated convergence theorem implies that $\mathbf{E}(\varphi(C_H(X))) \to \mathbf{E}(\varphi(X))$. This proves the claim.

In view of the definition of $C_H(f)$, which only involves finitely many Fourier coefficients, the equality of finite distributions of $\mathrm{FT}(X)$ and $\mathrm{FT}(Y)$ implies by composition that the $C([0,1])$-valued random variables $C_H(X)$ and $C_H(Y)$ have the same law for any $H \geqslant 1$. Since the previous argument implies that $C_H(X)$ converges in law to X and that $C_H(Y)$ converges in law to Y, it follows that X and Y have the same law. ☐

Finally, we can state our convergence criterion.

Proposition 3. *Let (X_n) be a sequence of $C([0,1])$-valued random variables and let X be a $C([0,1])$-valued random variable. Suppose that $\mathrm{FT}(X_n)$ converges to $\mathrm{FT}(X)$ in the sense of finite distributions. Then (X_n) converges in law to X in the sense of $C([0,1])$-valued random variables if and only if (X_n) is tight.*

Proof. It is elementary that if (X_n) converges in law to X, then the family (X_n) is tight, so we need only prove the converse assertion.

It suffices to prove that any subsequence of (X_n) has a further subsequence that converges in law to X (see [1, Th. 2.6]). Because (X_n) is tight, so is any of its subsequences. By Prokhorov's

Theorem ([1, Th. 5.1]), such a subsequence therefore contains a further subsequence, say $(X_{n_k})_{k\geqslant 1}$, that converges in law to some probability measure Y.

By Lemma 1, the sequence of Fourier coefficients $\mathrm{FT}(X_{n_k})$ converges in law to $\mathrm{FT}(Y)$. On the other hand, this sequence converges to $\mathrm{FT}(X)$ in the sense of finite distributions by assumption. Hence $\mathrm{FT}(X)$ and $\mathrm{FT}(Y)$ have the same finite distributions, which implies that X and Y have the same law by Lemma 2. ☐

Remark 4. A classical example that shows that convergence in finite distributions in $C([0,1])$ does not imply convergence in law (see [1, Ex. 2.5]) also shows that the convergence of finite distributions of $\mathrm{FT}(X_n)$ to $\mathrm{FT}(X)$ is not sufficient to conclude that (X_n) converges in law to X.

Indeed, in this example, we define the random variable X_n to be the *constant* random variable equal to the function f_n that is piecewise linear on $[0,1/n]$, $[1/n,1/(2n)]$ and $[1/(2n),1]$, and such that $0\mapsto 0$, $1/n\mapsto 1$, $1/(2n)\mapsto 0$ and $1\mapsto 0$. Then it is elementary that X_n converges to the constant zero random variable in the sense of finite distributions, but that X_n does not converge in law to 0 (because f_n does not converge uniformly to 0). On the other hand, for $n\geqslant 1$, the random variable X_n satisfies $X_n(0)=X_n(1)=0$, and by direct computation, its Fourier coefficients (are deterministic and) satisfy also $|\widetilde{X}_n(h)|\leqslant n^{-1}$ for all $h\in\mathbf{Z}$, which implies that $\mathrm{FT}(X_n)$ converges in the sense of finite distributions to the constant random variable equal to $0\in C_0(\widetilde{\mathbf{Z}})$.

References

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