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ON A THEOREM OF BOMBIERI–VINOGRADOV TYPE

E. FOUVRY AND H. IWANIEC

§1. *Introduction.* The celebrated theorem of Bombieri and A. I. Vinogradov states that

$$\sum_{q < x^{(1/2)-\varepsilon}} \max_{(a,q)=1} \left| \pi(x\,;q,a) - \frac{\pi(x)}{\phi(q)} \right| \ll x(\log x)^{-A}, \tag{1}$$

for any $\varepsilon > 0$ and $A > 0$, the implied constant in the symbol $\ll$ depending at most on ε and A (see [1] and [14]). The original proofs of Bombieri and Vinogradov were greatly simplified by P. X. Gallagher [4]. An elegant proof has been given recently by R. C. Vaughan [13]. For other references see H. L. Montgomery [10] and H. -E. Richert [12]. Estimates of type (1) are required in various applications of sieve methods. Having this in mind distinct generalizations have been investigated (see for example [15] and [2]). Y. Motohashi established a general theorem which, roughly speaking, says that if (1) holds for two arithmetic functions then it also holds for their Dirichlet convolution; for precise assumptions and statement see [11]. So far, all methods depend on the large sieve inequality (see [10])

$$\sum_{q \leqslant Q} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, q)} \left| \sum_{n \leqslant N} a_n \chi(n) \right|^2 \ll (Q^2+N) \sum_{n \leqslant N} |a_n|^2, \tag{2}$$

which sets the limit $x^{1/2}$ for the modulus q in (1) and in its generalizations.

It is the aim of this paper to present arguments which yield theorems of Bombieri–Vinogradov type with an extended range for q. We shall treat carefully

$$\pi(x, z\,; q, a) = \sum_{\substack{n \leqslant x \\ n \equiv a (\mathrm{mod}\, q)}} f_z(n),$$

where f_z is the characteristic function of the set of integers n having no prime factor less than z. Let us introduce also

$$\pi(x\,; z, q) = \sum_{\substack{n \leqslant x \\ (n,q)=1}} f_z(n).$$

We have proved the following

THEOREM. *Let $z \leqslant x^{1/883}$ and $1 \leqslant |a| \leqslant x$. Then, for any $A > 0$,*

$$\sum_{\substack{q < x^{11/21} \\ (q,a)=1}} \left| \pi(x, z; q, a) - \frac{1}{\phi(q)} \pi(x, z; q) \right| \ll x(\log x)^{-A}, \tag{3}$$

the implied constant depending only on A.

Our method applies to a wide class of arithmetic functions $f(n)$, for which the sum

$$\pi_f(x; q, a) = \sum_{\substack{n \leqslant x \\ n \equiv a (\mathrm{mod}\, q)}} f(n)$$

can be rearranged as a sum

$$\sum_{L < l \leqslant 2L} \sum_{\substack{M < m \leqslant 2M \\ lm \equiv a (\mathrm{mod}\, q)}} \alpha_l \beta_m \tag{4}$$

of bilinear forms, with the variables of summation l and m in appropriate intervals. Such a representation for $f_z(n)$ is obtained through a combinatorial sieve identity (see Lemma 1). We failed to obtain (3) in the most interesting case $z = x^{1/2}$, in other words for $f(n) = \Lambda(n)$. In the latter case, Vaughan's identity (see **[13]**) would serve as a bilinear form (4), but unfortunately with L and M not well enough controlled for our method to apply.

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§2. *Sketch of the main ideas.* The bilinear form (4) is approximated by

$$\frac{1}{\phi(q)} \sum_{L < l \leqslant 2L} \sum_{\substack{M < m \leqslant 2M \\ (lm, q) = 1}} \alpha_l \beta_m ,$$

with the total error less than

$$R(M, L; Q) = \sum_{\substack{Q < q \leqslant 2Q \\ (q,a)=1}} \sum_{\substack{M < m \leqslant 2M \\ (m,q)=1}} \left| \sum_{\substack{L < l \leqslant 2L \\ lm \equiv a (\mathrm{mod}\, q)}} \alpha_l - \frac{1}{\phi(q)} \sum_{\substack{L < l \leqslant 2L \\ (l,q)=1}} \alpha_l \right| .$$

The problem of bounding $R(M, L; Q)$ is reduced, by the Cauchy–Schwarz inequality

$$R(M, L; Q) \leqslant (QM)^{1/2} D^{1/2}(M, L; Q),$$

to that of bounding the dispersion

$$D(M, L; Q) = \sum_q \sum_m \Bigg(\sum_{\substack{L < l \leqslant 2L \\ lm \equiv a \pmod q}} \alpha_l - \frac{1}{\phi(q)} \sum_{\substack{L < l \leqslant 2L \\ (l, q) = 1}} \alpha_l \Bigg)^2$$

$$= W(M, L; Q) - 2V(M, L; Q) + U(M, L; Q),$$

say. Each of the terms U, V and W is evaluated separately, the most difficult being W. By definition

$$W(M, L; Q) = \sum_{\substack{Q < q \leqslant 2Q \\ (q, a) = 1}} \sum_{\substack{M < m \leqslant 2M \\ (m, q) = 1}} \sum_{\substack{L < l_1, l_2 \leqslant 2L \\ l_1 m \equiv l_2 m \equiv a \pmod q}} \alpha_{l_1} \alpha_{l_2}.$$

With an admissible error we may replace $W(M, L; Q)$ by $W^*(M, L; Q)$, which stands for the same sum with the range of summation restricted to $(l_1, l_2) = 1$, $l_1 \equiv l_2 \pmod q$. In particular the diagonal terms $l_1 = l_2$ disappear.

When treating $W^*(M, L; Q)$ we carry out the summation over m first. We use $\bar{l}_1$ to denote the reciprocal of l_1 modulo q, so that $l_1 \bar{l}_1 \equiv 1 \bmod q$. Writing

$$\sum_{\substack{M < m \leqslant 2M \\ m \equiv a\bar{l}_1 \pmod q}} 1 = \frac{M}{q} + r(q, a\bar{l}_1),$$

it is trivial that $|r(q, a\bar{l}_1)| \leqslant 1$, but this turns out to be not satisfactory. We obtain a great cancellation of the errors $r(q, a\bar{l}_1)$ in the sums over l_1, l_2 and q. By expanding each $r(q, a\bar{l}_1)$ into a Fourier series, a typical term to be considered is

$$W_h(M, L; Q) = \sum_{\substack{Q < q \leqslant 2Q \\ (q, a) = 1}} \sum_{\substack{L < l_1, l_2 \leqslant 2L \\ l_1 \equiv l_2 \pmod q \\ (l_1, l_2) = 1, (l_1 l_2, q) = 1}} \alpha_{l_1} \alpha_{l_2} e\left(h \frac{M - a\bar{l}_1}{q} \right),$$

with $h \neq 0$. Since Q will be nearly as large as L and $l_1 \equiv l_2 \pmod q$ there is not much room for summation over l_1 and l_2. For this reason we reinterpret the condition $l_1 \equiv l_2 \pmod q$ by writing

$$l_1 - l_2 = qr \quad \text{with } 0 < |r| \leqslant L/Q, \ (r, l_1 l_2) = 1.$$

Here r is rather small, so the condition $l_1 \equiv l_2 \pmod r$ constrains the variables l_1, l_2 less than does $l_1 \equiv l_2 \pmod q$. In addition,

$$\frac{M - a\bar{l}_1}{q} \equiv \left(M - \frac{a}{l_1} \right) \frac{r}{l_1 - l_2} - ar \frac{\bar{l}_2}{l_1} \pmod 1.$$

Therefore we arrive at sums of the type

$$\sum_{0<|r|<L/Q} \sum_{\substack{L<l_1,l_2\leqslant L_1\\(l_1,l_2)=1,(l_1-l_2,ar)=r}} \alpha_{l_1}\alpha_{l_2} e\left(-ahr\frac{\bar{l}_2}{l_1}\right),$$

with some $L_1 \in (L, 2L]$, the factor

$$e\left(\left(M-\frac{a}{l_1}\right)\frac{hr}{l_1-l_2}\right)$$

being removed by partial summation. A connection with the incomplete Kloosterman sums is suggested. By the Cauchy–Schwarz inequality,

$$\left|\sum_{l_1,l_2} \alpha_{l_1}\alpha_{l_2} e\left(-ahr\frac{\bar{l}_2}{l_1}\right)\right|^2 \leqslant L \sum_{l',l''} \alpha_{l'}\bar{\alpha}_{l''} \sum_{l} e\left(-ahr\frac{(l'-l'')\bar{l}}{l'l''}\right).$$

Using Weil's estimate of $\sum_l$ one just fails to get a non-trivial bound because the modulus $l'l''$ is as large as the square of the length of the incomplete Kloosterman sum $\sum_l$. Hooley's conjecture R^* (see [7]) would be helpful. In order to avoid any unproved hypothesis, we appeal to a particular property of the coefficients α_l to rearrange the sum $\sum_{l_1,l_2}$ into another bilinear form with variables of summation of a different order of magnitude. Then, the above procedure yields incomplete Kloosterman sums which are manageable by Weil's estimate. We doubt whether the elementary result of Kloosterman [9] is sufficient.

From the main terms in the dispersion $D(M, L; Q)$ we get

$$M \sum_{\substack{Q<q\leqslant 2Q\\(q,a)=1}} \phi^{-2}(q) \sum_{\substack{\lambda(\mathrm{mod}\, q)\\(\lambda,q)=1}} \left|\sum_{\substack{L<l\leqslant 2L\\l\equiv\lambda(\mathrm{mod}\, q)}} \alpha_l - \frac{1}{\phi(q)} \sum_{\substack{L<l\leqslant 2L\\(l,q)=1}} \alpha_l\right|^2.$$

We estimate this by applying the large sieve inequality and the Siegel–Walfisz theorem in a way familiar from the Barban and Davenport–Halberstam theorem.

§3. *Lemmas.* Let $P(z) = \prod_{p<z} p$ for $z \geqslant 2$. Let $F(n)$ be an arithmetic function vanishing for almost all n. By the Buchstab identity

$$\sum_n f_z(n)F(n) = \sum_n F(n) - \sum_{p<z}\left(\sum_{n\equiv 0(\mathrm{mod}\, p)} f_p(n)F(n)\right),$$

on applying the 'exclusion-inclusion principle' familiar from combinatorial sieve theory (see [5] and [8]) we obtain

LEMMA 1. *Let $D \geqslant z \geqslant 2$. Then*

$$\sum_n f_z(n)F(n) = \sum_{d\mid P(z)} \lambda_d\left(\sum_{n\equiv 0(\mathrm{mod}\, d)} F(n)\right) + \sum_{d\mid P(z)} \sigma_d\left(\sum_{n\equiv 0(\mathrm{mod}\, d)} f_{p(d)}(n)F(n)\right),$$

where $p(d)$ stands for the least prime factor of d, and, for a square-free $d = p_1 \dots p_r > 1$, $p_1 > \dots > p_r$, we define

$$\lambda_d = \begin{cases} (-1)^r, & \text{if } p_1 \dots p_l p_l < D \text{ for all } l \leqslant r\,, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\sigma_d = \begin{cases} (-1)^r, & \text{if } p_1 \dots p_l p_l < D \text{ for all } l < r \text{ and } p_1 \dots p_r p_r \geqslant D\,, \\ 0, & \text{otherwise.} \end{cases}$$

For $d = 1$ we define $\lambda_1 = 1$ and $\sigma_1 = 0$.

Note that if $\lambda_d \neq 0$ then $d < D$, and if $\sigma_d \neq 0$ then $D/p(d) \leqslant d < D$. Hence we obtain the

COROLLARY. *Let $D \geqslant z \geqslant 2$. Then*

$$\left|\sum_n f_z(n)F(n)\right| \leqslant \sum_{\substack{d<D \\ d \mid P(z)}} \left|\sum_l F(dl)\right| + \sum_{p<z} \sum_{\substack{D/p^2 \leqslant d < D/p \\ pd \mid P(z)}} \left|\sum_l f_p(l)F(dlp)\right|.$$

The following result is known in sieve theory as a 'fundamental lemma' (see [5]).

LEMMA 2. *For $R, z \geqslant 2$ and $(\alpha, q) = 1$ we have*

$$\sum_{\substack{n \leqslant x \\ n \equiv \alpha \pmod q}} f_z(n) = \frac{x}{q} \prod_{\substack{p<z \\ p \nmid q}} \left(1 - \frac{1}{p}\right)\{1 + O(e^{-s})\} + O(R)\,,$$

where $s = \log R/\log z$. The implied constants are absolute.

LEMMA 3. *If $\tau(q)$ is the number of divisors of q,*

$$\sum_{\substack{n \leqslant x \\ (n,q)=1}} 1 = \frac{\phi(q)}{q}x + O(\tau(q))\,.$$

LEMMA 4. *Let $\psi(\xi) = \xi - [\xi] - \frac{1}{2}$ and $\Delta > 0$. There are two functions $A(\xi)$ and $B(\xi)$ periodic in ξ (mod 1) such that*

$$|\psi(\xi) - A(\xi)| \leqslant B(\xi)$$

$$A(\xi) = \sum_{h \neq 0} \hat{A}(h)e(h\xi)$$

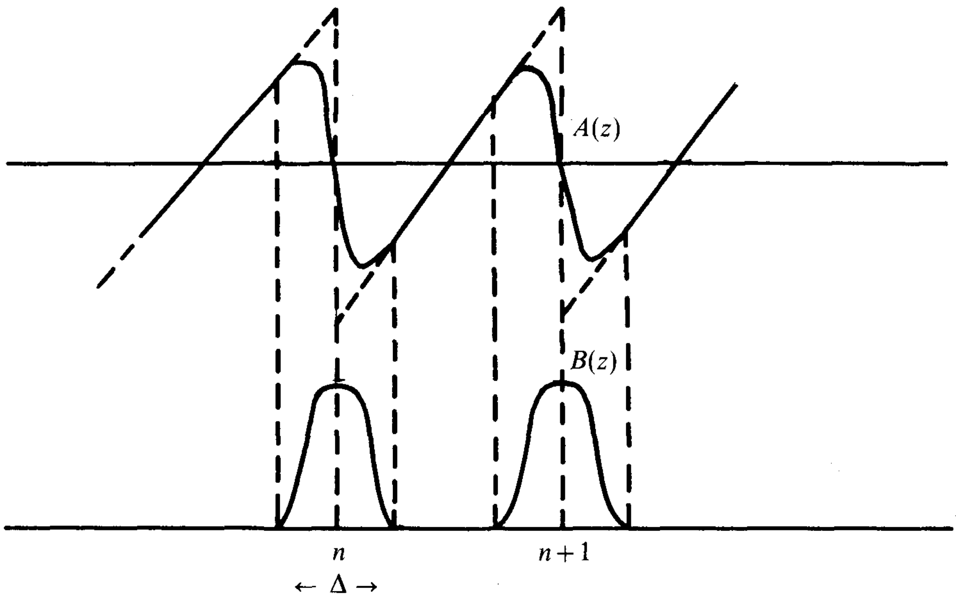
$$B(\xi) = \sum_{h \neq 0} \hat{B}(h)e(h\xi) + \Delta$$

with

$$\hat{A}(h), \hat{B}(h) \ll C_h = \min\left(\frac{1}{|h|}, \frac{1}{\Delta^3 h^4}\right),$$

and $e(z) = e^{2\pi i z}$.

Proof. Take functions $A(\xi)$ and $B(\xi)$ of class C^3 whose graphs are



and whose derivatives satisfy $A^{(p)}(\xi), B^{(p)}(\xi) \ll \Delta^{-p}$ for $p \leqslant 3$, (compare with Lemma 2 of [**3**]).

The next lemma is a consequence of Weil's estimate for Kloosterman's sums. The proof is similar to Lemma 3 of Hooley [**6**].

LEMMA 5. *Let* $0 < A_2 - A_1 \leqslant b$. *Then*

$$\sum_{\substack{A_1 < a \leqslant A_2 \\ (a,bc)=1 \\ a \equiv \lambda (\mathrm{mod}\, \Lambda)}} e\left(d\frac{\bar{a}}{b}\right) \ll (b,d)^{1/2} b^{1/2} \tau(bc) \log 2b.$$

The implied constant is absolute. The notation $\bar{a}$ *used when writing* $\bar{a}/b$ *or in a congruence* $(\mathrm{mod}\, b)$ *means that* $a\bar{a} \equiv 1 (\mathrm{mod}\, b)$.

LEMMA 6. *For any pair* a, b *of non-zero coprime integers,*

$$\frac{\bar{a}}{b} + \frac{\bar{b}}{a} \equiv \frac{1}{ab} (\mathrm{mod}\, 1).$$

LEMMA 7. *If* χ *is a non-principal character* $\mathrm{mod}\, d$, *and* $d \leqslant (\log \xi)^A$, *then*

$$\sum_{n \leqslant \xi} \chi(n) f_z(n) \ll \xi \exp\left(-(\log \xi)^{1/5}\right),$$

for any $A > 0$, *the implied constant depending only on* A.

Proof. By Buchstab's identity,

$$\sum_{n \leqslant \xi} \chi(n) f_z(n) = \sum_{n \leqslant \xi} \chi(n) - \sum_{p < z} \chi(p) \sum_{n \leqslant \xi/p} \chi(n) f_p(n)$$

$$\ll d + \sum_{p < z_1} \left| \sum_{n \leqslant \xi/p} \chi(n) f_p(n) \right| + \left| \sum_{z_1 \leqslant p < z} \chi(p) \sum_{n \leqslant \xi/p} \chi(n) f_p(n) \right|,$$

where $z_1 = \min(z, \exp\sqrt{\log \xi})$. Letting $R = \xi^{1/2}$ in the 'fundamental lemma' we obtain

$$\sum_{p < z_1} \left| \sum_{n \leqslant \xi/p} \chi(n) f_p(n) \right| = \sum_{p < z_1} \left| \sum_{l \, (\mathrm{mod}\, d)} \chi(l) \left[\frac{\xi}{pd} \prod_{\substack{p_1 < p \\ p_1 \nmid d}} \left(1 - \frac{1}{p_1}\right) \{1 + O(e^{-s_p})\} + O(R) \right] \right|$$

$$\ll \xi \exp\left(-\frac{\log R}{\log z_1}\right) + R d z_1 \ll \xi \exp\left(-\tfrac{1}{2}(\log \xi)^{1/2}\right),$$

where $s_p = \log R / \log p \geqslant \log R / \log z_1 \geqslant \frac{1}{2}(\log \xi)^{1/2}$. The second double sum is empty if $z \leqslant \exp(\sqrt{\log \xi})$; thus we assume that $z > \exp(\sqrt{\log \xi}) = z_1$, and we obtain

$$\sum_{z_1 \leqslant p < z} \chi(p) \sum_{n \leqslant \xi/p} \chi(n) f_p(n) = \sum_{n \leqslant \xi/z_1} \chi(n) \sum_{z_1 < p \leqslant \min(z, \xi/n, p(n))} \chi(p)$$

$$\ll \sum_{n \leqslant \xi/z_1} \frac{\xi}{n} \exp(-c\sqrt{\log z_1}) \ll \xi \exp\left(-(\log \xi)^{1/5}\right),$$

by the Siegel–Walfisz theorem. This completes the proof.

COROLLARY. *Under the same assumptions,*

$$\sum_{\substack{n < \xi \\ (n, e) = 1}} \chi(n) f_z(n) \ll \tau(e) \xi \exp\left(-(\log \xi)^{1/6}\right).$$

§4. *Reduction of the problem.* We split up the sum (3) into $\ll (\log x)^2$ sums of type

$$S(y, Q) = \sum_{\substack{Q < q \leqslant 2Q \\ (q, a) = 1}} \left| \sum_{\substack{y < n \leqslant 2y \\ n \equiv a \, (\mathrm{mod}\, q)}} f_z(n) - \frac{1}{\phi(q)} \sum_{\substack{y < n \leqslant 2y \\ (n, q) = 1}} f_z(n) \right|,$$

with $2y \leqslant x$ and $2Q \leqslant x^{11/21}$. It is sufficient to show that

$$S(y, Q) \ll x(\log x)^{-A-2}, \tag{5}$$

for $z \leqslant x^{1/883}$ and $1 \leqslant |a| \leqslant x$. We have trivially that

$$S(y, Q) \ll \sum_{Q < q \leqslant 2Q} \left(\frac{y}{\phi(q)} + 1\right) \ll y + Q,$$

so that (5) is obvious for $y \leqslant x(\log x)^{-A-2}$. In what follows we assume that

$$x(log\, x)^{-A-2} < y \leqslant x\,. \tag{6}$$

Now we want to rearrange $S(y, Q)$ as a sum of bilinear forms. For this, apply Lemma 1 twice to the characteristic function of the set of integers $n \in (y, 2y]$, $n \equiv a(\operatorname{mod} q)$ and to the characteristic function of the set of integers $n \in (y, 2y]$, $(n, q) = 1$. Then subtract $1/\phi(q)$ times the second inequality from the first, to obtain, as in the corollary to Lemma 1,

$$\left| \sum_{\substack{y < n \leqslant 2y \\ n \equiv a(\operatorname{mod} q)}} f_z(n) - \frac{1}{\phi(q)} \sum_{\substack{y < n \leqslant 2y \\ (n,q)=1}} f_z(n) \right|$$

$$\leqslant \sum_{\substack{d < D \\ d \mid P(z) \\ (d,q)=1}} \left| \sum_{\substack{y < n \leqslant 2y \\ n \equiv a(\operatorname{mod} q) \\ n \equiv 0(\operatorname{mod} d)}} 1 - \frac{1}{\phi(q)} \sum_{\substack{y < n \leqslant 2y \\ n \equiv 0(\operatorname{mod} d) \\ (n,q)=1}} 1 \right|$$

$$+ \sum_{p<z} \sum_{\substack{Dp^{-2} \leqslant d < Dp^{-1} \\ pd \mid P(z) \\ (pd,q)=1}} \left| \sum_{\substack{y < n \leqslant 2y \\ n \equiv a(\operatorname{mod} q) \\ n \equiv 0(\operatorname{mod} dp)}} f_p(n) - \frac{1}{\phi(q)} \sum_{\substack{y < n \leqslant 2y \\ n \equiv 0(\operatorname{mod} pd) \\ (n,q)=1}} f_p(n) \right|$$

$$= S_1(q, D) + \sum_{\substack{p<z \\ p \nmid q}} S_p(q, D)\,,$$

say. Hence, in the above notation

$$|S(y, Q)| \leqslant \sum_{\substack{Q < q \leqslant 2Q \\ (q,a)=1}} S_1(q, D) + \sum_{p<z} \sum_{\substack{Q < q \leqslant 2Q \\ (q,ap)=1}} S_p(q, D)$$

$$= S_1(y, D, Q) + \sum_{p<z} S_p(y, D, Q)\,, \tag{7}$$

say. The sums $S_p(y, D, Q)$ with $p < \min\left(z, \exp(\sqrt{\log x})\right) = z_0$, say, will be estimated easily by means of Lemmas 2 and 3 while those with larger p will be treated by a dispersion method.

§5. *Estimate of* $S_1(y, D, Q)$. For $(d, q) = 1$ we have

$$\sum_{\substack{y < n \leqslant 2y \\ n \equiv a(\operatorname{mod} q) \\ n \equiv 0(\operatorname{mod} d)}} 1 = \frac{y}{qd} + O(1)\,,$$

and, by Lemma 3,

$$\sum_{\substack{y < n \leqslant 2y \\ n \equiv 0(\operatorname{mod} d) \\ (n,q)=1}} 1 = \frac{\phi(q)}{q}\frac{y}{d} + O(\tau(q))\,.$$

Hence $S_1(q, D) \ll D$ and consequently

$$S_1(y, D, Q) \ll QD \ll x^{1-2\varepsilon}, \tag{8}$$

provided

$$QD \ll x^{1-2\varepsilon}, \tag{9}$$

which we henceforth assume.

§6. *Estimate of* $\sum_{p<z_0} S_p(y, D, Q)$. By Lemma 2, for each α with $(\alpha, q) = 1$, we have

$$\sum_{\substack{y < pmn \leqslant 2y \\ pmn \equiv \alpha \pmod q}} f_p(n) = \frac{y}{pmq} \prod_{\substack{p_1 < p \\ p_1 \nmid q}} \left(1 - \frac{1}{p_1}\right) \{1 + O(e^{-s_p})\} + O(R),$$

where R is any number $\geqslant p$ and $s_p = \log R/\log p$. Hence

$$S_p(q, D) \ll \sum_{Dp^{-2} \leqslant m \leqslant Dp^{-1}} \left(\frac{y}{pqm} e^{-s_p} + R\right) \ll \frac{y}{q} \frac{\log p}{p} e^{-s_p} + \frac{DR}{p}$$

and consequently

$$S_p(y, D, Q) \ll y \frac{\log p}{p} e^{-s_p} + \frac{DQR}{p}.$$

For $R = x^{\varepsilon/2}$ this bound yields

$$\sum_{p < z_0} S_p(y, D, Q) \ll x \exp\left(-(\log x)^{1/3}\right), \tag{10}$$

the implied constant depending on ε only.

Now we proceed to estimate $S_p(y, D, Q)$ with $z_0 \leqslant p < z$.

§7. *Rearrangement of* $S_p(y, D, Q)$. Let M take the values $2^{-\lambda} Dp^{-1}$ for $\lambda = 1, 2, \dots$ such that $Dp^{-2} \leqslant M < Dp^{-1}$, so that there are at most $2 \log p$ such M's. We split up $S_p(y, D, Q)$ into $\ll \log p$ sums of the type

$$E_p(y, M, Q) = \sum_{\substack{Q < q \leqslant 2Q \\ (q, ap) = 1}} \sum_{\substack{M < m \leqslant 2M \\ (m, q) = 1}} \left| \sum_{\substack{y < pmn \leqslant 2y \\ pmn \equiv a \pmod q \\ (p, n) = 1}} f_p(n) - \frac{1}{\phi(q)} \sum_{\substack{y < pmn \leqslant 2y \\ (n, pq) = 1}} f_p(n) \right|,$$

thus obtaining

$$S_p(y, D, Q) \leqslant \sum_M E_p(y, M, Q) + O(yp^{-2} \log x). \tag{11}$$

Here, the error term comes from the contribution of n's divisible by p^2. This error is admissible because

$$\sum_{z_0 \leqslant p < z} y p^{-2} \log x \ll y(\log x) \exp(-\sqrt{\log x}) .$$

§8. *The dispersion.* By the Cauchy–Schwarz inequality we obtain

$$E_p^2(y, M, Q) \leqslant MQD_p(M, Q),$$

where $D_p(M, Q)$, called the *dispersion*, stands for

$$D_p(M, Q) = \sum_{\substack{Q < q \leqslant 2Q \\ (q, ap) = 1}} \sum_{\substack{M < m \leqslant 2M \\ (m, q) = 1}} \Bigg(\sum_{\substack{y < pmn \leqslant 2y \\ pmn \equiv a (\operatorname{mod} q) \\ (n, p) = 1}} f_p(n) - \frac{1}{\phi(q)} \sum_{\substack{y < pmn \leqslant 2y \\ (n, pq) = 1}} f_p(n) \Bigg)^2$$

$$= W_p(M, Q) - 2V_p(M, Q) + U_p(M, Q),$$

say. Each term U_p, V_p and W_p will be evaluated separately. By definition,

$$U_p(M, Q) = \sum_{\substack{Q < q \leqslant 2Q \\ (q, ap) = 1}} \sum_{\substack{M < m \leqslant 2M \\ (m, q) = 1}} \Bigg(\frac{1}{\phi(q)} \sum_{\substack{y < pmn \leqslant 2y \\ (n, pq) = 1}} f_p(n) \Bigg)^2$$

§9. *Evaluation of $V_p(M, Q)$.* By definition,

$$V_p(M, Q) = \sum_{\substack{Q < q \leqslant 2Q \\ (q, ap) = 1}} \frac{1}{\phi(q)} \sum_{\substack{N/2 < n_1, n_2 < 2N \\ (n_1 n_2, pq) = 1 \\ 1/2 < n_1/n_2 < 2}} f_p(n_1) f_p(n_2) \sum_{\substack{M_1 < m \leqslant M_2 \\ m \equiv a\overline{pn_1} (\operatorname{mod} q)}} 1 ,$$

where for simplicity we have written $N = y/pM$, $M_1 = \max(M, y/(pn_1), y/(pn_2))$ and $M_2 = \min(2M, 2y/(pn_1), 2y/(pn_2))$. We carry out the summation over m first. Trivially we would take $(M_2 - M_1)q^{-1} + O(1)$ but this is useless because M is going to be smaller than Q. Therefore we are looking for an explicit formula for the error terms, with the expectation of obtaining substantial cancellations when summing them over q. We begin with

$$\sum_{\substack{M_1 < m \leqslant M_2 \\ m = a\overline{pn_1} (\operatorname{mod} q)}} 1 = \frac{M_2 - M_1}{q} + \psi\left(\frac{M_1 - a\overline{pn_1}}{q}\right) - \psi\left(\frac{M_2 - a\overline{pn_1}}{q}\right), \tag{12}$$

where $\psi(\theta) = \theta - [\theta] - \frac{1}{2}$. To arrive at $U_p(M, N)$ we replace $(M_2 - M_1)q^{-1}$, with the help of Lemma 3, by

$$\frac{M_2 - M_1}{q} = \frac{1}{\phi(q)} \sum_{\substack{M_1 < m \leqslant M_2 \\ (m, q) = 1}} 1 + O\left(\frac{\tau(q)}{\phi(q)}\right).$$

The first term above contributes to $V_p(M, Q)$ exactly $U_p(M, Q)$, while the error term

$O(\tau(q)/\phi(q))$ contributes

$$\ll \sum_{Q < q \leqslant 2Q} N^2 \frac{\tau(q)}{\phi^2(q)} \ll N^2 Q^{-1} \log Q .$$

Therefore we may write

$$V_p(M, Q) = U_p(M, Q) + \tilde{V}_p(M_1, Q) - \tilde{V}_p(M_2, Q) + O(N^2 Q^{-1} \log Q), \qquad (13)$$

where for $L = M_1$ or M_2 we define

$$\tilde{V}_p(L, Q) = \sum_{\substack{Q < q \leqslant 2Q \\ (q, ap) = 1}} \frac{1}{\phi(q)} \sum_{\substack{N/2 < n_1, n_2 < 2N \\ (n_1 n_2, pq) = 1 \\ 1/2 < n_1/n_2 < 2}} f_p(n_1) f_p(n_2) \psi\left(\frac{L - a\overline{pn_1}}{q}\right)$$

By Lemma 4 we approximate $\psi(\xi)$ by $A(\xi)$ with error $B(\xi)$ giving

$$\tilde{V}_p(L, Q) \ll \Delta N^2 + \sum_{h \neq 0} C_h \sum_{N/2 < n_1, n_2 < 2N} \left| \sum_{\substack{Q < q \leqslant 2Q \\ (q, apn_1 n_2) = 1}} \frac{1}{\phi(q)} e\left(h \frac{L - a\overline{pn_1}}{q}\right) \right| . \qquad (14)$$

We use $v \mid q^\infty$ to mean that each prime that divides v also divides q. Since $q/\phi(q) = \sum_{v \mid q^\infty} v^{-1}$, we obtain, by Lemmas 5 and 6 and by partial summation,

$$\sum_{\substack{Q < q \leqslant 2Q \\ (q, apn_1 n_2) = 1}} \frac{1}{\phi(q)} e\left(h \frac{L - a\overline{pn_1}}{q}\right) = \sum_{\substack{Q < q \leqslant 2Q \\ (q, apn_1 n_2) = 1}} \frac{1}{\phi(q)} e\left(h \frac{Lpn_1 - a}{qpn_1}\right) e\left(ah \frac{\bar{q}}{pn_1}\right)$$

$$\ll Q^{-1} \left(1 + \frac{|h| x}{pQN}\right) \sup_{Q < Q_1 \leqslant 2Q} \left| \sum_{\substack{Q < q \leqslant Q_1 \\ (q, apn_1 n_2) = 1}} \frac{q}{\phi(q)} l\left(ah \frac{\bar{q}}{pn_1}\right) \right|$$

$$\ll Q^{-1} \left(1 + \frac{|h| x}{pQN}\right) \sup_{Q < Q_1 \leqslant 2Q} \sum_{\substack{v \mid P(2Q)^\infty \\ (v, apn_1 n_2) = 1}} v^{-1} \left| \sum_{\substack{v^{-1} Q < r \leqslant v^{-1} Q_1 \\ (r, apn_1 n_2) = 1}} l\left(ah \frac{\overline{vr}}{pn_1}\right) \right|$$

$$\ll Q^{-1} \left(1 + \frac{|h| x}{pQN}\right) (ah, pn_1)^{1/2} (pN)^{1/2} x^\varepsilon .$$

Now, summing over $n_1, n_2 < 2N$ and $h \neq 0$ with weight C_h, (14) with $\Delta = MQ^{-1} x^{-2\varepsilon}$ yields

$$\tilde{V}_p(L, Q) \ll \Delta N^2 + Q^{-1} \left(1 + \frac{x}{\Delta pQN}\right) (a, p)^{1/2} (pN)^{1/2} N^2 x^{2\varepsilon}$$

$$\ll MN^2 Q^{-1} x^{-2\varepsilon} + (a, p)^{1/2} p^{1/2} N^{5/2} Q^{-1} x^{5\varepsilon} .$$

Hence relation (13) becomes

$$V_p(M,Q) = U_p(M,Q) + O(N^2Q^{-1}x^{\varepsilon}+MN^2Q^{-1}x^{-2\varepsilon}+(a,p)^{1/2}p^{1/2}N^{5/2}Q^{-1}x^{5\varepsilon}). \tag{15}$$

§10. *Rearrangement of* $W_p(M,Q)$. By definition,

$$W_p(M,Q) = \sum_{\substack{Q<q\leqslant 2Q\\(q,ap)=1}} \sum_{\substack{M<m\leqslant 2M\\(m,q)=1}} \sum_{\substack{(n_1n_2,p)=1\\ y/pm<n_1,n_2\leqslant 2y/pm\\ pmn_1\equiv pmn_2\equiv a(\mathrm{mod}\, q)}} f_p(n_1)f_p(n_2).$$

Here, if $(n_1,n_2)=d>1$ then $d\geqslant p$. Therefore, such terms contribute at most

$$O(MN^2Q^{-1}p^{-1}+MN\log x). \tag{16}$$

Now, let us consider $W_p^*(M,Q)$—the contribution to $W_p(M,Q)$ of terms with $(n_1,n_2)=1$. Notice that the range for n_1,n_2 is equal to

$$\mathscr{E}(q) = \{(n_1,n_2);(n_1n_2,pq)=1,\,(n_1,n_2)=1,\quad n_1\equiv n_2\,(\mathrm{mod}\, q), \frac{N}{2}<n_1,n_2<2N, \tfrac12<\frac{n_1}{n_2}<2\},$$

and for given q,n_1,n_2 the number of m's is given by (12) with the same notation for M_1 and M_2. We treat the main term $(M_2-M_1)/q$ as in Section 9. On replacing it by

$$\frac{1}{\phi(q)}\sum_{\substack{M_1<m\leqslant M_2\\(m,q)=1}} 1,$$

we make the total error $\ll N^2Q^{-1}\log Q$. Another error of order (16) is made when relaxing the condition $(n_1,n_2)=1$ in $\mathscr{E}(q)$. The latter operation is necessary to obtain

$$T_p(M,Q) = \sum_{\substack{Q<q\leqslant 2Q\\(q,ap)=1}} \frac{1}{\phi(q)} \sum_{\substack{l(\mathrm{mod}\, q)\\(l,q)=1}} \sum_{\substack{M<m\leqslant 2M\\(m,q)=1}} \Bigg(\sum_{\substack{y<pmn\leqslant 2y\\ pmn\equiv l(\mathrm{mod}\, q)\\(n,p)=1}} f_p(n)\Bigg)^2,$$

which we consider as a main term for $W_p(M,Q)$. It is clear, by the above discussion, that we obtain

$$W_p(M,Q) = T_p(M,Q)+2\tilde{W}_p(M_1,Q)-2\tilde{W}_p(M_2,Q) + O(MN^2Q^{-1}p^{-1}+N^2Q^{-1}\log Q+MN\log x),$$

where for $L = M_1$ or M_2 we define

$$\tilde{W}_p(L, Q) = \sum_{\substack{Q < q \leqslant 2Q \\ (q, ap) = 1}} \sum_{\substack{(n_1, n_2) \in \mathscr{E}(q) \\ n_1 > n_2}} f_p(n_1) f_p(n_2) \psi\left(\frac{L - a\bar{p}\bar{n}_2}{q}\right).$$

§11. *Rearrangement of* $\tilde{W}_p(L, Q)$. Now approximate $\psi(\xi)$ by $A(\xi)$, with error $B(\xi)$, and expand $A(\xi)$ and $B(\xi)$ into Fourier series (see Lemma 4) giving

$$\tilde{W}_p(L, Q) \ll \Delta N^2 \log N + \sum_{h=1}^{\infty} C_h |W_{p,h}(L, Q)| , \tag{17}$$

where

$$W_{p,h}(L, Q) = \sum_{\substack{Q < q \leqslant 2Q \\ (q, ap) = 1}} \sum_{\substack{(n_1, n_2) \in \mathscr{E}(q) \\ n_1 > n_2}} f_p(n_1) f_p(n_2) e\left(h \frac{L - a\bar{p}\bar{n}_2}{q}\right).$$

Replace the condition $(q, p) = 1$ by splitting up the summation over q into $p-1$ arithmetic progressions $\alpha(\bmod p)$, $1 \leqslant \alpha < p$, and detect $(q, a) = 1$ by the relation

$$\sum_{\substack{v | a \\ v | q}} \mu(v) = \begin{cases} 1, & \text{if } (q, a) = 1 , \\ 0, & \text{if } (q, a) > 1 , \end{cases}$$

to obtain

$$W_{p,h}(L, Q) = \sum_{1 \leqslant \alpha < p} \sum_{v | a} \mu(v) W_{p,h}^{\alpha, v}(L, Q) , \tag{18}$$

with

$$W_{p,h}^{\alpha, v}(L, Q) = \sum_{\substack{Q < q \leqslant 2Q \\ q \equiv \alpha (\bmod p) \\ q \equiv 0 (\bmod v)}} \sum_{\substack{(n_1, n_2) \in \mathscr{E}(q) \\ n_1 > n_2}} f_p(n_1) f_p(n_2) e\left(h \frac{L - a\bar{p}\bar{n}_2}{q}\right).$$

For $(n_1, n_2) \in \mathscr{E}(q)$, we reinterpret the condition $n_1 \equiv n_2 (\bmod q)$, by writing $n_1 - n_2 = qr$, so that $1 \leqslant r \leqslant NQ^{-1}$, $Qr < n_1 - n_2 \leqslant 2Qr$,

$$n_1 - n_2 \equiv \alpha r (\bmod pr),\ n_1 - n_2 \equiv 0 (\bmod vr) \quad \text{and } (n_1 n_2, pr) = 1 .$$

We may therefore write

$$W_{p,h}^{\alpha, v}(L, Q) = \sum_{1 \leqslant r \leqslant N/Q} \sum_{(n_1, n_2) \in \mathscr{F}(r)} f_p(n_1) f_p(n_2) e\left(h \frac{L - a\bar{p}\bar{n}_2}{q}\right), \tag{19}$$

where $q = (n_1 - n_2)/r$ and the range of summation in the inner sum is

$$\mathscr{F}(r) = \Big\{ (n_1, n_2) ; \frac{N}{2} < n_1, n_2 < 2N,\ n_2 < n_1 < 2n_2,$$

$$Qr < n_1 - n_2 \leqslant 2Qr,\ (n_1, n_2) = 1,$$

$$(n_1 n_2, prv) = 1,\ n_1 - n_2 \equiv \alpha r (\operatorname{mod} pr),$$

$$n_1 \equiv n_2 \ (\operatorname{mod} vr) \Big\}.$$

The variables n_2 and n_1 are of the same order of magnitude. Our intention is to spoil this 'symmetry' by an appeal to the combinatorial sieve identity. We apply Lemma 1, with some parameter $G > p$ in place of D, giving

$$\sum_{(n_1, n_2) \in \mathscr{F}(r)} f_p(n_1) f_p(n_2) e\left(h \frac{L - a\bar{p}\bar{n}_2}{q} \right)$$

$$\ll \sum_{g \leqslant G} \sum_{(n_2, prv) = 1} \left| \sum_{\substack{n_1 \equiv 0 (\operatorname{mod} g) \\ (n_1, n_2) \in \mathscr{F}(r)}} e\left(h \frac{L - a\bar{p}\bar{n}_2}{q} \right) \right|$$

$$+ \sum_{G/p \leqslant g \leqslant G} \sum_{(n_2, prv) = 1} \left| \sum_{\substack{n_1 \equiv 0 (\operatorname{mod} g) \\ (n_1, n_2) \in \mathscr{F}(r)}} f_{p(g)}(n_1) e\left(h \frac{L - a\bar{p}\bar{n}_1}{q} \right) \right|$$

$$= \sum_{g \leqslant G} \sum_{n_2} \left| \sum_{n_1} \right| + \sum_{G/p \leqslant g \leqslant G} \sum_{n_2} \left| \sum_{n_1} f \right|, \tag{20}$$

say.

§12. *Estimate of* $\sum_{g \leqslant G} \sum_{n_2} \left| \sum_{n_1} \right|$. Since $q \equiv \alpha (\operatorname{mod} p)$ and $q \equiv \bar{r} n_1 (\operatorname{mod} n_2)$ we obtain, by Lemma 6,

$$\frac{L - a\bar{p}\bar{n}_2}{q} \equiv \frac{Lpn_2 - a}{qpn_2} + a \frac{\bar{q}}{pn_2} \equiv \frac{r(Lpn_2 - a)}{(n_1 - n_2)pn_2} + a \frac{(r + \bar{\alpha} n_2)\bar{n}_1}{pn_2} \quad (\operatorname{mod} 1).$$

Insert this into the inner sum $\sum_{n_1}$ and remove the factor $e(hr(Lpn_2 - a)/pn_2(n_1 - n_2))$

by partial summation, to obtain

$$\sum_{n_1} = \sum_{\substack{n_1 \equiv 0 \pmod g \\ (n_1, n_2) \in \mathscr{F}(r)}} e\left(hr \frac{Lpn_2 - a}{pn_2(n_1 - n_2)}\right) e\left(ah \frac{(r + \bar{\alpha}n_2)\bar{n}_1}{pn_2}\right)$$

$$\ll \left(1 + \frac{hx}{pNQ}\right) \sup_{N/2 < N_1 < 2N} \left| \sum_{\substack{n_1 \leqslant N_1 \\ n_1 \equiv 0 \pmod g \\ (n_1, n_2) \in \mathscr{F}(r)}} e\left(ah \frac{(r + \bar{\alpha}n_2)\bar{n}_1}{pn_2}\right)\right|$$

$$\ll \left(1 + \frac{hx}{pNQ}\right)(ah, pn_2)^{1/2}(pN)^{1/2}x^{\varepsilon},$$

by Lemma 5. This yields

$$\sum_{g \leqslant G} \sum_{n_2} \left|\sum_{n_1}\right| \ll \left(1 + \frac{hx}{pNQ}\right)(ah, p)^{1/2} p^{1/2} \tau(h) G N^{3/2} x^{2\varepsilon}. \tag{21}$$

§13. *Estimate of* $\sum_{G/P \leqslant g \leqslant G} \sum_{n_2} \left|\sum_{n_1} f\right|$. Since $q \equiv \alpha \pmod p$ and $q \equiv -\bar{r}n_2 \pmod{n_1}$, Lemma 6 yields

$$\frac{L - a\overline{pn_1}}{q} \equiv \frac{Lpn_1 - a}{qpn_1} + a\frac{\bar{q}}{pn_1} \equiv \frac{(Lpn_1 - a)r}{pn_1(n_1 - n_2)} + a\frac{\alpha\bar{n}_1}{p} - ar\frac{\overline{n_2 p}}{n_1} \pmod 1.$$

Insert this into the inner sum $\sum_{n_1} f$ and remove the factor $e(hr(Lpn_1 - a)/pn_1(n_1 - n_2))$ by partial summation, to obtain

$$\sum_{n_2}\left|\sum_{n_1} f\right| \ll \left(1 + \frac{hx}{pNQ}\right) \sup_{N/2 < N_1 < 2N} \sum_{\substack{N/2 < n_2 < N \\ (n_2, prv) = 1}} \left| \sum_{\substack{n_1 \leqslant N_1 \\ n_1 \equiv 0 \pmod g \\ (n_1, n_2) \in \mathscr{F}(r)}} c(n_1) e\left(-ahr\frac{\overline{n_2 p}}{n_1}\right)\right|,$$

where $c(n_1) = f_{p(g)}(n_1) e\left(ah\frac{\alpha\bar{n}_1}{p}\right)$ is independent of n_2. By the Cauchy–Schwarz inequality,

$$\sum_{\substack{N/2 < n_2 < N \\ (n_2, prv) = 1}} \left| \sum_{\substack{n_1 \leqslant N_1 \\ n_1 \equiv 0 \pmod g \\ (n_1, n_2) \in \mathscr{F}(r)}} c(n_1) e\left(-ahr\frac{\overline{n_2 p}}{n_1}\right)\right|$$

$$\leqslant (2N)^{1/2} \left\{ \sum_{\substack{n', n'' \leqslant N_1 \\ n' \equiv n'' \equiv 0 \pmod g \\ (n'n'', p) = 1}} \left| \sum_{\substack{(n', n) \in \mathscr{F}(r) \\ (n'', n) \in \mathscr{F}(r)}} e\left(ahr\left(\frac{\overline{np}}{n'} - \frac{\overline{np}}{n''}\right)\right)\right| \right\}^{1/2}.$$

However,

$$\frac{\overline{np}}{n'} - \frac{\overline{np}}{n''} \equiv \frac{n'' - n'}{(n', n'')} \frac{\overline{np}}{[n', n'']} \pmod 1.$$

Hence, by Lemma 5,

$$\sum_{\substack{(n',n)\in\mathscr{F}(r)\\(n'',n)\in\mathscr{F}(r)}} e\left(ahr\left(\frac{\overline{np}}{n'}-\frac{\overline{np}}{n''}\right)\right) \ll (ah(n''-n'),n'n'')^{1/2}(n',n'')^{-1}Nx^{\varepsilon}.$$

Summation over n' and n'' yields

$$\sum_{\substack{n',n''<2N\\n'\equiv n''\equiv 0(\operatorname{mod} g)}} (ah(n''-n'),n'n'')^{1/2}(n',n'')^{-1}$$

$$\ll g^{-1}N+g^{-1/2}\sum_{l_1<l_2<2N/g}(ah(l_2-l_1),gl_1l_2)^{1/2}$$

$$\ll g^{-1}N+(ah,g)g^{-5/2}N^2x^{\varepsilon}.$$

Gathering all the above results together we finally obtain

$$\sum_{G/p<g\leqslant G}\sum_{n_2}\left|\sum_{n_1}f\right| \ll (G^{1/2}N^{3/2}+p^{1/4}G^{-1/4}N^2\tau(h))\left(1+\frac{hx}{pNQ}\right)x^{\varepsilon}. \qquad (22)$$

§14. *Estimate of* $\tilde{W}_p(L,Q)$. Collecting (17), (18), (19), (20), (21) and (22) we obtain

$$\tilde{W}_p(L,Q) \ll \Delta N^2\log N+\left(1+\frac{x}{pQN\Delta}\right)\frac{pN}{Q}((a,p)^{1/2}p^{1/2}GN^{3/2}+p^{1/4}G^{-1/4}N^2)x^{3\varepsilon}$$

$$\ll MN^2Q^{-1}x^{-2\varepsilon}+(a,p)^{1/10}p^{13/10}N^{29/10}Q^{-1}x^{7\varepsilon}, \qquad (23)$$

for $\Delta = MQ^{-1}x^{-3\varepsilon}$ and $G=(a,p)^{-2/5}p^{-1/5}N^{2/5}$.

§15. *Estimate of the dispersion* $D_p(M,Q)$. If we introduce results (15) and (23) into the definition of $D_p(M,Q)$ we obtain

$$D_p(M,Q) = X_p(M,Q)+R_p(M,Q),$$

where $X_p(M,Q)$ stands for the sum of the main terms, *i.e.*

$$X_p(M,Q) = T_p(M,Q)-2U_p(M,Q)+U_p(M,Q)$$

$$= \sum_{\substack{Q<q\leqslant 2Q\\(q,ap)=1}}\sum_{\substack{M<m\leqslant 2M\\(m,q)=1}}\frac{1}{\phi(q)}\sum_{\substack{l(\operatorname{mod} q)\\(l,q)=1}}\left(\sum_{\substack{y<pmn\leqslant 2y\\pmn\equiv l(\operatorname{mod} q)\\(n,p)=1}}f_p(n)-\frac{1}{\phi(q)}\sum_{\substack{y<pmn\leqslant 2y\\(n,pq)=1}}f_p(n)\right)^2,$$

and the error term $R_p(M,Q)$ is

$$R_p(M,Q) \ll MN^2Q^{-1}x^{-2\varepsilon}+N^2Q^{-1}x^{\varepsilon}+MN^2Q^{-1}p^{-1}$$

$$+MNx^{\varepsilon}+Qx^{\varepsilon}+(a,p)^{1/10}p^{13/10}N^{29/10}Q^{-1}x^{7\varepsilon}.$$

It remains to estimate $X_p(M, Q)$. For this, we appeal to the large sieve inequality (2). We first write

$$\sum_{\substack{l(\mathrm{mod}\, q)\\ (l,q)=1}} \Bigg(\sum_{\substack{y < pmn \leqslant 2y\\ pmn \equiv l(\mathrm{mod}\, q)\\ (n,p)=1}} f_p(n) - \frac{1}{\phi(q)} \sum_{\substack{y < pmn \leqslant 2y\\ (n,pq)=1}} f_p(n) \Bigg)^2 = \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \Bigg| \sum_{\substack{y < pmn \leqslant 2y\\ (n,pq)=1}} \chi(n) f_p(n) \Bigg|^2 .$$

Summing this over q and replacing each χ (mod q) by its induced primitive character $\chi^*(\mathrm{mod}\, d)$, $d \mid q$, we obtain

$$\sum_{Q < q \leqslant 2Q} \frac{1}{\phi^2(q)} \sum_{\chi \neq \chi_0} \Bigg| \sum_{\substack{y < pmn \leqslant 2y\\ (n,pq)=1}} \chi(n) f_p(n) \Bigg|^2$$

$$\ll Q^{-2}(\log Q)^2 \sum_{e < 2Q} \sum_{1 < d < 2Q/e} \sum_{\chi(\mathrm{mod}\, d)}{}^{*} \Bigg| \sum_{\substack{y < pmn \leqslant 2y\\ (n,ep)=1}} \chi(n) f_p(n) \Bigg|^2 .$$

If $Q/e \leqslant (\log x)^{2A+11} = H$, say, apply the Corollary to Lemma 7, and if $Q/e > H$ apply the large sieve inequality (2), to show that this expression is

$$\ll N^2 Q^{-1} H \exp\left(-\tfrac{1}{2}(\log x)^{1/6}\right) + \left(1 + \frac{N}{QH}\right) N (\log x)^2 ;$$

whence

$$X_p(M, Q) \ll MN^2 Q^{-1} (\log x)^{-2A-8} + MN(\log x)^2 ,$$

and finally

$$D_p(M, Q) \ll MN^2 Q^{-1}(\log x)^{-2A-8} + N^2 Q^{-1} x^{\varepsilon} + MNx^{\varepsilon} + Qx^{\varepsilon} + (a, p)^{1/10} p^{13/10} N^{29/10} Q^{-1} x^{7\varepsilon} . \qquad (24)$$

§16. *Conclusion.* By (11),

$$\sum_{z_0 \leqslant p < z} S_p(y, D, Q) \leqslant \sum_{z_0 \leqslant p < z} \sum_M E_p(y, M, Q) + O\left(y \exp\left(-\sqrt{\log x}\right)\right)$$

$$\ll \sum_{z_0 \leqslant p < z} \sum_M \left(QMD_p(M, Q)\right)^{1/2} + x \exp\left(-\sqrt{\log x}\right)$$

$$\ll \sum_{z_0 \leqslant p < z} \sum_M \{MN(\log x)^{-A-4} + M^{1/2} N x^{\varepsilon} + MN^{1/2} Q^{1/2} x^{\varepsilon} + M^{1/2} Q x^{\varepsilon} + (a, p)^{1/20} p^{13/20} M^{1/2} N^{29/20} x^{4\varepsilon}\} + x \exp\left(-\sqrt{\log x}\right)$$

$$\ll x(\log x)^{-A-2} + zD^{-1/2} x^{1+\varepsilon} + D^{1/2} Q^{1/2} x^{1/2+\varepsilon} + z^{1/2} D^{1/2} Q x^{\varepsilon} + z^{21/10} D^{-19/20} x^{29/20+5\varepsilon}$$

$$\ll x(\log x)^{-A-2} ,$$

on taking $D = x^{10/21-3\varepsilon}$ for $Q \leqslant x^{11/21}$ *and* $z \leqslant x^{1/883}$. Hence by (7), (8) and (10) we obtain (5). This completes the proof of the theorem.

Remark. The limit for the present method turns out to be $Q = x^{10/19-\eta}$ in which case $z \leqslant x^{\delta}$, with $\delta = \delta(\eta)$ a very small positive constant.

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Dr. E. Fouvry,
U.E.R. de Mathématiques et d'Informatique,
Université de Bordeaux I,
351, Cours de la Libération,
33405 Talence,
France.

Dr. H. Iwaniec,
U.E.R. de Mathématiques et d'Informatique,
Université de Bordeaux I,
351, Cours de la Libération,
33405 Talence,
France.

10H15: *NUMBER THEORY; Multiplicative theory; Distribution of primes and of integers with specified multiplicative properties.*

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