Fourier analysis over commutative algebraic groups and Frobenius distribution

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(Joint work with A. Forey & J. Frosán)
Outline

(1) A general equidistribution theorem

(2) Remarks

(3) A concrete example (lines on a smooth cubic threefold)

Notation: $k$ finite field, $\mathbb{K} \supset k \supset k_n \supset k$, $[k_n:k] = n$, $\ell$ prime invertible in $k$, $c : \overline{\mathbb{F}}_\ell \to C$
Theorem (Forey–Fresán–K.) \[ G/k \] connected commutative algebraic group

\[ \text{Ex. } \sum_{x \in X(k_n)} x(x) \cdot T(x) \cdot \ell_{x,h}(x) \]

\[ X_{\overline{k}} \subset G \text{ closed subvariety, irreducible, dim. } = d \]

\[ F \text{ lisse } \ell \text{-adic sheaf on } X, \text{ pure wt. } 0 \]

\[ \hat{G}(k_n) : \text{ character of } \hat{G}(k_n) \rightarrow \mathbb{C}^\times \iso \mathbb{C} \]

\[ S(F, \chi) = \frac{(-1)^d}{|k_n|^{d/2}} \sum_{x \in X(k_n)} x(x) \cdot t_f(x; k_n) \]

"Fourier transform"
Either $S(\mathcal{F}, x) = 0$ for "most" $x$

or there exist $n > 0$, $K \subseteq \mathcal{U}_r(\mathbb{C})$ compact subgroup.

such that for all $f: \mathbb{C} \longrightarrow \mathbb{C}$ continuous/bounded

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \in \mathbb{N}} \frac{1}{|\hat{G}(kn)|} \sum_{x \in \hat{G}(kn)} f(S(\mathcal{F}, x)) = \int_{K} f(Tr g) dg$$

(some times can be omitted)
Remarks:

(1) Version at the level of conjugacy classes in $K$ exists under suitable assumptions (on $G$, or on $X$).

(2) Previously known cases:
   (a) Deligne's equid. th. $\rightarrow \mathbb{G}_a$
   (b) Katz: $\mathbb{G}_m$ (2012)

(3) What is the link with Frobenius?
The Lang torsor construction associates to any \( x \in \mathcal{C}(\mathbb{A}) \) a sheaf \( L_x \) with trace function \( \chi(N_{k^n/k}(x)) \), so that

\[
S(F, \chi) = \frac{(-1)^d}{|k_n|^{d/2}} \sum_{x \in \mathcal{C}(k_n)} \text{Tr}(Fr_x | F \otimes L_x)
\]

(Trace formula)

\[
= \frac{(-1)^d}{|k_n|^{d/2}} \sum_{i=0}^{2d} (-1)^i \text{Tr}(Fr_{k_n} | H^i_c(\mathcal{O}_{k^n}, \mathcal{F}_{\mathfrak{O}}))
\]

Deligne's R.H.: eigenvalues of \( Fr_{k_n} \) have \( |\lambda| \leq |k_n|^{1/2} \)
so the normalization really only works if

\[
\begin{cases}
H_c^i (F \otimes L x) = 0 & \text{if } i \neq d \\
H_c^d (F \otimes L x) & \text{is "pure" of } \mu(1, d) \\
& |d| = |k_n|^{d/2}
\end{cases}
\]

When this occurs:

\[
S(F, x) = \frac{1}{|k_n|^{d/2}} \text{Tr} \left( Fr_{kn} \middle| H_c^d (F \otimes L x) \right)
\]

= trace of a unitary matrix
Tools of the proof

(1) Deligne's RH / formalism

(2) Tannakian formalism / convolution [Katz]

(3) Vanishing the. for cohomology
    $\rightarrow$ for most $x$, the previous result works
    [Gabber-Loeser, Krämer, Weissauer, ...]

(4) Sawin's Quantitative Sheaf Theory
A concrete example  

( inspired by Krümer over \( \mathbb{C} \) )

\( X_C \subset \mathbb{P}^k \) smooth cubic 3-fold

\( F \subset \text{Gr}(2,4) \) : lines in \( X \) [Fano; smooth irreducible surface]

\( A = \text{Alb}(F) \simeq \text{Pic}(F) \)

\( \simeq_c \) integral Jacobian

abelian of dim. 5

\[ i : F \hookrightarrow A \] [Beauville]

\( F \) : trivial sheaf on \( i(F) \)

\[ S(F, x) = \frac{1}{|nn|} \sum_{x \in F(n)} x(x) \]
"Th. " - The group $K$ in that case has connected component of exceptional type $E_6$.

Remark. $\dim E_6 = 78$ has a faithful rep. of $\dim 27$

Sketch:

Criterion - Let $K \subseteq U_{27}(\mathbb{C})$ be connected, semisimple compact group
such that

(1) $K$ is not self-dual

(2) $M_4(K) = \int_\mathcal{K} |Tr g|^4 dg = 3$

Then $K = E_6$ [compact].

The point is to check these; key is (2) which we do by computing

$$M_4(K) = \lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \frac{1}{|A(h_n)|} \sum |s(F, \chi)|^4$$

$\neq 3$