EXCLUDING CERTAIN BAD BEHAVIOR OF FOURIER COEFFICIENTS OF MODULAR FORMS

E. KOWALSKI

In his recent work on the Quantum Unique Ergodicity conjecture for quotients of the upper half-plane by $SL(2, \mathbf{Z})$ (or other congruence groups), R. Holowinsky encouters the following difficulty: for a Hecke-Maass cusp form f, with Laplace eigenvalue $\lambda_j \leqslant T$ and Hecke eigenvalues $\lambda_f(p)$, it might conceivably be the case that $|\lambda_f(p)|$ is very close to 1 for most primes $p \leqslant T^c$ for some constant $c > 0$. Such a phenomenon would render inoperant his estimates for shifted convolution sums of Fourier coefficients of modular forms.

In a similar way, in trying to generalize the lower bound of Garaev, Garcia and Konyagin for the number of distinct values of $\lambda_f(n)$, $n \leqslant T$, the possibility that the $\lambda_f(p)$ be all in geometric progression (or close to one) in such a range arises. The worst case there would be again if $\lambda_f(p) = 1$ for many primes.

It seems very difficult to prove that this type of bias does not occur in the current state of knowledge. In this note, we show at least that the worst case scenario (where $|\lambda_f(p)| = 1$ exactly) can not occur often if f is a holomorphic cusp form of even weight k. Unfortunately, the method (being algebraic) fails to say anything if equality is replaced with $|\lambda_f(p)|$ being very close to 1, and it doesn't work with forms of odd weight.[1]

To set up the notation, let $q \geqslant 1$ be an integer, $k \equiv 0 \,(\mathrm{mod}\, 2)$ an even integer $\geqslant 2$, $S_k(q)$ the vector space of cusp forms of weight k and level q and $S_k(q)^*$ the finite set of primitive forms (newforms) in $S_k(q)$. For $f \in S_k(q)^*$, let

$$f(z) = \sum_{n \geqslant 1} \lambda_f(n) n^{(k-1)/2} e(nz) = \sum_{n \geqslant 1} a_f(n) e(nz)$$

be its Fourier expansion at infinity. The normalized Hecke eigenvalues are $\lambda_f(n)$, and we prove:

Proposition 1. *Let k, q and f be as above. We have*

$$|\{p \textit{ prime} \mid |\lambda_f(p)| = 1\}| \leqslant \frac{\log(kq)}{\log 2} + C$$

for some absolute constant $C \geqslant 0$.

(Note that this is really taking into account all the primes, not just $p \leqslant T$ with T growing).

Proof. We will need some basic facts on Fourier coefficients of primitive forms, the first one of which is elementary while the others are essentially due to Shimura [1]:[2]
– The Fourier coefficients $a_f(n)$ are real numbers.
– The field

$$\mathbf{Q}_f = \mathbf{Q}(a_f(n))_{n \geqslant 1}$$

is a number field.

[1] The latter can probably give counterexamples since the Fourier coefficients at primes take only finitely many values.

[2] Page-numbered references to appear in the next version of this note.

– For any automorphism σ in the Galois group of $\bar{\mathbf{Q}}$ over $\mathbf{Q}$, the function

$$f^{\sigma}(z) = \sum_{n\geqslant 1} \sigma(a_f(n))e(nz)$$

is also an element of $S_k(q)^*$.

From the last two properties, we deduce first that

$$[\mathbf{Q}_f : \mathbf{Q}] \leqslant |S_k(q)^*| \leqslant \dim S_k(q).$$

Indeed, notice that we have $f^{\sigma} = f$ if and only if σ is in the subgroup of the Galois group of $\mathbf{Q}$ fixing $\mathbf{Q}_f$, so that the number of distinct conjugates f^{σ} is at most the index of this subgroup, or in other words the degree of the extension field $\mathbf{Q}_f$, while on the other hand there can be no more that $|S_k(q)^*|$ distinct conjugates by the third property.

Now since the Fourier coefficients are real numbers, we have $|\lambda_f(p)| = 1$ if and only if $a_f(p) = \pm p^{(k-1)/2}$. Since k is even, this implies in either case that $\sqrt{p} \in \mathbf{Q}_f$. Now for any distinct prime numbers $p_1 < p_2 < \cdots < p_d$, it is classical that

$$[\mathbf{Q}(\sqrt{p_1}, \ldots, \sqrt{p_d}) : \mathbf{Q}] = 2^d$$

(linear disjointness of quadratic extensions of $\mathbf{Q}$). Hence if those primes are such that $|\lambda_f(p_i)| = 1$, it follows that

$$2^d \leqslant [\mathbf{Q}_f : \mathbf{Q}] \leqslant \dim S_k(q).$$

Since moreover it is also well-known (see, e.g., [1,]) that

$$\dim S_k(q) \ll kq$$

for $k \geqslant 2$ and $q \geqslant 1$, with an absolute implied constant, we obtain the desired bound

$$d \leqslant \frac{\log kq}{\log 2} + O(1).$$

□

References

[1] G. Shimura: *Introduction to the arithmetic theory of automorphic functions*, Princeton Univ. Press, 1971.