EXAMPLES OF MOD-CAUCHY CONVERGENCE

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After mod-Gaussian [1] and mod-Poisson [3] convergence, it is natural to look for examples of similar limiting behavior involving other types of standard random variables. This note reports on three occurences of either mod-Cauchy convergence or of a slightly weaker notion. Two are related to arithmetics, and are re-interpretations of results of Vardi [6] and of Sarnak [4], and one is purely probabilistic, and is a re-interpretation of a result of Spitzer [5].

First, recall that a Cauchy variable with parameter $\gamma > 0$ is one with law given by

$$d\mu_\gamma = \frac{\gamma}{\pi}\frac{1}{\gamma^2 + x^2}dx,$$

and with characteristic function

$$\int_{\mathbf{R}} e^{itx} d\mu_\gamma(x) = e^{-\gamma|t|}, \qquad t \in \mathbf{R}.$$

The most natural definition of mod-Cauchy convergence would then be that (X_N) converges in mod-Cauchy sense with parameters (γ_N) and limiting function Φ if we have

$$\lim_{N\to+\infty} \exp(\gamma_N|t|)\boldsymbol{E}(e^{itX_N}) = \Phi(t)$$

and the limit is locally uniform in t (so Φ is continuous and $\Phi(0) = 1$). Let's say that we have *restricted* mod-Cauchy convergence if the limits above exist, locally uniformly, for $|t| < \delta$ for some $\delta > 0$. This restricted convergence is sufficient to ensure the following:

Fact. *If (X_N) converges in restricted mod-Cauchy sense with parameters (γ_N), then we have convergence in law*

$$\frac{X_N}{\gamma_N} \Longrightarrow \mu_1.$$

Here are first two examples of restricted mod-Cauchy convergence in number theory. Note that, although they seem to involve very different objects, they are in fact closely related through the way they are proved using spectral theory for certain differential operator involving complex multiplier systems on the modular surface $SL(2,\mathbf{Z})\backslash\mathbf{H}$.

Example 1 (Dedekind sums)**.** (See [6]) The *Dedekind sum* $s(d,c)$ is defined by

$$s(d,c) = \sum_{h=1}^{d-1}\Bigl(\Bigl(\frac{hd}{c}\Bigr)\Bigr)\Bigl(\Bigl(\frac{h}{c}\Bigr)\Bigr), \qquad ((x)) = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ x - \lfloor x\rfloor - 1/2, & \text{otherwise.}\end{cases}$$

for $1 \leqslant d < c$ integers with $(c,d) = 1$.

Vardi proved the existence of a renormalized Cauchy limit for $s(d,c)$: precisely, let

$$F_N = \{(c,d) \mid 1 \leqslant d < c < N, \quad (c,d) = 1\},$$

for $N \geqslant 1$, and give it the probability counting measure $\boldsymbol{P}_N$ and expectation denoted $\boldsymbol{E}_N(\cdot)$. For any $a < b$, we then have ([6, Theorem 1]) the limit

$$\lim_{N\to+\infty} \boldsymbol{P}_N\Bigl(a < \frac{s(d,c)}{(\log c)/(2\pi)} < b\Bigr) = \mu_1([a,b]).$$

Looking at Vardi's proof reveals that, in fact, he proves first (cf. [6, Prop. 1])

$$\lim_{N\to+\infty} \boldsymbol{P}_N\Bigl(a < \frac{s(d,c)}{(\log N)/(2\pi)} < b\Bigr) = \mu_1([a,b]),$$

and that the latter is obtained as consequence (using the Fact above) of a restricted mod-Cauchy convergence.

Theorem 1 (Vardi)**.** *Let D_N be the random variable defined on F_N by $(d,c)\mapsto s(d,c)$. Then, for any $\varepsilon>0$, we have*

$$\boldsymbol{E}_N(e^{itD_N}) = \exp(-\gamma_N|t|)\Phi(t) + O(N^{-2/3+\varepsilon})$$

uniformly for $|t|<2\pi$ where $\gamma_N=\frac{1}{2\pi}(\log N/4)$ and

$$\Phi(t) = (1-\tfrac{|t|}{4\pi})^{-1}\ \Bigl(\frac{3}{\pi}\int_{SL(2,\mathbf{Z})\backslash\mathbf{H}} (y|\eta(z)|^4)^{\frac{t}{2\pi}}\,\frac{dxdy}{y^2}\Bigr)^{-1}$$

the function $\eta(z)$ being the Dedekind eta function

$$\eta(z) = e^{i\pi z/12}\prod_{n\geqslant 1}(1-e^{2i\pi nz})$$

defined for $\mathrm{Im}(z)>0$.

Proof. This follows from [6, Prop. 2], after making minor notational adjustments. In particular: Vardi uses $2\pi r$ instead of t; the case $t=0$ is omitted in Vardi's statement, but it is trivial; only the case $0<r<1$ is mentioned, but there is a symmetry $r\leftrightarrow -r$ (see [6, p. 7]) that extends the result to $-1<r\leqslant 0$. □

Because

$$\exp(-|t|\gamma_N) = \Bigl(\frac{N}{4}\Bigr)^{-\frac{t}{2\pi}},$$

we see from the error term that the formula gives, in fact, only restricted convergence with a well-defined limit for $|t|<\frac{4\pi}{3}$. It is not clear on theoretical grounds whether this is optimal or not (note also the pole of the first factor of $\Phi(t)$ for $t=\pm 4\pi$), but the numerical experiments summarized in Figures 1 to 4, which illustrate the behavior of

$$\boldsymbol{E}_N(e^{itD_N})\exp(\gamma_N|t|)$$

for $N\leqslant 5000$ and $t\in\{\pi/2,\pi,2\pi,4\pi\}$, tend to indicate that there is no limit when t is large (note in particular the y-scale for the last picture).

Concerning the limiting function, recall that the measure

$$\frac{3}{\pi}\frac{dxdy}{y^2}$$

is a probability measure on the modular surface, so $\Phi(t)$ (surprisingly?) involves the inverse of a *Laplace transform* of the distribution function of $\log(y|\eta(z)|^4)$. It would be interesting to have some idea of the behavior of this limiting function.

Example 2 (Linking numbers of modular geodesics)**.** (See [4]). The second example looks very different, as it concerns issues of geometry and topology. More precisely, following Ghys, Sarnak considers the asymptotic behavior of a map

$$C\mapsto \mathrm{lk}(k_C),$$

where C runs over the set Π of prime closed geodedics in $SL(2,\mathbf{Z})\backslash\mathbf{H}$ and $\mathrm{lk}(k_C)$ is the linking number of a knot associated to C and the trefoil knot – the relation coming from an identification of the homogeneous space $SL(2,\mathbf{Z})\backslash SL(2,\mathbf{R})$ with the complement in $\mathbf{S}^3$ of the trefoil knot. This is also accessible more concretely by the classical identification of Π with the set of primitive (i.e.,

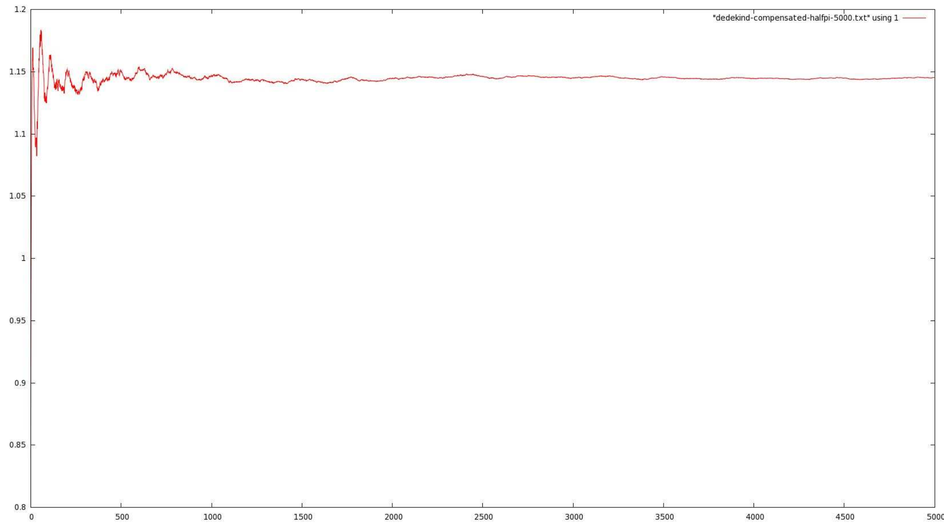

FIGURE 1. $t = \pi/2$

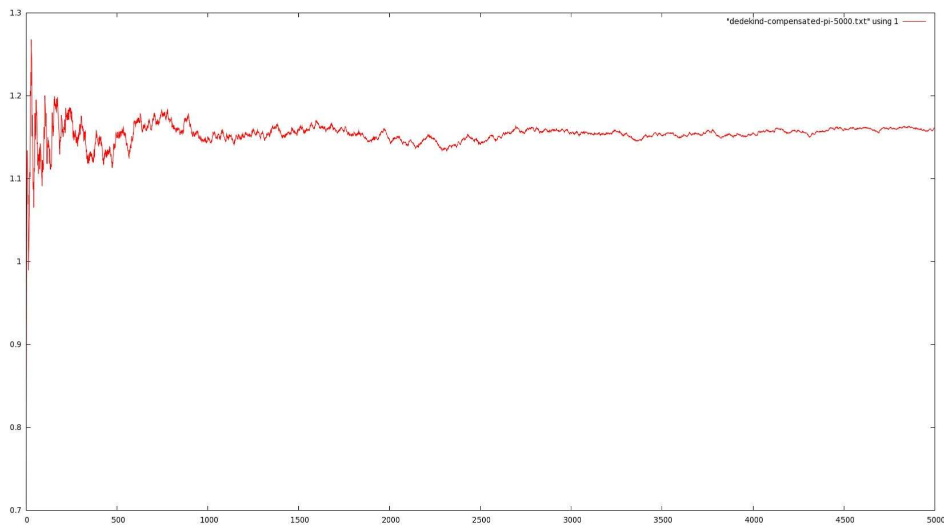

FIGURE 2. $t = \pi$

not of the form g^n, $n \geqslant 2$) hyperbolic (i.e., with $|\operatorname{Tr}(g)| > 2$) conjugacy classes in $SL(2, \mathbf{Z})$. In this identification $C \leftrightarrow g$, one has

$$\operatorname{lk}(k_C) = \psi(g),$$

where $\psi : PSL(2, \mathbf{Z}) \to \mathbf{Z}$ is a fairly classical map (called the Rademacher map), which is not a homomorphism but a "quasi-homomorphism". In turn, this ψ-function is related to the multiplier system for the η function.

Now, for $x > 0$, let

$$\Pi_x = \{g \in \Pi \mid N(g) \leqslant x\}$$

where the "norm" $N(g)$ is defined and related to the length $\ell(g)$ of the closed geodesic by

$$N(g) = \Big(\frac{\operatorname{Tr}(g) + \sqrt{\operatorname{Tr}(g)^2 - 4}}{2}\Big)^2, \qquad \ell(g) = \log N(g).$$

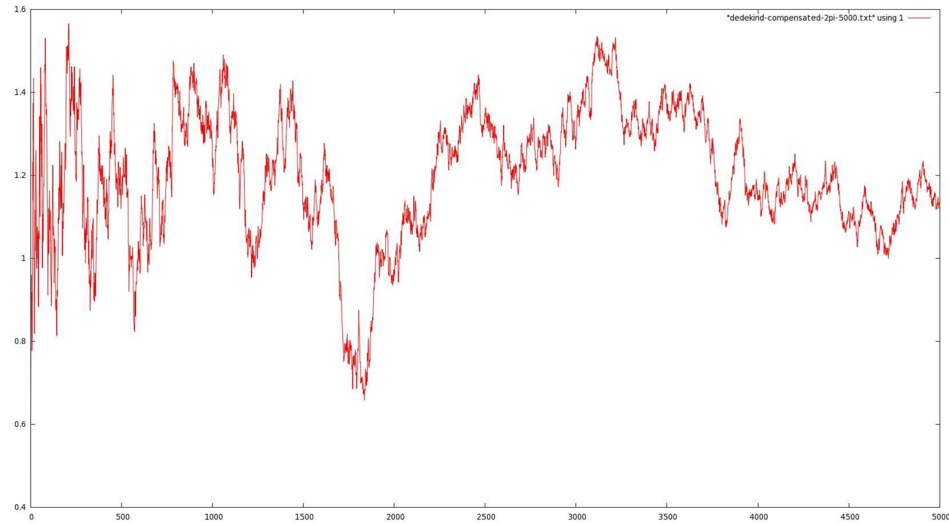

FIGURE 3. $t = 2\pi$

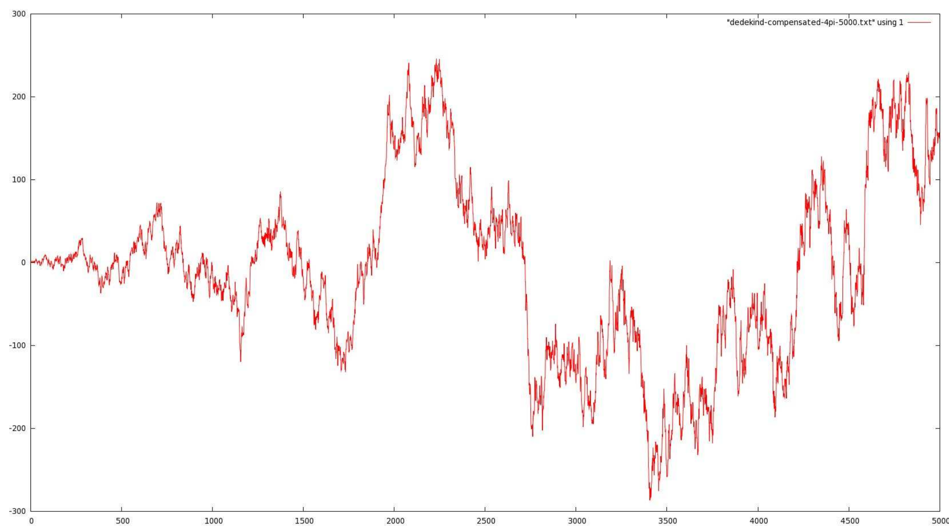

FIGURE 4. $t = 4\pi$

Let $\boldsymbol{P}_x$ denote the probability measure where each $g \in \Pi_x$ has weight proportional to $\ell(g)$; the normalizing factor to ensure that it is a probability measure is

$$\sum_{N(g) \leqslant x} \log N(g) \sim x$$

as $x \to +\infty$, by Selberg's Prime Geodesic Theorem (this can be made much more precise). Let $\boldsymbol{E}_x$ denote the corresponding expectation operator.

Sarnak [4, Theorem 3] proves a limiting Cauchy behavior:

$$\lim_{x \to +\infty} \boldsymbol{P}_x \Big(a < \frac{\mathrm{lk}(g)}{\ell(g)} < b \Big) = \mu_1([a, b])$$

for any $a < b$.

Again, if one looks at the proof, one sees that this is deduced from:

Theorem 2 (Sarnak)**.** *Let* lk_x *denote the random variable* $g \mapsto \mathrm{lk}(g) = \psi(g)$ *on* Π_x*. Then for* $|t| \leqslant \pi/12$*, we have*

$$\boldsymbol{E}_x(e^{it\mathrm{lk}_x}) = \exp(-|t|\gamma_x)\Phi_1(t) + O(x^{3/4})$$

where $\gamma_x = \frac{3}{\pi}(\log x)$ *and* $\Phi_1(t) = \frac{1}{1-\frac{3|t|}{\pi}}$*.*

Proof. Again, up to notational changes, this is given by [4, (16)] since the quantity $v_r(\gamma)$ there is given by

$$v_r(g) = e^{i\pi r\psi(g)/6}$$

for $g \in \Pi$ and $r \in \mathbf{R}$. So the r in loc. cit. is given by $r = 6t/\pi$ to recover our formulation. □

This is again an example of restricted mod-Cauchy convergence. Again, I do not know how far the restriction on t is necessary. One may of course perform a summation by parts to remove the weight $\log N(g) = \ell(g)$ from these results, if desired.

Here is now a probabilistic example of (unrestricted) mod-Cauchy convergence.

Example 3 (Argument of complex Brownian motion)**.** (See [5, §2]) Let $(B(t))_{t\geqslant 0}$ be a *complex* Brownian motion with starting point $B(0) = R > 0$ for some (fixed) real number R (recall that this means that

$$B(t) = B_1(t) + iB_2(t)$$

where B_1 and B_2 are independent real-valued Brownian motions, with $B_1(0) = R$, $B_2(0) = 0$; for an accessible introduction to Brownian motion, see e.g. [2]). Consider the *winding number*

$$\theta_R(t) = \mathrm{Im}\log B(t) \in \mathbf{R},$$

for $t \geqslant 0$, which is almost surely well-defined because $\boldsymbol{P}(B(t) = 0) = 0$ for any t. This is a (non-Markovian) continuous stochastic process. Spitzer [5, Th. 3] proves the convergence in law

$$\lim_{t\to+\infty} \boldsymbol{P}\Big(\frac{\theta_R(t)}{\frac{1}{2}\log t} \in [a,b]\Big) = \mu_1([a,b]),$$

but in fact this follows from a mod-Cauchy convergence result:

Theorem 3 (Spitzer)**.** *Let* $\theta_R(t)$ *be as above. For* $u \in \mathbf{R}$*, we have*

$$\boldsymbol{E}(e^{iu\theta_R(t)}) \sim \exp(-\gamma_t|u|)\Phi_2(u)$$

as $t \to +\infty$*, where* $\gamma_t = \log \frac{\sqrt{8t}}{R}$ *and*

$$\Phi_2(u) = \frac{\Gamma\big(\frac{1}{2}\big)}{\Gamma\Big(\frac{|u|+1}{2}\Big)}.$$

Proof. Spitzer [5, (2.10)] gives in fact the exact formula

$$\boldsymbol{E}(e^{iu\theta_R(t)}) = \sqrt{\pi}\frac{R}{\sqrt{8t}}\exp\Big(-\frac{R^2}{4t}\Big)\Big\{I_{(u-1)/2}\Big(\frac{R^2}{4t}\Big) + I_{(u+1)/2}\Big(\frac{R^2}{4t}\Big)\Big\}$$

for $u \geqslant 0$, and $\boldsymbol{E}(e^{iu\theta_R(t)}) = \boldsymbol{E}(e^{-iu\theta_R(t)})$, where $I_\nu(z)$ denotes the I-Bessel function, which can be described by its Taylor expansion

$$I_\nu(z) = \sum_{k\geqslant 0} \frac{1}{k!\Gamma(\nu+k+1)}\Big(\frac{z}{2}\Big)^{\nu+2k}.$$

In particular, we have

$$I_\nu(z) = \frac{1}{\Gamma(\nu+1)}\Big(\frac{z}{2}\Big)^{\nu} + O(z^{\nu+2}),$$

for $|z| \leqslant 1$, and therefore we deduce that

$$\boldsymbol{E}(e^{iu\theta_R(t)}) \sim \Bigl(\frac{R}{\sqrt{8t}}\Bigr)^{|u|} \frac{\Gamma(1/2)}{\Gamma((|u|+1)/2)}$$

for fixed $u \geqslant 0$ as $t \to +\infty$, which is exactly the result we claimed. □

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