The second moment theory of families of L-functions

The case of twisted Hecke L-functions

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Abstract

For a fairly general family of L-functions, we survey the known consequences of the existence of asymptotic formulas with power-saving error term for the (twisted) first and second moments of the central values in the family.

We then consider in detail the important special case of the family of twists of a fixed cusp form by primitive Dirichlet characters modulo a prime q, and prove that it satisfies such formulas. We derive arithmetic consequences:

- a positive proportion of central values $L(f \otimes \chi, 1/2)$ are non-zero, and indeed bounded from below;
- there exist many characters χ for which the central L-value is very large;
- the probability of a large analytic rank decays exponentially fast.

We finally show how the second moment estimate establishes a special case of a conjecture of Mazur and Rubin concerning the distribution of modular symbols.

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CHAPTER 1

The second moment theory of families of L-functions

1.1. General introduction

1.1.1. Families and moments. In the analytic theory of automorphic forms, many problems are out of reach, or make little sense, when specialized to single L-functions or modular forms. It has therefore been a very common theme of research to study *families* of L-functions, and to search for *statistical results* on average over families. This point of view has led to numerous insights. In fact, it also sometimes provides a viable approach to questions for individual objects, as in most works concerning the subconvexity problem for L-functions. An excellent survey of this point of view is that of Iwaniec and Sarnak [38].

When studying L-functions on average, it has emerged from a series of works in the last ten to fifteen years that a remarkable array of results can be obtained as soon as one has sufficiently strong information concerning the first and especially the *second* moment of the values of the L-functions on the critical line. More precisely, what is often the crucial input needed is "a bit more" than the second moment, which is most easily captured in practice by an asymptotic formula *with power saving* for the second moment, together with some basic information for individual L-functions (such as versions of the Prime Number Theorem, sometimes for auxiliary L-functions, or bounds for averages or mean-square averages of coefficients). This phenomenon is of course consistent with probabilistic intuition: recall for instance that the Law of Large Numbers only requires the first moment to exist, and that the Central Limit Theorem only depends on the second moment.

In this section, we will explain the basic principle and describe some of its applications in a fairly general and informal setting. In the next sections, we will introduce the particular family that will be the focus of the remainder of this book, and we will state the precise new results that we have obtained in that case.

Let $d \geqslant 1$ be an integer. We interpret here a *family* of cusp forms of rank $d \geqslant 1$ as the data, for any integer N, of a finite set $\mathcal{F}_N$ of cusp forms (cuspidal automorphic representations) on GL_d (over $\mathbf{Q}$ for simplicity). Given such a family $\mathcal{F}$, we obtain probability and average operators

$$\mathbf{P}_N(f \in A) = \frac{1}{|\mathcal{F}_N|} \sum_{\substack{f \in \mathcal{F}_N \\ f \in A}} 1 \qquad \mathbf{E}_N(T(f)) = \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} T(f)$$

for any N such that $\mathcal{F}_N$ is not empty. Here A is any subset of cusp forms on GL_d, and T is any complex-valued function defined on the set of cusp forms on GL_d. We will sometimes informally write $f \in \mathcal{F}$ to say that $f \in \mathcal{F}_N$ for some N (which might not be unique).

We also require that the size of $\mathcal{F}_N$ and the analytic conductors $q(f)$ of the cusp forms $f \in \mathcal{F}_N$ grow with N in a nice way, say $|\mathcal{F}_N| \asymp N^{\alpha}$ and $q(f) \asymp N^{\beta}$ for some $\alpha > 0$ and $\beta > 0$.

The basic invariants are the standard (Godement-Jacquet) L-functions

$$L(f,s) = \sum_{n \geqslant 1} \lambda_f(n) n^{-s}$$

associated to a cusp form $f \in \mathcal{F}$, that we always normalize in this book so that the center of the critical strip is $s = \frac{1}{2}$. These are indeed often so important that one speaks of *families of L-functions* instead of families of cusp forms.

For any *reasonable* family of L-functions, one can make precise conjectures for the asymptotic behavior as $N \to +\infty$ of the complex moments

$$\mathbf{E}_N\Bigl(L(f,\sigma+it)^k \overline{L(f,\sigma+it)}^l\Bigr),$$

at any point $\sigma+it \in \mathbf{C}$, where k and l are non-negative integers. Here, "reasonable" has no precise generally accepted formal definition. A minimal requirement is that the family should satisfy some form of "local spectral equidistribution" (see [47]), which means that for any fixed prime p, the local component at p of the cusp forms $f \in \mathcal{F}_N$ should become equidistributed with respect to some measure μ_p as $N \to +\infty$ in the unitary spectrum $\widehat{\mathrm{GL}}_d(\mathbf{Q}_p)$ of $\mathrm{GL}_d(\mathbf{Q}_p)$. Such a statement is more or less required to express for instance the arithmetic component of the leading term of the asymptotic of the moments at $s = \frac{1}{2}$.

Indeed, following the work of Keating and Snaith [42] and the ideas of Katz and Sarnak [41], for any integer $k \geqslant 0$, one expects an asymptotic formula of the type

$$\mathbf{E}_N\Bigl(|L(f,\tfrac{1}{2})|^{2k}\Bigr) \sim a_k g_k (\log N)^{c_k} \tag{1.1}$$

as $N \to +\infty$, where a_k is an arithmetic factor, whereas g_k and c_k are real numbers that depend only on the so-called "symmetry type" of the family, and have an interpretation in terms of Random Matrix Theory. In general, these invariants can be predicted, based on the local spectral equidistribution properties of the family. More precisely, one can often deduce the "symmetry type", in the sense of Katz-Sarnak, from the limiting behavior of the measures μ_p as $p \to +\infty$ (see [47, §9, §10], but note that this line of reasoning wouldn't always work in the case of "algebraic" families [73]). From this symmetry type, which is either unitary, symplectic, or orthogonal (with some variants in the orthogonal case related to root numbers), one can predict the values of g_k and c_k. These are related to the asymptotic beahvior of the moments of the value of the characteristic polynomials of random matrices in families of compact Lie groups of unitary, or symplectic or orthogonal matrices, in the limit when the size of the matrices increases. For instance, in a unitary family, we have $c_k = k^2$ and

$$g_k = \frac{G(1+k)}{G(1+2k)} = \prod_{j=0}^{k} \frac{j!}{(j+k)!}$$

where G is the Barnes function.

Using this information, a_k is an Euler product given by

$$a_k = \prod_p (1-p^{-1})^{c_k} \int |L_p(\pi, \tfrac{1}{2})|^{2k}\ d\mu_p(\pi)$$

where the integral is over the unitary spectrum of $\mathrm{GL}_d(\mathbf{Q}_p)$ and L_p is the local L-factor at p. (The value of c_k is exactly such that the Euler product converges).

Remark 1.1. Families can also be defined with non-uniform weights over finite sets, instead of the uniform measure, or can also be continuous families with a finite probability measure (typically to consider L-functions in the t-aspect on the critical line). The sets $\mathcal{F}_N$ might also be defined only for a subset of the integers N. This does not affect the general discussion. We will see certain variants of this type in the examples below.

For orientation, here are some examples of families that have been studied extensively, and that we will refer to in the list of applications below. One is given for each of the three basic symmetry types.

Example 1.2. (1) For $N \geqslant 1$, let $\mathcal{D}_N$ be the finite set of primitive Dirichlet characters modulo $q \leqslant N$. This is a unitary family.

(2) For $q \geqslant 17$ prime, let $\mathcal{C}_q$ be the finite set of primitive weight 2 cusp forms of level q. This family has particularly nice arithmetic applications, because the Eichler–Shimura formula implies that

$$\prod_{f \in \mathcal{F}_q} L(f, s)$$

is the (normalized) Hasse-Weil L-function of the jacobian $J_0(q)$ of the modular curve $X_0(q)$. This illustrates one way in which average studies of families of L-functions may have consequences for a single arithmetic object of natural interest. The family $\mathcal{C}_q$ is of orthogonal type. It is often of interest to restrict to cusp forms where $L(f,s)$ has a given root number 1 or -1, which would split into even and odd orthogonal types.

(3) For $N \geqslant 2$, let $\mathcal{Q}_N$ be the finite set of primitive real Dirichlet characters modulo $q \leqslant N$. This is a family of symplectic type.

(4) Finally, there have been a number of important works recently that show that many natural families of cusp forms on GL_d, or other groups, satisfy the basic local spectral equidistribution properties (see for instance the work of Shin and Templier [77] and surveys by Sarnak–Shin–Templier [73] and Matz [56]).

The assumption of the *second moment theory* of a family of L-functions is that the expected asymptotic formula holds for the first and second moments on the critical line, with a power-saving in the error terms with respect to N, and polynomial dependency with respect to the imaginary part. More precisely, we assume that there exists $\delta > 0$ and $A \geqslant 0$ such that

$$\mathbf{E}_N\Big(L(f, \tfrac{1}{2}+it)\Big) = \mathrm{MT}_1(N;t) + O((1+|t|)^A N^{-\delta}), \tag{1.2}$$

and

$$\mathbf{E}_N\Big(|L(f, \tfrac{1}{2}+it)|^2\Big) = \mathrm{MT}_2(N;t) + O((1+|t|)^A N^{-\delta}), \tag{1.3}$$

for $N \geqslant 1$ and $t \in \mathbf{R}$. The main terms are polynomials of some fixed degree in $\log N$, as also predicted by the precise forms of the moment conjectures (due to

Conrey, Farmer, Keating, Rubinstein and Snaith [12]). In fact, it is not required in practice to know that the main terms exactly fit the moment conjectures, provided they are given in sufficiently manageable form for the computations that will follow (the degree of the polynomial in $\log N$ is of crucial importance).

As we hinted at above when saying that one needs "a bit more", these estimates are in fact intermediate steps. The really crucial point is that, if they can be proved with almost any of the currently known techniques, then it is also possible to improve them to derive asymptotic formulas for the first and second moments *twisted* by the coefficients $\lambda_f(\ell)$ of the L-functions, namely

$$\mathbf{E}_N\Bigl(\lambda_f(\ell)L(f,\tfrac{1}{2}+it)\Bigr) = \mathrm{MT}_1(N;t,\ell) + O((1+|t|)^A L^B N^{-\delta}), \tag{1.4}$$

and

$$\mathbf{E}_N\Bigl(\lambda_f(\ell)|L(f,\tfrac{1}{2}+it)|^2\Bigr) = \mathrm{MT}_2(N;t,\ell) + O((1+|t|)^A L^B N^{-\delta}), \tag{1.5}$$

where $1 \leqslant \ell \leqslant L$ is an integer (maybe with some restrictions) and $B \geqslant 0$.

The consequences that follow from such asymptotic formulas are remarkably varied. We will now discuss some of them, with references to cases where the corresponding results have been established. The discussion is still informal. The ordering goes (roughly and not systematically) in increasing order of the amount of information required of the moments. We will make no attempt to be exhaustive.

1.1.2. Universality outside the critical line. One can generalize Bagchi's version of Voronin's Universality Theorem to establish a functional limit theorem for the distribution of the holomorphic functions $L(f,s)$ restricted to a fixed suitable compact subset D of the strip $0 < \mathfrak{Re}(s) < 1$ (see [81] for Voronin's original paper and [2] for Bagchi's probabilistic interpretation). This result is much softer than those that follow. It first requires an upper-bound of the right order of magnitude (with respect to N) of the untwisted second moment, which is used to get an upper bound for

$$\mathbf{E}_N\Bigl(|L(f,\tfrac{1}{2}+it)|\Bigr)$$

using the Cauchy-Schwarz inequality. Using this (and local spectral equidistribution), one proves a form of equidistribution of $L(f,s)$ restricted to D in a space of holomorphic functions on D. Then some form of the Prime Number Theorem (for an auxiliary L-function) is required to compute the support of the random holomorphic function that appeared in the first step, in order to deduce the universality statement.

For instance, in the case of the family $\mathcal{C}$ above, it is proved in [48] that the L-functions become distributed like the random Euler products

$$\prod_p \det(1 - X_p p^{-s})^{-1}$$

where (X_p) is a sequence of independent random variables that have the Sato-Tate distribution. The support of this random Euler product is (for D a small disc centered on the real axis and contained in the interior of the strip $\frac{1}{2} < \mathfrak{Re}(s) < 1$) the set of non-vanishing holomorphic functions φ on D, continuous on the boundary, that satisfy the real condition $\varphi(\bar{s}) = \overline{\varphi(s)}$.

1.1.3. Upper and lower bounds for integral moments. For a family $\mathcal{F}$ with a given symmetry type (in the Katz-Sarnak sense described above), the asymptotic formula from the moment conjectures (1.1) imply that the order of magnitude of $\mathbf{E}_N(|L(f,\frac{1}{2})|^{2k})$ should be $(\log N)^{c_k}$ for some constant c_k depending only on the symmetry type, with $c_k = k^2$ if the family is unitary, for instance. Although the asymptotic remains very mysterious, the order of magnitude is much better understood.

First, there exists a robust method due to Rudnick and Soundararajan [71] to derive *lower bounds* of the right form. We illustrate it in the case of a unitary family. The method involves evaluating the two averages

$$S_1 = \mathbf{E}_N\Big(L(f,\tfrac{1}{2})A(f)^{k-1}\overline{A(f)^k}\Big), \qquad S_2 = \mathbf{E}_N\Big(|A(f)|^{2k}\Big),$$

where

$$A(f) = \sum_{n\leqslant L}\frac{\lambda_f(n)}{\sqrt{n}}$$

for some parameter L. Hölder's inequality gives the lower bound

$$\mathbf{E}_N\Big(|L(f,\tfrac{1}{2})|^{2k}\Big) \geqslant \frac{|S_1|^{2k}}{S_2^{2k-1}},$$

and hence we obtain the desired lower bounds if we can prove that

$$S_2 \ll (\log N)^{k^2} \ll S_1.$$

After expanding the value of $A(f)$, and using multiplicativity, we see that S_1 is a combination of twisted first moments involving integers $\ell \leqslant L^{2k-1}$. It is therefore to be expected that we can evaluate S_1, provided we have an asymptotic formula for the twisted first moments (1.4) valid for the corresponding values of ℓ. We can expect to evaluate the first moment in such a range only when the "pure" first moment has an asymptotic formula with power saving. The evaluation of S_2 is, in principle, simpler. It can be expected (and turns out to be true when the method is applicable) that one requires L to be comparable to the conductor N^α in logarithmic scale for the bounds above to hold.

There is no corresponding unconditional upper bound. However, Soundararajan [78] devised a method to obtain almost sharp upper bounds when one assumes that the L-functions in the family $\mathcal{F}$ satisfy the Riemann Hypothesis (i.e., all zeros of $L(f,s)$ with positive real part have real part 1/2). Precisely, he obtained results like

$$\mathbf{E}_N\Big(|L(f,\tfrac{1}{2})|^{2k}\Big) \ll (\log N)^{c_k+\varepsilon},$$

for any $\varepsilon > 0$ for some important families (or the analogue for the k-th moment in symplectic and orthogonal families). His approach was refined by Harper [29], who obtained the upper-bound $(\log N)^{c_k}$ (still under the Riemann Hypothesis for the L-functions). We refer to the introductions to both papers for a description of the ideas involved.

1.1.4. Proportion of non-vanishing. Because of the Riemann Hypothesis, the problem of the location of zeros of L-functions is especially important. In particular, much interest has been concentrated on the special point $s = \frac{1}{2}$. This is obviously natural in families where the order of vanishing at this point has some arithmetic interpretation. This is the case, for instance, in the family $\mathcal{C}$ of cusp forms of weight 2: indeed, for any $f \in \mathcal{C}$, Shimura has constructed an abelian variety A_f over $\mathbf{Q}$, of dimension equal to the degree of the field generated by the coefficients $\sqrt{p}\lambda_f(p)$ for p prime, such that $L(f,s)$ is the Hasse-Weil L-function of A_f; then the Birch and Swinnerton-Dyer conjecture predicts that the order of vanishing of f at $\frac{1}{2}$ should be equal to the rank of the group $A_f(\mathbf{Q})$. However, there are also other applications to an understanding of the behavior of central values (see the highly influential study of Landau-Siegel zeros by Iwaniec and Sarnak [37, 38]).

If the L-function $L(f,s)$ is self-dual, and the sign of its functional equation of $L(f,s)$ is -1, then we get $L(f,\frac{1}{2}) = 0$ trivially. One may expect conversely that few L-functions satisfy $L(f,\frac{1}{2}) = 0$ otherwise (some do have this property, but they are not easy to come by; see, e.g., [36, Ch. 22–23] for an account of the construction of a single such L-function by Gross and Zagier, and how it completed Goldfeld's effective lower-bound for class numbers of imaginary quadratic fields).

Using ideas reminiscent of Markov's inequality in probability theory, one can obtain rather good information on the proportion of non-vanishing of central values. The basic observation is that, assuming asymptotic formulas (1.2) and (1.3), a simple application of the Cauchy-Schwarz inequality (or of Markov's inequality), leads to the lower bound

$$\mathbf{P}_N\Big(L(f,\tfrac{1}{2}) \neq 0\Big) \geqslant \frac{\mathbf{E}_N\big(L(f,\frac{1}{2})\big)^2}{\mathbf{E}_N\big(|L(f,\frac{1}{2})|^2\big)} \geqslant \frac{\mathrm{MT}_1(N;0)^2 + o(1)}{\mathrm{MT}_2(N;0) + o(1)}$$

as $N \to +\infty$. Since the main terms are polynomials in $\log N$, the lower-bound is of the form $(\log N)^{-k}$ for some integer $k \geqslant 0$. This suffices to obtain a large number of non-vanishing central critical values, but in practice, one finds that $k \geqslant 1$ (which can be guessed from the degrees of the polynomials, predicted by the moment conjectures), so we do not obtain an asymptotic positive proportion of non-vanishing.

The *mollification method*, pioneered by Selberg [75], exploits the twisted first and second moments to overcome this loss in the case where $k = 1$, which is the most common. This method introduces a *mollifier*

$$M(f) = \sum_{1 \leqslant \ell \leqslant L} \alpha(\ell)\lambda_f(\ell)$$

where the coefficients $\alpha(\ell)$ are chosen so that $M(f)$ approximates (in some sense) the inverse of $L(f,\frac{1}{2})$. Using the asymptotic formulas for the twisted first and second moment, one obtains asymptotic formulas for the mollified moments

$$\mathbf{E}_N\Big(M(f)L(f,\tfrac{1}{2})\Big) = \widetilde{\mathrm{MT}}_1(N,L) + O(L^B N^{-\delta}),$$

and

$$\mathbf{E}_N\Big(|M(f)L(f,\tfrac{1}{2})|^2\Big) = \widetilde{\mathrm{MT}}_2(N,L) + O(L^B N^{-\delta}).$$

The effect of the power-saving with respect to N is that we can select $L = N^\gamma$ to be a small enough (fixed) power of N so that the Cauchy-Schwarz inequality now leads to the lower bound

$$\mathbf{P}_N\Big(L(f,\tfrac{1}{2}) \neq 0\Big) \geqslant \frac{\widetilde{\mathrm{MT}}_1(N,L)^2}{\widetilde{\mathrm{MT}}_2(N,L)}(1 + o(1))$$

as $N \to +\infty$. It turns out that the leading term is now a positive constant (of size depending on γ), so we get a positive lower bound for the proportion of non-vanishing special values.

A version of this method was used by Selberg to prove his celebrated result on a positive proportion of critical zeros of the Riemann zeta function. It applies also, for instance, in proving that there is a positive proportion of non-vanishing critical values in the families $\mathcal{D}$ and $\mathcal{C}$ above (due to Iwaniec–Sarnak [37] and Kowalski–Michel [49], respectively), among other important families.

1.1.5. Existence of large values. The problem of the possible extreme sizes of values of L-functions is one of the most difficult and mysterious. This is due, in part, to the fact that their "typical" average behavior seems to be quite accurately predicted using various probabilistic models, but there is no particular reason to expect that such models can be reliable at the level of "large deviations". And even if one is convinced (rightly or wrongly!) that such a model is accurate, rigorous results are very difficult to come by. Soundararajan [79] introduced a tool called the "resonator method" to produce remarkably large values of $L(f, \frac{1}{2})$ for some $f \in \mathcal{F}_N$ (this method is also related to the ideas introduced by Goldston-Pintz-Yıldırım [26] to study gaps between primes). The idea is again to select coefficients $(\alpha(\ell))_{\ell \leqslant L}$ and form the corresponding sums

$$R(f) = \sum_{\ell \leqslant L} \alpha(\ell)\lambda_f(\ell),$$

(or some variations thereof, cf. Section 7.5.1) which are now called "resonators". Indeed, they are constructed so that the sizes of the two quantities

$$Q_1 = \mathbf{E}_N(|R(f)|^2)$$

and

$$Q_2 = \mathbf{E}_N\Bigl(|R(f)|^2 L(f, \tfrac{1}{2})\Bigr)$$

are such that $|Q_2|/Q_1$ is as large as possible. These sums can be evaluated asymptotically as quadratic forms (with variables $\alpha(\ell)$) if $L = N^\gamma$ with $\gamma > 0$ small enough, because of the asymptotic formula for twisted first moments (1.4). We then have

$$\max_{f \in \mathcal{F}_N} |L(f, \tfrac{1}{2})| \geqslant \sqrt{\frac{|Q_2|}{Q_1}}.$$

It remains a delicate issue to optimize the choice of the coefficients $\alpha(\ell)$, but as in the previous application, we see that we can certainly expect to find asymptotic formulas for Q_1 and Q_2, provided L is not too large, if we have access to an asymptotic formula for twisted moments for $\ell \leqslant L^2$. Once more, the dependency on L is such that we obtain really good results only if we can take L of size comparable to N in logarithmic scale, which is what power-savings in the first moment leads to.

1.1.6. Decay of probability of large order of vanishing. As we already indicated, the distribution of the order of vanishing $\mathrm{rk}_{an}(f)$ of an L-function $L(f, s)$ at the critical point $\frac{1}{2}$ (which is also called the *analytic rank*) has been extensively studied, often because of its links to arithmetic geometry in special cases. A method due to Heath-Brown and Michel [31] exploits a variant of the mollification method to study how often the analytic rank might be a very large integer. The starting point are the moments

$$\mathbf{E}_N\Bigl(\Bigl|\sum_p \frac{\lambda_f(p)(\log p)}{\sqrt{p}}\phi(\log p)\Bigr|^{2k}\Bigr),\qquad \mathbf{E}_N\Bigl(\Bigl|\sum_\varrho \widehat{\phi}(\varrho-\tfrac{1}{2})\Bigr|^{2k}\Bigr)$$

for integers $k \geqslant 0$, where ϱ runs over zeros of $L(f,s)$ "far away" from $\frac{1}{2}$ in some sense, and ϕ are suitable test functions. The first of these two can be studied relatively elementarily, if ϕ has sufficiently small support. The second is estimated by a delicate computation using the explicit formula (relating zeros of L-functions and their coefficients), and the asymptotic formulas for twisted second moments (1.5). From these bounds, one can deduce that the analytic rank cannot be large very often. In fact, one obtains exponentially-decaying tail-bounds: there exists a constant $c > 0$ (depending on the family) such that

$$\mathbf{E}_N(e^{c\,\mathrm{rk}_{an}(f)}) \ll 1$$

for $N \geqslant 1$, from which it follows that

$$\limsup_{N\to+\infty} \mathbf{P}_N(\mathrm{rk}_{an}(f) \geqslant r) \ll e^{-cr}$$

for $r \geqslant 0$.

1.1.7. Subgaussian bounds for critical values. Given a family $\mathcal{F}$ of L-functions, the most general moment conjectures of Keating-Snaith (including suitable complex exponents; see [**53**] for some general discussion of these) lead to the expectation that $\log|L(f,\frac{1}{2})|$ should have an approximately normal distribution as $N \to +\infty$, after a suitable normalization. There are currently very few results of this type. The first one is due to Selberg [**76**] (see also the short proof by Radziwiłł and Soundararajan in [**65**]), and applies to the Riemann zeta function. It states that for $T \geqslant 3$, and $t \in [T, 2T]$, the distribution of

$$\frac{\log\zeta(\frac{1}{2}+it)}{\sqrt{\frac{1}{2}\log\log T}}$$

converges as $T \to +\infty$ to a standard complex gaussian.

Radziwiłł and Soundararajan [**64**] have developed a robust method to prove *subgaussian upper bounds* in many families. For a given family, they show that such bounds hold whenever one has suitable asymptotic formulas for the twisted moments (1.4) when ℓ is as large as a small power of the conductor (which, in turn, usually follows once an asymptotic formula for the first moment is known with power-saving error term).

The results have a different form depending on the symmetry type of the family. In the orthogonal case, for instance, the method leads to

$$\mathbf{P}_N\Bigl(\frac{\log L(f,\frac{1}{2})+\frac{1}{2}\log\log N}{\sqrt{\log\log N}} \geqslant V\Bigr) \leqslant \frac{1}{\sqrt{2\pi}}\int_V^{+\infty} e^{-x^2/2}dx + o(1),$$

for any fixed $V \in \mathbf{R}$; in this case, the gaussian conjecture would be that the left-hand side is *equal* to the right-hand side.

The method is quite intricate. Roughly speaking, it starts with the proof that the sums over primes

$$P_N(f) = \sum_{p \leqslant P} \frac{\lambda_f(p)}{\sqrt{p}}$$

have a gaussian distribution if P is well-chosen, typically $P = N^{1/(\log\log N)^2}$, which in turn is an effect of quantitative local spectral equidistribution (with independence of the local components at distinct primes).

One expects that $P_N(f)$ is a good approximation to $\log L(f, \frac{1}{2}) + \frac{1}{2}\log\log N$ in some statistical sense (the additional term is the contribution of squares of primes, and the plus sign reflects the orthogonal symmetry). Fixing V, one distinguishes (again, roughly speaking; see [64, p. 1046] for a precise discussion) between three possibilities to compute the probability that

$$\frac{\log L(f, \frac{1}{2}) + \frac{1}{2}\log\log N}{\sqrt{\log\log N}} \geqslant V,$$

namely:

- It may be that $P_N(f) \geqslant (V - \varepsilon)\sqrt{\log\log N}$ for some small $\varepsilon > 0$, and the gaussian distribution of $P_N(f)$ gives a suitable gaussian bound for that event;
- It may be that $|P_N(f)| \geqslant \log\log N$, but the gaussian behavior shows that this is very unlikely;
- In the remaining case, we have

$$|P_N(f)| \leqslant \log\log N,$$
$$L(f, \tfrac{1}{2})(\log N)^{1/2} \exp(-P_N(f)) \geqslant \exp(\varepsilon\sqrt{\log\log N}).$$

To control this last critical case, one shows that it implies

$$L(f, \tfrac{1}{2})(\log N)^{1/2}\Bigl(\sum_{j=0}^{k} \frac{(-P_N(f))^j}{j!}\Bigr) \gg \exp(\varepsilon\sqrt{\log\log N})$$

for some suitable integer $k \geqslant 1$. But one can obtain an upper bound for

$$\mathbf{E}_N\Bigl(L(f, \tfrac{1}{2})\Bigl(\sum_{j=0}^{k} \frac{(-P_N(f))^j}{j!}\Bigr)\Bigr)$$

using the twisted first moments (where the power-saving gives as before the crucial control of a suitable value of the length P of $P_N(f)$). Then, by the Markov inequality, we get

$$\mathbf{P}_N(\text{third case}) \leqslant \exp(-\varepsilon\sqrt{\log\log N})\mathbf{E}_N\Bigl(L(f, \tfrac{1}{2})\Bigl(\sum_{j=0}^{k} \frac{(-P_N(f))^j}{j!}\Bigr)\Bigr)$$

which shows that the third event is also unlikely.

Radziwiłł and Soundararajan [66] have recently announced another method that leads to gaussian lower bounds for *conditional* probabilities that normalized values of $\log L(f, \frac{1}{2})$ belong to some interval, *knowing that they are non-zero.* These rely on (and in some sense incorporate) the proof of existence of a positive proportion of non-vanishing (discussed in Section 1.1.4). For an orthogonal family, the statements are of the type

$$\mathbf{P}_N\Big(\alpha \leqslant \frac{\log L(f,\frac{1}{2})+\frac{1}{2}\log\log N}{\sqrt{\log\log N}} \leqslant \beta\Big) \\ \geqslant \mathbf{P}_N\Big(L(f,\tfrac{1}{2})\neq 0\Big)\times\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-x^2/2}dx+o(1),$$

as $N\to+\infty$, when it is known that

$$\liminf_{N\to+\infty}\mathbf{P}_N(L(f,\tfrac{1}{2})\neq 0)>0.$$

1.1.8. Paucity of real zeros. The problem of possible existence of real zeros of L-functions on the right of the critical line is fascinating and difficult, especially in the case of self-dual L-functions, with the famous problem of Landau-Siegel zeros (concerning real zeros close to 1 of L-functions of real Dirichlet characters) remaining one of the key open problems of analytic number theory.

In this respect, Conrey and Soundararajan [**13**] discovered a very subtle variant of the mollification method (related to some of the techniques of Section 1.1.6, in particular a critical lemma of Selberg, see [**13**, Lemma 2.1]) that allowed them to prove that the specific family of real Dirichlet characters $\mathcal{Q}_N$ (which is of special interest in this respect) satisfies

$$\liminf_{N\to+\infty}\mathbf{P}_N\Big(L(\chi,s)\text{ has no real zero }s>0\Big)>0.$$

It is unclear how general this method is, because it ultimately depends on the numerical evaluation of a certain quantity. Conrey and Soundararajan [**13**, end of §2] explain that the success can be motivated by computations from Random Matrix Theory for symplectic families, but these assume (at least) the Generalized Riemann Hypothesis, and therefore are no guarantee of success in practice. In fact, we may note that Ricotta [**69**] obtained a similar result for families of Rankin-Selberg L-functions, but obtaining a positive proportion with at most three real zeros. This type of result is probably more robust.

1.2. The family of twists of a fixed modular form

We present in this section the (quite classical) family of L-functions that we will study in the remainder of the book.

We fix *throughout the book* a primitive cusp form (newform) f with respect to some congruence subgroup $\Gamma_0(r)$, with trivial central character, i.e., trivial nebentypus, which we will denote χ_r. The modular form f may be either a holomorphic cusp form of some weight k_f or a Maaß cusp form with Laplace eigenvalue $1/4+t_f^2$.

To simplify some computations, the following convention will be useful:

Convention 1.3. For a modular form f as above we define a quantity, the *signed level*, denoted by r, which is equal to the level of f (and thus positive) if f is holomorphic and equal to minus the level (and thus negative) if f is a Maaß form.

Remark 1.4. In general, the level is an ideal in the ring of integers of the underlying number field (in our case the number field is simply $\mathbf{Q}$), and the signed level should be thought of a suitable idele generating the ideal. For simplicity of notation, we continue to write χ_r for the trivial character modulo $|r|$.

We denote by $\lambda_f(n)$, for $n \geqslant 1$, the Hecke eigenvalues of f, normalized so that the mean square is 1 by Rankin-Selberg theory, or equivalently so that the standard L-function of f,

$$\sum_{n\geqslant 1} \lambda_f(n) n^{-s},$$

is absolutely convergent in $\mathfrak{Re}\, s > 1$.

From the point of view of cusp forms, we now consider the family parameterized by primes q, not dividing r, which is given by

$$\mathcal{F}_q = \{f \otimes \chi \mid \chi \pmod q,\ \chi \neq \chi_q\},$$

(where χ runs over the set of primitive Dirichlet characters modulo q). Since $(q, r) = 1$, this is a subset of the set of primitive cusp forms of level rq^2. The associated L-functions are the twisted L-functions

$$L(f \otimes \chi, s) = \sum_{n\geqslant 1} \frac{\lambda_f(n)\chi(n)}{n^s} = \prod_p \Bigl(1 - \frac{\chi(p)\lambda_f(p)}{p^s} + \frac{\chi_r \chi^2(p)}{p^{2s}}\Bigr)^{-1}.$$

Remark 1.5. **We emphasize that *throughout the remainder of this memoir, the modulus q will be assumed to be prime*, unless explicitly stated otherwise.**

We will usually think of the L-functions as simply parameterized by q, and write the probability and expectation explicitly as

$$\frac{1}{q-2} |\{\chi \bmod q \mid \chi \neq \chi_q,\ \chi \in A\}|, \qquad \frac{1}{q-2} \sideset{}{^*}\sum_{\chi \bmod q} T(\chi)$$

for any set A of Dirichlet characters, or any function T defined for Dirichlet characters; the notation $\sum^*$ restricts the sum to primitive characters. We will write $\varphi^*(q) = q - 2$ to clarify the notation.

This family has been studied in a number a papers (for instance by Duke, Friedlander, Iwaniec [16], Stefanicki [80], Chinta [10], Gao, Khan and Ricotta [23], Hoffstein and Lee [33] and in our own papers [4, 6, 52]). It is a very challenging family from the analytic point of view, and also has some very interesting algebraic aspects, at least when f is a holomorphic cusp form of weight 2. Indeed, if A_f is the abelian variety over $\mathbf{Q}$ constructed by Shimura with Hasse-Weil L-function equal to $L(f, s)$, then the product

$$\prod_{\chi \bmod q} L(f \otimes \chi, s) = L(f, s) \prod_{\substack{\chi \bmod q \\ \chi \neq \chi_q}} L(f \otimes \chi, s)$$

is the Hasse-Weil L-function of the base change of A_f to the cyclotomic field K_q generated by q-th roots of unity. According to the Birch and Swinnerton-Dyer conjecture, the vanishing (or not) of critical values of $L(f \otimes \chi, s)$ for $\chi \neq \chi_q$ is therefore related to the increase of rank of the Mordell-Weil group of A_f over K_q compared with that over $\mathbf{Q}$. (We will come back to this relation, as related to recent conjectures and questions of Mazur and Rubin).

The starting point of this book is that our recent papers [4, 6, 52] give access to the second moment theory of this family, in the sense sketched in the previous section. Precisely, the combination of these works provides a formula with power-saving error term for the second moment of the central value at $s = 1/2$, namely

$$\frac{1}{\varphi^*(q)} \sum_{\chi \,(\mathrm{mod}\, q)}^{*} |L(f \otimes \chi, \tfrac{1}{2})|^2.$$

From this, it is a relatively simple matter to derive asymptotic formulas including twists, and to include values at other points of the critical line with polynomial dependency on the imaginary part, giving formulas of the type (1.4) and (1.5). We can then attempt to implement the various applications of the previous section, and we will now list those which are found in this book.

Remark 1.6. (1) The family of twists is very simple from the point of view of local spectral equidistribution, which in that case amounts merely to an application of the orthogonality relations for Dirichlet characters modulo q. Precisely, for any prime p, the local components at p (in the sense of automorphic representations) of $f \otimes \chi$ become equidistributed in the unitary spectrum of $\mathrm{GL}_2(\mathbf{Q}_p)$ as $q \to +\infty$, with limit measure the uniform probability measure on the set of unramified twists $\pi_p(f) \otimes |\cdot|_p^{it}$, where $\pi_p(f)$ is the p-component of f. This fact will not actually play a role in our arguments so we skip the easy proof. It implies however (and this can easily be checked by means of the distribution of low-lying zeros) that the family is of unitary type.

(2) Because the orthogonality relations for Dirichlet characters are easier to manipulate when summing over all characters modulo q, we will sometimes use sums over all Dirichlet characters. This amounts to adding the L-function of f into our family (with the Euler factor at q removed) and changing the normalizing factor from $1/\varphi^*(q)$ to $1/\varphi(q)$, and has no consequence in the asymptotic picture.

In the next sections, we state precise forms of the results we will prove concerning this family. These concern non-vanishing properties and extremal values. We leave as an exercise to the interested reader the proof of the universality theorem (following [**48**]), which takes the following form:

Theorem 1.7. *Let $0 < R < 1/4$ be a real number and let D be the disc of radius R centered at $3/4$, which has closure $\bar{D}$ contained in the critical strip $1/2 < \mathfrak{Re}(s) < 1$. Let $\varphi\colon \bar{D} \to \mathbf{C}$ be a function that is continuous and holomorphic in D, and does not vanish in D. Then, for all $\varepsilon > 0$, we have*

$$\liminf_{q \to +\infty} \frac{1}{\varphi^*(q)} \Big| \{\chi \,(\mathrm{mod}\, q) \text{ non-trivial} \mid \sup_{s \in \bar{D}} |L(f \otimes \chi, s) - \varphi(s)| < \varepsilon\} \Big| > 0.$$

We do not consider lower-bounds for integral moments, but refer to the earlier paper of Blomer and Milićević [**6**, Th. 4] where a special case is treated.

1.3. Positive proportion of non-vanishing

We will first use the mollification method to show that the central value $L(f \otimes \chi, \frac{1}{2})$ is not zero for a positive proportion of $\chi \,(\mathrm{mod}\, q)$. In fact, as is classical, we will obtain a quantitative lower-bound. Moreover, inspired by the work of B. Hough [**34**], we will establish a result of this type with an additional constraint on the argument of the L-value.

For χ such that $L(f \otimes \chi, \frac{1}{2}) \neq 0$, we let

$$\theta(f \otimes \chi) = \arg(L(f \otimes \chi, \tfrac{1}{2})) \in \mathbf{R}/2\pi\mathbf{Z} \tag{1.6}$$

be the argument of $L(f\otimes\chi,\frac{1}{2})$. We will also use the same notation for the reduction of this argument in $\mathbf{R}/\pi\mathbf{Z}$ (we emphasize that this is $\mathbf{R}/\pi\mathbf{Z}$, and not $\mathbf{R}/2\pi\mathbf{Z}$; as we will see, our method is not sensitive enough to detect angles modulo 2π).

We say that a subset $I\subset\mathbf{R}/\pi\mathbf{Z}$ (or $I\subset\mathbf{R}/2\pi\mathbf{Z}$) is an *interval* if it is the image of an interval of $\mathbf{R}$ under the canonical projection.

THEOREM 1.8. *Let $I\subset\mathbf{R}/\pi\mathbf{Z}$ be an interval of positive measure. There exists a constant $\eta>0$, depending only on I, such that*

$$\frac{1}{\varphi^*(q)}|\{\chi\,(\mathrm{mod}\,q)\text{ non-trivial}\mid |L(f\otimes\chi,\tfrac{1}{2})|\geqslant(\log q)^{-1},\ \theta(f\otimes\chi)\in I\}|\geqslant\eta+o_{f,I}(1)$$

as $q\to\infty$ among the primes.

REMARK 1.9. (1) Our proof will show that one can take $\eta=\frac{\mu(I)^2}{1443\zeta(2)}$, where $\mu(I)$ denotes the (Haar probability) measure of I. It also shows that the lower bound $(\log q)^{-1}$ can be replaced with $(\log q)^{-1/2-\varepsilon}$ for any $\varepsilon>0$. For more details see §6.6 below.

(2) When f is a holomorphic form with rational coefficients (i.e., it is the cusp form associated to an elliptic curve over $\mathbf{Q}$), Chinta [10] has proved the following very strong non-vanishing result: for any $\varepsilon>0$, we have

$$\frac{1}{\varphi(q)}|\{\chi\,(\mathrm{mod}\,q)\mid L(f\otimes\chi,\tfrac{1}{2})\neq0\}|=1+O_{f,\varepsilon}(q^{-1/8+\varepsilon}).\tag{1.7}$$

His argument uses ideas of Rohrlich, and in particular the fact that in this case, the vanishing or non-vanishing of $L(f\otimes\chi,\frac{1}{2})$ depends only on the orbit of χ under the action of the absolute Galois group of $\mathbf{Q}$. The Galois invariance of the non-vanishing of $L(f\otimes\chi,\frac{1}{2})$ is not known if f is a Maaß form, and it is not known either whether a lower bound such as $|L(f\otimes\chi,\frac{1}{2})|\geqslant(\log q)^{-1}$ is Galois-invariant, even when f is holomorphic.

Theorem 1.8 can be seen as a special case of a more general class of new non-vanishing results for $L(f\otimes\chi,1/2)$ under additional constraints on χ. In Section 6.7, we combine the mollification method with Katz's work on the equidistribution of Mellin transforms of trace functions over finite fields (see [40]) to prove a very general theorem of this type (see Theorem 6.7). We state a representative special case here.

For any $\chi\,(\mathrm{mod}\,q)$, the *Evans sum* is defined as

$$\widetilde{t}_e(\chi)=\frac{1}{\sqrt{q}}\sum_{x\in\mathbf{F}_q^\times}\chi(x)e\Big(\frac{x-\bar{x}}{q}\Big).$$

By Weil's bound for exponential sums in one variable, the Evans sums are real numbers in the interval $[-2,2]$. A result of Katz [40, Th. 14.2] implies that they become equidistributed, as $q\to+\infty$, with respect to the Sato–Tate measure on $[-2,2]$. We then have:

THEOREM 1.10. *Let $I\subset[-2,2]$ be a set of positive measure with non-empty interior. There exists a constant $\eta>0$, depending only on I, such that*

$$\frac{1}{\varphi(q)}|\{\chi\,(\mathrm{mod}\,q)\ \mid |L(f\otimes\chi,\tfrac{1}{2})|\geqslant(\log q)^{-1},\ \widetilde{t}_e(\chi)\in I\}|\geqslant\eta+o_{f,I}(1)$$

as $q\to\infty$ among the primes.

1.4. Large central values

Our next result exhibits large central values of twisted L-functions in our family, using Soundararajan's resonator method. More precisely, we first prove a result that includes an angular constraint, similar to that in the previous section.

THEOREM 1.11. *Let $I \subset \mathbf{R}/\pi\mathbf{Z}$ be an interval of positive measure. There exists a constant $c > 0$ such for all primes q large enough, depending on I and f, there exists a non-trivial character $\chi \pmod q$ such that*

$$L(f \otimes \chi, \tfrac{1}{2}) \geqslant \exp\Bigl(\Bigl(\frac{c \log q}{\log\log q}\Bigr)^{1/2}\Bigr) \quad \textit{and} \quad \theta(f \otimes \chi) \in I.$$

We will also prove a second version which involves a product of twisted L-functions (and thus a slightly different family of L-functions), without angular restriction.

THEOREM 1.12. *Let g be a fixed primitive cusp form of level r' and trivial central character. There exists a constant $c > 0$, depending only on f and g, such that for all primes q large enough in terms of f and g, there exists a non-trivial character χ mod q such that*

$$|L(f \otimes \chi, \tfrac{1}{2}) L(g \otimes \chi, \tfrac{1}{2})| \geqslant \exp\Bigl(\Bigl(\frac{c \log q}{\log\log q}\Bigr)^{1/2}\Bigr).$$

Note that because we have a product of two special values, the resonator method is now not a "first moment" method, but will involve the average of these products, which is of the level of difficulty of the second moment for a single cusp form f, and once more, a power-saving in the error term is crucial for success.

1.5. Bounds on the analytic rank

Our third result concerns the order of vanishing (the analytic rank)

$$\mathrm{rk}_{an}(f \otimes \chi) = \operatorname*{ord}_{s=1/2} L(f \otimes \chi, s)$$

of the twisted L-functions at the central point. Using the methods of [31, 49, 50] (as in Section 1.1.6) we prove the exponential decay of the probability that the analytic rank exceeds a certain value:

THEOREM 1.13. *There exist constants $R \geqslant 0$, $c > 0$, depending only on f, such that*

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \exp(c\, \mathrm{rk}_{an}(f \otimes \chi)) \leqslant \exp(cR)$$

for all primes q. In particular, by the inequality of arithmetic and geometric means, we have

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \mathrm{rk}_{an}(f \otimes \chi) \leqslant R,$$

for all primes q, and for any $t \geqslant 0$ we have

$$\limsup_{q \to +\infty} \frac{1}{\varphi^*(q)} |\{\chi \,(\mathrm{mod}\, q) \text{ non-trivial} \mid \mathrm{rk}_{an}(f \otimes \chi) \geqslant t\}| \ll_f \exp(-ct).$$

REMARK 1.14. If f is holomorphic with rational coefficients, an immediate consequence of Chinta's bound (1.7) (using the bound $\mathrm{rk}_{an}(f \otimes \chi) \ll_f \log q$, for which see, e.g., [36, Th. 5.7]) is that

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \mathrm{rk}_{an}(f \otimes \chi) \ll_{f,\varepsilon} q^{-1/8+\varepsilon}.$$

1.6. A conjecture of Mazur-Rubin concerning modular symbols

Suppose that f is a holomorphic form of weight 2. For any integers $q \geqslant 1$ coprime with r and a coprime to q, the corresponding *modular symbol* (associated to f) is defined by

$$\Big\langle \frac{a}{q} \Big\rangle_f = 2\pi i \int_{i\infty}^{a/q} f(z)dz = 2\pi \int_0^{\infty} f\Big(\frac{a}{q} + iy\Big)\, dy,$$

where the path of integration can be taken as the vertical line joining $i\infty$ to a/q in the upper half-plane. This quantity, as a function of a, depends only on $a \,(\mathrm{mod}\, q)$.

It turns out that modular symbols are closely related to the special values $L(f \otimes \chi, \frac{1}{2})$ for Dirichlet characters $\chi \,(\mathrm{mod}\, q)$, by means of a formula due to Birch and Stevens (cf. [58, (8.6)]).

Recently, Mazur and Rubin [57] have investigated the variation of the rank of a fixed elliptic curve $E/\mathbf{Q}$ in abelian extensions of $\mathbf{Q}$ (including infinite extensions). This has led them (via the Birch–Swinnerton-Dyer conjecture and the Birch–Stevens formula) to a number of questions and conjectures concerning the modular symbols of the cusp form f attached to E (i.e., the cusp form whose L-function coincides with the Hasse-Weil L-function of E by the modularity theorem). In particular, they raised a number of problems concerning the distribution of these modular symbols.

Many of these questions have now been solved by Petridis and Risager [62] on average over q. In Chapter 9, we will study the distribution of modular symbols associated to an individual prime modulus q (see also the recent work [46] by Kim and Sun for a more arithmetic/algebraic perspective on modular symbols). Among other things, we will solve a conjecture of Mazur and Rubin (see [62, Conj. 1.2]) concerning their variance. Let

$$M_f(q) = \frac{1}{\varphi(q)} \sum_{\substack{a \bmod q \\ (a,q)=1}} \Big\langle \frac{a}{q} \Big\rangle_f$$

be the mean value, which will be computed in Theorem 9.2.

THEOREM 1.15. *For q a prime, the variance of modular symbols*

$$V_f(q) = \frac{1}{\varphi(q)} \sum_{\substack{a \bmod q \\ (a,q)=1}} \Big| \Big\langle \frac{a}{q} \Big\rangle_f - M_f(q) \Big|^2$$

satisfies

$$V_f(q) = 2 \prod_{p \mid r} (1 + p^{-1})^{-1} \frac{L^*(\mathrm{Sym}^2 f, 1)}{\zeta(2)} \log q + \beta_f + O(q^{-1/145})$$

for q prime, where $\beta_f \in \mathbf{C}$ is a constant and $L^*(\mathrm{Sym}^2 f, s)$ denotes the imprimitive *symmetric square L-function of f* (cf. Section 2.3.3).

1.7. Twisted moment estimates

As we have explained in Section 1.1, the proofs of most of these results rely on the *amplification method* and the *resonator method*, and involve various asymptotic formulas for moments and twisted moments of the L-functions in the family.

In our case, since the Fourier coefficients of $f \otimes \chi$ are $\lambda_f(n)\chi(n)$, and the first factor is fixed, it is most natural to consider moments twisted simply by character values $\chi(\ell)$ for some integers ℓ. Moreover, in order to incorporate angular restrictions on the central values, as in Theorems 1.8 and 1.11, it is useful to also consider twists by powers of the Gauß sums of the characters, at least in the first moment.

Hence, our basic sums of interests are

$$
\begin{aligned}
\mathcal{L}(f,s;\ell,k) &:= \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} L(f\otimes\chi,s)\varepsilon_\chi^k \chi(\ell),\\
\mathcal{Q}(f,s;\ell,\ell') &:= \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \bigl|L(f\otimes\chi,s)\bigr|^2 \chi(\ell)\overline{\chi(\ell')}
\end{aligned}
\tag{1.8}
$$

where s is a complex parameter (with real part close to $\frac{1}{2}$ in practice), ℓ and ℓ' are coprime integers, $k \in \mathbf{Z}$ and

$$
\varepsilon_\chi = \frac{1}{\sqrt{q}} \sum_{h \,(\mathrm{mod}\, q)} \chi(h) e\Bigl(\frac{h}{q}\Bigr) \tag{1.9}
$$

is the normalized Gauß sum of χ. If $s = 1/2$, we will drop it from the notation and write $\mathcal{L}(f;\ell,k) = \mathcal{L}(f,1/2;\ell,k)$, $\mathcal{Q}(f;\ell,\ell') = \mathcal{Q}(f,1/2;\ell,\ell')$.

Using these, we can build the mollified moments (or resonating moments, depending on the application), namely

$$
\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} L(f\otimes\chi,s) e(2k\theta(f\otimes\chi)) M(f\otimes\chi,s;\boldsymbol{x}_L)
$$

and

$$
\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \bigl|L(f\otimes\chi,s)\bigr|^2 \bigl|M(f\otimes\chi,s;\boldsymbol{x}_L)\bigr|^2
$$

where $M(f\otimes\chi,s;\boldsymbol{x}_L)$ is a finite sum

$$
M(f\otimes\chi,s;\boldsymbol{x}_L) = \sum_{\ell\leqslant L} x_\ell \frac{\chi(\ell)}{\ell^s}
$$

involving complex parameters $\boldsymbol{x}_L = (x_\ell)_{\ell\leqslant L}$ that we select carefully depending on each application.

We need to evaluate the first moment only for $s = \frac{1}{2}$, and by the functional equation of $L(f\otimes\chi,s)$ is it sufficient to do so when $k \geqslant -1$.

THEOREM 1.16. *For $k \geqslant -1$, $\ell \in (\mathbf{Z}/q\mathbf{Z})^\times$ and any $\varepsilon > 0$, we have*

$$
\mathcal{L}(f;\ell,k) = \delta_{k=0} \frac{\lambda_f(\overline{\ell}_q)}{\overline{\ell}_q^{1/2}} + O_{f,\varepsilon,k}(q^{-1/8+\varepsilon}),
$$

for q prime, where $\overline{\ell}_q$ denotes the unique integer in the interval $[1,q]$ satisfying the congruence $\ell\overline{\ell}_q \equiv 1\,(\mathrm{mod}\, q)$.

The proof of this theorem is rather elementary when $k = 0$, but it requires the results of Fouvry, Kowalski and Michel [20] on twists of Fourier coefficients by trace functions otherwise.

The evaluation of the second moment is significantly more challenging. The combination of our three papers [4, 6, 52] successfully handles the case $\ell = \ell' = |r| = 1$. Precisely, by [52, Th. 1.5] (which relies on the previous papers), we have:

THEOREM 1.17. *Assume that the level of f is $r = 1$. For any $\delta < 1/144$, we have*

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} |L(f \otimes \chi, \tfrac{1}{2})|^2 = P_f(\log q) + O_{f,\delta}(q^{-\delta}),$$

for q prime, where $P_f(X)$ is a polynomial of degree 1 depending on f only with leading coefficient $2L(\mathrm{Sym}^2 f, 1)/\zeta(2)$.

As we discussed above, this is in a certain sense the main case, and from there it is possible to evaluate the more general second moments $\mathcal{Q}(f, s; \ell, \ell')$, which we do here for f of general level. In fact, for the proof of Theorem 1.12, we will require an estimate involving two cusp forms (which of course may be equal!).

THEOREM 1.18. *Let f, g be primitive cusp forms of signed levels r and r' coprime to q, with trivial central character. Define*

$$\mathcal{Q}(f, g, s; \ell, \ell') = \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} L(f \otimes \chi, s)\overline{L(g \otimes \chi, s)}\chi(\ell)\overline{\chi(\ell')}$$

for integers $1 \leqslant \ell, \ell' \leqslant L$, with $(\ell\ell', qrr') = (\ell, \ell') = 1$, and $s \in \mathbf{C}$.

Then, for $s = \frac{1}{2} + \beta + it$, $\beta, t \in \mathbf{R}$ with $|\beta| \leqslant 1/\log q$, we have the asymptotic formula

$$\mathcal{Q}(f, g, s; \ell, \ell') = \mathrm{MT}(f, g, s; \ell, \ell') + O_{f,g,\varepsilon}(|s|^{O(1)} L^{3/2} q^{-1/144+\varepsilon})$$

for q prime, where

$$\mathrm{MT}(f, g, s; \ell, \ell') = \frac{1}{2}\mathrm{MT}^+(f, g, s; \ell, \ell') + \frac{1}{2}\mathrm{MT}^-(f, g, s; \ell, \ell')$$

is a "main term" whose even and odd parts $\mathrm{MT}^\pm(f, g, s; \ell, \ell')$ are given in (5.2).

The main terms as we express them here are well-suited to further transformations for our main applications. If one is interested in the second moment $\mathcal{Q}(f, g, \frac{1}{2}; 1, 1)$ only (as in Theorem 1.15), then one can express the main term more concretely, but there are a number of cases to consider.

If $f = g$ is of squarefree level r, then

$$\mathrm{MT}(f, f, \tfrac{1}{2}; 1, 1) = 2\prod_{p|r}(1 + p^{-1})^{-1}\frac{L(\mathrm{Sym}^2 f, 1)}{\zeta(2)}(\log q) + \beta_f + O(q^{-2/5})$$

for some constant β_f, where $\mathrm{Sym}^2 f$ is the symmetric square of f (cf. Section 2.3.3).

If $f \neq g$, it may be that $\mathcal{Q}(f, g, \frac{1}{2}; 1, 1)$ is *exactly* zero for "trivial" reasons. This happens if f and g have the same signed level $r = r'$ (recall that, with the convention 1.3, this implies that either both are holomorphic, or that both are non-holomorphic) and their root numbers $\varepsilon(f)$ and $\varepsilon(g)$ satisfy $\varepsilon(f)\varepsilon(g) = -1$. In that case, computations with root numbers show that

$$L(f \otimes \chi, \tfrac{1}{2})\overline{L(g \otimes \chi, \tfrac{1}{2})} = -L(f \otimes \overline{\chi}, \tfrac{1}{2})\overline{L(g \otimes \overline{\chi}, \tfrac{1}{2})}$$

so the second moment cancels by pairing each character with its conjugate (see Remark 2.4).

If, on the other hand, we have $\varepsilon(f)\varepsilon(g) = 1$ and f, g are of the same type, then we have

$$\mathrm{MT}(f, g, \tfrac{1}{2}; 1, 1) = 2\gamma_{f,g}\frac{L(f \otimes g, 1)}{\zeta(2)} + O(q^{-2/5})$$

where $L(f \otimes g, s)$ is the Rankin-Selberg convolution of f and g (cf. Section 2.3.1) and $\gamma_{f,g}$ is some non-zero constant depending on f and g. We defer a more detailed discussion to Proposition 5.2.

REMARK 1.19. (1) Let d be the usual divisor function. Then $d(n)$ is the n-th Hecke eigenvalue of a non-holomorphic Eisenstein series $E(s)$, and the identity

$$L(\chi, s)^2 = L(E \otimes \chi, s)$$

shows that the problem of estimating the second moment of twists of E is equivalent to the problem of estimating the fourth moment of the values of the Dirichlet L-functions $L(\chi, s)$. This remark shows that several parts of this memoir have obvious links with the beautiful work of Young [83] (later improved in [4]) where he proves the existence of a polynomial P_4 of degree 4 and of a constant $\delta > 0$ such that, for all $q \geqslant 2$, we have

$$\frac{1}{\varphi^*(q)} \sum_{\chi \mod q} |L(\chi, 1/2)|^4 = P_4(\log q) + O(q^{-\delta}).$$

(2) Recently Zacharias [85] used the evaluation of the mollified second moment $\mathcal{Q}(f, s; \ell, \ell')$ of this memoir together with his own evaluation of the mixed twisted moment

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} L(\chi, \tfrac{1}{2})L(f \otimes \chi, \tfrac{1}{2})\chi(\ell)$$

to establish the existence of a positive proportion of primitive $\chi \,(\mathrm{mod}\, q)$ such that

$$L(\chi, \tfrac{1}{2})L(f \otimes \chi, \tfrac{1}{2}) \neq 0.$$

He also obtained similar results when f is an Eisenstein series (in which case the L-function $L(f \otimes \chi, s)$ is a product of Dirichlet L-functions): using his evaluation of the fourth mollified moment of Dirichlet L-functions ([84]) he shows that for any pair of characters $\chi_1, \chi_2 \,(\mathrm{mod}\, q)$, there exists a positive proportion of primitive characters $\chi \,(\mathrm{mod}\, q)$ for which $L(\chi, \frac{1}{2})L(\chi \cdot \chi_1, \frac{1}{2})L(\chi \cdot \chi_2, \frac{1}{2}) \neq 0$.

(3) The approach of Hoffstein and Lee [33] towards the second moment, based on multiple Dirichlet series, reduces a proof of Theorem 1.17 (with some power-saving exponent) to a non-trivial estimate for a certain special value of a double Dirichlet series, which is denoted $\tilde{Z}_q(1-k/2, 1/2; f, f)$ in loc. cit. When q is prime, our theorem therefore indirectly provides such an estimate

$$\tilde{Z}_q(1 - k/2, 1/2; f, f) \ll q^{-1/144}.$$

Outline of the book

This book is organized as follows:

(1) Chapter 2 is preliminary to the main results; we set up the notation, and recall a number of important facts concerning Hecke L-functions (such as those of our family), as well as auxiliary L-functions that arise during the proofs of the main results (such as Rankin-Selberg L-functions). We require, in particular, some forms of the Prime Number Theorem and zero-free regions for these L-functions, and since the literature is not fully clear in this matter, we discuss some of these in some detail. We also discuss briefly a shifted convolution bound that is a slight adaptation of one of Blomer and Milićević [6].
(2) Chapter 3 gives an account of the algebraic exponential and character sums that occur in the book; on the one hand, these are the elementary orthogonality properties of character sums, and the averages of Gauß sums that give rise to hyper-Kloosterman sums, and on the other hand, we state a number of deep bounds for various sums of Kloosterman sums. Although we do not need to develop new bounds of this type, we give a quick sketch of the arguments that lead to them, with references to the original proofs. It is worth mentioning that these proofs rely in an absolutely essential way on the most general form of the Riemann Hypothesis over finite fields, due to Deligne, as well as on works of Katz. In Sections 3.4 and 3.5, we present some background on trace functions and discuss the results of Katz on discrete Mellin transforms over finite fields that are involved in the proof of (the general form of) Theorem 1.10.
(3) In Chapter 4, we prove the necessary asymptotic estimates for the first twisted moment of our family. The proof is very short, which illustrates the principle that the complexity of moment computations in families of L-functions increases steeply as the order of moment increases.
(4) In turn Chapter 5 gives the proof of the required twisted second moment estimates. Although this is much more involved than the first moment, most of the necessary ingredients are found in our previous works, and the chapter is relatively short.
(5) Finally, Chapters 6, 7, 8 and 9 are devoted to the proofs of our main results: positive proportion of non-vanishing (including Theorem 1.8), existence of large values, bounds for the analytic rank and the variance of modular symbols, respectively. These chapters are essentially independent of each other (the last one is extremely short, as the proof of Theorem 1.15 is mostly a direct translation of the second moment estimate), and many readers will find it preferable to start reading one of them, and to refer to the required results of the previous chapters only as needed.

    Since the theory of trace functions and its required background involve prerequisites that may be unfamiliar to some readers, the corresponding statements and results are isolated in independent sections (besides the background sections 3.4 and 3.5, they are in Sections 4.3 and 6.7).

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CHAPTER 2

Preliminaries

We collect in this chapter some preliminary material. Most of it is well-known, however some cases of the Prime Number Theorem (Proposition 2.11) are difficult to locate in the literature, and the computation of the ramified factors of the symmetric square L-function in Section 2.3.3 are even more problematic.

2.1. Notation and conventions

– We use the notation $\delta_{x,y}$ or $\delta(x,y)$ or $\delta_{x=y}$ for the Kronecker delta symbol.

– The notation $A \asymp B$ means

$$A \ll B \ll A,$$

where $\ll$ denotes the Vinogradov symbol.

- In this book, we will denote generically by W, sometimes with subscripts, some smooth complex-valued functions, compactly supported on $[1/2, 2]$ and possibly depending on a finite set $\mathscr{S}$ of complex numbers, whose derivatives satisfy

$$W^{(j)}(x) \ll_j \prod_{s\in\mathscr{S}} (1+|s|)^{cj} \tag{2.1}$$

for some fixed constant $c > 0$ and any $j \geqslant 0$ (as usual, an empty product is defined to be equal to 1). In practice, $\mathscr{S}$ may be empty, or may contain the (signed) levels r, r' of two cusp forms, their weight/spectral parameter, and/or a complex number s on or close to the $\frac{1}{2}$-line. Of course, the set $\mathscr{S}$ must not contain our basic parameter q, but no harm is done if some $s \in \mathscr{S}$ grows like $(\log q)^2$, say, since all our estimates contain a q^{ε}-valve. To lighten the notation, we will not the display the dependence on parameters $s \in \mathscr{S}$ in implied constants and just keep in mind that it is polynomial.

– Throughout this book, we will use the ε-convention, according to which a statement involving ε holds for all sufficiently small $\varepsilon > 0$ (with implied constants depending on ε) and the value of ε may change from line to line. A typical example is (4.4), where the various ε's in (2.29) and (2.16) combine to a new ε.

– For $z \in \mathbf{C}$, we denote $e(z) = e^{2\pi i z}$. We recall that if $q \geqslant 1$ is an integer, then $x \mapsto e(x/q)$ is a well-defined additive character modulo q.

– For an integer $c \geqslant 1$ and $a \in \mathbf{Z}$ coprime to c, we often write $\bar{a}$ for the inverse of a modulo c in $(\mathbf{Z}/c\mathbf{Z})^{\times}$. The value of c will always be clear from the context.

- For any (polynomially bounded) multiplicative function $a(n)$, we define a Dirichlet series
$$A(s) = \sum_{n \geqslant 1} a(n) n^{-s},$$
we denote by $A_p(s)$ the p-factor of the corresponding Euler product, so that
$$A(s) = \prod_p A_p(s)$$
in the region of absolute convergence. For any integer r, we also write $A^{(r)}(s)$ for the Euler product restricted to primes $p \nmid r$.
- Let $c \geqslant 1$ and let a, b be integers. We denote
$$S(a, b; c) = \sum_{\substack{d \, (\mathrm{mod}\, c) \\ (d,c)=1}} e\Big(\frac{ad + b\bar{d}}{c}\Big)$$
the Kloosterman sum modulo c. We also denote
$$\mathrm{Kl}(a; c) = \frac{1}{\sqrt{c}} S(a, 1; c),$$
the normalized Kloosterman sum.
- As we already mentioned, ***unless otherwise specified,*** q ***will be a prime number***.

2.2. Hecke L-functions

Let f be a primitive cusp form (holomorphic or Maaß) of signed level r (i.e. for the group $\Gamma_0(|r|)$) with trivial central character χ_r. The Hecke L-function of f is a degree 2 Euler product absolutely convergent for $\mathfrak{Re}\, s > 1$:
$$\begin{aligned} L(f, s) := \prod_p L_p(f, s) &= \prod_p \prod_{i=1}^{2} \Big(1 - \frac{\alpha_{f,i}(p)}{p^s}\Big)^{-1} \\ &= \prod_p \Big(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_r(p)}{p^{2s}}\Big)^{-1} = \sum_{n \geqslant 1} \frac{\lambda_f(n)}{n^s}, \quad \mathfrak{Re}\, s > 1. \end{aligned}$$
The factor $L_p(f, s)$ is the local L-factor at the prime p and the coefficients $\alpha_{f,i}(p)$ for $i = 1, 2$ are called the local parameters of f at p. The coefficients of this Dirichlet series $(\lambda_f(n))_{n \geqslant 1}$ have a simple expression in terms of these parameters: for any prime p, we have
$$\lambda_f(p) = \alpha_{f,1}(p) + \alpha_{f,2}(p), \ \ \alpha_{f,1}(p)\alpha_{f,2}(p) = \chi_r(p),$$
and we have the multiplicativity relations
$$\begin{aligned} \lambda_f(m)\lambda_f(n) &= \sum_{d \mid (m,n)} \chi_r(d) \lambda_f\left(\frac{mn}{d^2}\right), \\ \lambda_f(mn) &= \sum_{d \mid (m,n)} \chi_r(d)\mu(d) \lambda_f\left(\frac{m}{d}\right) \lambda_f\left(\frac{n}{d}\right). \end{aligned}$$
For a primitive form, the Dirichlet coefficient $\lambda_f(n)$ is the eigenvalue of f for the n-th Hecke operator.

The local L-factors $(L_p(f,s))_p$ are completed by an archimedean local factor which is a product of shifted Gamma functions

$$L_\infty(f,s)=\Gamma_{\mathbf{R}}(s-\mu_{f,1})\Gamma_{\mathbf{R}}(s-\mu_{f,2}),\ \Gamma_{\mathbf{R}}(s)=\pi^{-s/2}\Gamma(s/2). \tag{2.2}$$

The coefficients $\mu_{f,i},\ i=1,2$ are called the local archimedean parameters of f and are related to the classical invariants of f as follows:

$$\mu_{f,1}=-\frac{k-1}{2},\ \mu_{f,2}=-\frac{k}{2}$$

if f is holomorphic of weight $k\geqslant 2$ and

$$\mu_{f,1}=\frac{1-\kappa_f}{2}+it_f,\ \mu_{f,2}=\frac{1-\kappa_f}{2}-it_f$$

if f is a Maaß form with Laplace eigenvalue $\lambda_f(\infty)=(\frac{1}{2}+it_f)(\frac{1}{2}-it_f)$ and $\kappa_f\in\{\pm1\}$ is the eigenvalue of f under the involution $f\mapsto f(-\bar z)$. The completed product

$$\Lambda(f,s)=|r|^{s/2}L_\infty(f,s)L(f,s)$$

admits a holomorphic continuation to the whole complex plane and satisfies a functional equation of the shape

$$\Lambda(f,s)=\varepsilon(f)\overline{\Lambda(f,1-\overline{s})}$$

where $\varepsilon(f)$ (the root number) is a complex number satisfying $|\varepsilon(f)|=1$.

2.2.1. Character twists. Let χ be a non-trivial Dirichlet character of prime modulus q also coprime with r. The twisted L-function

$$\begin{aligned}L(f\otimes\chi,s)&=\prod_p L_p(f\otimes\chi,s)=\prod_p\prod_{i=1,2}\Big(1-\frac{\alpha_{f,i}(p)\chi(p)}{p^s}\Big)^{-1}\\&=\prod_p\Big(1-\frac{\lambda_f(p)\chi(p)}{p^s}+\frac{\chi_r(p)\chi^2(p)}{p^{2s}}\Big)^{-1}=\sum_{n\geqslant1}\frac{\lambda_f(n)\chi(n)}{n^s},\quad \mathfrak{Re}\,s>1.\end{aligned}$$

is in fact the Hecke L-function of a primitive cusp form $f\otimes\chi$ for the group $\Gamma_0(q^2|r|)$ with central character $\chi^2\chi_r$ (see [**36**, Propositions 14.19 & 14.20], for instance, in the holomorphic case which carries over to the general case).

LEMMA 2.1. *Let f be a primitive (holomorphic or Maaß) cusp form of signed level r and trivial central character, and let χ be a primitive character modulo q, not necessarily prime. Then the twisted L-function satisfies the functional equation*

$$\Lambda(f\otimes\chi,s)=\varepsilon(f\otimes\chi)\overline{\Lambda(f\otimes\chi,1-\overline{s})}=\varepsilon(f\otimes\chi)\Lambda(f\otimes\overline{\chi},1-s)$$

with

$$\Lambda(f\otimes\chi,s)=(q^2|r|)^{s/2}L_\infty(f\otimes\chi,s)L(f\otimes\chi,s).$$

Setting

$$\mathfrak{a}=\frac{1-\kappa_f\chi(-1)}{2}=\begin{cases}0, & \textit{if } \chi \textit{ and } f \textit{ have the same parity,}\\ 1, & \textit{if } \chi \textit{ and } f \textit{ have different parity,}\end{cases}$$

we have

$$L_\infty(f\otimes\chi,s)=\begin{cases}L_\infty(f,s) \textit{ if } f \textit{ is holomorphic of weight } k,\\ L_\infty(f,s+\mathfrak{a}) \textit{ if } f \textit{ is an even Maaß form,}\\ L_\infty(f,s-1+\mathfrak{a}) \textit{ if } f \textit{ is an odd Maaß form,}\end{cases}$$

and

$$\varepsilon(f \otimes \chi) = \varepsilon(f)\chi(r)\varepsilon_\chi^2 \tag{2.3}$$

where $\varepsilon(f)$ is the root number of $L(f,s)$ and ε_χ is the normalized Gauß sum, cf. (1.9).

Recall Convention 1.3 that r can be positive or negative depending on whether f is holomorphic or not. Observe that $L_\infty(f \otimes \chi, s)$ depends at most on the parity of χ, and is independent of χ if f is holomorphic. The following notation will be useful: for $\chi(-1) = \pm 1$ we write

$$L_\infty(f, \pm, s) := L_\infty(f \otimes \chi, s).$$

Proof. This is standard (see, e.g., [36, Th. 14.17 and Prop. 14.20] in the holomorphic case). We did not find a reference for the explicit root number computation (2.3) in the Maaß case, so for the reader's convenience we include the details. We start with some general "converse type" computations. Let

$$F(z) = \sqrt{y}\sum_{n\neq 0} a(n)K_{it}(2\pi|n|y)e(nx), \quad G(z) = \sqrt{y}\sum_{n\neq 0} b(n)K_{it}(2\pi|n|y)e(nx)$$

be two Maaß form that are both even or both odd and satisfy

$$F(-1/Nz) = \bar{\eta}G(z)$$

for some integer $N \geqslant 1$ and some complex number η of modulus 1. Differentiating both sides of the functional equation with respect to x, we obtain

$$\bar{\eta}G_x(z) = \frac{\partial}{\partial x}F\left(\frac{-x+iy}{N(x^2+y^2)}\right) = F_x\left(-\frac{1}{Nz}\right)\frac{1}{Nz^2}.$$

If both F and G are even, we compute

$$\begin{aligned} 2\int_0^\infty F(iy)y^{s-1/2}\frac{dy}{y} &= 4\sum_{n>0} a(n)\int_0^\infty K_{it}(2\pi ny)y^s\frac{dy}{y} \\ &= 4\sum_{n>0}\frac{a(n)}{n^s}\frac{\pi^{-s}\Gamma(1/2(s-it))\Gamma(1/2(s+it))}{4} = L(F,s)L_\infty(F,s). \end{aligned}$$

On the other hand, by the functional equation, this equals

$$2\bar{\eta}\int_0^\infty G(i/Ny)y^{s-1/2}\frac{dy}{y} = \bar{\eta}N^{1/2-s}2\int_0^\infty G(iy)y^{-s+1/2}\frac{dy}{y}$$

so that by the above computation we have

$$L(F,s)L_\infty(F,s) = \bar{\eta}N^{1/2-s}L(G,1-s)L_\infty(G,1-s).$$

If both F and G are odd, we compute

$$\begin{aligned} \frac{1}{i}\int_0^\infty F_x(iy)y^{s+1/2}\frac{dy}{y} &= 4\pi\sum_{n>0} a(n)n\int_0^\infty K_{it}(2\pi ny)y^{s+1}\frac{dy}{y} \\ &= 4\pi\sum_{n>0}\frac{a(n)}{n^s}\frac{\pi^{-s-1}\Gamma(1/2(s+1-it))\Gamma(1/2(s+1+it))}{4} = L(F,s)L_\infty(F,s). \end{aligned}$$

On the other hand, by the functional equation for the derivative, this equals

$$\frac{1}{i}\int_0^\infty \bar{\eta} G_x(i/Ny)\frac{1}{N(iy)^2}y^{s+1/2}\frac{dy}{y} = -\frac{\bar{\eta}}{iN}\int_0^\infty G_x(i/Ny)y^{s-3/2}\frac{dy}{y}$$
$$= -\frac{\bar{\eta}}{i}N^{1/2-s}\int_0^\infty G_x(iy)y^{3/2-s}\frac{dy}{y},$$

so that by the above computation we have

$$L(F,s)L_\infty(F,s) = -\bar{\eta}N^{1/2-s}L(G,1-s)L_\infty(G,1-s).$$

After these general considerations, we return to the functional equation of twisted L-functions. Let f be a Maaß form of parity $\kappa_f \in \{\pm 1\}$ and signed level $r<0$ and trivial central character. Then $f\otimes\chi$ has parity $\kappa_f\chi(-1)\in\{\pm1\}$. Write

$$f(-1/(|r|z)) = \bar{\eta} f(z)$$

so that by the above computation we have $\varepsilon(f)=\bar{\eta}\kappa_f$ for the root number. By a formal matrix computation [35, Theorem 7.5] we see that

$$(f\otimes\chi)\left(-\frac{1}{q^2|r|z}\right) = \varepsilon_\chi^2\chi(|r|)\bar{\eta}\cdot(f\otimes\bar{\chi})(z)$$

So by the above computation, the root number of $f\otimes\chi$ is indeed

$$\varepsilon_\chi^2\chi(|r|)\bar{\eta}\cdot\kappa_f\chi(-1) = \varepsilon_\chi^2\chi(r)\varepsilon(f)$$

as claimed. ☐

Remark 2.2. One could also recover the root number from general principles of automorphic representation theory, since it is given by a product over all places, and the behavior of local root numbers under twisting is relatively straightforward. The above "classical" treatment doesn't require knowledge of, say, the classification of local representations at infinity.

In particular, taking $s=1/2$ and recalling that $\theta(f\otimes\chi)$ is the argument of $L(f\otimes\chi,1/2)$ (if the latter is non-zero), cf. (1.6), we obtain:
(2.4)

$$\text{If } L(f\otimes\chi,1/2)\neq 0, \text{ one has } \exp(2i\theta(f\otimes\chi)) = \frac{L(f\otimes\chi,1/2)}{L(f\otimes\overline{\chi},1/2)} = \varepsilon(f)\chi(r)\varepsilon_\chi^2.$$

From the above discussion, and from the formula $\varepsilon_{\overline{\chi}}^2=\overline{\varepsilon_\chi}^2$, we can derive an explicit form of the functional equation for a product of twisted L-functions.

Lemma 2.3. *Let f, g be primitive cusp forms with trivial central character, of signed levels r and r' respectively, both coprime to q. We have*

$$\Lambda(f\otimes\chi,s)\Lambda(g\otimes\overline{\chi},\bar{s}) = \varepsilon(f)\varepsilon(g)\chi(r\overline{r'})\Lambda(f\otimes\overline{\chi},1-s)\Lambda(g\otimes\chi,1-\bar{s}).$$

We recall our convention 1.3: if f is a Maaß form, then its level is defined to be the opposite of the arithmetic conductor.

Remark 2.4. If we assume that $r=r'$ (so f and g are of the same type) and $s=1/2$, then using the fact that $\lambda_f(n)$ and $\lambda_g(n)$ are real-valued, we obtain from the functional equation the relation

$$L(f\otimes\chi,\tfrac12)\overline{L(g\otimes\chi,\tfrac12)} = \varepsilon(f)\varepsilon(g)L(f\otimes\overline{\chi},\tfrac12)\overline{L(g\otimes\overline{\chi},\tfrac12)}.$$

In particular, if furthermore $\varepsilon(f)\varepsilon(g)=-1$, it follows that

$$L(f\otimes\chi,\tfrac12)\overline{L(g\otimes\chi,\tfrac12)} = -L(f\otimes\overline{\chi},\tfrac12)\overline{L(g\otimes\overline{\chi},\tfrac12)}. \tag{2.5}$$

2.2.2. The explicit formula. In Chapter 8 we will obtain upper bounds for the analytic rank of $L(f \otimes \chi, s)$ (i.e. the order of vanishing at $s = 1/2$) on average over χ. For this we will need the explicit formula in this specific situation. Define Λ_f and $\Lambda_{f\otimes\chi}$ by the formulas

$$-\frac{L'}{L}(f, s) = \sum_{n\geqslant 1} \frac{\Lambda_f(n)}{n^s},$$

$$-\frac{L'}{L}(f \otimes \chi, s) = \sum_{n\geqslant 1} \frac{\Lambda_{f\otimes\chi}(n)}{n^s} = \sum_{n\geqslant 1} \frac{\Lambda_f(n)\chi(n)}{n^s}.$$

The explicit formula for $L(f \otimes \chi, s)$ is:

Proposition 2.5. *Let $\varphi :]0, +\infty[\to \mathbf{C}$ be smooth and compactly supported, and let*

$$\widetilde{\varphi}(s) = \int_0^\infty \varphi(x) x^s \frac{dx}{x}$$

be its Mellin transform and $\psi(x) = x^{-1}\varphi(x^{-1})$ so that $\widetilde{\psi}(s) = \widetilde{\varphi}(1 - s)$. One has

$$\sum_{n\geqslant 1} \Big(\Lambda_f(n)\chi(n)\varphi(n) + \Lambda_f(n)\overline{\chi}(n)\psi(n) \Big) = \tag{2.6}$$

$$\varphi(1) \log(q^2|r|) + \frac{1}{2i\pi} \int_{(1/2)} \Big(\frac{L'_\infty(f, \pm, s)}{L_\infty(f, \pm, s)} + \frac{L'_\infty(f, \pm, 1 - s)}{L_\infty(f, \pm, 1 - s)} \Big) \widetilde{\varphi}(s) ds - \sum_{\varrho} \widetilde{\varphi}(\varrho)$$

where ϱ ranges over the multiset of zeros of $\Lambda(f \otimes \chi, s)$ in the strip $0 < \mathfrak{Re}\, s < 1$.

See [36, §5.5] for the proof.

2.3. Auxiliary L-functions

In addition to Hecke L-functions and their twists by characters, several auxiliary L-functions will play an important role in this memoir. They will arise as *individual* L-functions (not in a family), typically in expressions for leading terms of various asymptotic formulas. As a consequence, it is their behavior close to $\mathfrak{Re}(s) = 1$ that is of most interest.

We review in this section the definitions of these L-functions, and summarize their analytic properties. We then list some useful consequences.

2.3.1. Ranking-Selberg L-functions on GL_2. We recall the basic theory of Rankin-Selberg convolution for GL_2. Given two primitive modular forms f and g of level r and r' respectively with trivial central character, the Rankin-Selberg L-function of f and g is a degree 4 Euler product

$$L(f \otimes g, s) = \prod_p L_p(f \otimes g, s) = \sum_{n\geqslant 1} \frac{\lambda_{f\otimes g}(n)}{n^s}, \ \mathfrak{Re}\, s > 1$$

such that, for $p \nmid rr'$, we have

$$L_p(f \otimes g, s) = \prod_{i,j=1}^{2} \Big(1 - \frac{\alpha_{f,i}(p)\alpha_{g,j}(p)}{p^s} \Big)^{-1}$$

and in general

$$L_p(f\otimes g,s)=\prod_i^4\Bigl(1-\frac{\alpha_{f\otimes g,i}(p)}{p^s}\Bigr)^{-1}.$$

In particular, $\lambda_{f\otimes g}(n)=\lambda_f(n)\lambda_g(n)$ for any n squarefree coprime with rr'. An exact description of all Dirichlet coefficients is given by Winnie Li in [54], but this is rather complicated.

By Rankin-Selberg theory, $L(f\otimes g,s)$ admits analytic continuation to $\mathbf{C}$ with at most one simple pole at $s=1$, which occurs if and only if $f=g$. This L-function satisfies a functional equation of the shape

$$\Lambda(f\otimes g,s)=\varepsilon(f\otimes g)\Lambda(f\otimes g,1-s)$$

with

$$\Lambda(f\otimes g,s)=r(f\otimes g)^{s/2}L_\infty(f\otimes g,s)L(f\otimes g,s)$$

where $r(f\otimes g)$ is a positive integer, $L_\infty(f\otimes g,s)$ is a product of Gamma factors and $\varepsilon(f\otimes g)=\pm1$. Moreover, as a consequence of the descriptions above and of the approximation to the Ramanujan-Petersson conjecture (cf. Section 2.3.5), for any prime p the local factor $L_p(f\otimes g,s)$ has no poles for $\mathfrak{Re}\,s\geqslant 1/2$.

In some of our applications, we will also encounter the Dirichlet series

$$L^*(f\otimes g,s)=\sum_{n\geqslant1}\frac{\lambda_f(n)\lambda_g(n)}{n^s},\tag{2.7}$$

initially defined in $\mathfrak{Re}\,s>1$. By the above discussion, it has holomorphic continuation to $\mathfrak{Re}\,s>1/2$, except for a pole at $s=1$ which exists if and only if $f=g$. If $f\neq g$, then

$$L^*(f\otimes g,s)=\frac{L(f\otimes g,s)}{\zeta^{(rr')}(2s)}\prod_{p\mid rr'}A_p(f,g;s),$$

for some correction factors $A_p(f,g;s)$ which have been computed explicitly by Winnie Li [54, §2, Th. 2.2] when f and g are both holomorphic. Here and throughout this book a superscript (r) denotes the removal of the Euler factors at primes dividing r.

Lemma 2.6. *For two newforms $f\neq g$ we have $L^*(f\otimes g,1)\neq0$.*

Proof. For $p\mid rr'$ we have

$$A_p(f,g;s)=L_p^*(f\otimes g,s)L_p(f\otimes g,s)^{-1}.$$

The factor $L_p^*(f\otimes g,s)$ is the inverse of a polynomial at p^{-s} (by multiplicativity), so doesn't vanish. On the other hand, it follows e.g. from results of Gelbart and Jacquet [25, Prop. 1.2, 1.4] that $L_p(f\otimes g,s)$ has no poles in $\mathfrak{Re}(s)\geqslant1$, so that $A_p(f,g;1)\neq0$. (More precisely, if one of the local representations of f or g at p is supercuspidal, then Prop. 1.2 in [25], and the fact that the central character is unitary, imply that all poles of $L_p(f\otimes g,s)$ satisfy $\mathfrak{Re}(s)=0$; on the other hand, if none of the local representation is supercuspidal, then Prop. 1.4 of [25] implies that $L_p(f\otimes g,s)$ is a product of GL_2-local factors, which have no poles for $\mathfrak{Re}(s)=1$, e.g. because they are products of at most two factors $1/(1-\alpha p^{-s})$ where $|\alpha|<p^{1/2}$ by elementary results towards the Ramanujan-Petersson conjecture, as recalled in Section 2.3.5). The lemma now follows from the fact that $L(f\otimes g,1)\neq0$ (see Proposition 2.11). $\square$

If $f = g$, then we define $L^*(\mathrm{Sym}^2 f, s)$ through the relation

$$\zeta^{(r)}(s)L^*(\mathrm{Sym}^2 f, s) = \zeta^{(r)}(2s)L^*(f \otimes f, s). \tag{2.8}$$

In particular, we have

$$L^*(\mathrm{Sym}^2 f, 1) = \zeta(2) \prod_{p|r}(1 + p^{-1}) \operatorname*{res}_{s=1} L^*(f \otimes f, s). \tag{2.9}$$

Using the formulas in [54, p. 145, Example 1], it follows that if r is squarefree, then we have

$$L^*(\mathrm{Sym}^2 f, 1) = L(\mathrm{Sym}^2 f, 1)$$

where $\mathrm{Sym}^2 f$ is the symmetric square; we will explain how to recover this fact (and describe the corresponding formulas if r is not squarefree) in Section 2.3.3, using the local Langlands correspondence.

2.3.2. Rankin-Selberg convolutions on GL_d. The previous examples are special cases of Rankin-Selberg L-functions attached to two general automorphic representations of $\mathrm{GL}_d(\mathbf{A_Q})$. The general theory is due to Jacquet–Piatetskii-Shapiro–Shalika [39], and we recall it briefly here.

Let $d, e \geqslant 1$ be integers, and let π, π' be automorphic cuspidal representations of $\mathrm{GL}_d(\mathbf{A_Q})$ and $\mathrm{GL}_e(\mathbf{A_Q})$, respectively, whose central characters ω, ω' are trivial on $\mathbf{R}_{>0}$. We denote by $\widetilde{\pi}$ and $\widetilde{\pi}'$ their contragredient representations.

The Rankin-Selberg L-function associated to π and π' is an Euler product, absolutely convergent for $\mathfrak{Re}\, s > 1$, of the form

$$\begin{aligned} L(\pi \otimes \pi', s) = \prod_p L_p(\pi \otimes \pi', s) &= \prod_p \prod_{i=1}^d \prod_{j=1}^e \Big(1 - \frac{\alpha_{\pi\otimes\pi',(i,j)}(p)}{p^s}\Big)^{-1} \\ &= \sum_{n \geqslant 1} \frac{\lambda_{\pi\otimes\pi'}(n)}{n^s},\ \mathfrak{Re}\, s > 1 \end{aligned}$$

such that, for p not dividing the product of the conductors $q(\pi)q(\pi')$, we have

$$\alpha_{\pi\otimes\pi',(i,j)}(p) = \alpha_{\pi,i}(p)\alpha_{\pi',j}(p)$$

where $\alpha_{\pi,i}(p)$ and $\alpha_{\pi',j}(p)$ are the local parameters of π, π' at the place p, i.e.

$$L_p(\pi, s) = \prod_{i=1}^d \Big(1 - \frac{\alpha_{\pi,i}(p)}{p^s}\Big)^{-1},\ L_p(\pi', s) = \prod_{j=1}^e \Big(1 - \frac{\alpha_{\pi',j}(p)}{p^s}\Big)^{-1}$$

are the local factors of the standard L-functions of π, π'.

When $e = 1$ and $\pi' = 1$ is the trivial representation, this Rankin-Selberg L-function is the standard L-function: we have then $L(\pi \otimes \pi', s) = L(\pi, s)$.

The Rankin-Selberg L-functions admit meromorphic continuation to $\mathbf{C}$, and satisfy a functional equations of the shape

$$\Lambda(\pi \otimes \pi', s) = \varepsilon(\pi \otimes \pi')\Lambda(\widetilde{\pi} \otimes \widetilde{\pi}', 1 - s)$$

with $|\varepsilon(\pi \otimes \pi')| = 1$ and

$$\Lambda(\pi \otimes \pi', s) = q(\pi \otimes \pi')^{s/2} L_\infty(\pi \otimes \pi', s) L(\pi \otimes \pi', s)$$

where $q(\pi\otimes\pi')\geqslant 1$ is an integer and

$$L_\infty(\pi\otimes\pi',s)=\prod_{i=1}^{d}\prod_{j=1}^{e}\Gamma_{\mathbf{R}}(s+\mu_{\pi\otimes\pi',(i,j)})$$

is a product of Gamma factors. The completed L-function $\Lambda(\pi\otimes\pi',s)$ is holomorphic on $\mathbf{C}$, unless $\pi'\simeq\widetilde{\pi}$, in which case it has simple poles at $s=0,1$.

If π and π' are not necessarily cuspidal, but are isobaric sums of cuspidal representations π_i and π'_j (whose central characters are trivial on $\mathbf{R}_{>0}$), say

$$\pi=\boxplus_i\mu_i\pi_i,\ \ \pi'=\boxplus_j\nu_j\pi'_j,\ \ \mu_i,\nu_j\geqslant 1,$$

then the Rankin-Selberg L-function exists and is given as the product

$$L(\pi\otimes\pi',s)=\prod_{i,j}L(\pi_i\otimes\pi'_j,s)^{\mu_i\nu_j}$$

so that analytic properties in the isobaric case are deduced immediately from the purely cuspidal case.

The local parameters of the Rankin-Selberg L-function enjoy the following additional properties:

(1) For a prime $p\nmid q(\pi)q(\pi')$, we have

$$\lambda_{\pi\otimes\pi'}(p)=\lambda_\pi(p)\lambda_{\pi'}(p).$$

(2) If $d=e$ and $\pi'=\widetilde{\pi}$ is the contragredient of π, then we have

$$\lambda_{\pi\otimes\widetilde{\pi}}(n)\geqslant 0$$

for all $n\geqslant 1$ (cf. [70, p. 318]).

(3) The archimedean local factor $L_\infty(\pi\otimes\pi',s)$ has no poles in the half-plane $\mathfrak{Re}\, s>1$, and likewise for any of the local factors $L_p(\pi\otimes\pi',s)$ for p prime, because of the absolute convergence of the series $L(\pi\otimes\pi',s)$ in this region.

To measure the complexity of an L-function, we use the *analytic conductor*, which is defined as the function

$$Q(\pi\otimes\pi',s)=q(\pi\otimes\pi')\prod_{i,j}(1+|\mu_{\pi\otimes\pi',(i,j)}+s|).$$

The conductor of a Rankin-Selberg L-function is controlled by that of the factors, more precisely we have

$$q(\pi\otimes\pi')\leqslant q(\pi)^{n'}q(\pi')^{n},\quad Q(\pi\otimes\pi',0)\leqslant Q(\pi,0)^{n'}Q(\pi',0)^{n} \tag{2.10}$$

for some n,n' (due to Bushnell and Henniart [8] for the non-archimedean part). Analogously, we will use the notation $Q(\pi)$ for the analytic conductor of π.

2.3.3. The symmetric square L-function. We return to the case $d=2$. When $f=g$ (of level r and with trivial central character), it is possible to factor the Rankin-Selberg L-function

$$L(f\otimes f,s)=\zeta(s)L(\mathrm{Sym}^2 f,s)$$

where $L(\mathrm{Sym}^2 f, s)$ is the symmetric square L-function of f. This is an Euler product of degree three given by

$$L(\mathrm{Sym}^2 f, s) = \prod_p L_p(\mathrm{Sym}^2 f, s) = \prod_p \prod_{i=1}^{3} \Big(1 - \frac{\alpha_{\mathrm{Sym}^2 f, i}(p)}{p^s}\Big)^{-1} = \sum_{n \geqslant 1} \frac{\lambda_{\mathrm{Sym}^2 f}(n)}{n^s},$$

for $\mathfrak{Re}\, s > 1$. For all $p \nmid r$, we have

$$\begin{aligned} L_p(\mathrm{Sym}^2 f, s) &= \frac{L_p(f \otimes f, s)}{(1 - p^{-s})^{-1}} \\ &= \Big(1 - \frac{\alpha_{f,1}(p)^2}{p^s}\Big)^{-1} \Big(1 - \frac{\alpha_{f,2}(p)^2}{p^s}\Big)^{-1} \Big(1 - \frac{\alpha_{f,1}\alpha_{f,2}(p)}{p^s}\Big)^{-1}. \end{aligned}$$

This L-function admits analytic continuation to $\mathbf{C}$ and satisfies a functional equation of the shape

$$\Lambda(\mathrm{Sym}^2 f, s) = \varepsilon(\mathrm{Sym}^2 f)\Lambda(\mathrm{Sym}^2 f, 1 - s)$$

with $\varepsilon(\mathrm{Sym}^2 f) = +1$ and

$$\Lambda(\mathrm{Sym}^2 f, s) = q(\mathrm{Sym}^2 f)^{s/2} L_\infty(\mathrm{Sym}^2 f, s) L(\mathrm{Sym}^2 f, s)$$

where $L_\infty(\mathrm{Sym}^2 f, s)$ is a product of Gamma factors. In fact, it was proved by Gelbart-Jacquet [25] that $L(\mathrm{Sym}^2 f, s)$ is the L-function of an automorphic representation on GL_3 over $\mathbf{Q}$, which we denote $\mathrm{Sym}^2 f$, and that $L(\mathrm{Sym}^2 f, s)$ is entire. This result also implies that the Rankin-Selberg L-function $L(f \otimes f, s)$ is the L-function of a (non-cuspidal) $\mathrm{GL}_4(\mathbf{A_Q})$-automorphic representation.

In some applications, as in our Chapter 9, it is of some importance to understand the precise relation between the automorphic symmetric square L-function of Gelbart-Jacquet and the "imprimitive" version $L^*(\mathrm{Sym}^2 f, s)$ defined by (2.8), i.e.

$$\zeta^{(r)}(2s) L^*(f \otimes f, s) = \zeta^{(r)}(s) L^*(\mathrm{Sym}^2 f, s).$$

Since it is quite complicated to track the literature concerning this point (especially when the level of f is not squarefree), we record the result in our case of interest, and sketch the proof using the local Langlands correspondance.

Let f be a primitive cusp form with trivial central character and level r. For any prime p, let π_p be the local representation of the automorphic representation corresponding to f. The following list enumerates the possibilities for π_p, the corresponding inverse L-factors at p, namely $L_p(\pi_p, s)^{-1}$ for the standard L-function, and $L_p(\mathrm{Sym}^2 f, s)^{-1}$ for the automorphic symmetric square L-function, and finally the "correction factor"

$$C_p = \frac{L_p^*(\mathrm{Sym}^2 f, s)}{L_p(\mathrm{Sym}^2 f, s)}.$$

(1) Unramified:

$L_p(\pi_p, s)^{-1}$	$(1 - \alpha_p p^{-s})(1 - \beta_p p^{-s}), \qquad \alpha_p \beta_p = 1$
$L_p(\mathrm{Sym}^2 f, s)^{-1}$	$(1 - \alpha_p^2 p^{-s})(1 - p^{-s})(1 - \beta_p^2 p^{-s})$
C_p	1

(2) Unramified up to quadratic twist ($\pi_p = \pi'_p \otimes \eta$ for some ramified quadratic character η and some unramified representation π'_p):

$$\begin{array}{c|l} L_p(\pi_p,s)^{-1} & 1 \\ L_p(\mathrm{Sym}^2 f,s)^{-1} & (1-(\alpha'_p)^2p^{-s})(1-p^{-s})(1-(\beta'_p)^2p^{-s}) \\ C_p & (1-(\alpha'_p)^2p^{-s})(1-p^{-s})(1-(\beta'_p)^2p^{-s}) \end{array}$$

(3) Steinberg:

$$\begin{array}{c|l} L_p(\pi_p,s)^{-1} & 1-\alpha_p p^{-s}, \qquad \alpha_p^2 = p^{-1} \\ L_p(\mathrm{Sym}^2 f,s)^{-1} & 1-p^{-1-s} \\ C_p & 1 \end{array}$$

(4) Steinberg up to a quadratic twist ($\pi_p = \sigma \otimes \eta$ for σ the Steinberg representation and some ramified quadratic character η):

$$\begin{array}{c|l} L_p(\pi_p,s)^{-1} & 1 \\ L_p(\mathrm{Sym}^2 f,s)^{-1} & 1-p^{-1-s} \\ C_p & 1-p^{-1-s} \end{array}$$

(5) Ramified principal series and not of Type (2):

$$\begin{array}{c|l} L_p(\pi_p,s)^{-1} & 1 \\ L_p(\mathrm{Sym}^2 f,s)^{-1} & 1-p^{-s} \\ C_p & 1-p^{-s} \end{array}$$

(6) Supercuspidal equal to its twist by the unramified quadratic character:

$$\begin{array}{c|l} L_p(\pi_p,s)^{-1} & 1 \\ L_p(\mathrm{Sym}^2 f,s)^{-1} & 1+p^{-s} \\ C_p & 1+p^{-s} \end{array}$$

(7) Supercuspidal not equal to its twist by the unramified quadratic character:

$$\begin{array}{c|l} L_p(\pi_p,s)^{-1} & 1 \\ L_p(\mathrm{Sym}^2 f,s)^{-1} & 1 \\ C_p & 1 \end{array}$$

REMARK 2.7. (1) In some references, only “twist minimal” representations are considered, i.e, those f which have minimal conductor among all their twists $f \otimes \chi$ by (all) Dirichlet characters. Cases (2), (4) and (5) cannot happen for such representations.

(2) All cases may happen for elliptic curves. Case (1) comes from good reduction, case (2) from good reduction up to a quadratic twist, (3) from semistable reduction, (4) from semistable reduction up to a quadratic twist, while (5), (6) and (7) can all come from potentially good reduction, with (7) only occurring at primes 2 and 3.

PROPOSITION 2.8. *The above list is correct and complete.*

PROOF. We use the local Langlands correspondance (due to Harris and Taylor [30]), and its compatibility with the symmetric square (due to Henniart [32]). The Langlands parameter corresponding to π_p is a two-dimensional Weil-Deligne representation representation V of $\mathbf{Q}_p$ with trivial determinant. The local L-factor is then

$$\det\bigl(1 - p^{-s}F \mid (\operatorname{Sym}^2 V)^{I_p,N}\bigr)^{-1},$$

where F is the Frobenius automorphism of $\mathbf{Q}_p$, I_p is the inertia subgroup of $W_{\mathbf{Q}_p}$, and N is the monodromy operator (whose invariant subspace is defined to be its kernel). The restriction of V to I_p is semisimple, and thus can be of three possible types:

(a) A sum of two copies of the same character.
(b) A sum of two different characters.
(c) A single irreducible character.

Only in case (a) can N act non-trivially, as N is nilpotent and commutes with I_p, and we will handle that separately.

We now compute the symmetric square and the Frobenius action on the inertia invariants in each case. It will be convenient to recall in some of the cases that, because V has trivial determinant, we can canonically identify $\operatorname{Sym}^2 V$ with the space of endomorphisms of V with trace zero.

Case (a) – N trivial. Because the determinant is trivial, the inertia character that appears must be either quadratic or trivial. In this case the representation is unramified, potentially after a quadratic twist. This gives cases (1) and (2); the L-factor calculation is well-known (see, e.g., [**25**, Section 3.5], or [**11**, p. 107, case 1] on the Galois side).

Case (a) – N nontrivial. By the same logic, the character is quadratic or trivial. Then the inertia representation is trivial, potentially after a quadratic twist, and the associated smooth representation is Steinberg, potentially after a quadratic twist. This gives cases (3) and (4), and the L-factor calculation is also well-known (e.g., the GL_2 case can be found in [**24**, Th. 6.15], and the symmetric square factor is computed, on the Galois side, in [**11**, p. 107, case 2], recalling that the symmetric square is unchanged by a quadratic twist).

Case (b). The inertia invariants of the adjoint representation form a one-dimensional space. The Frobenius action on this space defines a one-dimensional unramified character η, which is either trivial or nontrivial. It is trivial if and only if there is a non-scalar endomorphism of the whole representation, i.e., if it fails to be irreducible, or in other words if the corresponding smooth representation is a principal series. In this case, the Frobenius action on inertia invariants is trivial, so the factor is $1/(1-p^{-s})$. This is case (5). If η is non-trivial, then we have an isomorphism $V \otimes \eta \to V$, which taking determinants implies that η is quadratic. Hence we have the corresponding isomorphism on the automorphic side, and the L-factor is $1/(1-p^{-s})$. This is case (6).

Case (c). The space of inertia invariants of the adjoint representation vanishes. Then the local L-factor is 1, and so the representation has no nontrivial endomorphisms and thus is irreducible, hence the corresponding automorphic representation is supercuspidal, and has L-factor 1. This is case (7). □

2.3.4. Symmetric power L-functions. More generally, for any integer $k \geqslant 1$, one can form the symmetric k-th power L-function $L(\operatorname{Sym}^k f, s)$, which is an

Euler product of degree $k+1$, namely

$$L(\mathrm{Sym}^k f, s) = \prod_p L_p(\mathrm{Sym}^k, f, s) = \prod_p \prod_{i=0}^{k} (1 - \alpha_{\mathrm{Sym}^k f, i} p^{-s})^{-1} = \sum_{n \geqslant 1} \frac{\lambda_{\mathrm{Sym}^k f}(n)}{n^s}$$

and for $p \nmid r$,

$$L_p(\mathrm{Sym}^k, f, s) = \prod_{i=0}^{k} \Big(1 - \frac{\alpha_{f,1}^i(p) \alpha_{f,2}^{k-i}(p)}{p^s}\Big)^{-1}.$$

The analytic continuation of these Euler products is not known in general. For $k = 3$ and $k = 4$, Kim and Shahidi [44, 45] have proven that $L(\mathrm{Sym}^k f, s)$ is the L-function of a self-dual automorphic (not-necessarily cuspidal) representation of GL_{k+1} and in particular it admits analytic continuation to $\mathbf{C}$ and satisfies a functional equation of the usual shape:

$$\Lambda(\mathrm{Sym}^k f, s) = \varepsilon(\mathrm{Sym}^k f) \Lambda(\mathrm{Sym}^k f, 1-s)$$

where $\varepsilon(\mathrm{Sym}^k f) = \pm 1$ and

$$\Lambda(\mathrm{Sym}^2 f, s) = q(\mathrm{Sym}^k f)^{s/2} L_\infty(\mathrm{Sym}^k f, s) L(\mathrm{Sym}^k f, s),$$

and again $L_\infty(\mathrm{Sym}^k f, s)$ is a product of Gamma factors.

We summarize the results of Kim and Shahidi, as well as those of Gelbart and Jacquet that were already mentioned, as follows.

For $k \leqslant 4$ the L-function $L(\mathrm{Sym}^k f, s)$ is the L-function of an automorphic representation $\mathrm{Sym}^k f$ of $\mathrm{GL}_{k+1}(\mathbf{A_Q})$. The representation $\mathrm{Sym}^k f$ decomposes into an isobaric sum

$$\mathrm{Sym}^k f \simeq \boxplus_{j=1}^{n_{f,k}} \mu_j \pi_j \tag{2.11}$$

where $n_{f,k} \geqslant 1$ and (π_j) are cuspidal automorphic representations on $\mathrm{GL}_{d_j}(\mathbf{A_Q})$. This implies

$$L(\mathrm{Sym}^k f, s) = \prod_j L(\pi_j, s)^{\mu_j}.$$

The decomposition (2.11) satisfies $\sum_j \mu_j d_j = k+1$. The automorphic representation $\mathrm{Sym}^k \pi$ is self-dual, hence its decomposition into isotypical components is invariant by taking contragredient, i.e., the multiset $\{(\mu_j, \pi_j) \mid j \leqslant n_{f,k}\}$ is invariant under contragredient. Moreover, for every $1 \leqslant j \leqslant n_{f,k}$, we have

$$\pi_j \simeq \tilde{\pi}_j \quad \text{or} \quad d_j \leqslant 2.$$

We now list the precise possibilities for the decomposition. Let π be the automorphic representation associated to f. It is self-dual with trivial central character, and $\mathrm{Sym}^k f = \mathrm{Sym}^k \pi$. If π is of CM-type, then $\pi \otimes \eta \simeq \pi$ for a nontrivial quadratic Dirichlet character η, which determines a quadratic extension $E/\mathbf{Q}$, and there exists a Größencharacter χ of E such that $\pi = \pi(\chi)$ (the automorphic induction of χ). Write χ' for the conjugate of χ by the nontrivial element of $\mathrm{Gal}(E/\mathbf{Q})$. Then we have:

- $\mathrm{Sym}^2 \pi = \pi(\chi^2) \boxplus \eta$,
- $\mathrm{Sym}^3 \pi = \pi(\chi^3) \boxplus \pi(\chi^2 \chi')$,
- $\mathrm{Sym}^4 \pi = \pi(\chi^4) \boxplus \pi(\chi^3 \chi') \boxplus \mathbf{1}$.

The individual terms $\pi(\chi^a\chi'^b)$ either remain cuspidal, and have unitary central character, or split into the Eisenstein series of two unitary characters.

If π is not of CM-type, then the automorphic representations $\mathrm{Sym}^2\pi$, $\mathrm{Sym}^3\pi$ and $\mathrm{Sym}^4\pi$ are all cuspidal and self-dual, with central character trivial on $\mathbf{R}_{>0}$.

In either case, we conclude

COROLLARY 2.9. *The automorphic representations π_j on $\mathrm{GL}_{d_j}(\mathbf{A_Q})$ of the isobaric decomposition of $\mathrm{Sym}^k\pi$ have unitary central characters trivial on $\mathbf{R}_{>0}$. They satisfy either $\pi_j \simeq \tilde{\pi}_j$, or $d_j \leqslant 2$.*

REMARK 2.10. One can check that this corollary remains true for any cuspidal automorphic representation π, even if the central character of π is non-trivial. However, checking this requires the consideration of more cases, since π could be of polyhedral type.

2.3.5. The Ramanujan-Petersson conjecture and its approximation. The *Ramanujan-Petersson conjecture* at unramified places predicts optimal bounds for the local parameters of f (equivalently a pole free region for the local L-factors), namely it predicts that

$$\begin{aligned} |\alpha_{f,i}(p)| &\leqslant 1,\ \ i=1,2, \\ \mathfrak{Re}\,\mu_{f,i} &\leqslant 0,\ \ i=1,2. \end{aligned} \tag{2.12}$$

This would imply that for any $(n, q(f)) = 1$ one has

$$|\lambda_f(n)| \leqslant d(n) \tag{2.13}$$

where $d(n)$ is the divisor function.

If f is holomorphic, the Ramanujan-Petersson conjecture is known by the work of Deligne [14]. Moreover, it is known that (2.12) holds for any prime $p \mid q(f)$, and so (2.13) holds for all integers n.

The results on the functoriality of the symmetric power L-functions $L(\mathrm{Sym}^k f, s)$ mentioned above together with Rankin-Selberg theory imply that the Ramanujan-Petersson conjecture is true on average in a strong form: for any $x \geqslant 1$ and $\varepsilon > 0$, we have

$$\sum_{n\leqslant x}\Big(|\lambda_f(n^4)|^2 + |\lambda_f(n^2)|^4 + |\lambda_f(n)|^8\Big) \ll_f x^{1+\varepsilon}, \tag{2.14}$$

where the implied constant depends on ε, and also

$$\sum_{n\leqslant x}|\lambda_f(n)|^2 \ll_f x \tag{2.15}$$

This implies that

$$|\alpha_{f,i}(p)| \leqslant p^{1/8},\ \ i=1,2.$$

With additional more sophisticated arguments, Kim and Sarnak [43] have obtained the currently best approximation to the Ramanujan-Petersson conjecture. For $\theta = 7/64$, we have

$$\begin{aligned} |\alpha_{f,i}(p)| &\leqslant p^{\theta} \\ \mathfrak{Re}\,\mu_{f,i} &\leqslant \theta,\ \ i=1,2, \end{aligned}$$

and therefore, for any $n \geqslant 1$, we have

$$|\lambda_f(n)| \leqslant d(n)n^{\theta}. \tag{2.16}$$

On the other hand, for $p \mid r$, we have (cf. e.g. [60, Theorem 4.6.17] or more generally [24])

$$|\lambda_f(p)| = p^{-1/2} \text{ or } \lambda_f(p) = 0. \tag{2.17}$$

For the rest of the book the letter $\theta \leqslant 7/64$ is reserved for an admissible exponent towards the Ramanujan-Petersson conjecture.

2.4. Prime Number Theorems

By "Prime Number Theorems" we mean the problem of evaluating asymptotically certain sums over the primes of arithmetic functions associated to Hecke eigenvalues of f and g. The main tool for this is the determination of zero-free regions of the relevant L-functions. We first state a general result concerning the zero-free domain for Rankin-Selberg L-functions.

PROPOSITION 2.11. *Let π and π' be irreducible cuspidal automorphic representations of $\mathrm{GL}_d(\mathbf{A_Q})$ and $\mathrm{GL}_e(\mathbf{A_Q})$. Assume that the central characters ω_π and $\omega_{\pi'}$ are unitary and trivial on $\mathbf{R}_{>0}$ and either:*

(1) *At least one of π or π' is a GL_1-twist of a self-dual representation, possibly the trivial one, or*
(2) *$d \leqslant 3$ and $e \leqslant 2$, or vice-versa.*

Then, there is an explicitly computable constant $c = c(d, e) > 0$ such that the Rankin–Selberg L-function $L(\pi \times \pi', \sigma + it)$ has no zeros in the region

$$\sigma > 1 - \frac{c}{\log(Q(\pi)Q(\pi')(|t| + 2))} \tag{2.18}$$

except for at most one exceptional simple Landau-Siegel real zero < 1. Such a zero may only occur if $\pi \otimes \pi'$ is self-dual, i.e., if $\tilde{\pi} \otimes \tilde{\pi}' \simeq \pi \otimes \pi'$ as admissible representations.

PROOF. If π and π' are both self-dual, then this is a result of Moreno [61, Theorem 3.3]. If only one of the two is self-dual it was observed by Sarnak that Moreno's method extends [72]. However we could not find a proof of this in the literature and we take this opportunity to report a proof kindly provided by F. Brumley. We assume that π' is self-dual and that π is not (in particular $\pi' \not\simeq \pi, \tilde{\pi}$). Given a non-zero real number t, consider the isobaric representation

$$\Pi = (\pi \otimes |\cdot|^{-it}) \boxplus (\tilde{\pi} \otimes |\cdot|^{it}) \boxplus \pi'$$

and its Rankin-Selberg L-function

$$L(s) = L(\Pi \otimes \tilde{\Pi}, s).$$

This L-function factors as a product of the following nine L-functions:

$$\begin{gathered} L(\pi \otimes \tilde{\pi}, s), \quad L(\pi \otimes \tilde{\pi}, s), \quad L(\pi' \otimes \pi', s), \\ L(\pi \otimes \pi, s + 2it), \quad L(\pi \otimes \pi', s + it), \\ L(\tilde{\pi} \otimes \tilde{\pi}, s - 2it), \quad L(\tilde{\pi} \otimes \pi', s - it), \\ L(\pi \otimes \pi', s + it), \quad L(\tilde{\pi} \otimes \pi', s - it). \end{gathered}$$

Also by construction the coefficients of $-L'/L(s)$ are non-negative so that we can use the Goldfeld-Hoffstein-Lieman Lemma [36, Lemma 5.9].

The L-function $L(s)$ has a pole of order 3 at $s = 1$. On the other hand, suppose that $L(\pi \otimes \pi', \sigma + it) = 0$ for $\sigma < 1$ satisfying (2.18). Then $L(s)$ vanishes to order at least 4 at σ (the two factors $L(\pi\otimes\pi', s+it)$ and the two factors $L(\tilde{\pi}\otimes\pi', s-it)$), thus contradicting the Goldfeld-Hoffstein-Lieman Lemma if c is small enough, depending on d, e.

Suppose now that neither π nor π' are self-dual up to GL_1-twists. If $d = e = 2$, then the result follows from the functorial lift $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_4$ of Ramakrishnan [67, Theorem M], according to which there exists an isobaric automorphic representation $\pi \boxtimes \pi'$ of $\mathrm{GL}_4(\mathbf{A}_{\mathbf{Q}})$ (with unitary central character trivial on $\mathbf{R}^+$) such that

$$L(s, \pi \boxtimes \pi') = L(s, \pi \times \pi').$$

If $d = 2$ and $e = 3$ this follows from the functorial lift

$$\mathrm{GL}_2 \times \mathrm{GL}_3 \to \mathrm{GL}_6$$

established by Kim and Shahidi [44]. $\square$

Remark 2.12. (1) This result covers the case when at least one of π or π' is a GL_1-twist of the self-dual representation by passing the twist to the other factor.

In particular, this contains the case where (say) $\pi' = 1$ is the (self-dual) trivial representation, that is the standard zero-free region

$$\sigma > 1 - \frac{c}{\log(Q(\pi)(|t| + 2))}$$

for the standard L-function $L(\pi, s)$ of any cuspidal representation, except for the possible Landau–Siegel zero if $\pi \simeq \tilde{\pi}$.

(2) In our actual applications in this book, we will apply the result only to a finite set of auxiliary L-functions (depending on the given cusp forms f and g, which are fixed), hence the issue of Landau-Siegel zeros is not an important one, as long as we have a standard zero-free region in t-aspect.

From this, we deduce the next result.

Proposition 2.13. *Let f, g be primitive cusp forms of levels r, r' with trivial central character. There exists an absolute constant $c > 0$ such that for $k, k' \leqslant 4$ the Rankin-Selberg L-function $L(\mathrm{Sym}^k f \otimes \mathrm{Sym}^{k'} g, s)$ has no zeros in the domain*

$$\mathfrak{Re}\, s \geq 1 - \frac{c}{\log(Q(f)Q(g)(2 + |s|))}$$

except for possible real zeros < 1.

Proof. In terms of the isobaric decompositions (2.11) of f and g given in Section 2.3.4, we have

$$L(\mathrm{Sym}^k f \otimes \mathrm{Sym}^{k'} g, s) = \prod_{i,j} L(\pi_i \otimes \pi'_j, s)^{\mu_i \nu_j}.$$

It will then be sufficient to prove the result for each factor $L(\pi_i \otimes \pi'_j, s)$, since

$$Q(\pi_i \otimes \pi'_j) \leqslant (Q(\pi_i)Q(\pi'_j))^{O(1)} \leqslant (Q(\mathrm{Sym}^k f)Q(\mathrm{Sym}^{k'} f))^{O(1)} \leqslant (Q(f)Q(g))^{O(1)}$$

by (2.10) (and [44, 45]). By Corollary 2.9 we see that at least one of the two sufficient conditions of Proposition 2.11 is always satisfied. $\square$

We now spell out several corollaries which are deduced from these zero-free domains by standard techniques. The first one concerns upper and lower bounds for values of this L-function in the zero-free region:

COROLLARY 2.14. *Let f, g be primitive cusp forms of levels r, r' with trivial central character. For $0 \leqslant k, k' \leqslant 4$, there exist two constants $0 < c = c_{f,g} < 1/10$ and $A = A_{f,g} \geqslant 0$ such that for s satisfying*

$$\mathfrak{Re}\, s \geqslant 1 - \frac{c}{\log(2+|s|)}$$

the following bounds hold:

$$\log^{-A}(2+|s|) \ll \Big|\frac{s-1}{s}\Big|^{\varrho} L(\mathrm{Sym}^k f \otimes \mathrm{Sym}^{k'} g, s) \ll \log^{A}(2+|s|)$$

where

$$\varrho = \varrho_{f,g,k,k'} = \operatorname*{ord}_{s=1} L(\mathrm{Sym}^k f \otimes \mathrm{Sym}^{k'} g, s) \geqslant 0$$

is the order of the pole of $L(\mathrm{Sym}^k f \otimes \mathrm{Sym}^{k'} g, s)$ at $s = 1$ and the implicit constants depends on f and g only. Here we also make the convention that for $k = k' = 0$, we have $L(\mathrm{Sym}^k f \otimes \mathrm{Sym}^{k'} g, s) = \zeta(s)$.

The second corollary concerns the versions of the Prime Number Theorem that can be deduced from these zero-free regions:

COROLLARY 2.15. *Let f, g be primitive cusp forms of levels r, r' with trivial central character. Let $0 \leqslant k, k' \leqslant 4$. There exists a constant $C > 0$ such that:*

(1) *There exist $\gamma_{k,k'} \in \mathbf{R}$ and an integer $m_{k,k'} \geqslant 0$ (possibly also depending on f, g) such that for any $x \geqslant 2$, we have*

$$\sum_{p \leqslant x} \lambda_{\mathrm{Sym}^k f}(p)\lambda_{\mathrm{Sym}^{k'} g}(p)\log p = m_{k,k'}x + O(x\exp(-C\sqrt{\log x}))$$

$$\sum_{p \leqslant x} \lambda_{\mathrm{Sym}^k f}(p)\lambda_{\mathrm{Sym}^{k'} g}(p)\frac{\log p}{p} = m_{k,k'}\log x + \gamma_{k,k'} + O\Big(\frac{1}{\log x}\Big).$$

(2) *There exists $\gamma'_{k,k'} \in \mathbf{R}$ and an integer $n_{k,k'} \geqslant 0$ (possibly depending on f, g) such that*

$$\sum_{p \leqslant x} \lambda_f(p)^k \lambda_g(p)^{k'} \log p = n_{k,k'}x + O(x\exp(-C\sqrt{\log x})), \tag{2.19}$$

$$\sum_{p \leqslant x} \lambda_f(p)^k \lambda_g(p)^{k'} \frac{\log p}{p} = n_{k,k'}\log x + \gamma'_{k,k'} + O\Big(\frac{1}{\log x}\Big), \tag{2.20}$$

and for $2 \leqslant x \leqslant y/2$ we have

$$\sum_{x \leqslant p \leqslant y} \frac{\lambda_f(p)^k \lambda_g(p)^{k'}}{p\log p} = \Big(n_{k,k'} + O\Big(\frac{1}{\log x}\Big)\Big)\Big(\frac{1}{\log x} - \frac{1}{\log y}\Big). \tag{2.21}$$

In these estimates, the implied constants depend on f and g only.

PROOF. The first two equalities are deduced from the zero free region for $L(\mathrm{Sym}^k f \otimes \mathrm{Sym}^{k'} g, s)$ (see for instance Liu–Ye [55]).

The remaining ones follow by partial summation, using the decompositions

$$\lambda_f(p) = \lambda_{\mathrm{Sym}^1 f}(p),$$
$$\lambda_f(p)^2 = \lambda_f(p^2) + 1 = \lambda_{\mathrm{Sym}^2 f}(p) + 1,$$
$$\lambda_f(p)^3 = \lambda_f(p^3) + 2\lambda_f(p) = \lambda_{\mathrm{Sym}^3 f}(p) + 2\lambda_{\mathrm{Sym}^1 f}(p),$$
$$\lambda_f(p)^4 = \lambda_f(p^4) + 3\lambda_f(p^2) + 2 = \lambda_{\mathrm{Sym}^4 f}(p) + 3\lambda_{\mathrm{Sym}^2 f}(p) + 2$$

for $p \nmid rr'$, which reflect the decomposition of tensor powers of the standard representation of SL_2 in terms of irreducible representation (in particular, all coefficients are non-negative integers). $\square$

REMARK 2.16. From

$$\begin{aligned}\lambda_f(p)^2\lambda_g(p)^2 &= (\lambda_{\mathrm{Sym}^2 f}(p)+1)(\lambda_{\mathrm{Sym}^2 g}(p)+1)\\ &= \lambda_{\mathrm{Sym}^2 f}(p)\lambda_{\mathrm{Sym}^2 g}(p)+\lambda_{\mathrm{Sym}^2 f}(p)+\lambda_{\mathrm{Sym}^2 g}(p)+1\end{aligned}$$

for $p \nmid rr'$, we see that

$$n_{2,2} = m_{2,2}+m_{2,0}+m_{0,2}+1 \geqslant 1,$$

and similarly

$$n_{4,4} = m_{4,4}+3m_{4,2}+3m_{2,4}+2m_{4,0}+2m_{0,4}+9m_{2,2}+3m_{2,0}+3m_{0,2}+4 \geqslant 4.$$

We will also need a variant. We denote by λ_f^* and λ_g^* any multiplicative functions such that

$$\lambda_f^*(p) = \lambda_f(p)+O(p^{\theta-1}), \quad \lambda_g^*(p) = \lambda_g(p)+O(p^{\theta-1}), \tag{2.22}$$

where the implied constants depend on f and g. (Note that these functions may depend on both f and g).

COROLLARY 2.17. *The estimates (2.19), (2.20) and (2.21) are valid with λ_f, λ_g replaced by λ_f^*, λ_g^*, with the same integers $n_{k,k'}$, but with possibly different values for $\gamma_{k,k'}$.*

PROOF. It suffices to verify (2.19). Since $|\lambda_f(p)|, |\lambda_g(p)| \leqslant 2p^\theta$, we have

$$\lambda_f^*(p)^k\lambda_g^*(p)^{k'} = \lambda_f(p)^k\lambda_g(p)^{k'}+O(p^{(k+k')\theta-1})$$

and, since $(k+k')\theta \leqslant 8\theta < 1$, the difference between

$$\sum_{p\leqslant x}\lambda_f^*(p)^k\lambda_g^*(p)^{k'}\log p \quad\text{and}\quad \sum_{p\leqslant x}\lambda_f(p)^k\lambda_g(p)^{k'}\log p$$

is $\ll x\exp(-C\sqrt{\log x})$. $\square$

2.5. Consequences of the functional equations

The functional equation satisfied by an L-function makes it possible to obtain (by inverse Mellin transform) either a representation of its values by rapidly converging smooth sums (this is called, somewhat improperly, the "approximate functional equation"), or identities between rapidly converging smooth sums of these coefficients (an example is the Voronoi summation formula). We discuss the versions of these identities that we need in this section.

2.5.1. Approximate functional equations. The following proposition is obtained by specializing [36, Thm. 5.3, Prop. 5.4] to twisted L-functions and to the product of two twisted L-functions, using the functional equations of Lemmas 2.1 and 2.3. Again, we recall that we use the convention 1.3 about the signed level of a Maaß form.

PROPOSITION 2.18. *Let f, g be two primitive cusp forms of signed levels r and r' coprime to q, where $f = g$ is possible. Given any $A > 2$, let $G = G_A$ be the holomorphic function defined in the strip $|\mathfrak{Re}\, u| < 2A$ by*

$$G(u) = \Big(\cos \frac{\pi u}{4A}\Big)^{-16A}. \tag{2.23}$$

Let $s \in \mathbf{C}$ be such that $1/4 < \mathfrak{Re}\, s < 3/4$, and let χ be a primitive Dirichlet character modulo q, with parity $\chi(-1) = \pm 1$.

(1) *We have*

$$\begin{aligned} L(f \otimes \chi, s) = \sum_{m \geqslant 1} \frac{\lambda_f(m)}{m^s} \chi(m) V_{f,\pm,s}\Big(\frac{m}{q\sqrt{|r|}}\Big) \\ + \varepsilon(f, \pm, s) \sum_{m \geqslant 1} \frac{\lambda_f(m)}{m^{1-s}} \overline{\chi}(m) V_{f,\pm,1-s}\Big(\frac{m}{q\sqrt{|r|}}\Big), \end{aligned} \tag{2.24}$$

where

$$\varepsilon(f, \pm, s) = \varepsilon(f \otimes \chi)(q^2|r|)^{\frac{1}{2}-s} \frac{L_\infty(f, \pm, 1-s)}{L_\infty(f, \pm, s)}$$

and

$$V_{f,\pm,s}(y) = \frac{1}{2\pi i} \int_{(2)} \frac{L_\infty(f, \pm, s+u)}{L_\infty(f, \pm, s)} G(u) y^{-u} \frac{du}{u}.$$

(2) *We have*

$$\begin{aligned} L(f \otimes \chi, s)\overline{L(g \otimes \chi, s)} = \sum_{m,n \geqslant 1} \frac{\lambda_f(m)\lambda_g(n)}{m^s n^{\overline{s}}} \chi(m)\overline{\chi}(n) W_{f,g,\pm,s}\left(\frac{mn}{q^2|rr'|}\right) \\ + \varepsilon(f, g, \pm, s)\chi(r\overline{r}') \sum_{m,n \geqslant 1} \frac{\lambda_f(m)\lambda_g(n)}{m^{1-s} n^{1-\overline{s}}} \overline{\chi}(m)\chi(n) W_{f,g,\pm,1-s}\left(\frac{mn}{q^2|rr'|}\right), \end{aligned} \tag{2.25}$$

where

$$\varepsilon(f, g, \pm, s) = \varepsilon(f)\varepsilon(g)(q^2 r)^{\frac{1}{2}-s}(q^2 r)^{\frac{1}{2}-\overline{s}} \frac{L_\infty(f, \pm, 1-s)}{L_\infty(f, \pm, s)} \frac{L_\infty(g, \pm, 1-\overline{s})}{L_\infty(g, \pm, \overline{s})} \tag{2.26}$$

(again these expressions depend only on the parity of χ) and

$$W_{f,g,\pm,s}(y) = \frac{1}{2\pi i} \int_{(2)} \frac{L_\infty(f, \pm, s+u)}{L_\infty(f, \pm, s)} \frac{L_\infty(g, \pm, \overline{s}+u)}{L_\infty(g, \pm, \overline{s})} G(u) y^{-u} \frac{du}{u}. \tag{2.27}$$

Note that the Hecke eigenvalues $\lambda_f(n), \lambda_g(n)$ are real. Also note the special cases

$$\varepsilon(f, \pm, \tfrac{1}{2}) = \varepsilon(f \otimes \chi) = \varepsilon(f)\chi(r)\varepsilon_\chi^2, \qquad \varepsilon(f, g, \pm, \tfrac{1}{2}) = \varepsilon(f)\varepsilon(g). \tag{2.28}$$

We need to record some decay properties for $V_{f,\pm,s}, W_{f,g,\pm,s}$ and their derivatives.

Shifting the contour to $\mathfrak{Re}\, u = A$ or $\mathfrak{Re}\, u = -(\sigma - \theta) + \varepsilon$ for $\varepsilon > 0$ and using Stirling's formula, we have

LEMMA 2.19. *Assume that $\sigma = \mathfrak{Re}\, s \in]1/4, 3/4[$. For any integer $j \geqslant 0$ any $y > 0$, we have*

$$V_{f,\pm,s}(y) - 1 \ll (y/|s|)^{\sigma-\theta-\varepsilon}, \ W_{f,g,\pm,s}(y) - 1 \ll (y/|s|^2)^{\sigma-\theta-\varepsilon} \tag{2.29}$$

and

$$y^j V_{f,\pm,s}^{(j)}(y) \ll (1 + y/|s|)^{-A}, \ y^j W_{f,g,\pm,s}^{(j)}(y) \ll (1 + y/|s|^2)^{-A}$$

where the constant implied depends on f, g, ε and j (where applicable).

CONVENTION 2.20. In most of this book, we will only treat in detail averages over the *even* characters, since the odd case is entirely similar. To simplify notation, we may then write $\varepsilon(f,s)$, $V_{f,s}$ and $W_{f,g,s}$ in place of $\varepsilon(f,+,s)$, $V_{f,+,s}$ and $W_{f,g,+,s}$. Moreover, for $s=1/2$, we may simplify further, and write V_f and $W_{f,g}$ in place of $V_{f,1/2}$ and $W_{f,g,1/2}$.

2.5.2. The Voronoi summation formula. The next lemma is a version of the Voronoi formula.

LEMMA 2.21. *Let q be a positive integer and a an integer coprime to q, and let W be a smooth function compactly supported in $]0,\infty[$. Let f a primitive cusp form of signed level r coprime with q and trivial central character. For any real number $N>0$, we have*

$$\sum_{n\geqslant 1}\lambda_f(n)W\Big(\frac{n}{N}\Big)e\Big(\frac{an}{q}\Big)=\varepsilon(f)\sum_{\pm}\frac{N}{q|r|^{1/2}}\sum_{n\geqslant 1}\lambda_f(n)e\Big(\mp\frac{\overline{a|r|}n}{q}\Big)\widetilde{W}_{\pm}\Big(\frac{Nn}{q^2|r|}\Big)$$

with

$$\widetilde{W}_{\pm}(y)=\int_0^\infty W(u)\mathcal{J}_{\pm}(4\pi\sqrt{uy})du,$$

where (1) *for f holomorphic of weight k_f we write*

$$\mathcal{J}_+(u)=2\pi i^{k_f}J_{k_f-1}(u),\quad \mathcal{J}_-(u)=0;$$

(2) *for f a Maaß form with Laplace eigenvalue $(\frac12+it_f)(\frac12-it_f)$ and reflection eigenvalue $\varepsilon_f=\pm1$ we write*

$$\mathcal{J}_+(u)=\frac{-\pi}{\sin(\pi t_f)}(J_{2it_f}(u)-J_{-2it_f}(u)),\quad \mathcal{J}_-(u)=4\varepsilon_f\cosh(\pi t_f)K_{2it_f}(u).$$

See [51, Theorem A.4] for the proof. Note that $\widetilde{W}_{\pm}$ depends on the archimedean parameters of f, which we suppress from the notation. In particular, the passage from a smooth weight function W to $\widetilde{W}_{\pm}$ may increase the set of parameters $\mathscr{S}$, cf. Section 2.1.

Let $K\colon \mathbf{Z}\to\mathbf{C}$ be a q-periodic function. Its *normalized Fourier transform* is the q-periodic function defined by

$$\widehat{K}(h)=\frac{1}{\sqrt{q}}\sum_{n\,(\mathrm{mod}\,q)}K(n)e\Big(\frac{nh}{q}\Big)$$

for $h\in\mathbf{Z}$. The *Voronoi transform* of K is the q-periodic function defined by

$$\widecheck{K}(n)=\frac{1}{\sqrt{q}}\sum_{\substack{h\,(\mathrm{mod}\,q)\\(h,q)=1}}\widehat{K}(h)e\Big(\frac{\bar{h}n}{q}\Big)$$

for $n\in\mathbf{Z}$ (see [21, §2.2]). Combining the Voronoi formula above with the discrete Fourier inversion formula

$$K(n)=\frac{1}{\sqrt{q}}\sum_{a\,(\mathrm{mod}\,q)}\widehat{K}(a)e\Big(-\frac{an}{q}\Big),$$

we deduce:

COROLLARY 2.22. *Let q be a prime number. Let W be a smooth function compactly supported in $]0,\infty[$. Let f be a primitive cusp form of signed level r coprime with q. For any real number $N>0$, we have*

$$\sum_n \lambda_f(n)K(n)W\Big(\frac{n}{N}\Big) = \frac{\widehat{K}(0)}{q^{1/2}}\sum_{n\geqslant 1}\lambda_f(n)W\Big(\frac{n}{N}\Big)+ \varepsilon(f)\sum_{\pm}\frac{N}{q|r|^{1/2}}\sum_{n\geqslant 1}\lambda_f(n)\widecheck{K}(\pm\overline{|r|}n)\widetilde{W}_\pm\Big(\frac{nN}{q^2|r|}\Big)$$

In particular, for any integer a coprime to q, if we take

$$K(n)=q^{1/2}\delta_{n\equiv a\,(\mathrm{mod}\,q)}=\begin{cases} q^{1/2} & \text{if } n\equiv a\,(\mathrm{mod}\,q)\\ 0 & \text{otherwise,}\end{cases}$$

then we have

$$q^{1/2}\sum_{n\equiv a\,(\mathrm{mod}\,q)}\lambda_f(n)W\Big(\frac{n}{N}\Big) = \frac{1}{q^{1/2}}\sum_{n\geqslant 1}\lambda_f(n)W\Big(\frac{n}{N}\Big)+ \varepsilon(f)\sum_{\pm}\frac{N}{q|r|^{1/2}}\sum_{n\geqslant 1}\lambda_f(n)\widetilde{W}_\pm\Big(\frac{nN}{q^2|r|}\Big)\mathrm{Kl}(\pm a\overline{|r|}n;q).$$

Finally, we recall the decay properties of the Bessel transforms $\widetilde{W}_\pm$ which follow from repeated integration by parts and the decay properties of of Bessel functions and their derivatives. These are proved in [4, Lemma 2.4].

LEMMA 2.23. *Let W be a smooth function compactly supported in $[1/2,2]$ and satisfying (2.1). In the Maaß case set $\vartheta=|\mathfrak{Re}\, it|$, otherwise set $\vartheta=0$. For $M\geqslant 1$ let $W_M(x)=W(x/M)$. For any ε, for any $i,j\geqslant 0$ and for all $y>0$, we have*

$$y^j\,\widetilde{(W_M)}_\pm^{(j)}(y)\ll_{i,j,\varepsilon} M(1+My)^{j/2}\big(1+(My)^{-2\vartheta-\varepsilon}\big)\big(1+(My)^{1/2}\big)^{-i}.$$

In particular, the functions $\widetilde{(W_M)}_\pm(y)$ decay rapidly when $y\gg 1/M$.

We recall that all implied constants may depend polynomially on the parameters $s\in\mathscr{S}$ that W and $\tilde{W}$ depend on.

2.6. A factorization lemma

To shorten notations, let us write

$$T(s)=L(f\otimes f,s). \tag{2.30}$$

We denote by $(\mu_f(n))_{n\geqslant 1}$ the convolution inverse of $(\lambda_f(n))_{n\geqslant 1}$, which is given by

$$L(f,s)^{-1}=\prod_p\Big(1-\frac{\lambda_f(p)}{p^s}+\frac{\chi_r(p)}{p^{2s}}\Big)=\sum_{n\geqslant 1}\frac{\mu_f(n)}{n^s},\quad \mathfrak{Re}\, s\geqslant 1. \tag{2.31}$$

We then define an auxiliary function of six complex variables by

$$L(s,z,z',u,v,w)=\sum\sum_{\substack{d,\ell_1,\ell_2,n\\(\ell_1,\ell_2)=(d\ell_1\ell_2,r)=1}}\frac{\mu_f(d\ell_1)\lambda_f(\ell_1 n)\mu_f(d\ell_2)\lambda_f(\ell_2 n)}{\ell_1^{s+z+u+v}\ell_2^{s+z'+u+w}d^{z+z'+v+w}n^{2s+2u}}. \tag{2.32}$$

In Chapters 6 and 8, we will use the following lemma.

LEMMA 2.24. *For $\eta \in \mathbf{R}$, let $\mathcal{R}(\eta)$ be the open subset of $(s, z, z', u, v, w) \in \mathbf{C}^6$ defined by the inequalities*

$$\mathcal{R}(\eta) := \begin{cases} \mathfrak{Re}\, s > \frac{1}{2} - \eta, \ \mathfrak{Re}\, z > \frac{1}{2} - \eta, & \mathfrak{Re}\, z' > \frac{1}{2} - \eta, \\ \mathfrak{Re}\, u > -\eta, \ \mathfrak{Re}\, v > -\eta, & \mathfrak{Re}\, w > -\eta. \end{cases}$$

There exists $\eta > 0$ and a holomorphic function $D(s, z, z', u, v, w)$ defined on $\mathcal{R}(\eta)$ such that D is absolutely bounded on $\mathcal{R}(\eta)$, and such that the holomorphic function $L(s, z, z', u, v, w)$ admits meromorphic continuation to $\mathcal{R}(\eta)$ and satisfies the equality

$$L(s, z, z', u, v, w) = \frac{T(2s + 2u)T(z + z' + v + w)}{T(s + z + u + v)T(s + z' + u + w)} D(s, z, z', u, v, w).$$

As a special case:

COROLLARY 2.25. *The function $(u, v, w) \mapsto L(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, u, v, w)$ initially defined as a convergent holomorphic series over a domain of the shape*

$$\mathfrak{Re}\, u, \mathfrak{Re}\, v, \mathfrak{Re}\, w \gg 1$$

extends meromorphically to the domain

$$\mathfrak{Re}\, u, \mathfrak{Re}\, v, \mathfrak{Re}\, w > -\eta$$

for some absolute constant $\eta > 0$ and satisfies

$$\begin{aligned} L(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, u, v, w) &= \frac{T(1 + 2u)T(1 + v + w)}{T(1 + u + v)T(1 + u + w)} D(u, v, w) \\ &= \eta_3(f, u, v, w) \frac{(u + v)(u + w)}{u(v + w)}. \end{aligned}$$

where

- *D is an Euler product absolutely convergent for $\mathfrak{Re}\, u, \mathfrak{Re}\, v, \mathfrak{Re}\, w \geqslant -\eta$,*
- *η_3 is holomorphic and non-vanishing in a neighborhood of $(u, v, w) = (0, 0, 0)$.*

PROOF OF LEMMA 2.24. The function μ_f is multiplicative and satisfies

$$\mu_f(p) = -\lambda_f(p), \ \mu_f(p^2) = \chi_r(p), \ \mu_f(p^k) = 0 \text{ for } k \geqslant 3.$$

By (2.14), the series (2.32) is absolutely convergent in the intersection

$$\mathcal{C} := \begin{cases} \mathfrak{Re}(s + z + u + v) > 1, & \mathfrak{Re}(s + z' + u + w) > 1, \\ \mathfrak{Re}(z + z' + v + w) > 1, & \mathfrak{Re}(2s + 2u) > 1 \end{cases}$$

of four half spaces of $\mathbf{C}^6$. In particular, the region $\mathcal{C}$ contains the region $\mathcal{R}(0)$ of $\mathbf{C}^6$.

In this region we have the factorization

$$L(s, z, z', u, v, w) = \prod_p L_p(s, z, z', u, v, w)$$

where

$$L_p(s, z, z', u, v, w) = \sum\sum_{\substack{\delta, \lambda_1, \lambda_2, \nu \geqslant 0 \\ \lambda_1 \lambda_2 = 0}} \frac{\mu_f(p^{\delta + \lambda_1})\lambda_f(p^{\lambda_1 + \nu})\mu_f(p^{\delta + \lambda_2})\lambda_f(p^{\lambda_2 + \nu})}{p^{\lambda_1(s+z+u+v) + \lambda_2(s+z'+u+w) + \delta(z+z'+v+w) + \nu(2s+2u)}}$$

for $p \nmid r$. For $\alpha \geqslant 0$ and $\theta = 7/64$, we have

$$|\lambda_f(p^\alpha)| \leqslant (\alpha+1)p^{\alpha\theta}, \tag{2.33}$$

hence the factor $L_p(s,z,z',u,v,w)$ is absolutely convergent for (s,z,z,u,v,w) such that

$$\mathfrak{Re}(z+z'+v+w) > 2\theta, \qquad \mathfrak{Re}(s+z+u+v) > 2\theta,$$
$$\mathfrak{Re}(s+z'+u+w) > 2\theta, \qquad \mathfrak{Re}(s+u) > \theta.$$

This includes the region $\mathcal{R}(\eta)$ with $\eta := \theta/4$. Now splitting the summation over the set of $(\delta, \lambda_1, \lambda_2, \nu)$ with $0 \leqslant \delta + \lambda_1 + \lambda_2 + \nu \leqslant 1$ and the complementary set, we see that for $(s,z,z',u,v,w) \in \mathcal{R}(\theta/4)$ we have the equality

$$\begin{aligned} (2.34)\quad & L_p(s,z,z',u,v,w) \\ &= 1 + \frac{\lambda_f(p)^2}{p^{z+z'+v+w}} - \frac{\lambda_f(p)^2}{p^{s+z+u+v}} - \frac{\lambda_f(p)^2}{p^{s+z'+u+w}} + \frac{\lambda_f(p)^2}{p^{2s+2u}} + \mathrm{EL}_p(s,z,z',u,v,w) \end{aligned}$$

with $\mathrm{EL}_p(s,z,z',u,v,w)$ holomorphic in that region and satisfying

$$\mathrm{EL}_p(s,z,z',u,v,w) = O\Big(\frac{p^{2\theta}}{p^{2(1-\theta)}}\Big) = O\Big(\frac{1}{p^{2-4\theta}}\Big) \tag{2.35}$$

where the implied constant is absolute.

We consider now the multivariable Dirichlet series

$$M(s,z,z',u,v,w) := \frac{T(2s+2u)T(z+z'+v+w)}{T(s+z+u+v)T(s+z'+u+w)}.$$

In the region $\mathcal{C}$, it is absolutely convergent and factors as

$$M(s,z,z',u,v,w) = \prod_p M_p(s,z,z',u,v,w)$$

where

$$M_p(s,z,z',u,v,w) = \frac{T_p(2s+2u)T_p(z+z'+v+w)}{T_p(s+z+u+v)T_p(s+z'+u+w)}.$$

Let us recall that for any p we have

$$T_p(s) = \zeta_p(s)\prod_{i=1}^{3}\Big(1 - \frac{\alpha_{\mathrm{Sym}^2 f,i}(p)}{p^s}\Big)^{-1}.$$

with

$$|\alpha_{\mathrm{Sym}^2 f,i}(p)| \leqslant p^{2\theta}; \tag{2.36}$$

in particular $T_p(s)$ is holomorphic and non-vanishing for $\mathfrak{Re}\, s > 2\theta$. Moreover, for $p \nmid r$, we have

$$T_p(s) = (1 - \frac{1}{p^{2s}})^{-1}\sum_{\alpha\geqslant 0}\frac{\lambda_f(p^\alpha)^2}{p^{\alpha s}} = 1 + \frac{\lambda_f(p)^2}{p^s} + \sum_{\alpha\geqslant 2}\frac{\xi_f(p^\alpha)}{p^{\alpha s}}$$

where the coefficients $\xi_f(p^\alpha)$ satisfy

$$|\xi_f(p^\alpha)| = |\lambda_f(p^\alpha)^2 + \lambda_f(p^{\alpha-2})^2 + \cdots| \leqslant (\alpha+1)^3 p^{2\alpha\theta},$$

by (2.33). Hence, by the same reasoning as before, we have for $p \nmid r$ the equality

$$\begin{aligned}(2.37)\quad & M_p(s,z,z',u,v,w)\\ &= 1+\frac{\lambda_f(p)^2}{p^{z+z'+v+w}}-\frac{\lambda_f(p)^2}{p^{s+z+u+v}}-\frac{\lambda_f(p)^2}{p^{s+z'+u+w}}+\frac{\lambda_f(p)^2}{p^{2s+2u}}\\ &\qquad + \mathrm{EM}_p(s,z,z',u,v,w)\end{aligned}$$

with $\mathrm{EM}_p(s,z,z',u,v,w)$ holomorphic in $\mathcal{R}(\theta/4)$ and satisfying

$$\mathrm{EM}_p(s,z,z',u,v,w)=O\Big(\frac{p^{2\theta}}{p^{2(1-\theta)}}\Big)=O\Big(\frac{1}{p^{2-4\theta}}\Big). \tag{2.38}$$

Let $P\geqslant 1$ be a parameter to be chosen sufficiently large; given some converging Euler product

$$L=\prod_p L_p$$

we set

$$L_{\leqslant P}=\prod_{p\leqslant P}L_p,\ \ L_{>P}=\prod_{p>P}L_p$$

so that

$$L=L_{\leqslant P}L_{>P}.$$

We apply this decomposition to $L(s,z,z',u,v,w)$ for $P>r$. In the region of absolute convergence, we have

$$L(s,z,z',u,v,w)=L_{\leqslant P}(s,z,z',u,v,w)L_{>P}(s,z,z',u,v,w).$$

We write

$$L_{>P}(s,z,z',u,v,w)=M_{>P}(s,z,z',u,v,w)D_{>P}(s,z,z',u,v,w)$$

where

$$D_{>P}(s,z,z',u,v,w)=\prod_{p>P}\frac{L_p(s,z,z',u,v,w)}{M_p(s,z,z',u,v,w)}$$

By (2.37) and (2.38) we can choose $P>|r|$ sufficiently large so that for $p>P$, $M_p(s,z,z',u,v,w)^{-1}$ is holomorphic in the region $\mathcal{R}(\theta/4)$, then by (2.34), (2.35) (2.37) and (2.38) we have, in that same region the equality

$$\frac{L_p(s,z,z',u,v,w)}{M_p(s,z,z',u,v,w)}=1+O\Big(\frac{1}{p^{2(1-3\theta)}}+\frac{1}{p^{2-4\theta}}\Big).$$

Since $2-6\theta>1$ the product

$$D_{>P}(s,z,z',u,v,w)=\prod_{p>P}\Big(1+O\Big(\frac{1}{p^{2(1-3\theta)}}\Big)\Big)$$

is absolutely convergent and uniformly bounded in the region $\mathcal{R}(\theta/4)$. We now write the finite product

$$L_{\leqslant P}(s,z,z',u,v,w)=M_{\leqslant P}(s,z,z',u,v,w)D_{\leqslant P}(s,z,z',u,v,w).$$

By (2.36) the finite product

$$D_{\leqslant P}(s,z,z',u,v,w)=\prod_{p\leqslant P}\frac{L_p(s,z,z',u,v,w)}{M_p(s,z,z',u,v,w)}$$

is holomorphic and uniformly bounded in the region $\mathcal{R}(\theta/4)$ and

$$D(s,z,z',u,v,w)=D_{\leqslant P}(s,z,z',u,v,w)D_{>P}(s,z,z',u,v,w)$$

has the required properties. □

2.7. A shifted convolution problem

The objective of this section is to adapt the work of Blomer and Milićević [6] to prove a variant of the shifted convolution problem that is required in this book. The following result is proved in loc. cit. in the case of cusp forms of level one. Since, the generalization to arbitrary fixed (signed) level r is straightforward, we will only briefly indicate the changes that are required.

Most of the notation in this section is borrowed from [6], except that the modulus which is denoted q in this book is denoted d in loc. cit.

PROPOSITION 2.26. *Let $\ell_1,\ell_2\geqslant 1$ two integers, $q\geqslant 1$ and $N\geqslant M\geqslant 1$. Let f_1,f_2 be two primitive cusp forms of signed levels r_1 and r_2 and Hecke eigenvalues $(\lambda_1(n))_{n\geqslant 1}$ and $(\lambda_2(m))_{m\geqslant 1}$, respectively. Assume that $(\ell_1\ell_2,r_1r_2)=1$. Let V_1,V_2 be fixed smooth weight functions satisfying (2.1). Then for $\theta=7/64$, we have*

$$\sum_{\substack{\ell_2m-\ell_1n\equiv 0\,(\mathrm{mod}\,q)\\ \ell_2m-\ell_1n\neq 0}}\lambda_1(m)\lambda_2(n)V_1\left(\frac{\ell_2m}{M}\right)V_2\left(\frac{\ell_1n}{N}\right)$$
$$\ll (qN)^{\varepsilon}\left(\left(\frac{N}{q^{1/2}}+\frac{N^{3/4}M^{1/4}}{q^{1/4}}\right)\left(1+\frac{(NM)^{1/4}}{q^{1/2}}\right)+\frac{M^{3/2+\theta}}{q}\right)$$

uniformly in ℓ_1,ℓ_2, with an implied constant depending on f_1,f_2 and the parameters $\mathscr{S}$ that V_1,V_2 depend on. The same bound holds if the congruence condition $\ell_2m-\ell_1n\equiv 0\,(\mathrm{mod}\ q)$ is replaced by

$$\ell_2m+\ell_1n\equiv 0\,(\mathrm{mod}\ q).$$

PROOF. If $N\asymp M$, we write $\ell_2m\pm\ell_1n=hq$ with $0\neq h\ll M/q$. For each value of h, we use [3] to bound the corresponding shifted convolution sum by $M^{1/2+\theta+\varepsilon}$, so that we get a total contribution of

$$\ll q^{-1}M^{3/2+\theta+\varepsilon}.$$

If $N\geqslant 20M$, say, then the bound is a straightforward adaptation of [6, Proposition 8] to cusp forms with general level. The key observation is that Jutila's circle method allows us to impose extra conditions on the moduli c. It is easiest to work with moduli c such that $r_1r_2\mid c$ (the condition $(c,r_1r_2)=1$ would also do the job). With this in mind, we follow the argument and the notation of [6, Sections 7 and 8]. We replace the definition [6, (7.1)] with $Q=(N|r_1r_2|)^{1000}$ (note that this has no influence on the dependency of the implied constant on the levels, since an important feature of Jutila's method is the fact that Q enters the final bound only as Q^{ε}). The definition of the weight function w in [6, (7.5)] is non-trivial only for $\ell_1\ell_2r_1r_2\mid c$, so that

$$\Lambda\gg C^2\frac{\varphi(\ell_1\ell_2|r_1r_2|)}{(\ell_1\ell_2|r_1r_2|)^2}$$

in [6, (7.6)]. From there, the argument proceeds identically with the Voronoi summation formula and the Kuznetsov formula for level $|r_1r_2|\ell_1\ell_2$. In [6, (8.1)], we put $\beta=\mathrm{lcm}(\ell_1,\ell_2,d,r_1,r_2)$. Again the argument proceeds verbatim as before. Wilton's bound in [6, Section 8.2] is polynomial in the level, see [28, Proposition 5]. The rest of the argument remains unchanged, except that the level of the relevant subgroup for the spectral decomposition in [6](7.14) and below is $\Gamma_0(\ell_1\ell_2|r_1r_2|)$ instead of $\Gamma_0(\ell_1\ell_2)$; as a consequence, the sum over δ before and after [6, (8.8)] must be over $\delta \mid \ell_1\ell_2r_1r_2$.

The changes that are required to handle the congruence $\ell_2 m + \ell_1 n \equiv 0 \bmod q$ are explained in Section 11 of [6]. $\square$

2.8. Partition of unity

We will use partitions of unity repeatedly in order to decompose a long sum over integers into smooth localized sums (see e.g. [17, Lemme 2]).

Lemma 2.27. *There exists a smooth non-negative function $W(x)$ supported on $[1/2,2]$ and satisfying (2.1) such that*

$$\sum_{k\geqslant 0} W\Big(\frac{x}{2^k}\Big) = 1$$

for any $x \geqslant 1$.

CHAPTER 3

Algebraic exponential sums

In this chapter, we will first summarize elementary orthogonality properties of Dirichlet characters, then state and sketch some ideas of the proofs of bilinear estimates with Kloosterman sums. These are the core results that we use in all main results of this book. In Sections 3.4 and 3.5, which are only used later in Sections 4.3 and 6.7, we discuss briefly trace functions over finite fields, and the equidistribution properties of their discrete Mellin transforms (following Katz [40]).

3.1. Averages over Dirichlet characters

Let q be an odd prime. Given a function τ defined on Dirichlet characters modulo q, we will write

$$\begin{aligned}
\sideset{}{^+}\sum_{\chi \,(\mathrm{mod}\, q)} \tau(\chi) &= \sum_{\chi \,(\mathrm{mod}\, q)} \frac{1+\chi(-1)}{2}\tau(\chi),\\
\sideset{}{^-}\sum_{\chi \,(\mathrm{mod}\, q)} \tau(\chi) &= \sum_{\chi \,(\mathrm{mod}\, q)} \frac{1-\chi(-1)}{2}\tau(\chi),\\
\sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \tau(\chi) &= \sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi \text{ primitive}}} \tau(\chi)
\end{aligned}$$

for the sum of τ over even (resp. odd, primitive) Dirichlet characters modulo q.

We recall the basic orthogonality relations

$$\begin{aligned}
\frac{1}{\varphi(q)}\sum_{\chi} \chi(m)\overline{\chi(n)} &= \delta_{(mn,q)=1}\delta_{m\equiv n \,(\mathrm{mod}\, q)},\\
\frac{2}{\varphi(q)}\sideset{}{^+}\sum_{\chi} \chi(m)\overline{\chi(n)} &= \delta_{(mn,q)=1}\delta_{m\equiv \pm n \,(\mathrm{mod}\, q)}.
\end{aligned} \tag{3.1}$$

As in (1.9), we denote

$$\varepsilon_\chi = \frac{1}{q^{1/2}}\sum_{h \,(\mathrm{mod}\, q)} \chi(h) e\Big(\frac{h}{q}\Big)$$

the normalized Gauß sum of a character χ modulo q. If $\chi = \chi_q$ is the trivial character, then we have $\varepsilon_{\chi_q} = -q^{-1/2}$.

Since we are interested in the distribution of root numbers, we will need to handle moments of the Gauß sums. These are well-known (see, e.g., [36, Proof of Th. 21.6]): for any integer $k \geqslant 1$ and $(m,q)=1$, we have

$$\frac{1}{\varphi(q)}\sum_{\chi} \chi(m)\varepsilon_\chi^k = q^{-1/2}\mathrm{Kl}_k(\overline{m};q),$$

where

$$\mathrm{Kl}_k(m;q) = \frac{1}{q^{\frac{k-1}{2}}} \sum_{x_1\cdots x_k = m \,(\mathrm{mod}\, q)} e\Big(\frac{x_1+\cdots+x_k}{q}\Big) \tag{3.2}$$

is the normalized hyper-Kloosterman sum modulo q. Consequently, we have

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\, q)} \chi(m)\varepsilon_\chi^k = q^{-1/2}\mathrm{Kl}_k(\overline{m};q) + O(q^{-1-|k|/2}). \tag{3.3}$$

This formula remains true for $k=0$ if we define

$$\mathrm{Kl}_0(m;q) = q^{1/2}\delta_{m=1\,(\mathrm{mod}\, q)}.$$

Moreover, since

$$\varepsilon_\chi = \overline{\varepsilon_\chi}^{-1} = \chi(-1)\varepsilon_{\overline{\chi}}^{-1}$$

for primitive characters, the formula (3.3) extends to negative k when we define

$$\mathrm{Kl}_k(m;q) := \mathrm{Kl}_{|k|}((-1)^k\overline{m};q), \quad k \leqslant -1.$$

Similarly, we obtain

$$\frac{2}{\varphi^*(q)} \sideset{}{^+}\sum_{\chi \text{ primitive}} \chi(m)\varepsilon_\chi^k = \frac{1}{q^{1/2}}\sum_{\pm}\mathrm{Kl}_k(\pm\overline{m};q) + O(q^{-1-|k|/2}). \tag{3.4}$$

for the sum restricted to even characters only.

The following deep bound of Deligne is essential at many points, in particular it implies the equidistribution of angles of Gauß sums.

Proposition 3.1 (Deligne). *Let k be a non-zero integer. For any prime q and any integer m coprime to q, we have*

$$|\mathrm{Kl}_k(m;q)| \leqslant |k|.$$

Remark 3.2. To avoid confusion, we will never use the notation k to refer to a finite field.

This was proved by Deligne [14,15] as a consequence of his general form of the Riemann Hypothesis over finite fields. For $k=2$, it is simply the Weil bound for classical Kloosterman sums.

3.2. Bounds for Kloosterman sums

In this section we recall various bounds for sums of Kloosterman sums which will be required in some of our applications. For a prime q and an integer a coprime with q, we define

$$B(\mathrm{Kl}_2,\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{m\leqslant M,}\sum_{n\leqslant N} \alpha_m\beta_n\,\mathrm{Kl}_2(amn;q), \tag{3.5}$$

where $\boldsymbol{\alpha} = (\alpha_m)_{1\leqslant m\leqslant M}$, $\boldsymbol{\beta} = (\beta_n)_{1\leqslant n\leqslant N}$ are sequences of complex numbers. We write

$$\|\boldsymbol{\alpha}\|_2^2 = \sum_{m\leqslant M}|\alpha_m|^2, \quad \|\boldsymbol{\beta}\|_2^2 = \sum_{n\leqslant N}|\beta_n|^2.$$

The following bound is a special case of a result of Fouvry, Kowalski and Michel [18, Thm. 1.17].

Proposition 3.3. *For any $\varepsilon > 0$, we have*

$$B(\mathrm{Kl}_2, \boldsymbol{\alpha}, \boldsymbol{\beta}) \ll_\varepsilon q^\varepsilon \|\boldsymbol{\alpha}\|_2 \|\boldsymbol{\beta}\|_2 (MN)^{1/2} \Big(\frac{1}{q^{1/4}} + \frac{1}{M^{1/2}} + \frac{q^{1/4}}{N^{1/2}}\Big),$$

uniformly for $(a, q) = 1$.

We will also need the following bound which is a special case of another result of Fouvry, Kowalski and Michel [20, Thm. 1.2 and (1.3)]:

Proposition 3.4. *For any primitive cusp form f with trivial central character and level r, and any smooth function W satisfying (2.1), we have*

$$\sum_{n \geqslant 1} \lambda_f(n) \mathrm{Kl}_k(an; q) W\Big(\frac{n}{N}\Big) \ll_\varepsilon q^{\varepsilon + 1/2 - 1/8} N^{1/2} \Big(1 + \frac{N}{q}\Big)^{1/2}$$

for any $(a, q) = 1$, any $k \in \mathbf{Z} - \{0\}$, any integer $N \geqslant 1$ and any $\varepsilon > 0$, where the implied constant depends polynomially on f and k (and the parameters $\mathscr{S}$ that W depends on).

The last estimate we require was conjectured by Blomer, Fouvry, Kowalski, Michel and Milićević in [4], and was proved by Kowalski, Michel and Sawin [52, Thm. 1.1]:

Proposition 3.5. *Suppose that $M, N \geqslant 1$ satisfy*

$$1 \leqslant M \leqslant N q^{1/4}, \quad MN \leqslant q^{5/4}.$$

For any $\varepsilon > 0$, we have

$$B(\mathrm{Kl}_2, \boldsymbol{\alpha}, \boldsymbol{\beta}) \ll_\varepsilon q^\varepsilon \|\boldsymbol{\alpha}\|_2 \|\boldsymbol{\beta}\|_2 (MN)^{\frac{1}{2}} \left(\frac{1}{M^{1/2}} + q^{-\frac{1}{64}} \Big(\frac{q}{MN}\Big)^{\frac{3}{16}}\right), \tag{3.6}$$

uniformly for $(a, q) = 1$.

Remark 3.6. (1) The point of this result, in comparison with Proposition 3.3 (which applies in much greater generality than [52]) is that it is non-trivial even in ranges where $MN < q$. More precisely, Proposition 3.5 gives a non-trivial estimate as soon as $MN \geqslant q^{7/8+\delta}$ for some $\delta > 0$.

(2) Notice that we may always assume in addition that $MN > q^{1/4}$ (as in [52]), since otherwise the bound (3.6) is implied by the trivial bound

$$B(\mathrm{Kl}_2, \boldsymbol{\alpha}, \boldsymbol{\beta}) \ll \|\boldsymbol{\alpha}\|_2 \|\boldsymbol{\beta}\|_2 (MN)^{1/2}.$$

3.3. Sketch of the arguments

We summarize here the key ideas of the proofs of the estimates of the previous section. We hope that this informal discussion will be helpful to readers yet unfamiliar with the tools involved in the use of trace functions and of Deligne's form of the Riemann Hypothesis over finite fields.

Trace functions modulo a prime q are functions on $\mathbf{F}_q$ attached to ℓ-adic sheaves on the affine line or on the multiplicative group over $\mathbf{F}_q$ (here ℓ is a prime number different from q). We will not recall precise definitions of trace functions (see [19] for an accessible survey), but we note that Kl_2 is a fundamental example of a trace function, and will give further examples later.

Proposition 3.3 is proved by the use of the Cauchy-Schwarz inequality to eliminate the arbitrary coefficients β_n, and then by completing the sum in the n variable to obtain sums over $\mathbf{F}_q$. This reduces the proof to the estimation of correlation sums

$$\sum_{n\in\mathbf{F}_q} \mathrm{Kl}_2(am_1n;q)\,\mathrm{Kl}_2(am_2n;q)e\left(\frac{hn}{q}\right).$$

In the generalization of this problem considered in [18], where Kl_2 is replaced with a general trace function, such sums are estimated using Deligne's most general form of the Riemann hypothesis over finite fields [15]. This argument gives square-root cancellation for sums of trace functions over algebraic curves, as long as an associated cohomology group vanishes, and this vanishing reduces to an elementary problem of representation theory for the geometric monodromy group of the sheaves associated to these trace functions (in fact, one that does not require knowing precisely what the monodromy group is).

However, a number of special cases, including this one, have a more elementary proof. In this case, the bound follows directly from Weil's bound for Kloosterman sums, since one can check elementarily that

$$\sum_{n\in\mathbf{F}_q} \mathrm{Kl}_2(am_1n;q)\,\mathrm{Kl}_2(am_2n;q)e\left(\frac{hn}{q}\right) = \sum_{y\in\mathbf{F}_q^\times} e\Big(\frac{am_2\bar{y}-am_1\overline{(h+y)}}{q}\Big)$$

which, for $h\in\mathbf{F}_q^\times$, is equal to

$$q^{1/2}e\left(\frac{-a\bar{h}(m_1+m_2)}{q}\right)\mathrm{Kl}_2\left(\frac{a^2m_1m_2}{h^2};q\right)$$

(such an identity is to be expected since $\mathrm{Kl}_2(a;q)$ is a discrete Fourier transform of $x\mapsto e(\bar{x}/q)$, so that the correlation sum can be evaluated by the discrete Plancherel formula).

Proposition 3.4 is proven using the amplification method. This involves amplifying over modular forms, which means that the sum is enlarged dramatically to a sum of similar expressions over a basis $\mathcal{B}$ of Hecke eigenforms g of the space of modular forms of the same weight as f and of level pr (both holomorphic and non-holomorphic), so that the Petersson-Kuznetsov formula may be applied. Here we view f, a form of level r, as being of level pr, and we can assume that $f\in\mathcal{B}$.

To get a nontrivial bound using this approach, it is necessary to insert an amplifier $A(g)$, which is here a weighted sum of Hecke eigenvalues, of the form

$$A(g)=\sum_{\ell\leqslant L}\alpha_\ell\lambda_g(\ell),$$

for $g\in\mathcal{B}$, chosen so that $A(g)$ is "large". We hope to get an upper bound for

$$\Sigma=\sum_{g\in\mathcal{B}}|A(g)|^2\Big|\sum_{n\geqslant 1}\lambda_g(n)\mathrm{Kl}_k(an;q)W\Big(\frac{n}{N}\Big)\Big|^2,$$

in order to claim by positivity that

$$\Big|\sum_{n\geqslant 1}\lambda_f(n)\mathrm{Kl}_k(an;q)W\Big(\frac{n}{N}\Big)\Big|^2\leqslant\frac{\Sigma}{|A(f)|^2}.$$

The application of the Kuznetsov formula produces a complicated sum on the arithmetic side. In the off-diagonal terms of the amplified sums, we see that correlation sums of the following shapes

$$\mathcal{C}(\mathrm{Kl}_2;\gamma)=\sum_{z\in\mathbf{F}_q}\widehat{\mathrm{Kl}_2}(\gamma\cdot z;q)\overline{\widehat{\mathrm{Kl}_2}(z;q)},$$

appear, for certain quite specific $\gamma\in\mathrm{PGL}_2(\mathbf{F}_q)$. In the generalized version, one again uses Deligne's Theorem to estimate such sums; this involves separating the possible γ for which there is no square-root cancellation, and exploiting the fact that they are very rare, except for very special input sheaves, and cannot coincide too often with the "special" γ that occur in the application of the Kuznetsov formula.

Here also, in the special case of the Kloosterman sums that we are dealing with, whose Fourier transform is $x\mapsto e(\bar{x}/q)$, the estimate for correlation sums reduce to Weil's bound for Kloosterman sums (see [20, 1.5(3)]). In the simplest case of Dirichlet characters, this method was pioneered by Bykovskiĭ [9], cf. also [5]. However, for general hyper-Kloosterman sums Kl_k, it seems very unlikely that a similarly elementary argument exists to prove this bound.

Finally, the proof of Proposition 3.5 is by far the most difficult and involves highly non-trivial algebraic geometry. In particular, it uses heavily some special properties of Kloosterman sums, and does not apply to an arbitrary trace function (although one can certainly expect that a similar result should be true for any trace function that is not an additive character times a multiplicative character, in which case it is trivially false).

The completion step in this case should be thought of as primarily an analogue of the Burgess bound [7] for short sums of Dirichlet characters. Similarly to the standard proof of the Burgess bound, we use the multiplicative structure of the function $\mathrm{Kl}_2(amn)$ to bound sums over intervals of length smaller than $\sqrt{q}$ by reducing them to high moments of sums over even shorter intervals, which themselves can be controlled by more complicated complete sums.

More precisely, we begin in the same way as in the proof of Proposition 3.3 by applying the Cauchy-Schwarz inequality to eliminate the coefficients β_n. However, the resulting sum over n is now too short to be usefully completed directly. Instead, we apply the Burgess argument in the form of the "shift by ab" trick of Karatsuba and Vinogradov. This ends up reducing the problem to the estimation of certain complete exponential sums in three variables. The key result that we need to prove (a special case of [52, Theorem 2.6]) is the following:

THEOREM 3.7. *For a prime q, for $r\in\mathbf{F}_q$, $\lambda\in\mathbf{F}_q$ and $\boldsymbol{b}=(b_1,\ldots,b_4)\in\mathbf{F}_q^4$, let*

$$R(r,\lambda,\boldsymbol{b})=\sum_{s\in\mathbf{F}_q^\times}e\left(\frac{\lambda a}{q}\right)\prod_{i=1}^{2}\mathrm{Kl}_2(s(r+b_i))\,\mathrm{Kl}_2(s(r+b_{i+2})).$$

Then we have

$$\sum_{r\in\mathbf{F}_q}R(r,\lambda_1,\boldsymbol{b})\overline{R(r,\lambda_2,\boldsymbol{b})}=q^2\delta(\lambda_1,\lambda_2)+O(q^{3/2})\tag{3.7}$$

for all $\boldsymbol{b}$, except those that satisfy a certain non-trivial polynomial equation $Q(\boldsymbol{b})=0$ of degree bounded independently of q.

A key difference with the Burgess bound is that, whereas the Weil bound for multiplicative character sums over curves which is used there gives square-root cancellation outside of an explicit and very small set of diagonal parameters, the exceptional set of parameter values $\boldsymbol{b}$ in Theorem 3.7 is not explicit, and is also relatively large (it has codimension one). This is the main difficulty in generalizing the bound (3.6) to shorter ranges, since in order to do so, we must take higher moments of short sums, leading to complete sums of more variables, for which even best-possible estimates are not helpful *unless* one can show that the codimension of the diagonal locus diminishes proportionally to the exponent.

However, this difficulty is not significant for the applications in this book, since we need the bound (3.6) only in the case where M and N are very close to $\sqrt{q}$. In this case, using higher moments would not give better results, even if the analogue of Theorem 3.7 was obtained with an exceptional locus of the highest possible codimension (as in the Burgess case).

We now give a longer but still informal summary of the techniques behind Theorem 3.7, which involve simpler exponential sum estimates, topology, elementary representation theory, and simple arguments with Galois representations, as well as more technical steps based on vanishing cycles.

The proof of the theorem begins by constructing (in [52, §4.1]) a sheaf $\mathcal{R}$ on $\mathbf{A}^6_{\mathbf{F}_q}$ whose trace function exponential sum R, which depends on 6 variables $r, \lambda, \boldsymbol{b}$. This is proven using the ℓ-adic machinery in a relatively formal way, exploiting known sheaf-theoretic analogues of the algebraic operations involved in the definition of R. One begins with a fundamental result of Deligne (related to Proposition 3.1) which implies that there is a sheaf (of conductor bounded in terms of q only) with trace function equal to hyper-Kloosterman sums; then taking tensor products of two sheaves multiplies their trace functions, and the sum over s is obtained by computing sheaf cohomology (precisely, computing a higher direct image with compact support of a 7-variable sheaf). Both the result of Deligne and the step where we sum over s involve key results in étale cohomology, such as the Grothendieck–Lefschetz trace formula.

At this point, we apply Katz's Diophantine Criterion for Irreducibility and Deligne's Riemann Hypothesis (in [52, Theorem 4.11 and (3.4)]) . These imply that the bound (3.7) holds for a given $\boldsymbol{b}$ *if and only if* the sheaves $\mathcal{R}_{\boldsymbol{b},\lambda}$ in one variable r obtained by specializing the parameters $(\boldsymbol{b}, \lambda)$ of the $\mathcal{R}$ are geometrically irreducible, and that $\mathcal{R}_{\boldsymbol{b},\lambda_1}$ and $\mathcal{R}_{\boldsymbol{b},\lambda_2}$ are geometrically non-isomorphic for $\lambda_1 \neq \lambda_2$. We will use this equivalence in both directions. We note that proving the second part is easier, because there are in general many ways to prove that two sheaves are non-isomorphic, and we can in fact handle most cases using byproducts of the arguments involved in the proof of irreducibility.

For this irreducibility statement, we begin by computing directly the diagonal average over λ of the sum R in the case $\lambda_1 = \lambda_2 = \lambda$, and the average of R over all $\boldsymbol{b}$ in the special case $\lambda_1 = \lambda_2 = 0$ (in [52, Proposition 4.3 and (3.4)]). These computations reveal that the restrictions of the sheaf $\mathcal{R}$ to certain higher dimensional spaces are irreducible.

In general, estimating the average value of a sum such as R will give very little concrete information on any of it specific values. The *geometric* analogue of this operation here is to show that the restriction of an irreducible sheaf on some variety to a proper subvariety remains irreducible, and this turns out to be often tractable.

The proof of irreducibility requires different methods in the $\lambda = 0$ and $\lambda \neq 0$ cases.

For $\lambda = 0$, an elementary computation shows that the values of the R sum are independent of the choice of additive character used to define Kloosterman sums. The geometric analogue of this fact is that the sheaf $\mathcal{R}$ (specialized to $\lambda = 0$) may be defined without the use of additive character sheaves [52, Lemma 4.27]. Since it turns out that this is the only part of the construction that requires working in positive characteristic q, we deduce that the sheaf $\mathcal{R}_{\lambda=0}$ can actually be constructed over the integers and over the complex numbers. Over $\mathbf{C}$, we may apply topological arguments to study the irreducibility of the sheaf, and it is then possible to derive the same conclusion for sufficiently large prime characteristic q. (It is actually ultimately more convenient to apply the argument in characteristic q, using only the intuitions from topology; the integrality property of the sheaf is used in [52, §4.4] to show that it is tamely ramified, and the topological properties and arguments carry over to the tamely ramified case).

To be a bit more precise we may view the complex version of the sheaf $\mathcal{R}_{\lambda=0}$ as a representation of the fundamental group of the open subset X of $\mathbf{C}^5$ (with coordinates $(r, \boldsymbol{b})$) where it is lisse. We can think of this space as a family of punctured Riemann surfaces parametrized by $\boldsymbol{b}$. Over the open subset of this parameter space X where the punctures do not collide, we can "follow" a loop in one Riemann surface into a loop in any other, so their fundamental groups are equal (as subgroups of the fundamental group $\pi_1(X)$ of the total space) and thus have the same action on the sheaf. An immediate consequence of this is that, if one fiber of the sheaf over some $\boldsymbol{b}$ is irreducible, then *all* fibers are irreducible. However, we can do better, because the (common) fundamental group of our Riemann surfaces is a normal subgroup of $\pi_1(X)$, with quotient isomorphic to the fundamental group of the base, minus the set Y of points whose fibers are empty. We can show that the variety Y of points whose fibers are empty has codimension 2, so the quotient is in fact the fundamental group of $\mathbf{C}^4$, which is trivial, hence our subgroup is in fact equal to the whole group $\pi_1(X)$. Since the representation associated to the sheaf $\mathcal{R}_{\lambda=0}$ is an irreducible representation of $\pi_1(X)$, it is therefore irreducible on each fiber, away from the points where the punctures collide. (Note that in practice, the argument is phrased using Galois theory, instead of loops, but the conclusion is the same.)

Remark 3.8. In the special case of Kl_2, one can compute that the rank of $\mathcal{R}$ over points where $\lambda = 0$ is 2. As a rank 2 sheaf whose trace function takes values in $\mathbf{Q}$, it looks very much like the sheaf of Tate modules of a family of elliptic curves, and it is possible that there exists an argument reducing the R-sum for $\lambda = 0$ to the number of points on a family of elliptic curves. If this is so, then checking the irreducibility property would be the same as checking that the j-invariant of this family is nonconstant. However, such an argument is unlikely to apply for $k \geqslant 3$.

For $\lambda \neq 0$, the sum R depends on the choice of additive character, and is in general an element of $\mathbf{Q}(\mu_p)$ and not $\mathbf{Q}$. Geometrically, the associated sheaf has wild ramification. This causes difficulties if one tries a direct analogue of the previous argument. Indeed, the argument that, if one fiber of the sheaf is irreducible, then all are irreducible, is not valid in the wildly ramified setting without additional work. What's more, our previous argument that one fiber of the sheaf is irreducible also does not generalize to the $\lambda \neq 0$ setting.

Instead, we use (in [**52**, §4.5]) arguments from the theory of *vanishing cycles*. After interpreting the irreducibility at a given $\lambda \neq 0$ in terms of the rank of the stalks of a suitable auxiliary sheaf $\mathcal{E}$ (namely, the sheaf $\mathcal{R}$ tensored by its dual), Deligne's semicontinuity theorem gives a tool to check that the irreducibility is independent of $\lambda \neq 0$. The key input that is needed is the proof that the Swan conductors of the local monodromy representations associated to $\mathcal{E}$, which are numerical invariants of wild ramification, are themselves independent of λ. (In the tame case, the Swan conductors are always zero, which explains partly why it is easier to handle).

In order to check this constancy property, we must compute the local monodromy representations at every singular point. These are known for Kloosterman sheaves (by work of Katz) and for additive character sheaves (by elementary means) and it is easy to combine this information when taking tensor products. The main difficulty is to understand the local monodromy representations after taking cohomology (which amounts to computing the sum over s that defines the R-sum). This is precisely what the theory of vanishing cycles achieves in situations where the local geometry is sufficiently "nice".

In our case of interest when $\lambda \neq 0$, the singularities of the one-variable specialized sheaf $\mathcal{R}_{\boldsymbol{b},\lambda}$ are those r where we can "see" that the sum degenerates in an obvious way, namely those r such that $r + b_i = 0$, and $r = \infty$. One can then compute that the local monodromy representation where $r + b_i = 0$ is tame (so has Swan conductor 0), and the local monodromy at ∞ is wild, with large but constant Swan conductor [**52**, Lemma 4.32, Corollary 4.37].

The computation of the local monodromy representation at ∞ also allows us to prove irreducibility for generic λ, because the problem still involves restricting an irreducible representation to a normal subgroup, making it isotypic (up to conjugacy). Because of this, if it were not irreducible, then the unique isomorphism class of its irreducible components would be repeated with multiplicity at least two. Then, when we restrict further to the local monodromy group at ∞, each irreducible component must have multiplicity at least two. But the explicit computation (using vanishing cycles) allows us to detect an irreducible component of multiplicity one, which is not conjugate to any other.

Finally, combining these arguments, we prove irreducibility for every value of λ. We require some fairly elementary arguments to conclude the proof by excluding that some specialized sheaves are isomorphic for different values of λ. The most difficult case is when $\lambda_2 = -\lambda_1$ and we are dealing with the generalization of (3.6) to hyper-Kloosterman Kl_k with k odd, in which case some extra steps are needed.

3.4. Trace functions and their Mellin transforms

Let ℓ be a prime distinct from q. Let $\mathcal{F}$ be a geometrically irreducible ℓ-adic sheaf on $\mathbf{A}^1_{\mathbf{F}_q}$, which we assume to be a middle-extension of weight 0. The complexity of $\mathcal{F}$ is measured by its conductor $\mathbf{c}(\mathcal{F})$, in the sense of [**20**]. Among its properties, we mention that $|t(x)| \leqslant c(\mathcal{F})$ for all $x \in \mathbf{F}_q$.

An important property is that if we denote

$$\widehat{t}(x) = \frac{1}{\sqrt{q}} \sum_{a \in \mathbf{F}_q} t(a) e\Big(\frac{ax}{q}\Big)$$

the discrete Fourier transform of a function $t\colon \mathbf{F}_q \to \mathbf{C}$, then unless the trace function t is proportional to $e(ax/p)$ for some a, then we have

$$|\widehat{t}(x)| \ll 1$$

where the implied constant depends only on $\mathbf{c}(\mathcal{F})$, as a consequence of Deligne's general form of the Riemann Hypothesis over finite fields; see the statement and references in [19, Th. 4.1]. More precisely, if $\mathcal{F}$ is not geometrically isomorphic to an Artin-Schreier sheaf, then $\widehat{t}$ is itself the trace function of a geometrically irreducible middle-extension ℓ-adic sheaf of weight 0, whose conductor is bounded (polynomially) in terms of $\mathbf{c}(\mathcal{F})$ only (see the survey previously mentioned and [20, Prop. 8.2] for the bound on the conductor), so that $\widehat{t} \ll 1$ is a special case of the assertion that a trace function is bounded by its conductor.

Similarly, if we define the Mellin transform of t by

$$\widetilde{t}(\chi) = \frac{1}{\sqrt{q}} \sum_{a \in \mathbf{F}_q^\times} t(a)\chi(a)$$

for any Dirichlet character χ modulo q, then we have

$$|\widetilde{t}(\chi)| \ll 1$$

where the implied constant depends only on $\mathbf{c}(\mathcal{F})$, unless t is itself proportional to a Dirichlet character (loc. cit.).

Example 3.9. (1) Let $k \geqslant 2$ be an integer. The function $x \mapsto \mathrm{Kl}_k(x; q)$ defined by (3.2) is a trace function (for any $\ell \neq q$) of a sheaf $\mathcal{K}\ell_k$ with conductor bounded by a constant depending only on k. These sheaves, constructed by Deligne and extensively studied by Katz, are called *Kloosterman sheaves*; they are fundamental in the proof of Theorem 3.7.

(2) Let $f \in \mathbf{Z}[X]$ be a polynomial and $\chi \pmod q$ a non-trivial Dirichlet character. Define

$$t_1(x) = e\Big(\frac{f(x)}{q}\Big), \qquad t_2(x) = \chi(f(x)).$$

Then t_1 and t_2 are trace functions, with conductor depending only on $\deg(f)$. If f has degree 1, then t_1 is associated to an *Artin-Schreier sheaf*, and if $f = aX$ for some $a \neq 0$, then f is associated to a *Kummer sheaf*.

Below we will use the following definition:

Definition 3.10. A *Mellin sheaf* over $\mathbf{F}_q$ is a geometrically irreducible, geometrically non-constant, middle-extension sheaf of weight 0 on $\mathbf{G}_{m,\mathbf{F}_q}$ that is not geometrically isomorphic to a Kummer sheaf.

By orthogonality of characters, we have the discrete Mellin inversion formula

$$\sum_{\chi \pmod q} \widetilde{t}(\chi)\chi(x) = \frac{q-1}{\sqrt{q}} t(x^{-1})$$

for $x \in \mathbf{F}_q^\times$. Similarly, we get

$$\sum_{\chi \pmod q} \widetilde{t}(\chi)\chi(x)\varepsilon_\chi^2 = \frac{q-1}{\sqrt{q}} (t \star \mathrm{Kl}_2)(x)$$

by opening the Gauß sums (this is also a case of the discrete Plancherel formula), where

$$(t_1 \star t_2)(x) = \frac{1}{\sqrt{q}} \sum_{ab=x} t_1(a) t_2(b)$$

is the multiplicative convolution of two functions on $\mathbf{F}_q^\times$.

We will need:

LEMMA 3.11. *Let $\mathcal{F}$ be a Mellin sheaf with trace function t. Then one of the following two conditions holds:*

(1) *There exists a Mellin sheaf $\mathcal{G}$ with conductor $\ll \mathbf{c}(\mathcal{F})^4$ with trace function τ such that*

$$(t \star \mathrm{Kl}_2)(x) = \tau(x) + O(q^{-1/2})$$

for $x \in \mathbf{F}_q^\times$, where the implied constant depends only on $\mathbf{c}(\mathcal{F})$.

(2) *The sheaf $\mathcal{F}$ is geometrically isomorphic to a pullback $[x \mapsto a/x]^* \mathcal{K}\ell_2$ of a Kloosterman sheaf for some $a \in \mathbf{F}_q^\times$, in which case there exists $\alpha \in \mathbf{C}$ with modulus 1 such that*

$$t(x) = \alpha \mathrm{Kl}_2(a\bar{x}; q)$$

for all $x \in \mathbf{F}_q^\times$. We then have

$$\widetilde{t}(\chi) = \alpha \chi(a) \varepsilon_\chi^{-2}$$

for all χ.

PROOF. If $\mathcal{F}$ is not geometrically isomorphic to a pullback $[x \mapsto a/x]^* \mathcal{K}\ell_2$ of a Kloosterman sheaf, then the "shriek" convolution $\mathcal{F} \star_! \mathcal{K}\ell_2$ has trace function $t \star \mathrm{Kl}_2$, and the middle-convolution $\mathcal{G} = \mathcal{F} \star_{mid} \mathcal{K}\ell_2$ of $\mathcal{F}$ and $\mathcal{K}\ell_2$ is a sheaf with trace function $t \star \mathrm{Kl}_2 + O(q^{-1/2})$, as a consequence of the properties of middle-convolution [40, Ch. 2].

The middle-convolution is a Mellin sheaf in this case: indeed, it is geometrically irreducible because $\mathcal{K}\ell_2$ is of "dimension" one in the Tannakian sense, so $(\mathcal{F} \star_{mid} \mathcal{K}\ell_2)[1]$ is an irreducible object in the Tannakian sense, which implies the result by [40, p. 20]). In that case, we obtain (1), where the conductor bound is a special case of the results from the Appendix by Fouvry, Kowalski and Michel to P. Xi's paper [82].

□

REMARK 3.12. (1) The "error term" in Case (1) of this lemma is linked to the possible existence of Frobenius eigenvalues of weight $\leqslant -1$ in the "naive" convolution. One can think of the middle-convolution here as the "weight 0" part of this naive convolution.

(2) Using a more intrinsic definition of the conductor than the one in [20], one could obtain a better exponent that $\mathbf{c}(\mathcal{G}) \ll \mathbf{c}(\mathcal{F})^4$ (see [74]).

In Chapter 9, we will use the following variant of Lemma 3.11.

LEMMA 3.13. *Let $\mathcal{F}$ be a geometrically irreducible ℓ-adic that is not geometrically isomorphic to a Kummer sheaf, an Artin-Schreier sheaf or the pull-back of an Artin-Schreier sheaf by the map $x \mapsto x^{-1}$. There exists a Mellin sheaf $\mathcal{G}$ with conductor bounded polynomially in terms of $\mathbf{c}(\mathcal{F})$, not geometrically isomorphic to*

$[x \mapsto a/x]^*\mathcal{K}\ell_2$ for any $a \in \mathbf{F}_q^\times$, such that the trace function τ of $\mathcal{G}$ satisfies

$$\tau(x) = \frac{1}{\sqrt{q}} \sum_{y \in \mathbf{F}_q^\times} \overline{t(y)} e\Big(-\frac{\bar{x}y}{p}\Big) + O(q^{-1/2}) \tag{3.8}$$

where the implied constant depends only on $\mathbf{c}(\mathcal{F})$.

Proof. The principle is the same as in Lemma 3.11. We denote by $\mathcal{L}$ the Artin-Schreier sheaf with trace function $x \mapsto e(x/p)$ and its Tannakian dual $\mathcal{L}^\vee$ with trace function $x \mapsto e(-\bar{x}/p)$. We consider the middle-convolution object $\mathcal{F} \star_{mid} \mathcal{L}^\vee$. Because $\mathcal{F}$ is not an Artin-Schreier sheaf, this object is associated to a middle-extension sheaf $\mathcal{G}$ of weight 0, which is geometrically irreducible because $\mathcal{L}^\vee$ is of Tannakian dimension 1.

Middle-convolution of $\mathcal{G}$ with $\mathcal{L}$ gives back the input sheaf $\mathcal{F}$ (again because $\mathcal{L}$ is of dimension 1 with dual $\mathcal{L}^\vee$ in the Tannakian sense). Thus, because $\mathcal{F}$ is not geometrically isomorphic to a Kummer sheaf, so is $\mathcal{G}$. And because $\mathcal{F}$ is not the pull-back of an Artin-Schreier sheaf by the map $x \mapsto x^{-1}$, the sheaf $\mathcal{G}$ is not of the form $[x \mapsto a/x]^*\mathcal{K}\ell_2$.

Finally, as in Lemma 3.11, the sheaf $\mathcal{G}$ has trace function satisfying (3.8) and has conductor bounded in terms of the conductor of $\mathcal{F}$. □

3.5. The equidistribution group of a Mellin transform

In a remarkable recent work, Katz [40] has shown that the discrete Mellin transforms of quite general trace functions satisfy equidistribution theorem similar to those known for families of exponential sums indexed by points of an algebraic variety.

Katz's work relies in an essential way on deep algebraic-geometric ideas, especially on the so-called Tannakian formalism. We will minimize what background is needed by presenting this as a black-box, with examples. We refer, besides Katz's book, to the recent Bourbaki report of Fresán [22] for an accessible survey.

Let $\mathcal{F}$ be a Mellin sheaf over $\mathbf{F}_q$ as in Definition 3.10. Katz [40, p. 11] defines two linear algebraic groups related to $\mathcal{F}$, its arithmetic and geometric Tannakian monodromy groups, the geometric one being a normal subgroup of the arithmetic one under our assumptions [40, Th. 6.1]. In equidistribution statements, it is often simpler to assume that they are equal, and Katz frequently does so.

Definition 3.14. We say that $\mathcal{F}$ has Property EAGM ("Equal Arithmetic and Geometric Monodromy") if the two groups defined by Katz in [40, p. 11] are equal. We then call a maximal compact subgroup K of (the base change from $\overline{\mathbb{Q}}_\ell$ to $\mathbb{C}$ of) this common group the *equidistribution group* of the Mellin transform of $\mathcal{F}$. We denote by $K^\sharp$ the space of conjugacy classes in K.

Assuming that $\mathcal{F}$ has EAGM, Katz [40, p. 12–13] defines a subset X_q of the set of characters of $\mathbf{F}_q^\times$, of cardinality $\leqslant 2\,\mathrm{rk}(\mathcal{F}) \leqslant 2\,\mathbf{c}(\mathcal{F})$, and for any $\chi \notin X_q$, he defines a conjugacy class $\theta_\chi \in K^\sharp$ such that the Mellin transform $\widetilde{t}$ of the trace function of $\mathcal{F}$ satisfies

$$\widetilde{t}(\chi) = \mathrm{tr}(\theta_\chi)$$

for $\chi \notin X_q$. It will be convenient for us to enlarge X_q to always include the trivial character.

The key result that we need is the following further consequence of the work of Katz. It can be considered as a black box in the next section.

THEOREM 3.15. *Let π be an irreducible representation of the equidistribution group K of the Mellin sheaf $\mathcal{F}$, which is assumed to have Property EAGM. Then one of the following properties holds:*

(1) *There exists a Mellin sheaf $\pi(\mathcal{F})$, as in Definition 3.10, with Property* EAGM*, such that for any $\chi \notin X_q$, we have*

$$\mathrm{tr}(\pi(\theta_\chi)) = \widetilde{t}_\pi(\chi),$$

where t_π is the trace function of $\pi(\mathcal{F})$, and such that the conductor of $\pi(\mathcal{F})$ is bounded in terms of π and $\mathbf{c}(\mathcal{F})$ only.

(2) *There exists $a \in \mathbf{F}_q^\times$ such that*

$$\mathrm{tr}(\pi(\theta_\chi)) = \chi(a)$$

for all χ. Moreover, if d is the order of the finite group of characters of finite order of K, then a is a d-th root of unity in $\mathbf{F}_q^\times$.

PROOF. The existence of the sheaf $\pi(\mathcal{F})$ as an object in the Tannakian category associated to $\mathcal{F}$ by Katz is part of the Tannakian formalism [40, Ch. 2]. By construction, this object is irreducible in the Tannakian sense over $\bar{\mathbf{F}}_q$. By the classification of the geometrically irreducible objects [40], it is either "punctual", in which case we are in Case (2) (because π is then a finite order charcter of K), or there exists a geometrically irreducible ℓ-adic sheaf $\mathcal{G}$ on $\mathbf{G}_{m,\mathbf{F}_q}$ such that $\pi(\mathcal{F}) = \mathcal{G}$. By construction of the Tannakian category, this sheaf is not geometrically isomorphic to a Kummer sheaf (loc. cit.) and it is of weight 0. Hence it is a Mellin sheaf.

Still in this second case, the Tannakian groups of $\pi(\mathcal{F})$ are the image by (the algebraic representation corresponding to π) of the groups associated to $\mathcal{F}$, and are therefore equal, so that $\pi(\mathcal{F})$ has Property EAGM. The bound for the conductor follow easily from the computations in [40, Ch. 28, Th. 28.2]. □

REMARK 3.16. (1) The second case will be called the "punctual" case.

(2) The conductor bounds resulting from [40, Th. 28.2] are relatively weak because they rely on bounds for tensor products and on embedding the representation π in a tensor product of tensor powers of the standard representation and its dual. A much stronger estimate (which would be essential for strong quantitative applications, such as "shrinking targets" problems) has been proved by Sawin [74]: we have

$$\mathbf{c}(\mathcal{G}) \leqslant 2 + \dim(\pi)(1 + w(\pi)\,\mathrm{rk}(\mathcal{F}))$$

where $w(\pi)$ is the minimum of $a+b$ over pairs (a,b) of non-negative integers such that π embeds in $\varrho^{\otimes a} \otimes \varrho^{\vee \otimes b}$, where ϱ is the representation of K corresponding to $\mathcal{F}$ itself. In turn $w(\pi)$ is bounded by an affine function of the norm of the highest weight vectors of the restriction of π to K^0.

EXAMPLE 3.17. (1) (The Evans sums, see [40, Ch. 14]). Let

$$t(x) = e\Big(\frac{x - \bar{x}}{q}\Big)$$

for $x \in \mathbf{F}_q^\times$. Then t is the trace function of a Mellin sheaf $\mathcal{F}_e$ of conductor bounded independently of q, and Katz [40, Th. 14.2] proves that $\mathcal{F}_e$ has Property EAGM, and that its equidistribution group K is $\mathrm{SU}_2(\mathbf{C})$ with $X_q = \emptyset$. By definition, each value of the Mellin transform

$$\widetilde{t}_e(\chi) = \frac{1}{\sqrt{q}} \sum_{x \in \mathbf{F}_q^\times} \chi(x) e\Big(\frac{x - \bar{x}}{q}\Big)$$

is the *Evans sum* associated to χ.

(2) (The Rosenzweig-Rudnick sums, see [40, Ch. 14]). Let

$$t(x) = e\Big(\frac{(x+1)\overline{(x-1)}}{q}\Big)$$

for $x \in \mathbf{F}_q^\times$ such that $x \neq 1$ and $t(1) = 0$. Then t is the trace function of a Mellin sheaf $\mathcal{F}_{rr}$ of conductor bounded independently of q, and Katz [40, Th. 14.5] proves that $\mathcal{F}_{rr}$ has Property EAGM, and that its equidistribution group K is also $\mathrm{SU}_2(\mathbf{C})$ with $X_q = \emptyset$. Each value of the Mellin transform

$$\widetilde{t}_{rr}(\chi) = \frac{1}{\sqrt{q}} \sum_{x \in \mathbf{F}_q^\times} \chi(x) e\Big(\frac{(x+1)\overline{(x-1)}}{q}\Big)$$

is the Rosenzweig-Rudnick sum associated to χ.

(3) (Unitary examples, see [40, Ch. 17]) Katz gives many examples where the equidistribution group is $\mathrm{U}_N(\mathbf{C})$ for some integer $N \geqslant 1$. For instance, fix a non-trivial multiplicative character η modulo q, of order d. Let $n \geqslant 2$ be an integer coprime to d, and let $P \in \mathbf{F}_q[X]$ be a monic polynomial of degree n with distinct roots in $\bar{\mathbf{F}}_q$, and with $P(0) \neq 0$. Write

$$P = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0,$$

and assume that $\gcd(\{i : a_i \neq 0\}) = 1$. 1. Then $t(x) = \eta(P(x))$ is the trace function of a Mellin sheaf with Property EAGM, for which the equidistribution group is $\mathrm{U}_n(\mathbf{C})$ by [40, Th. 17.5].

If, on the other hand, P has degree coprime to q, and if P' has $n-1$ distinct roots $\alpha \in \bar{\mathbf{F}}_q$, if the set $S = \{P(\alpha) \mid P'(\alpha) = 0\}$ has $n-1$ distinct points (i.e., P is "weakly super-morse"), and if in addition S is not invariant by multiplication by any constant $\neq 1$, then the "solution counting" function

$$t(x) = \sum_{P(y)=x} 1 - 1$$

is the trace function of a Mellin sheaf with Property EAGM and with equidistribution group $\mathrm{U}_{n-1}(\mathbf{C})$ (see [40, Th. 17.6]). The discrete Mellin transform in that case is

$$\widetilde{t}(\chi) = \frac{1}{\sqrt{q}} \sum_{y \in \mathbf{F}_q} \chi(P(y))$$

(see [40, Remark 17.7]).

(4) For further examples including groups like $\mathrm{SU}_n(\mathbf{C})$ for some n, $\mathrm{O}_{2n}(\mathbf{C})$, G_2 or products, see [40].

CHAPTER 4

Computation of the first twisted moment

Besides stating and proving the general form of the first moment formulas twisted by characters that we will need in our main result, we will also consider in this chapter the first moment twisted by more general discrete Mellin transforms of trace functions over finite fields, in the sense of Section 3.4. We present these last results in a separate section for greater readability; it may be safely omitted in a first reading and is only used in Section 6.7.

4.1. Introduction

In this chapter, we will prove Theorem 1.16, which we first recall. We fix f as in Section 1.2, i.e. a primitive (holomorphic or Maaß) cusp form for $\Gamma_0(|r|)$ with trivial nebentypus, and recall Convention 1.3 on signed levels.

Given $\ell \in (\mathbf{Z}/q\mathbf{Z})^{\times}$ and $k \in \mathbf{Z}$, we consider

$$\mathcal{L}(f;\ell,k) = \frac{1}{\varphi^*(q)} \sideset{}{^\star}\sum_{\chi \,(\mathrm{mod}\, q)} \varepsilon_\chi^k \chi(\ell) L(f\otimes\chi, 1/2), \tag{4.1}$$

(cf. (1.8) and our convention to drop the parameter s if it equals $1/2$). We first observe that

$$\mathcal{L}(f;\ell,k) = \varepsilon(f)\mathcal{L}\left(f;(-1)^k\overline{\ell r}, -(k+2)\right). \tag{4.2}$$

Indeed, for any non-trivial character $\chi \,(\mathrm{mod}\, q)$, we have $\varepsilon_\chi = \overline{\varepsilon_{\overline{\chi}}}^{-1} = \chi(-1)\varepsilon_{\overline{\chi}}^{-1}$, and moreover

$$L(f\otimes\chi,1/2) = \varepsilon(f)\chi(r)\varepsilon_\chi^2 L(f\otimes\overline{\chi},1/2) = \varepsilon(f)\chi(r)\varepsilon_{\overline{\chi}}^{-2} L(f\otimes\overline{\chi},1/2),$$

by the functional equation (cf. (2.3)), which implies the formula.

It is therefore sufficient to evaluate $\mathcal{L}(f;\ell,k)$ for $k \geqslant -1$ in order to handle all values of k. In this case we will prove a slightly more precise version of Theorem 1.16.

Theorem 4.1. *There exists an absolute constant $B \geqslant 0$ such that for $k \geqslant -1$, $\ell \in (\mathbf{Z}/q\mathbf{Z})^{\times}$ and any $\varepsilon > 0$, we have*

$$\mathcal{L}(f;\ell,k) = \delta_{k=0}\frac{\lambda_f(\overline{\ell}_q)}{\overline{\ell}_q^{1/2}} + O_{f,\varepsilon}((1+|k|)^B q^{-1/8+\varepsilon}),$$

where $\overline{\ell}_q$ denotes the unique integer in the interval $[1,q]$ satisfying the congruence

$$\ell\overline{\ell}_q \equiv 1 \,(\mathrm{mod}\, q).$$

Combining this theorem with the formula (4.2), we obtain

COROLLARY 4.2. *There exists an absolute constant $B \geqslant 0$ such that, for $k \in \mathbf{Z}$ and $\ell \in (\mathbf{Z}/q\mathbf{Z})^{\times}$, we have*

$$\mathcal{L}(f;\ell,k) = \delta_{k=0}\frac{\lambda_f(\overline{\ell}_q)}{\overline{\ell}_q^{1/2}} + \delta_{k=-2}\varepsilon(f)\frac{\lambda_f((\ell r)_q)}{(\ell r)_q^{1/2}} + O_{f,\varepsilon}((1+|k|)^B q^{-1/8+\varepsilon}).$$

where we denote by $(\ell r)_q$ the unique integer in $[1,q]$ representing the congruence class $\ell r \pmod q$.

REMARK 4.3. Observe that for $1 \leqslant \ell < q^{1/2}$, we have $\overline{\ell}_q \geqslant q^{1/2}$ unless $\ell = 1$, so that the main term for $k=0$ can be absorbed in the error term. Hence for $k \geqslant -1$, and $(\ell,q)=1$ with $1 \leqslant \ell < q^{1/2}$, we obtain

$$\mathcal{L}(f;\ell,k) = \delta_{k=0}\delta_{\ell=1} + O_{f,\varepsilon}((1+|k|)^B q^{-1/8+\varepsilon}).$$

4.2. Proof

The first moment decomposes into the sum over the even and odd characters

$$\mathcal{L}(f;\ell,k) = \frac{1}{2}\mathcal{L}^{+}(f;\ell,k) + \frac{1}{2}\mathcal{L}^{-}(f;\ell,k)$$

where

$$\mathcal{L}^{\pm}(f;\ell,k) = \frac{2}{\varphi^*(q)} \sideset{}{^\pm}\sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi \text{ primitive}}} \varepsilon_\chi^k \chi(\ell) L(f\otimes\chi, 1/2).$$

We evaluate the "even" first moment $\mathcal{L}^{+}(f;\ell,k)$ in detail, the odd part is entirely similar.

The approximate functional equation (2.24), (3.4) and (2.3) give (using notations of Convention 2.20)

$$\begin{aligned}\mathcal{L}^{+}(f;\ell,k) = &\sum_{\pm} q^{-1/2}\sum_{(n,q)=1} \mathrm{Kl}_k(\pm\overline{\ell n};q)\frac{\lambda_f(n)}{n^{1/2}}V_{f,1/2}\Big(\frac{n}{q\sqrt{|r|}}\Big)\\ &+\varepsilon(f)\sum_{\pm} q^{-1/2}\sum_{(n,q)=1}\mathrm{Kl}_{k+2}(\pm\overline{|r|\ell}n;q)\frac{\lambda_f(n)}{n^{1/2}}V_{f,1/2}\Big(\frac{n}{q\sqrt{|r|}}\Big)\\ &+O\Big(\sum_n \frac{|\lambda_f(n)|}{n^{1/2}}\Big|V_{f,1/2}\Big(\frac{n}{q\sqrt{|r|}}\Big)\Big|q^{-1-|k|/2}\Big).\end{aligned} \tag{4.3}$$

The error term is $O_f(q^{-(1+|k|)/2})$. Since $k \geqslant -1$, we have $k+2 \neq 0$, so that it follows from Proposition 3.4 (after applying a smooth partition of unity into dyadic intervals to the n-sum) that

$$q^{-1/2}\sum_{\substack{n\geqslant 1\\(n,q)=1}} \mathrm{Kl}_{k+2}(\pm\overline{|r|\ell}n;q)\frac{\lambda_f(n)}{n^{1/2}}V_{f,1/2}\Big(\frac{n}{q\sqrt{|r|}}\Big) \ll_{\varepsilon,f} (1+|k|)^B q^{-1/8+\varepsilon}.$$

If $k \neq 0$, the same bound holds for the first term on the right-hand side of (4.3), and otherwise this term equals

$$\sum_{\pm}\sum_{n\equiv\pm\overline{\ell}\,(\mathrm{mod}\,q)} \frac{\lambda_f(n)}{n^{1/2}}V_{f,1/2}\left(\frac{n}{q\sqrt{|r|}}\right) = \sum_{\pm}\frac{\lambda_f(\overline{\ell}_q^{\pm})}{(\overline{\ell}_q^{\pm})^{1/2}}V_{f,1/2}\left(\frac{\overline{\ell}_q^{\pm}}{q\sqrt{|r|}}\right) + O_f(q^{\varepsilon+\theta-1/2}),$$

where $\overline{\ell}_q^{\pm}$ denotes the unique solution n of the equation

$$\ell n \equiv \pm 1 \,(\mathrm{mod}\, q)$$

contained in the interval $[1, q]$. By (2.29) and (2.16) we have

$$\frac{\lambda_f(\overline{\ell}_q^{\pm})}{(\overline{\ell}_q^{\pm})^{1/2}} V_{f,1/2}\left(\frac{\overline{\ell}_q^{\pm}}{q\sqrt{|r|}}\right) = \frac{\lambda_f(\overline{\ell}_q^{\pm})}{(\overline{\ell}_q^{\pm})^{1/2}} + O_{f,\varepsilon}(q^{\varepsilon+\theta-1/2}). \tag{4.4}$$

We have therefore proven that

$$\mathcal{L}^{+}(f;\ell,k) = \delta_{k=0} \sum_{\pm} \frac{\lambda_f(\overline{\ell}_q^{\pm})}{(\overline{\ell}_q^{\pm})^{1/2}} + O_{f,\varepsilon}(q^{-1/8+\varepsilon}).$$

for $k \geqslant -1$. Similarly we have

$$\mathcal{L}^{-}(f;\ell,k) = \delta_{k=0} \sum_{\pm} (\pm 1) \frac{\lambda_f(\overline{\ell}_q^{\pm})}{(\overline{\ell}_q^{\pm})^{1/2}} + O_{f,\varepsilon}(q^{-1/8+\varepsilon}).$$

Combining the two equations above, we obtain Theorem 4.1.

4.3. First moment with trace functions

Let q be a prime, and $\mathcal{F}$ a Mellin sheaf over $\mathbf{F}_q$ as in Definition 3.10. Let $t\colon \mathbf{F}_q^{\times} \to \mathbf{C}$ be its trace function and $\widetilde{t}$ its discrete Mellin transform. Fix f as in Section 1.2.

THEOREM 4.4. *Assume that we are not in case* (2) *of* Lemma 3.11. *Then, for any integer $\ell \geqslant 1$, we have*[1]

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)} L(f\otimes\chi, \tfrac{1}{2})\chi(\ell)\widetilde{t}(\chi) \ll q^{-1/8+\varepsilon} \tag{4.5}$$

for any $\varepsilon > 0$, where the implied constant depends on f and ε, and polynomially on $\mathbf{c}(\mathcal{F})$.

PROOF. We may assume that ℓ is coprime to q. For any non-trivial character χ, by the approximate functional equation (2.24) and (2.26), we have

$$L(f\otimes\chi, 1/2) = \sum_{n\geqslant 1} \frac{\lambda_f(n)}{n^{1/2}}\chi(n) V\Big(\frac{n}{q\sqrt{|r|}}\Big) + \varepsilon(f)\chi(r)\varepsilon_\chi^2 \sum_{n\geqslant 1} \frac{\lambda_f(n)}{n^{1/2}}\overline{\chi}(n) V\Big(\frac{n}{q\sqrt{|r|}}\Big)$$

where $V = V_{f,\chi(-1),1/2}$. Distinguishing according to the parity of χ, the left-hand side of (4.5) is

$$\frac{1}{\varphi^*(q)} \sideset{}{^+}\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq 1}} L(f\otimes\chi, \tfrac{1}{2})\chi(\ell)\widetilde{t}(\chi) + \frac{1}{\varphi^*(q)} \sideset{}{^-}\sum_{\chi\,(\mathrm{mod}\,q)} L(f\otimes\chi, \tfrac{1}{2})\chi(\ell)\widetilde{t}(\chi).$$

[1] Here, the integer ℓ is not related to the auxiliary prime used in defining the sheaf $\mathcal{F}$.

We consider the sum over non-trivial even characters, since the case of odd characters is (as usual) entirely similar. We have

$$\frac{1}{\varphi^*(q)} \sideset{}{^+}\sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \neq 1}} L(f \otimes \chi, \tfrac{1}{2}) \chi(\ell) \widetilde{t}(\chi)$$
$$= \sum_{n \geqslant 1} \frac{\lambda_f(n)}{n^{1/2}} V\Big(\frac{n}{q\sqrt{r}}\Big) T_1(n, \ell) + \varepsilon(f) \sum_{n} \frac{\lambda_f(n)}{n^{1/2}} V\Big(\frac{n}{q\sqrt{r}}\Big) T_2(n, \ell, r)$$

where

$$T_1(n, \ell) = \frac{1}{\varphi^*(q)} \sideset{}{^+}\sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \neq 1}} \widetilde{t}(\chi) \chi(n\ell)$$
$$T_2(n, \ell, r) = \frac{1}{\varphi^*(q)} \sideset{}{^+}\sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \neq 1}} \varepsilon_\chi^2 \chi(r\ell) \bar{\chi}(n) \widetilde{t}(\chi).$$

By (3.1) and discrete Mellin inversion, we compute

$$T_1(n, \ell) = \frac{\alpha(q)}{2} \Big(t(\overline{n\ell}) + t(-\overline{n\ell}) \Big) - \frac{1}{\varphi^*(q)} \widetilde{t}(1) \delta_{(n,q)=1}$$
$$T_2(n, \ell, r) = \frac{\alpha(q)}{2} \Big((t \star \mathrm{Kl}_2)(r\ell\overline{n}) + (t \star \mathrm{Kl}_2)(-r\ell\overline{n}) \Big) - \frac{\varepsilon_1^2}{\varphi^*(q)} \widetilde{t}(1) \delta_{(n,q)=1}$$

where

$$\alpha(q) = \frac{\varphi(q)}{\varphi^*(q)\sqrt{q}} \sim \frac{1}{\sqrt{q}}.$$

Since $\widetilde{t}(1) \ll 1$, the contribution of the trivial character to the first moment is $\ll q^{-1}$, where the implied constant depends only on $\mathbf{c}(\mathcal{F})$.

Since we are in Case (1) of Lemma 3.11, we see that, up to negligible error, the even part of the first moment is the sum of four expressions of the type

$$\gamma \alpha(q) \sum_{\substack{n \geqslant 1 \\ (n,q)=1}} \frac{\lambda_f(n)}{n^{1/2}} V_{f,+,1/2}\Big(\frac{n}{q\sqrt{|r|}}\Big) \tau(n)$$

with $\gamma = 1$ or $\gamma = \varepsilon(f)$, where τ is (by Lemma 3.11 in the cases involving $t \star \mathrm{Kl}_2$) a trace function of a geometrically irreducible middle-extension sheaf of weight 0 with conductor $\ll 1$. By [20, Th. 1.2], each of these sums is $\ll q^{-1/8+\varepsilon}$ (cf. Proposition 3.4 for a special case), where the implied constant depends only on ε, f and polynomially on $\mathbf{c}(\mathcal{F})$. □

REMARK 4.5. Case (2) in Lemma 3.11 leads to a first moment

$$\frac{1}{\varphi^*(q)} \sideset{}{^\star}\sum_{\chi \,(\mathrm{mod}\, q)} L(f \otimes \chi, 1/2) \chi(a\ell) \varepsilon_\chi^{-2},$$

which is evaluated asymptotically in Theorem 4.1.

CHAPTER 5

Computation of the second twisted moment

5.1. Introduction

In this chapter, we prove Theorem 1.18, which we will now state with precise main terms.

We fix f as in Section 1.2. Let g be a primitive cusp form of signed level r' coprime to q and trivial central character; we allow the possibility that $g = f$. We recall that we use Convention 1.3 concerning the signed levels of cusp forms. We will use the approximate functional equation (2.25), and the corresponding test functions $W_{f,g,\pm,s}$ (see (2.27)) and "signs" $\varepsilon(f, g, \pm, s)$ (see (2.26)).

Recall that we consider the twisted second moments

$$\mathcal{Q}(f, g, s; \ell, \ell') = \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} L(f \otimes \chi, s)\overline{L(g \otimes \chi, s)}\chi(\ell)\overline{\chi(\ell')} \tag{5.1}$$

for integers $1 \leqslant \ell, \ell' \leqslant L \leqslant q^{1/2}$ (say), with $(\ell\ell', qrr') = (\ell, \ell') = 1$, and $s \in \mathbf{C}$ with $|\,\mathfrak{Re}\, s - 1/2| < (\log q)^{-1}$. We write $\sigma = \mathfrak{Re}\, s$, and we may assume without loss of generality that $q \geqslant e^{10}$, say.

We write $|r| = \varrho\delta$, $|r'| = \varrho'\delta$ with $\delta = (r, r') \geqslant 1$ and $(\varrho, \varrho') = 1$. In particular, note that ϱ and ϱ' are positive.

Theorem 5.1. *We have*

$$\mathcal{Q}(f, g, s; \ell, \ell') = \mathrm{MT}(f, g, s; \ell, \ell') + O(|s|^{O(1)} L^{3/2} q^{-1/144+\varepsilon}),$$

where the main term is given by

$$\mathrm{MT}(f, g, s; \ell, \ell') = \frac{1}{2}\mathrm{MT}^+(f, g, s; \ell, \ell') + \frac{1}{2}\mathrm{MT}^-(f, g, s; \ell, \ell')$$

with

$$\begin{aligned}\mathrm{MT}^{\pm}(f, g; \ell, \ell') = \frac{1}{2}\sum_{n \geqslant 1} \frac{\lambda_f(\ell' n)\lambda_g(\ell n)}{\ell'^s \ell^{\overline{s}} n^{2\sigma}} W_{f,g,\pm,s}\Big(\frac{\ell\ell' n^2}{q^2|rr'|}\Big) \\ + \frac{\varepsilon(f, g, \pm, s)\lambda_f(\varrho)\lambda_g(\varrho')}{2\varrho^{1-s}(\varrho')^{1-\overline{s}}} \sum_{n \geqslant 1} \frac{\lambda_f(\ell n)\lambda_g(\ell' n)}{\ell^{1-s}(\ell')^{1-\overline{s}} n^{2-2\sigma}} W_{f,g,\pm,1-s}\Big(\frac{\ell\ell' n^2}{q^2\delta^2}\Big).\end{aligned} \tag{5.2}$$

If $r = r'$ and $\varepsilon(f)\varepsilon(g) = -1$, then $\mathcal{Q}(f, g, s; 1, 1) = 0$.

In the rest of this chapter, to simplify notation, we will not display the s dependency and will write $\mathcal{Q}(f, g; \ell, \ell')$ for $\mathcal{Q}(f, g, s; \ell, \ell')$. Moreover if $f = g$, we will just write $\mathcal{Q}(f; \ell, \ell')$.

We first justify the last assertion of the theorem concerning the exact vanishing of the untwisted second moment when $r = r'$ and $\varepsilon(f)\varepsilon(g) = -1$. Indeed, in that case we have

$$L(f\otimes\chi,\tfrac{1}{2})\overline{L(g\otimes\chi,\tfrac{1}{2})}=-L(f\otimes\overline{\chi},\tfrac{1}{2})\overline{L(g\otimes\overline{\chi},\tfrac{1}{2})}$$

for all χ, by (2.5). If χ is real, this shows that $L(f\otimes\chi,\frac{1}{2})\overline{L(g\otimes\chi,\frac{1}{2})}=0$, and otherwise, the sum of the values for χ and $\overline{\chi}$ is zero.

5.2. Isolating the main term

This moment decomposes as the sum of its even and odd part

$$\mathcal{Q}(f,g;\ell,\ell')=\frac{1}{2}\sum_{\pm}\mathcal{Q}^{\pm}(f,g;\ell,\ell')$$

where

$$\mathcal{Q}^{\pm}(f,g;\ell,\ell')=\frac{2}{\varphi^*(q)}\sideset{}{^{\pm}}\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\text{ primitive}}}L(f\otimes\chi,s)\overline{L(g\otimes\chi,s)}\chi(\ell/\ell').$$

We give the details for the even second moment $\mathcal{Q}^{+}(f,g,\ell,\ell')$; the treatment of the odd second moment is identical.

We apply the approximate functional equation (2.25). A simple large sieve argument shows that $\mathcal{Q}^{+}(f,g,\ell,\ell')=O_{f,g}(|s|^{O(1)})$, so that replacing $\varphi^*(q)$ by $\varphi(q)$ introduces an error of $O_{f,g}(|s|^{O(1)}q^{-1})$. Adding and subtracting the contribution of the trivial character using the bound

$$\sum_{(nm,q)=1}\frac{\lambda_f(m)\lambda_g(n)}{(mn)^{1/2}}W_{f,g,+,s}\Big(\frac{nm}{q^2|rr'|}\Big)\ll_{f,g}|s|^{O(1)},$$

and applying orthogonality (3.1), we obtain

$$\begin{aligned}\mathcal{Q}^{+}(f,g;\ell,\ell')=&\frac{1}{2}\sum_{\pm}\sum\sum_{\substack{\ell m\equiv\pm\ell' n\,(\mathrm{mod}\,q)\\ (mn,q)=1}}\frac{\lambda_f(m)\lambda_g(n)}{m^s n^{\overline{s}}}W_{f,g,+,s}\left(\frac{mn}{q^2|rr'|}\right)\\ &+\varepsilon(f,g,+,s)\frac{1}{2}\sum_{\pm}\sum_{\substack{r\ell n\equiv\pm r'\ell' m\,(\mathrm{mod}\,q)\\ (mn,q)=1}}\frac{\lambda_f(m)\lambda_g(n)}{m^{1-s}n^{1-\overline{s}}}W_{f,g,+,1-s}\left(\frac{mn}{q^2|rr'|}\right)\\ &+O\left(\frac{|s|^{O(1)}}{q}\right).\end{aligned}$$

The contribution of the terms such that $q\mid m$ (and therefore $q\mid n$) is bounded trivially by $\ll|s|^{O(1)}q^{-1+2\theta+o(1)}$ so we can remove the constraint $(mn,q)=1$.

Let $\varepsilon=\pm1$ be the sign of rr'. The contribution of the terms satisfying

$$\ell m=\ell' n$$

in the first sum, and

$$r\ell n=\varepsilon r'\ell' m$$

in the second sum, forms the main term, and is denoted $\mathrm{MT}^{+}(f,g;\ell,\ell')$. (Note that the corresponding equations with opposite signs have no solutions). We write as above $|r|=\varrho\delta$, $|r'|=\varrho'\delta$ with $\delta=(r,r')\geqslant1$ and $(\varrho,\varrho')=1$. Since ℓ,ℓ' are coprime and coprime to $\varrho\varrho'$, the solutions $n\geqslant1$, $m\geqslant1$, of the second equation are parameterized in all cases by

$$n=\varrho'\ell' k,\qquad m=\varrho\ell k,$$

where $k \geqslant 1$. The main term is then

$$\frac{1}{2}\sum_{n\geqslant 1}\frac{\lambda_f(\ell' n)\lambda_g(\ell n)}{\ell'^s\ell^{\overline{s}}n^{2\sigma}}W_{f,g,+,s}\Big(\frac{\ell\ell' n^2}{q^2|rr'|}\Big) + \frac{\varepsilon(f,g,+,s)}{2}\sum_{n\geqslant 1}\frac{\lambda_f(\varrho\ell n)\lambda_g(\varrho'\ell' n)}{(\varrho\ell)^{1-s}(\varrho'\ell')^{1-\overline{s}}n^{2-2\sigma}}W_{f,g,+,1-s}\Big(\frac{\ell\ell' n^2}{q^2\delta^2}\Big).$$

Moreover, since $\varrho \mid r$ and $\varrho' \mid r'$, we have $\lambda_f(\varrho\ell n) = \lambda_f(\varrho)\lambda_f(\ell n)$ and $\lambda_g(\varrho'\ell' n) = \lambda_g(\varrho')\lambda_g(\ell' n)$, hence this main term becomes

$$\frac{1}{2}\sum_{n\geqslant 1}\frac{\lambda_f(\ell' n)\lambda_g(\ell n)}{\ell'^s\ell^{\overline{s}}n^{2\sigma}}W_{f,g,+,s}\Big(\frac{\ell\ell' n^2}{q^2|rr'|}\Big) + \frac{\varepsilon(f,g,+,s)}{2}\frac{\lambda_f(\varrho)\lambda_g(\varrho')}{\varrho^{1-s}(\varrho')^{1-\overline{s}}}\sum_{n\geqslant 1}\frac{\lambda_f(\ell n)\lambda_g(\ell' n)}{\ell^{1-s}(\ell')^{1-\overline{s}}n^{2-2\sigma}}W_{f,g,+,1-s}\Big(\frac{\ell\ell' n^2}{q^2\delta^2}\Big). \tag{5.3}$$

The first term of this sum equals

$$\frac{1}{2}\frac{1}{2i\pi}\int\limits_{(2)}\frac{L_\infty(f,+,s+u)}{L_\infty(f,+,s)}\frac{L_\infty(g,+,\overline{s}+u)}{L_\infty(g,+,\overline{s})}L(f\times g,2s,u;\ell',\ell)G(u)(q^2|rr'|)^u\frac{du}{u}, \tag{5.4}$$

where

$$L(f\times g,2s,u;\ell',\ell)=\sum_{n\geqslant 1}\frac{\lambda_f(\ell' n)\lambda_g(\ell n)}{\ell'^{s+u}\ell^{\overline{s}+u}n^{2\sigma+2u}},\ \mathfrak{Re}(2\sigma+2u)>1, \tag{5.5}$$

and the second term equals

$$\frac{\varepsilon(f,g,+,s)\lambda_f(\varrho)\lambda_g(\varrho')}{2\varrho^{1-s}(\varrho')^{1-\overline{s}}}\frac{1}{2i\pi}\int\limits_{(2)}\frac{L_\infty(f,+,1-s+u)}{L_\infty(f,+,1-s)} \times\frac{L_\infty(g,+,1-\overline{s}+u)}{L_\infty(g,+,1-\overline{s})}L(f\times g,2-2s,u;\ell,\ell')G(u)(q^2\delta^2)^u\frac{du}{u}. \tag{5.6}$$

Similarly, the odd part of the second moment $\mathcal{Q}^-(f,g,\ell,\ell')$ yields the second part of the main term, namely

$$\mathrm{MT}^-(f,g;\ell,\ell')=\frac{1}{2}\sum_{n\geqslant 1}\frac{\lambda_f(\ell' n)\lambda_g(\ell n)}{\ell'^s\ell^{\overline{s}}n^{2\sigma}}W_{f,g,-,s}\Big(\frac{\ell\ell' n^2}{q^2|rr'|}\Big) + \frac{\varepsilon(f,g,-,s)\lambda_f(\varrho)\lambda_g(\varrho')}{2\varrho^{1-s}(\varrho')^{1-\overline{s}}}\sum_{n\geqslant 1}\frac{\lambda_f(\ell n)\lambda_g(\ell' n)}{\ell^{1-s}(\ell')^{1-\overline{s}}n^{2-2\sigma}}W_{f,g,-,1-s}\Big(\frac{\ell\ell' n^2}{q^2\delta^2}\Big),$$

where the first term equals

$$\frac{1}{2}\frac{1}{2i\pi}\int\limits_{(2)}\frac{L_\infty(f,-,s+u)}{L_\infty(f,-,s)}\frac{L_\infty(g,-,\overline{s}+u)}{L_\infty(g,-,\overline{s})}L(f\times g,2s,u;\ell',\ell)G(u)(q^2|rr'|)^u\frac{du}{u} \tag{5.7}$$

and the second

$$\text{(5.8)}\quad \frac{\varepsilon(f,g,-,s)\lambda_f(\varrho)\lambda_g(\varrho')}{2\varrho^{1-s}(\varrho')^{1-\bar{s}}}\frac{1}{2i\pi}\int_{(2)}\frac{L_\infty(f,-,1-s+u)}{L_\infty(f,-,1-s)}$$
$$\times\frac{L_\infty(g,-,1-\bar{s}+u)}{L_\infty(g,-,1-\bar{s})}L(f\times g,2-2s,u;\ell,\ell')G(u)(q^2\delta^2)^u\frac{du}{u}.$$

At this stage, we can therefore write

$$\mathcal{Q}(f,g,s;\ell,\ell')=\mathrm{MT}(f,g,s;\ell,\ell')+(\text{error term})$$

as in Theorem 5.1, with

$$\mathrm{MT}(f,g,s;\ell,\ell')=\frac{1}{2}\mathrm{MT}^+(f,g,s;\ell,\ell')+\frac{1}{2}\mathrm{MT}^-(f,g,s;\ell,\ell'),$$

and with an error term that we will estimate in the next sections to conclude the proof of the theorem.

It is not necessary (or, indeed, useful) to evaluate the mains terms very precisely in general, since in most applications (as in later chapters) we will perform further averages or combinations of them.

However, the special case $\ell=\ell'=1$ and $s=\frac{1}{2}$ (i.e., the "pure" second moment) is important, so we transform the main term in that case. We recall the notation $L^*(f\otimes g,1)$ from (2.7) and (2.9), and recall in particular that these are non-zero.

Proposition 5.2. *If $f=g$, then we have*

$$\mathrm{MT}(f,f;\tfrac{1}{2},1,1)=2\prod_{p\mid r}(1+p^{-1})^{-1}\frac{L^*(\mathrm{Sym}^2 f,1)}{\zeta(2)}\log q+\beta_f+O(q^{-2/5})$$

for some constant β_f. If $f\neq g$, then

$$\mathrm{MT}(f,g,\tfrac{1}{2};1,1)=\Big(1+\varepsilon(f)\varepsilon(g)\frac{\lambda_f(\varrho)\lambda_g(\varrho')}{(\varrho\varrho')^{1/2}}\Big)L^*(f\otimes g,1)+O(q^{-2/5}),$$

where the leading constant has modulus $\leqslant 2$, and is non-zero unless $\varrho=\varrho'=1$ and $\varepsilon(f)\varepsilon(g)=-1$.

Proof. The formulas follow easily from shifting the contour in (5.4), (5.6), (5.7), (5.8) to $\mathfrak{Re}\,u=-1/5$ and applying the residue theorem, involving only a pole at $u=0$ occurs, since by definition (see (2.7) and (2.8)), we have

$$L(f\times g,1,u;1,1)=L^*(f\otimes g,1+2u),$$

and

$$L(f\times f,1,u;1,1)=\frac{\zeta^{(r)}(1+2u)}{\zeta^{(r)}(2+4u)}L^*(\mathrm{Sym}^2 f,1+2u),$$

and moreover $\varepsilon(f,g,\pm,\frac{1}{2})=\varepsilon(f)\varepsilon(g)$ (see (2.28)).

If $f\neq g$, then since $\varrho\mid r$ and $\varrho'\mid r'$, we have

$$\frac{|\lambda_f(\varrho)\lambda_g(\varrho')|}{\sqrt{\varrho\varrho'}}\leqslant\frac{1}{\sqrt{\varrho\varrho'}}.$$

We deduce first that

$$\Big|1+\varepsilon(f)\varepsilon(g)\frac{\lambda_f(\varrho)\lambda_g(\varrho')}{(\varrho\varrho')^{1/2}}\Big|\leqslant 2,$$

and next that the leading constant can only be zero if $\varrho=\varrho'=1$, and then only if $\varepsilon(f)\varepsilon(g)=-1$ (recall that $L^*(f\otimes g,1)\neq 0$ by Lemma 2.6). ∎

5.3. The error term

The contributions to the error term are of the form

$$\mathrm{ET}(f,g;\ell,\pm\ell')=\sum\sum_{\substack{\ell m\equiv\pm\ell' n\,(\mathrm{mod}\,q)\\ \ell m\neq\ell' n}}\frac{\lambda_f(m)\lambda_g(n)}{m^s n^{\overline{s}}}W_{f,g,\pm,s}\left(\frac{mn}{q^2}\right)$$

or $\mathrm{ET}(f,g;r\ell,\pm r'\ell')$. The following bound then implies the theorem.

THEOREM 5.3. *Let $s\in\mathbf{C}$ be a complex number such that $|\sigma-1/2|\leqslant(\log q)^{-1}$. Let $L\leqslant q^{1/2}$. For any coprime integers ℓ and ℓ' such that $1\leqslant\ell,\ell'\leqslant L$, we have*

$$\mathrm{ET}(f,g;\ell,\pm\ell')\ll(rr'|s|)^{O(1)}L^{3/2}q^{-1/144+\varepsilon}.$$

The proof proceeds as in [4]. Using a partition of unity on the m,n variables and a Mellin transform to separate the variables m and n in the weight function, we reduce to the evaluation of $O(\log^2 q)$ sums of the shape

$$\mathrm{ET}(M,N;\ell,\pm\ell')=\frac{1}{(MN)^{1/2}}\sum_{\substack{\ell m\equiv\pm\ell' n\,(\mathrm{mod}\,q)\\ \ell m\neq\ell' n}}\lambda_f(m)\lambda_g(n)W_1\Big(\frac{m}{M}\Big)W_2\Big(\frac{n}{N}\Big)$$

test functions for W_1,W_2 satisfying (2.1) and for parameters $1\leqslant M,N$ such that $MN\leqslant q^{2+\varepsilon}$ (where we have removed f,g from the notation for simplicity). As in [4] the weight functions W_1,W_2 depend on a parameter of size $(\log q)^2$.

We will explain the proof of the estimate for $\mathrm{ET}(M,N;\ell,\ell')$. The case of $\mathrm{ET}(M,N;\ell,-\ell')$ is very similar, and left to the reader.

We first note that we may assume that $|r|+|r'|+|s|\ll q^{\varepsilon}$, since otherwise Theorem 5.3 holds trivially.

We will bound the sums $\mathrm{ET}(M,N;\ell,\ell')$ in different ways, according to the relative sizes of M,N. We may assume without loss of generality that $M\leqslant N$. Then we have at our disposal the following three bounds.

5.4. The trivial bound

Using (2.16) and (2.14), and distinguishing the cases $4LN<q$ (in which case the equation $\ell m\equiv\pm\ell' n\,(\mathrm{mod}\,q)$, $\ell m\neq\ell' n$ has no solutions) and $2LN\geqslant q$, we have

$$\mathrm{ET}(M,N;\ell,\ell')\ll q^{\varepsilon}M^{\theta}\frac{M}{(MN)^{1/2}}\frac{LN}{q}\ll q^{\varepsilon}LN^{\theta}\frac{(MN)^{1/2}}{q}. \tag{5.9}$$

5.5. The shifted convolution bound

Next we appeal to the shifted convolution estimate of Proposition 2.26. Setting

$$M'=\min(\ell M,\ell' N)\leqslant LM,\quad N'=\max(\ell M,\ell' N)\leqslant LN,$$

and using the bounds

$$M'N'\leqslant L^2q^{2+\varepsilon},\qquad M\leqslant\min(N,q^{1+\varepsilon}),$$

we see that $\mathrm{ET}(M,N;\ell,\ell')$ is

$$\begin{aligned}
&\ll \frac{q^{\varepsilon}}{(MN)^{1/2}}\left(\left(\frac{N'}{q^{1/2}}+\frac{N'^{3/4}M'^{1/4}}{q^{1/4}}\right)\left(1+\frac{(M'N')^{1/4}}{q^{1/2}}\right)+\frac{M'^{3/2+\theta}}{q}\right)\\
&\ll \frac{q^{\varepsilon}}{(MN)^{1/2}}\left(\left(\frac{N}{q^{1/2}}+\frac{N^{3/4}M^{1/4}}{q^{1/4}}\right)L^{3/2}+\frac{(LM)^{3/2+\theta}}{q}\right)\\
&\ll q^{\varepsilon}L^{3/2}\left(\left(\frac{N}{qM}\left(1+\frac{N}{qM}\right)\right)^{1/4}+\frac{L^{\theta}}{q^{1/2-\theta}}\right). \qquad (5.10)
\end{aligned}$$

5.6. Bilinear sums of Kloosterman sums

This is similar to [4, §6.2]. We apply the Voronoi summation formula of Corollary 2.22 to the n variable. We will do so only under the assumption

$$N > 4LM, \qquad (5.11)$$

so that the summation condition $\ell m \neq \ell' n$ is automatically satisfied. This expresses $\mathrm{ET}(M,N;\ell,\ell')$ as the sum of two terms. The first one is

$$\frac{1}{q(MN)^{1/2}}\sum_{m,n}\sum \lambda_f(m)\lambda_g(n)W_1\left(\frac{m}{M}\right)W_2\left(\frac{n}{N}\right) \ll \frac{q^{\varepsilon}}{q(MN)^{1/2}},$$

which is very small, and the second is

$$\varepsilon(g)\sum_{\pm}\frac{N}{q|r|^{1/2}}\frac{1}{(qMN)^{1/2}}\sum_{m,n\geqslant 1}\lambda_f(m)\lambda_g(n) \\ \times W_1\left(\frac{m}{M}\right)\widetilde{W}_{2,\pm}\left(\frac{n}{N^*}\right)\mathrm{Kl}_2(\pm\overline{|r'|\ell}\ell' mn;q),$$

where $N^* = q^2|r|/N$.

By Lemma 2.23, the function $\widetilde{W}_{2,\pm}(y)$ has rapid decay for $y \geqslant q^{\varepsilon}$. By a further partition of unity, we reduce to bounding quantities of the shape

$$E = \frac{(1+N^*/N')^{2\theta+\varepsilon}}{(qMN^*)^{1/2}}B(\mathrm{Kl}_2,\boldsymbol{\alpha}.\boldsymbol{\beta})$$

with coefficient sequences

$$\boldsymbol{\alpha} = \big(\lambda_f(m)W_1(m/M)\big)_{m\leqslant 2M} \quad \text{and} \quad \boldsymbol{\beta} = \big(\lambda_g(n)\widetilde{W}_{2,\pm}(n/N*)\big)_{n\leqslant N'}$$

that are supported on $[1,2M]$ and $[1,2q^{\varepsilon}N^*]$, respectively, and where

$$B(\mathrm{Kl}_2,\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{m,n}\sum \alpha_m\beta_n\,\mathrm{Kl}_2(amn;q)$$

with $a = \pm\overline{|r'|\ell}\ell'$ coprime to q, as in (3.5).

Bounding the Kloosterman sums trivially and using (2.14) we obtain first

$$E \ll q^{\varepsilon}\frac{MN^*}{(qMN^*)^{1/2}} = q^{\varepsilon}\left(\frac{qM}{N}\right)^{1/2}. \qquad (5.12)$$

Using instead Proposition 3.5, with the sequence $\boldsymbol{\beta}$ viewed as a sequence of length N^*, we obtain

$$\begin{aligned} E &\ll q^\varepsilon \frac{MN^*}{(qMN^*)^{1/2}} \left(\frac{1}{M} + q^{-\frac{1}{32}}\Big(\frac{q}{MN^*}\Big)^{3/8}\right)^{1/2} \\ &\ll q^\varepsilon \left(\frac{qM}{N}\right)^{1/2} \left(\frac{1}{M} + q^{-1/32}\Big(\frac{N}{qM}\Big)^{3/8}\right)^{1/2}, \end{aligned} \tag{5.13}$$

under the assumptions that

$$M \ll q^{\varepsilon+1/4}N^*,\ MN^* \ll q^{5/4}$$

or equivalently

$$MN \leqslant q^{\varepsilon+9/4},\ q^{3/4} \leqslant N/M. \tag{5.14}$$

Observe that the first inequality is always satisfied.

5.7. Optimization

Set $\eta = 1/144$. We have now derived the four basic bounds (5.9), (5.10), (5.12) and (5.13), all of which provide estimates for $\mathrm{ET}(M,N,\ell,\ell')$. We define β,λ,μ,ν so that the identities

$$\begin{gathered} M = q^\mu,\ N = q^\nu,\ L = q^\lambda,\ \mathrm{ET}(M,N,\ell,\ell') = q^\beta, \\ \mu^* = 2-\mu,\ \nu^* = 2-\nu. \end{gathered}$$

Our objective is to prove that

$$\beta \leqslant \varepsilon + \frac{3}{2}\lambda - \eta, \tag{5.15}$$

which will conclude the proof of Theorem 5.3. We use the same method as in [4, §6.2].

We have

$$\begin{gathered} 0 \leqslant \mu \leqslant \nu, \quad \mu+\nu \leqslant 2+\varepsilon, \quad \mu \leqslant 1+\varepsilon, \\ -1-\varepsilon \leqslant 1+\mu-\nu \leqslant 1, \end{gathered}$$

and assume that $\lambda \leqslant \frac{1}{2}$. By the trivial bound we have

$$\beta \leqslant \varepsilon + \lambda + \theta\nu + \frac{1}{2}(\mu+\nu-2).$$

We may therefore assume that

$$2 - 2\eta - 2\theta\nu \leqslant \mu+\nu \leqslant 2+\varepsilon$$

(otherwise (5.15) holds) and therefore

$$-2\eta - 2\theta\nu \leqslant \mu - \nu^* \leqslant \varepsilon. \tag{5.16}$$

Applying the shifted convolution estimate (5.10), we obtain

$$\beta \leqslant \varepsilon + \frac{3}{2}\lambda + \sup\Big(\frac{1}{4}(\nu-\mu-1), \frac{1}{2}(\nu-\mu-1), (\lambda+1)\theta - \frac{1}{2}\Big),$$

so that (5.15) holds unless

$$1 - 4\eta \leqslant \nu - \mu \text{ or equivalently } \mu + \nu^* \leqslant 1 + 4\eta. \tag{5.17}$$

This inequality implies that (5.11) and (5.14) both hold. We may therefore apply (5.12) and (5.13).

Applying (5.12) we obtain

$$\beta \leqslant \varepsilon + \frac{1}{2}(1+\mu-\nu) = \varepsilon + \frac{1}{2}(\mu+\nu^*-1)$$

which establishes (5.15) unless

$$1-2\eta \leqslant \mu+\nu^*.$$

This inequality together with (5.16) implies that

$$\mu \geqslant \frac{1}{2} - 2\eta - \theta\nu. \tag{5.18}$$

Applying now (5.13) we obtain

$$\begin{aligned}\beta &\leqslant \varepsilon + \frac{1}{2}(\mu+\nu^*-1) + \max\Big(-\frac{1}{2}\mu, -\frac{1}{64} - \frac{3}{16}(\mu+\nu^*-1)\Big)\\ &\leqslant \varepsilon + \max\Big(3\eta + \frac{1}{2}\theta\nu - \frac{1}{4}, \frac{5}{16}4\eta - \frac{1}{64}\Big)\end{aligned}$$

by (5.17) and (5.18), resp. (5.17)). This concludes the proof of (5.15), since the first term in the maximum is $< -\eta$ (recall that $\theta \leqslant 7/64$ and $\nu \leqslant 2$) and the second is equal to $-\eta$.

CHAPTER 6

Non-vanishing at the central point

6.1. Introduction

In this chapter, we will prove Theorem 1.8 using the mollification method. We fix f as in Section 1.2. Recall that by "interval" in $\mathbf{R}/\pi\mathbf{Z}$, we mean the image of an interval of $\mathbf{R}$ under the canonical projection, and its measure is the probability Haar measure μ of this image. The statement to prove is:

THEOREM 6.1. *Let $I \subset \mathbf{R}/\pi\mathbf{Z}$ be an interval of positive measure $\mu(I)$. There exists a constant $\eta = \eta_I > 0$ such that, as $q \to +\infty$ among the primes, we have*

$$\frac{1}{\varphi^*(q)}\Big|\Big\{\chi \,(\mathrm{mod}\, q) \text{ non-trivial} \;\Big|\; |L(f\otimes\chi, 1/2)| \geqslant \frac{1}{\log q},\ \theta(f\otimes\chi) \in I\Big\}\Big| \geqslant \eta + o_{f,I}(1).$$

Furthermore this formula remains true with the following choice of η

$$\eta = \eta_I = \frac{\mu(I)^2}{1443\,\zeta(2)}.$$

In Section 6.7, which may be omitted in a first reading, we will also prove a positive proportion of non-vanishing for central values of the twisted L-functions with characters satisfying conditions on the discrete Mellin transform of a quite general trace function.

6.2. The Cauchy-Schwarz inequality

Let $I \subset \mathbf{R}/\pi\mathbf{Z}$ be an interval with positive measure $\mu(I)$ and characteristic function χ_I. Let $\chi \mapsto M(f\otimes\chi, \boldsymbol{x}_L)$ be a function defined for Dirichlet characters modulo q (later, it will be the "mollifier"), depending on the fixed cusp form f and on some additional data $\boldsymbol{x}_L$.

By the Cauchy-Schwarz inequality, we have

$$\Big|\frac{1}{\varphi^*(q)}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)} \delta_{\substack{|L(f\otimes\chi,1/2)|\geqslant(\log q)^{-1}\\ \theta(f\otimes\chi)\in I}} L(f\otimes\chi,1/2)M(f\otimes\chi,\boldsymbol{x}_L)\Big|^2 \leqslant \mathcal{Q}(f;\boldsymbol{x}_L) \times \Big(\frac{1}{\varphi^*(q)}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)} \delta_{\substack{|L(f\otimes\chi,1/2)|\geqslant(\log q)^{-1}\\ \theta(f\otimes\chi)\in I}}\Big), \tag{6.1}$$

where

$$\mathcal{Q}(f;\boldsymbol{x}_L) := \frac{1}{\varphi^*(q)}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)} |L(f\otimes\chi,1/2)M(f\otimes\chi,\boldsymbol{x}_L)|^2. \tag{6.2}$$

On the left-hand side of (6.1), we remove the condition $|L(f\otimes\chi,1/2)|\geqslant(\log q)^{-1}$ in a trivial manner,

$$(6.3)\qquad \frac{1}{\varphi^*(q)}\sum_{\chi\,(\mathrm{mod}\,q)}^{*}\delta_{\substack{|L(f\otimes\chi,1/2)|\geqslant(\log q)^{-1}\\ \theta(f\otimes\chi)\in I}}L(f\otimes\chi,1/2)M(f\otimes\chi,\boldsymbol{x}_L)=$$
$$\mathscr{L}(f;\boldsymbol{x}_L,\chi_I)+O\Big(\frac{1}{q\log q}\sum_{\chi\,(\mathrm{mod}\,q)}^{*}|M(f\otimes\chi,\boldsymbol{x}_L)|\Big)$$

where, for any function $\psi:\mathbf{R}/\pi\mathbf{Z}\to\mathbf{C}$, we have defined

$$(6.4)\qquad \mathscr{L}(f;\boldsymbol{x}_L,\psi):=\frac{1}{\varphi^*(q)}\sum_{\chi\,(\mathrm{mod}\,q)}^{*}\psi(\theta(f\otimes\chi))L(f\otimes\chi,1/2)M(f\otimes\chi,\boldsymbol{x}_L).$$

Given δ with $0<\delta<1$, let

$$\psi(\theta)=\psi_{I,\delta}(\theta)=\sum_{|k|\leqslant k_\delta}\widehat{\psi}(2k)e^{2ik\theta}$$

be a trigonometric polynomial of period π such that

$$(6.5)\qquad \int_0^\pi|\chi_I-\psi|^2\,d\theta\leqslant\delta.$$

Since Gauß sums are equidistributed in $\mathbf{R}/2\pi\mathbf{Z}$ (as follows by Weyl's criterion from the formula (3.3) and Deligne's bound for hyper-Kloosterman sums), we have in particular

$$\frac{1}{\varphi^*(q)}\sum_{\chi\,(\mathrm{mod}\,q)}^{*}|\chi_I(\theta(f\otimes\chi))-\psi(\theta(f\otimes\chi))|^2\ll\delta.$$

By the definitions (6.2) and (6.4), the Cauchy-Schwarz inequality provides us with the approximation

$$\mathscr{L}(f;\boldsymbol{x}_L,\chi_I)=\mathscr{L}(f;\boldsymbol{x}_L,\psi)+O\left(\delta^{1/2}\mathcal{Q}(f;\boldsymbol{x}_L)^{1/2}\right).$$

6.3. Choosing the mollifier

As customary in the mollification method, we choose the function $M(f\otimes\chi,\boldsymbol{x}_L)$ to be a suitable Dirichlet polynomial of length $L=q^\lambda$, for some sufficiently small absolute positive constant $\lambda\leqslant 1/2$. Given some complex tuple $\boldsymbol{x}_L=(x_\ell)_{\ell\leqslant L}$, we set

$$(6.6)\qquad M(f\otimes\chi,\boldsymbol{x}_L)=\sum_{\ell\leqslant L}x_\ell\frac{\chi(\ell)}{\ell^{1/2}}.$$

We assume throughout that $x_\ell=0$ unless $(\ell,r)=1$. We recall that $(\mu_f(n))$ denotes the convolution inverse of the Hecke eigenvalues $(\lambda_f(n))$. We consider coefficients $(x_\ell)_{\ell\leqslant L}$ of the shape

$$(6.7)\qquad x_\ell=\mu_f(\ell)P\Big(\frac{\log(L/\ell)}{\log L}\Big)\,\delta_{\ell\leqslant L}\delta_{(\ell,r)=1},$$

where $P:[0,1]\to\mathbf{C}$ is a real-valued polynomial satisfying $P(1)=1$, $P(0)=0$.

In particular, as a consequence of (2.15) and (2.16), we have the inequalities

$$|x_\ell|\ll_\varepsilon\ell^{\theta+\varepsilon}$$

for any $\varepsilon > 0$, and

$$\sum_{\ell \leqslant L} |x_\ell|^2 \ll_f L. \tag{6.8}$$

Indeed, to prove (6.8) we first notice that the definition of the multiplicative function $n \mapsto \mu_f(n)$ given in (2.31) implies that for $p \nmid r$, we have the equalities

$$|\mu_f(p)| = |\lambda_f(p)|,\ \ \mu_f(p^2) = 1,\ \ \mu_f(p^k) = 0\ (k \geqslant 3).$$

In the sum studied in (6.8), we factorize each ℓ as $\ell = \ell_1\ell_2$ where ℓ_1 and ℓ_2 are coprime, ℓ_1 is squarefull and ℓ_2 is squarefree. We can now write the inequalities

$$\begin{aligned}\sum_{\ell \leqslant L} |x_\ell|^2 &\ll \sum_{\substack{\ell_1 \leqslant L\\ (\ell_1, r)=1}} |\mu_f(\ell_1)|^2 \sum_{\substack{\ell_2 \leqslant L/\ell_1\\ (\ell_2, r)=1}} |\mu_f(\ell_2)|^2\\ &\ll \sum_{\substack{\ell_1 \leqslant L\\ (\ell_1, r)=1}} \sum_{\substack{\ell_2 \leqslant L/\ell_1\\ (\ell_2, r)=1}} |\lambda_f(\ell_2)|^2\\ &\ll \sum_{\substack{\ell_1 \leqslant L\\ (\ell_1, r)=1}} (L/\ell_1) \ll L,\end{aligned}$$

by appealing to (2.15).

Next we have

Lemma 6.2. *We have*

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} |M(f \otimes \chi, \boldsymbol{x}_L)| \ll (\log q)^{1/2}.$$

Proof. By the Cauchy-Schwarz inequality we have

$$\sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} |M(f \otimes \chi, \boldsymbol{x}_L)| \leqslant \varphi^*(q)^{1/2} \Big(\sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} |M(f \otimes \chi, \boldsymbol{x}_L)|^2 \Big)^{1/2}$$

and, as a consequence of (6.8), we have

$$\begin{aligned}\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} |M(f \otimes \chi, \boldsymbol{x}_L)|^2 &\leqslant \frac{\varphi(q)}{\varphi^*(q)} \sum_{\ell \equiv \ell' \,(\mathrm{mod}\, q)} \frac{|x_\ell x'_\ell|}{(\ell\ell')^{1/2}}\\ &= \frac{\varphi(q)}{\varphi^*(q)} \sum_{\ell \leqslant L} \frac{|x_\ell|^2}{\ell} \ll \log L\end{aligned}$$

since $L < q$. ∎

We conclude from (6.1), (6.3) and Lemma 6.2 that

$$\begin{aligned}&\frac{1}{\varphi^*(q)} |\{\chi \,(\mathrm{mod}\, q) \text{ non-trivial} \mid |L(f \otimes \chi, 1/2)| \geqslant (\log q)^{-1}, \theta(f \otimes \chi) \in I\}|\\ &\qquad\qquad \geqslant \frac{\big|\mathscr{L}(f; \boldsymbol{x}_L, \psi_{I,\delta}) + O_f((\log q)^{-1/2} + \delta^{1/2}\mathcal{Q}(f; \boldsymbol{x}_L)^{1/2})\big|^2}{\mathcal{Q}(f, \boldsymbol{x}_L)}.\end{aligned} \tag{6.9}$$

6.4. Computation of the first mollified moment

In this section we evaluate $\mathscr{L}(f;\boldsymbol{x}_L,\psi_{I,\delta})$. Since

$$\exp(2i\theta(f\otimes\chi))=\varepsilon(f)\chi(r)\varepsilon_\chi^2 \tag{6.10}$$

by (2.4), we have

$$\mathscr{L}(f;\boldsymbol{x}_L,\psi)=\sum_{|k|\leqslant k_\delta}\widehat{\psi}(2k)\varepsilon(f)^k\sum_{\ell\leqslant L}\frac{x_\ell}{\ell^{1/2}}\mathcal{L}(f;r^k\ell,2k) \tag{6.11}$$

where $\mathcal{L}(f;r^k\ell,2k)$ is the first moment defined in (4.1).

REMARK 6.3. It is at this point that our restriction to intervals modulo π instead of modulo 2π intervenes: because of the factor 2 on the left-hand side of (6.10), we are not able to evaluate a first moment of $L(f\otimes\chi,\frac{1}{2})$ twisted by $e(k\theta(f\otimes\chi))$ for k odd.

By (4.2), Theorem 4.1 and (6.8), the total contribution, denoted by $S_{k\neq 0,-1}$, of the terms $k\neq 0,-1$ to the sum (6.11) satisfies

$$S_{k\neq 0,-1}\ll_{\delta,I,f,\varepsilon} q^{\varepsilon+\lambda/2-1/8}. \tag{6.12}$$

In particular, this contribution is negligibly small if $\lambda<1/4$, which we assume from now on.

The contribution $S_{k=0}$ of the term $k=0$ to (6.11) is equal to

$$S_{k=0}:=\widehat{\psi}(0)\sum_{\ell\leqslant L}\frac{x_\ell}{\ell^{1/2}}\mathcal{L}(f;\ell,0)=\widehat{\psi}(0)\sum_{\ell\leqslant L}\frac{x_\ell}{\ell^{1/2}}\Big\{\frac{\lambda_f(\bar{\ell}_q)}{\bar{\ell}_q^{1/2}}+O_{f,\varepsilon}(q^{\varepsilon-1/8})\Big\},$$

by Theorem 4.1. We treat separately the cases $\ell=1$ and $2\leqslant\ell\leqslant L$. The first case contributes by

$$\widehat{\psi}(0)x_1+O(q^{-1/8+\varepsilon})=\mu(I)+O_{f,\varepsilon}(\delta^{1/2}+q^{\varepsilon-1/8}), \tag{6.13}$$

by (6.5).

To deal with the cases $2\leqslant\ell\leqslant L$ we exploit the inequalities

$$|\lambda_f(\bar{\ell}_q)\bar{\ell}_q^{-1/2}|\leqslant\bar{\ell}_q^{\theta-1/2}\leqslant(q/\ell)^{\theta-1/2},$$

since ℓ satisfies $1<\ell<q^{1/2}$. From this we deduce that the contribution of the $2\leqslant\ell\leqslant L$ satisfies

$$\ll q^{\theta-1/2}L^{1-\theta}+q^{\varepsilon-1/8}L^{1/2}\ll q^{\varepsilon-1/8}L^{1/2}.$$

Gathering with (6.13), we proved the equality

$$S_{k=0}=\mu(I)+O_{f,\varepsilon}(\delta^{1/2}+q^{\varepsilon+\lambda/2-1/8}) \tag{6.14}$$

Next, by Corollary 4.2, if q is large enough (depending on the level r), the contribution $S_{k=-1}$ of the term $k=-1$ equals

$$\begin{aligned} S_{k=-1}&=\widehat{\psi}(-2)\sum_{\ell\leqslant L}\frac{x_\ell\lambda_f(\ell_q)}{\ell^{1/2}\ell_q^{1/2}}+O_{\delta,I,f,\varepsilon}(q^{\varepsilon+\lambda/2-1/8})\\ &=\widehat{\psi}(-2)\sum_{\substack{\ell\leqslant L\\(\ell,r)=1}}\frac{\mu_f(\ell)\lambda_f(\ell_q)}{\ell^{1/2}\ell_q^{1/2}}P\Big(\frac{\log(L/\ell)}{\log L}\Big)+O_{\delta,I,f,\varepsilon}(q^{\varepsilon+\lambda/2-1/8})\end{aligned} \tag{6.15}$$

where as before ℓ_q denotes the unique integer in $[1,q]$ representing the congruence class $\ell \pmod q$. Since $1\leqslant\ell\leqslant L<q$, we have $\ell_q=\ell$.

We now use the following lemma, which is stated in slightly greater generality than needed here, for later reference in Section 6.7.

LEMMA 6.4. *Let $a \geqslant 1$ be an integer. There is some constant $c > 0$, depending only on f, such that*

$$\sum_{\substack{\ell \leqslant L \\ (\ell,r)=1}} \frac{x_\ell}{\sqrt{\ell}} \frac{\lambda_f(a\ell)}{\sqrt{a\ell}} = \sum_{\substack{\ell \leqslant L \\ (\ell,r)=1}} \frac{\mu_f(\ell)\lambda_f(a\ell)}{a^{1/2}\ell} P\Big(\frac{\log(L/\ell)}{\log L}\Big) \ll_f \exp(-c\sqrt{\lambda \log q})$$

uniformly in a.

PROOF. We can write

$$\sum_{(\ell,r)=1} \mu_f(\ell)\lambda_f(a\ell)\ell^{-s} = \sum_{\substack{d \mid a \\ (d,r)=1}} \mu(d)\lambda_f(a/d)d^{-s} \sum_{(\ell,r)=1} \lambda_f(\ell)\mu_f(d\ell)\ell^{-s}.$$

In turn, since μ_f is supported on cubefree numbers, for d squarefree and coprime to r we have

$$\sum_{(\ell,r)=1} \lambda_f(\ell)\mu_f(d\ell)\ell^{-s} = \prod_{p \nmid dr} \Big(1 - \frac{\lambda_f(p)^2}{p^s} + \frac{\lambda_f(p^2)}{p^{2s}}\Big) \prod_{p \mid d,\, p \nmid r} (-\lambda_f(p))\Big(1 - \frac{1}{p^s}\Big).$$

We can therefore write the equality

$$\sum_{(\ell,r)=1} \lambda_f(\ell)\mu_f(d\ell)\ell^{-s} = \frac{H_d(s)}{T(s)}$$

where T is defined in (2.30) and where $H_d(s)$ is an Euler product that converges absolutely for $\mathfrak{Re}(s) > 3/4$ and is bounded by d^θ. This gives analytic continuation of the Dirichlet series

$$\sum_{(\ell,r)=1} \lambda_f(\ell)\mu_f(d\ell)\ell^{-s}$$

in the zero-free region of the Rankin-Selberg L-function (Proposition 2.11), and moreover the value of this function at $s = 1$ is zero. We then obtain the result of Lemma 6.4 by a standard contour integration and by a partial summation. $\square$

Returning to (6.15), we deduce that $S_{k=-1}$ satisfies the bound

$$S_{k=-1} \ll \exp(-c\sqrt{\log q}).$$

Gathering this bound with (6.12) and (6.14) and supposing that $0 < \lambda < 1/4$, we deduce the equality

$$\mathscr{L}(f; \boldsymbol{x}_L, \psi) = \mu(I) + O\big(\delta^{1/2} + \exp(-c\sqrt{\log q})\big). \tag{6.16}$$

We obtain now the lower bound

$$\frac{1}{\varphi^*(q)} |\{\chi \,(\mathrm{mod}\, q) \text{ non-trivial} \mid |L(f \otimes \chi, 1/2)| \geqslant (\log q)^{-1},\ \theta(f \otimes \chi) \in I\}| \tag{6.17}$$
$$\geqslant \frac{\mu(I)^2 + O_{f,\delta,I}\left((\log q)^{-1/2}\right) + O_f\left(\delta^{1/2}(1 + \mathcal{Q}(f; \boldsymbol{x}_L))\right)}{\mathcal{Q}(f; \boldsymbol{x}_L)}.$$

using (6.9). It remains to evaluate $\mathcal{Q}(f; \boldsymbol{x}_L)$.

6.5. Computation of the second mollified moment

In this section we compute

$$\begin{aligned}\mathcal{Q}(f;\boldsymbol{x}_L) &= \sum_{\ell,\ell'\leqslant L}\frac{x_\ell\overline{x_{\ell'}}}{(\ell\ell')^{1/2}}\mathcal{Q}(f,f,1/2,\ell,\ell')\\ &= \sum_d\sum_{(\ell,\ell')=1}\frac{x_{d\ell}\overline{x_{d\ell'}}}{d(\ell\ell')^{1/2}}\mathcal{Q}(f,f,1/2;\ell,\ell'),\end{aligned}$$

which is enough for our purpose, since $\mathcal{Q}(f,f,1/2;d\ell,d\ell')=\mathcal{Q}(f,f,1/2;\ell,\ell')$ (recall that the twisted second moment is defined in (5.1)).

Proposition 5.2 is not sufficient for our purpose since we will sum over ℓ and ℓ'. So we use Theorem 1.18 which evaluates $\mathcal{Q}(f,f,1/2;\ell,\ell')$ for $(\ell,\ell')=1$ with two main terms given in (5.2). Since $\varepsilon(f,f,\pm,1/2)=1$ and $\varrho=\varrho'=1$, $\delta=|r|=|r'|$, we obtain by (5.4), (5.6), (5.7), (5.8) that

$$\begin{aligned}(6.18)\quad \mathcal{Q}(f;\boldsymbol{x}_L) = \frac{1}{2i\pi}\int_{(2)}\Big(&\frac{L_\infty(f,1/2+u)^2}{L_\infty(f,1/2)^2}+\frac{L_\infty(f,3/2+u)^2}{L_\infty(f,3/2)^2}\Big)G(u)\\ &\times\Big(\sum_{d,(\ell,\ell')=1}\frac{x_{d\ell}\overline{x_{d\ell'}}}{d(\ell\ell')^{1/2}}L(f\times f,1,u;\ell,\ell')\Big)(|r|q)^{2u}\frac{du}{u}+O\left(L^{5/2}q^{-1/144+\varepsilon}\right),\end{aligned}$$

where

$$L(f\times f,1,u;\ell,\ell')=\sum_{n\geqslant 1}\frac{\lambda_f(\ell n)\lambda_f(\ell' n)}{(\ell\ell' n^2)^{1/2+u}},$$

(see (5.5)). We apply Mellin inversion to the sum over ℓ and ℓ'. For any polynomial

$$Q(X)=\sum_k a_kX^k,$$

and any $L>1$, we introduce the polynomial $\widehat{Q_L}(v)$ defined by

$$(6.19)\qquad \frac{L^v}{v}\widehat{Q_L}(v)=\int_0^L Q\left(\frac{\log(L/x)}{\log L}\right)x^{v-1}dx=\sum_k a_k\frac{k!L^v}{v^{k+1}(\log L)^k}.$$

With this notation, the main term of $\mathcal{Q}(f,\boldsymbol{x}_L)$ equals

$$\begin{aligned}(6.20)\quad \frac{1}{(2\pi i)^3}\int_{(2)}\int_{(2)}\int_{(2)}&\Big(\frac{L_\infty(f,1/2+u)^2}{L_\infty(f,1/2)^2}+\frac{L_\infty(f,3/2+u)^2}{L_\infty(f,3/2)^2}\Big)\\ &\times G(u)L(f,u,v,w)\widehat{P}_L(v)\overline{\widehat{P}}_L(w)L^{v+w}(|r|q)^{2u}\frac{du}{u}\frac{dv}{v}\frac{dw}{w}\end{aligned}$$

where
(6.21)

$$L(f,u,v,w)=\sum\sum_{\substack{d,\ell,\ell',n\\(\ell,\ell')=(d\ell\ell',r)=1}}\frac{\mu_f(d\ell)\lambda_f(\ell n)\mu_f(d\ell')\lambda_f(\ell' n)}{\ell^{1+u+v}\ell'^{1+u+w}d^{1+v+w}n^{1+2u}}=L(\tfrac12,\tfrac12,\tfrac12,u,v,w),$$

in terms of the auxiliary function introduced in (2.32).

Recall that $T(s)$ is defined in (2.30) as $T(s)=L(f\otimes f,s)$. From Corollary 2.25, we obtain the meromorphic continuation of this function to the domain

$$\mathfrak{Re}\,u,\mathfrak{Re}\,v,\mathfrak{Re}\,w>-\eta$$

for some $\eta > 0$, given by

$$\begin{aligned} L(f,u,v,w) &= \frac{T(1+2u)T(1+v+w)}{T(1+u+v)T(1+u+w)} D(u,v,w) \\ &= \eta_3(u,v,w)\frac{(u+v)(u+w)}{u(v+w)}, \end{aligned}$$

where D is an Euler product absolutely convergent for $\mathfrak{Re}\, u, \mathfrak{Re}\, v, \mathfrak{Re}\, w \geqslant -\eta$ and η_3 is holomorphic and non-vanishing in a neighborhood of $(u,v,w) = (0,0,0)$.

We shift the v,w-contours and then u-contour to the left of $u = v = w = 0$, using again the standard zero-free regions for Rankin-Selberg L-functions (Proposition 2.11) together with the rapid decay of Gamma-quotients. In this way we see that the triple integral in (6.20) equals

$$2\eta_3(0,0,0) \operatorname*{res}_{u=v=w=0} \widehat{P_L}(v)\widehat{P_L}(w)L^{v+w}(|r|q)^{2u}\frac{(u+v)(u+w)}{u^2(v+w)vw} + O\left(\frac{1}{\log L}\right). \tag{6.22}$$

We write

$$\frac{(u+v)(u+w)}{u^2(v+w)vw} = \frac{1}{(v+w)vw} + \frac{1}{uvw} + \frac{1}{u^2(v+w)}. \tag{6.23}$$

Our plan is now to compute the residue coming from each of the three terms on the right–hand side of (6.23). For this purpose, we gather in one lemma the contents of [50, Lemma 9.1-Corollary 9.4]. We have

Lemma 6.5. *For $M > 1$, P and Q polynomials, let $\widehat{P_M}$ and $\widehat{Q_M}$ be defined by (6.19). We then have the equalities*[1]

$$\operatorname*{res}_{s=0} \frac{M^s\widehat{Q_M}(s)}{s} = Q(1),$$

and

$$\operatorname*{res}_{s_1=s_2=0} \frac{M^{s_1+s_2}\widehat{Q_M}(s_1)\widehat{Q_M}(s_2)}{s_1s_2(s_1+s_2)} = (\log M)\Big(\int_0^1 P(x)Q(x)\,dx\Big).$$

Up to terms of size $O((\log L)^{-1})$, the contribution from the first term of the right side of (6.23) is zero while the contribution from the second term is equal to $2\eta_3(0,0,0)P(1)^2 = 2\eta_3(0,0,0)$, as a consequence of the first part of Lemma 6.5.

To deal with the residue coming from the third term, we first note the equality (recall that $P(0) = a_0 = 0$)

$$\begin{aligned} vw\widehat{P_L}(v)\widehat{P_L}(w) &= \frac{1}{\log^2 L}\Big(\sum_{k\geqslant 1} ka_k\frac{(k-1)!}{(v\log L)^{k-1}}\Big)\Big(\sum_{k\geqslant 1} ka_k\frac{(k-1)!}{(w\log L)^{k-1}}\Big) \\ &= \frac{\widehat{P_L'}(v)\widehat{P_L'}(w)}{\log^2 L}, \end{aligned}$$

[1]We take this opportunity to mention a misprint in the statement of [50, Lemma 9.4] (in that paper the formula was used in its correct form): the formula should read

$$\operatorname*{res}_{s_1,s_2=0} \frac{M^{s_1+s_2}\widehat{P_M}(s_1)\widehat{Q_M}(s_2)}{s_1s_2(s_1+s_2)} = \Big(\int_0^1 P(x)Q(x)dx\Big)(\log M).$$

The second part of Lemma 6.5 implies that we have

$$2\eta_3(0,0,0)\operatorname*{res}_{u=v=w=0}\frac{\widehat{P_L}(v)\widehat{P_L}(w)L^{v+w}(|r|q)^{2u}}{u^2(v+w)}$$
$$=2\eta_3(0,0,0)\frac{2\log q}{\log L}\Big(\int_0^1 P'(x)^2dx\Big)+O\Big(\frac{1}{\log L}\Big).$$

Altogether, (6.22) equals

$$2\eta_3(0,0,0)\Big(P(1)^2+\frac{2\log q}{\log L}\int_0^1 P'(x)^2dx\Big)+O\Big(\frac{1}{\log L}\Big).$$

Taking $L=q^\lambda$ with $0<\lambda<2/5\times 1/144=1/360$ to deal with the error term in (6.18) and $P=X$, it remains to recall (6.17) to obtain:

Proposition 6.6. *Let $\boldsymbol{x}_L$ be defined as above with $P(X)=X$, let $0<\lambda<1/360$ be fixed. For any $\delta>0$, we have*

$$\mathcal{Q}(f;\boldsymbol{x}_L)=2\eta_3(0,0,0)(1+2\lambda^{-1})+O(\log^{-1}q)$$

and

$$\frac{1}{\varphi^*(q)}|\{\chi\,(\mathrm{mod}\,q)\text{ non-trivial}\mid |L(f\otimes\chi,1/2)|\geqslant(\log q)^{-1},\theta(f\otimes\chi)\in I\}|$$
$$\geqslant\frac{1}{2\eta_3(0,0,0)}\cdot\frac{\mu(I)^2}{1+2\lambda^{-1}}+O_{f,\delta,I}\Big(\frac{1}{\log^{1/2}q}\Big)+O_f(\delta^{1/2}).$$

To conclude the proof of Theorem 1.8, it remains to observe that

$$\eta_3(0,0,0)\leqslant\zeta(2). \tag{6.24}$$

Indeed, this follows from the factorization

$$\eta_3(0,0,0)=\prod_p L_p(f,0,0,0)$$

where the local factors $L_p(f,u,v,w)$ of $L(f,u,v,w)$ are defined in (6.21) and satisfy

$$L_p(f,0,0,0)\leqslant(1-p^{-2})^{-1}\text{ if }p\mid r, \tag{6.25}$$

and

$$L_p(f,0,0,0)=1\text{ if }p\nmid r. \tag{6.26}$$

To prove (6.25) and (6.26) we will use the following identities where p is arbitrary:

$$\lambda_f(p)=\alpha+\beta,\ \text{where }\alpha\beta=\chi_r(p)$$
$$\lambda_f(p^k)=\alpha^k+\alpha^{k-1}\beta+\cdots+\alpha\beta^{k-1}+\beta^k$$
$$\mu_f(1)=1,\ \mu_f(p)=-\lambda_f(p),\ \mu_f(p^2)=\chi_r(p),\ \mu_f(p^k)=0,\ k\geqslant 3.$$

In the case of (6.25) we see that the definition (6.21) and the condition $p\mid r$ imply the equality

$$L_p(f,0,0,0)=\sum_{n\mid p^\infty}\frac{\lambda_f^2(n)}{n}=\sum_{k\geqslant 0}\frac{\lambda_f^{2k}(p)}{p^k}=(1-\lambda_f^2(p)/p)^{-1}.$$

It remains to appeal to (2.17) to complete the proof of (6.25).

The proof of (6.26) requires more attention. When $p\nmid r$ we write the local factor as

$$
\begin{aligned}
L_p(f,0,0,0) &= \sum\sum_{\substack{d,\ell,\ell',n\mid p^\infty\\(\ell,\ell')=1}} \frac{\mu_f(d\ell)\lambda_f(\ell n)\mu_f(d\ell')\lambda_f(\ell' n)}{\ell\ell' dn}\\
&= \sum\sum_{\substack{\delta,\lambda,\lambda',\nu\geqslant 0\\ \lambda.\lambda'=0}} \frac{\mu_f(p^{\delta+\lambda})\lambda_f(p^{\lambda+\nu})\mu_f(p^{\delta+\lambda'})\lambda_f(p^{\lambda'+\nu})}{p^{\delta+\lambda+\lambda'+\nu}}\\
&= 2\sum\sum_{\delta,\lambda,\nu\geqslant 0} \frac{\mu_f(p^{\delta+\lambda})\lambda_f(p^{\lambda+\nu})\mu_f(p^{\delta})\lambda_f(p^{\nu})}{p^{\delta+\lambda+\nu}} - \sum\sum_{\delta,\nu\geqslant 0}\frac{\mu_f(p^\delta)^2\lambda_f(p^\nu)^2}{p^{\delta+\nu}}. \qquad (6.27)
\end{aligned}
$$

As a consequence of the vanishing of the function μ_f, the first multiple sum can be restricted to the six subcases

$$(\delta,\lambda) = (0,0),\ (0,1),\ (0,2),\ (1,0),\ (1,1),\ (2,0),$$

and the second one to the three subcases

$$\delta = 0,\ 1,\ 2.$$

For instance, the contribution of the terms with $(\delta,\lambda)=(0,0)$ to the first multiple sum on the right–hand side of (6.27) can be expressed as

$$2\sum_{\nu\geqslant 0}\frac{1}{p^\nu}\Big(\frac{\alpha^{\nu+1}-\beta^{\nu+1}}{\alpha-\beta}\Big)^2 = \frac{2}{(\alpha-\beta)^2}\Big(\frac{\alpha^2}{1-\alpha^2/p}+\frac{\beta^2}{1-\beta^2/p}-\frac{2}{1-1/p}\Big).$$

The other contributions are computed similarly. By straightforward computations (most easily performed by computer–assisted symbolic calculations) we obtain (6.26). This completes the proof of Theorem 6.1.

6.6. Improvement of Theorem 1.8

We quickly explain Remark 1.9 (1) which asserts that the lower bound $(\log q)^{-1}$ in the statement of Theorem 1.8 and Theorem 6.1 can be improved in $(\log q)^{-\alpha}$ for every $\alpha > 1/2$. Indeed if, in the left–hand side of (6.1), we replace the condition $|L(f\otimes\chi,1/2)| \geqslant (\log q)^{-1}$ by $|L(f\otimes\chi,1/2)| \geqslant (\log q)^{-\alpha}$, then the coefficient $1/(q\log q)$ in the error term of the right–hand side of the equality (6.3) has to be replaced by $1/(q\log^\alpha q)$. A direct application of Lemma 6.2 bounds this error term by $O((\log q)^{1/2-\alpha})$. In order to make the method work, we only require to this error term to be negligible when compared with the main term $\mathscr{L}(f;\boldsymbol{x}_L,\chi_I)$ of (6.3) as q tends to infinity. The order of magnitude of $\mathscr{L}(f;\boldsymbol{x}_L,\chi_I)$ is known by the equality (6.16) and we are led to the sufficient condition $\alpha > 1/2$.

6.7. Non-vanishing with Mellin constraints

The goal of this section is to generalize the positive proportion of non-vanishing to incorporate certain conditions on χ, which are roughly of the form

$$\widetilde{t}(\chi)\in A,$$

where $A\subset\mathbf{C}$ and

$$\widetilde{t}(\chi) = \frac{1}{\sqrt{q}}\sum_{x\in\mathbf{F}_q^\times}\chi(x)t(x)$$

is the discrete Mellin transform of some suitable function $t\colon \mathbf{F}_q^\times \to \mathbf{C}$. The functions t that we can handle are some of the trace functions over $\mathbf{F}_q$ described in Section 3.4.

Let $C \geqslant 1$ be a real number and K be a compact Lie group. For each prime q (large enough), fix a Mellin sheaf $\mathcal{F}_q$ over $\mathbf{F}_q$ as in Definition 3.10 with conductor $\leqslant C$, with Property EAGM and with equidistribution group K (see Definition 3.14). Examples of such families, with $K = \mathrm{SU}_2(\mathbf{C})$ and $C = 5$, are provided by the sheaves related to Evans or Rosenzweig-Rudnick sums, see Example 3.17.

We denote by X_q the set of exceptional characters modulo q as described in Section 3.5; we recall that its size is bounded independently of q. For a Dirichlet character $\chi \notin X_q$, we denote by $\theta_{q,\chi} \in K^\sharp$ (or simply θ_χ) the conjugacy class associated to the Mellin transform of $\mathcal{F}_q$ at χ.

THEOREM 6.7. *With assumptions as above, let $A \subset K^\sharp$ be a measurable set with non-empty interior. Then*

$$\liminf_{q\to+\infty} \frac{1}{\varphi^*(q)}|\{\chi \notin X_q \mid |L(f\otimes\chi,\tfrac{1}{2})| \geqslant (\log q)^{-1} \text{ and } \theta_\chi \in A\}| > 0.$$

PROOF. Let $d \geqslant 1$ be the order of the finite group of finite-order characters of K. Let $\phi_0\colon K^\sharp \to [0,1]$ be a non-zero continuous function supported in an open subset contained in A. Let further ϕ be a finite linear combination of characters of irreducible representations of K such that $\|\phi-\phi_0\| < \delta$, where $\delta > 0$ will be specified later; such a function exists by the Peter-Weyl Theorem.

Fix q so that $\mathcal{F} = \mathcal{F}_q$ is defined. Let $L = q^\lambda$ with

$$0 < \lambda < \min\Bigl(\frac{1/2-\theta}{2d}, \frac{1}{360}\Bigr) \tag{6.28}$$

and consider the mollifier

$$M(f\otimes\chi, \boldsymbol{x}_L) = \sum_{\ell\leqslant L} \frac{x_\ell}{\sqrt{\ell}}\chi(\ell)$$

as in Section 6.3 (see (6.6) and (6.7)). Let

$$\mathcal{L} = \frac{1}{\varphi^*(q)} \sum_{\chi\notin X_q} \phi_0(\theta_\chi) L(f\otimes\chi,\tfrac{1}{2}) M(f\otimes\chi, \boldsymbol{x}_L).$$

We then have

$$\mathcal{L} = \mathcal{L}' + O((\log q)^{-1/2})$$

by Lemma 6.2, where

$$\mathcal{L}' = \frac{1}{\varphi^*(q)} \sum_{\substack{\chi\notin X_q\\ |L(f\otimes\chi,1/2)|\geqslant(\log q)^{-1}}} \phi_0(\theta_\chi) L(f\otimes\chi,\tfrac{1}{2}) M(f\otimes\chi, \boldsymbol{x}_L).$$

On the other hand, we have

$$|\mathcal{L}'|^2 \leqslant \mathcal{N}\mathcal{Q},$$

where

$$\mathcal{N} = \frac{1}{\varphi^*(q)}|\{\chi \notin X_q \mid |L(f\otimes\chi,\tfrac{1}{2})| \geqslant (\log q)^{-1} \text{ and } \theta_\chi \in A\}$$

(since $\phi_0(\theta_\chi) \neq 0$ implies that $\theta_\chi \in A$) and

$$\begin{aligned}\mathcal{Q} &= \frac{1}{\varphi^*(q)} \sum_{\chi \notin X_q} |\phi_0(\theta_\chi)|^2 |L(f \otimes \chi, \tfrac{1}{2})|^2 |M(f \otimes \chi, \boldsymbol{x}_L)|^2 \\ &\leqslant \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} |L(f \otimes \chi, \tfrac{1}{2})|^2 |M(f \otimes \chi, \boldsymbol{x}_L)|^2 \ll 1\end{aligned}$$

by Proposition 6.6 since $\lambda < 1/360$.

Hence it suffices to find a lower bound for $\mathcal{L}'$. Let

$$\mathcal{L}'' = \frac{1}{\varphi^*(q)} \sum_{\chi \notin X_q} \phi(\theta_\chi) L(f \otimes \chi, \tfrac{1}{2}) M(f \otimes \chi, \boldsymbol{x}_L).$$

Then

$$|\mathcal{L}'' - \mathcal{L}| \leqslant \|\phi - \phi_0\|_\infty \times \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} |L(f \otimes \chi, \tfrac{1}{2}) M(f \otimes \chi, \boldsymbol{x}_L)| \ll \|\phi - \phi_0\|_\infty$$

by Proposition 6.6 again. Write

$$\phi(x) = \int_K \phi + \sum_{\pi \neq 1} \widehat{\phi}(\pi) \,\mathrm{tr}(\pi(x)) \tag{6.29}$$

where the sum ranges over a finite set of non-trivial irreducible representations of K. Then, if δ is small enough, we have

$$\widehat{\phi}(1) = \int_K \phi > 0.$$

The equality (6.29) decomposes $\mathcal{L}''$ into

$$\mathcal{L}'' = \mathcal{L}''_{\mathrm{MT}} + \mathcal{L}''_{\mathrm{ET}}, \tag{6.30}$$

where the main term is given by

$$\begin{aligned}\mathcal{L}''_{\mathrm{MT}} &= \frac{\widehat{\phi}(1)}{\varphi^*(q)} \sum_{\chi \notin X_q} L(f \otimes \chi, \tfrac{1}{2}) M(f \otimes \chi, \boldsymbol{x}_L) \\ &= \frac{\widehat{\phi}(1)}{\varphi^*(q)} \sum_{\chi \bmod q} L(f \otimes \chi, \tfrac{1}{2}) M(f \otimes \chi, \boldsymbol{x}_L) + O(q^{\lambda/2 - 1/2 + \varepsilon}).\end{aligned}$$

In the line above, the error term is deduced from the fact that $|X_q| = O(1)$, from the classical bound of $L(f \otimes \chi, \frac{1}{2})$ and from (6.8). Finally by the computation in Section 6.4 with $k = 0$, we obtain the equality

$$\mathcal{L}''_{\mathrm{MT}} = \widehat{\phi}(1)\big(1 + o_\lambda(1)\big),$$

as q tends to infinity provided that λ satisfies (6.28).

Returning to (6.30), we have the equality

$$\mathcal{L}''_{\mathrm{ET}} = \frac{1}{\varphi^*(q)} \sum_{\pi \neq 1} \widehat{\phi}(\pi) \sum_{\chi \notin X_q} \mathrm{tr}(\pi(\theta_\chi)) L(f \otimes \chi, \tfrac{1}{2}) M(f \otimes \chi, \boldsymbol{x}_L). \tag{6.31}$$

Fix $\pi \neq 1$ in the sum. We have

$$\frac{1}{\varphi^*(q)} \sum_{\chi \notin X_q} \operatorname{tr}(\pi(\theta_\chi)) L(f \otimes \chi, \tfrac{1}{2}) M(f \otimes \chi, \boldsymbol{x}_L) = \frac{1}{\varphi^*(q)} \sum_{\ell \leqslant L} \frac{x_\ell}{\ell^{1/2}} \sum_{\chi \notin X_q} \operatorname{tr}(\pi(\theta_\chi)) L(f \otimes \chi, \tfrac{1}{2}) \chi(\ell). \tag{6.32}$$

We recall first that (6.8) implies

$$\sum_{\ell \leqslant L} \frac{|x_\ell|}{\sqrt{\ell}} \ll L^{1/2+\varepsilon} \tag{6.33}$$

for any $\varepsilon > 0$ and, for the end of the proof, we distinguish three cases.

Case 1. Let d be the order of the finite group of characters of finite order of K. There exists a such that $1 \leqslant a \leqslant q-1$ and such that $\operatorname{tr}(\pi(\theta_\chi)) = \chi(a)$ for all $\chi \notin X_q$, and moreover a is a non-trivial d-th root of unity modulo q since π is a non-trivial character of finite order of K. This is the "punctual" case (2) of Theorem 3.15. By Theorem 4.1, we have

$$\frac{1}{\varphi^*(q)} \sum_{\chi \notin X_q} L(f \otimes \chi, \tfrac{1}{2}) \chi(a\ell) = \frac{\lambda_f((\overline{a\ell})_q)}{((\overline{a\ell})_q)^{1/2}} + O(q^{-1/8+\varepsilon})$$

for any $\varepsilon > 0$.

In particular that there are at most $d-1$ possible values of a. Let a be such a root of unity. Write $b = \overline{a}_q \in [1, q-1]$ with the notation as in Theorem 4.1. Since $b^d \equiv 1 \pmod q$, and $b \neq 1$, we have $b \geqslant q^{1/d}$.

We write $m = (\overline{\ell})_q = (1+\alpha q)/\ell$ for some $\alpha \geqslant 0$. We then have

$$0 \leqslant \alpha = \frac{\ell m - 1}{q} < \frac{\ell m}{q} < \ell.$$

Write further $b\alpha = \delta \ell + \varrho$ where $0 \leqslant \varrho < \ell$. Then

$$bm = b\frac{1+\alpha q}{\ell} = \frac{b + \varrho q}{\ell} + \delta q,$$

and since

$$0 < \frac{b+\varrho q}{\ell} < \frac{q}{\ell} + \Big(1 - \frac{1}{\ell}\Big) q = q,$$

we get

$$(\overline{a\ell})_q = (bm)_q = \frac{b+\varrho q}{\ell} \geqslant \frac{b}{\ell} \geqslant q^{1/d-\lambda} \geqslant q^{1/(2d)}.$$

Therefore the contribution of this representation to the first moment is

$$\ll q^{1/(2d)(-1/2+\theta)} \sum_{\ell \leqslant L} \frac{|x_\ell|}{\ell^{1/2}} \ll q^{\lambda/2 + 1/(2d)(-1/2+\theta)+\varepsilon} \to 0$$

as $q \to +\infty$ by (6.28).

If we are not in Case 1, we denote by $\pi(\mathcal{F}_q)$ the Mellin sheaf obtained from Theorem 3.15. Two more cases appear.

Case 2. Assume that there exists a such that $1 \leqslant a \leqslant q-1$ and $\pi(\mathcal{F}_q)$ is geometrically isomorphic to $[x \mapsto a/x]^* \mathcal{K}\ell_2$, so that $\operatorname{tr}(\pi(\theta_\chi))$ is proportional to

$\varepsilon_\chi^{-2}\chi(a)$, with the proportionality constant of modulus 1 (see Lemma 3.11). Then, up to such a constant of modulus 1, the sum (6.32) is equal to

$$\sum_{\ell\leqslant L}\frac{x_\ell}{\ell^{1/2}}\frac{1}{\varphi^*(q)}\sum_{\chi\notin X_q}L(f\otimes\chi,\tfrac{1}{2})\varepsilon_\chi^{-2}\chi(a\ell)=\sum_{\ell\leqslant L}\frac{x_\ell}{\ell^{1/2}}\frac{\lambda_f((a\ell r)_q)}{(a\ell r)_q^{1/2}}+O(L^{1/2}q^{-1/8+\varepsilon})$$

for any $\varepsilon>0$ by Corollary 4.2, where $m=(a\ell r)_q$ is the representative between 1 and q of the residue class of $a\ell r$ modulo q. If there doesn't exist ℓ_0 such that $1\leqslant\ell_0\leqslant L$ and $(a\ell_0 r)_q\leqslant L^2$, then, by (6.33), we get

$$\sum_{\ell\leqslant L}\frac{x_\ell}{\ell^{1/2}}\frac{\lambda_f((a\ell r)_q)}{(a\ell r)_q^{1/2}}\ll\frac{1}{L^{1-\theta}}\sum_{\ell\leqslant L}\frac{|x_\ell|}{\ell^{1/2}}\ll L^{-1/2+\theta}.$$

If there does exist ℓ_0 such that $1\leqslant\ell_0\leqslant L$ and $1\leqslant m_0=(a\ell_0 r)_q\leqslant L^2$, then we get

$$a\equiv m_0\overline{\ell_0 r}.$$

We can write

$$a=\frac{m_0+\alpha_0 q}{\ell_0 r}$$

for some $\alpha_0\geqslant 0$. If $\alpha_0=0$, then we have $\ell_0 r\mid m_0$ and $a\leqslant L^2$. Then $(a\ell r)_q=a\ell r$ for all $\ell\leqslant L$, if q is large enough. Thus

$$\sum_{\ell\leqslant L}\frac{x_\ell}{\ell^{1/2}}\frac{\lambda_f((a\ell r)_q)}{(a\ell r)_q^{1/2}}=\sum_{\ell\leqslant L}\frac{x_\ell}{\ell^{1/2}}\frac{\lambda_f(a\ell r)}{(a\ell r)^{1/2}}\ll\exp(-c\sqrt{\log L})$$

for some $c>0$, by Lemma 6.4.

Now assume that $\alpha_0\geqslant 1$. Let $\ell\leqslant L$. Then

$$a\ell r=\frac{m_0\ell}{\ell_0}+\frac{\alpha_0 q\ell}{\ell_0}.$$

If ℓ_0 divides $\alpha_0\ell$, it follows that $\ell_0\mid m_0\ell$ and

$$(a\ell r)_q=\frac{m_0\ell}{\ell_0}.$$

Otherwise, write $\ell=\beta\ell_0+\delta$ where $1\leqslant\delta<\ell_0$. We get

$$a\ell r=\beta m_0+\frac{m_0\delta}{\ell_0}+\beta\alpha_0 q+\frac{\alpha_0\delta q}{\ell_0}=\beta m_0+\frac{\delta(m_0+\alpha_0 q)}{\ell_0}+\alpha_0\beta q.$$

Now write $\alpha_0\delta=\gamma\ell_0+\varrho$ with $0\leqslant\varrho<\ell_0$. We derive

$$a\ell r=\beta m_0+\frac{\delta m_0+\varrho q}{\ell_0}+\gamma q+\alpha_0\beta q.$$

We have $\varrho\neq 0$, since otherwise $\ell_0\mid\alpha_0\ell$. Since $\varrho<\ell_0$, for q large enough, we have

$$\beta m_0+\frac{\delta m_0+\varrho q}{\ell_0}\leqslant\Big(1-\frac{1}{\ell_0}\Big)q+O(L^3)<q,$$

and hence

$$(a\ell r)_q=\beta m_0+\frac{\delta m_0+\varrho q}{\ell_0}\geqslant\frac{\delta m_0+\varrho q}{\ell_0}\geqslant\frac{q}{\ell_0}\geqslant qL^{-1}$$

We conclude that

$$\sum_{\ell\leqslant L}\frac{x_\ell}{\ell^{1/2}}\frac{\lambda_f((a\ell r)_q)}{(a\ell r)_q^{1/2}}=\sum_{\substack{\ell\leqslant L\\ \ell_0\mid\alpha_0\ell}}\frac{x_\ell}{\ell^{1/2}}\frac{\lambda_f(m_0\ell/\ell_0)}{(m_0\ell/\ell_0)^{1/2}}+O\Big(\frac{L^{1-\theta+\varepsilon}}{q^{1/2-\theta}}\Big).$$

Write $\alpha_0 = \alpha_1\alpha_2$ where $\alpha_1 \mid \ell_0^\infty$. Define $\ell_1 = \ell_0/(\ell_0, \alpha_1)$. Then $\ell_0 \mid \alpha_0\ell$ if and only if $\ell_1 \mid \ell$. Moreover, since this condition holds for $\ell = \ell_1$, we have $\ell_0 \mid m_0\ell_1$, which implies that $(\ell_0, \alpha_1) \mid m_0$. Let $m_1 = m_0/(\ell_0, \alpha_1)$. Then, by applying Lemma 6.4, we have

$$\begin{aligned}\sum_{\ell \leqslant L} \frac{x_\ell}{\ell^{1/2}} \frac{\lambda_f((a\ell r)_q)}{(a\ell r)_q^{1/2}} &= \sum_{\ell \leqslant L/\ell_1} \frac{x_{\ell\ell_1}}{(\ell\ell_1)^{1/2}} \frac{\lambda_f(m_1\ell)}{(m_1\ell)^{1/2}} + O\Big(\frac{L^{1-\theta+\varepsilon}}{q^{1/2-\theta}}\Big) \\ &\ll \ell_1^{-1/2} \exp(-c\sqrt{\log(L/\ell_1)}) \to 0\end{aligned}$$

as $q \to +\infty$.

Case 3. In the final case, let $t_\pi(x)$ be the trace function of Theorem 3.15, so that $\widetilde{t_\pi}(\chi) = \operatorname{tr}(\pi(\theta_\chi))$ for $\chi \notin X_q$. Since we are not in Case 2, the sheaf $\pi(\mathcal{F}_q)$ is not of the type of Case (2) of Lemma 3.11.

For each individual character χ modulo q for every positive ε, we have the bound

$$|\widetilde{t_\pi}(\chi)L(f \otimes \chi, \tfrac{1}{2})\chi(\ell)| \ll_{f,\pi,\varepsilon} q^{3/8+\varepsilon}$$

by the subconvexity estimate of Blomer and Harcos [5, Th. 2] since $|\widetilde{t_\pi}(\chi)| \ll 1$. Hence we can add the characters in X_q to the sum and obtain

$$\begin{aligned}&\frac{1}{\varphi^*(q)} \sum_{\chi \notin X_q} \operatorname{tr}(\pi(\theta_\chi))L(f \otimes \chi, \tfrac{1}{2})\chi(\ell) \\ &\qquad\qquad = \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \widetilde{t_\pi}(\chi)L(f \otimes \chi, \tfrac{1}{2})\chi(\ell) + O(q^{-5/8+\varepsilon}).\end{aligned}$$

Since we are in Case (1) in Lemma 3.11, we deduce by Theorem 4.4,

$$\frac{1}{\varphi^*(q)} \sum_{\chi \notin X_q} \operatorname{tr}(\pi(\theta_\chi))L(f \otimes \chi, \tfrac{1}{2})\chi(\ell) \ll q^{-1/8+\varepsilon}$$

for any $\varepsilon > 0$, where the implied constant depends on f, C, π and ε. Using these bounds in (6.32) we deduce that

$$\frac{1}{\varphi^*(q)} \sum_{\chi \notin X_q} \operatorname{tr}(\pi(\theta_\chi))L(f \otimes \chi, \tfrac{1}{2})M(f \otimes \chi, \boldsymbol{x}_L) \ll_{f,\pi,\varepsilon} L^{1/2+\varepsilon}q^{-1/8+\varepsilon} \to 0$$

as $q \to +\infty$.

Collecting the bounds from all the three cases above, we conclude (see (6.31)) that under assumption (6.28)

$$\mathcal{L}''_{\mathrm{ET}} \to 0$$

as $q \to +\infty$. If we choose $\delta > 0$ small enough, depending only on ϕ_0, it follows from (6.30) that

$$\liminf_{q \to +\infty} |\mathcal{L}''| > 0,$$

hence the result. $\square$

CHAPTER 7

Extreme values of twisted L-functions

7.1. Introduction

In this chapter we prove Theorems 1.11 and 1.12, which establish the existence of very large values of twisted L-functions. We fix f as in Section 1.2.

More precisely, we will prove the following refined statements:

THEOREM 7.1. *Let $I \subseteq \mathbf{R}/\pi\mathbf{Z}$ be an interval of positive measure. Then, for every sufficiently large prime modulus q, there exist primitive characters χ of conductor q such that*

$$\left|L\big(f \otimes \chi, \tfrac{1}{2}\big)\right| \geqslant \exp\left(\Big(\frac{1}{\sqrt{8}} + o(1)\Big)\sqrt{\frac{\log q}{\log\log q}}\right) \quad \textit{and} \quad \theta(f \otimes \chi) \in I.$$

In fact, for every $3 \leqslant V \leqslant \frac{3}{14}\sqrt{\log q/\log\log q}$, we have

$$\begin{aligned} &\left|\left\{\chi \bmod q \;\middle|\; \left|L\big(f\otimes\chi, \tfrac{1}{2}\big)\right| \geqslant e^V \ \textit{and}\ \theta(f\otimes\chi) \in I\right\}\right| \\ &\qquad\qquad \geqslant \frac{\varphi(q)}{\log^2 q} \exp\left(-(12+o(1))\frac{V^2}{\log\big(\log q/(16V^2\log V)\big)}\right). \end{aligned}$$

We can also consider a product of twisted L-functions of two different cusp forms.

THEOREM 7.2. *Let $g \neq f$ be a fixed primitive cusp of conductor r' and trivial central character, holomorphic or not. There exists a constant $C > 0$ such that for every sufficiently large prime modulus q, there exists a primitive character χ of conductor q that satisfies*

$$\left|L\big(f\otimes\chi, \tfrac{1}{2}\big)L\big(g\otimes\chi,\tfrac{1}{2}\big)\right| \geqslant \exp\left(\big(C + o(1)\big)\sqrt{\frac{\log q}{\log\log q}}\right).$$

REMARK 7.3. The constant C depends on f and g and is effective. In particular, "generically", we can take $C = (6\sqrt{10})^{-1}$ (see Remark 7.20, which explains what is meant by generic). Note that we assumed that $g \neq f$, since otherwise the first theorem gives a stronger result.

We prove Theorems 7.1 and 7.2 using Soundararajan's method of resonators. We draw inspiration for Theorem 7.1 from Hough's paper [34]; however, our results are more modest (in that we are unable to detect angles in $\mathbf{R}/2\pi\mathbf{Z}$) due to our inability to evaluate second moments twisted by powers of Gauß sums and more in line with the previously available results on extreme values with angular restrictions.

We develop the method of resonators in a form ready for use in general arithmetic situations in Section 7.2. Section 7.3 combines this input with our evaluations of moments of twisted L-functions to prove asymptotics for moments of

L-functions twisted by resonator and amplifier polynomials and we then use the results of Section 2.4 to evaluate the resulting main terms and prove the existence of large L-values. Theorem 7.1 is proved in Section 7.4, while Theorem 7.2 is proved in Section 7.5.

7.2. Background on the resonator polynomial

The resonator method, originally introduced by Soundararajan [79], is a flexible tool that has been used in many contexts including (for extreme values in the t-aspect) the entire Selberg class (see, for example, [1]), subject as usual to the Ramanujan conjecture. The method itself is by now standard and relies on a specific multiplicative arithmetic function, the "resonator sequence", which can take slightly different forms depending on the range of large values aimed for.

We refer the reader to Section 1.1.5 in the introduction for a general description of the resonator method for a family of forms $f \in \mathcal{F}_N$. This relies on the comparison of the sizes of quantities

$$Q_1 = \mathbf{E}_N(|R(f)|^2) \quad \text{and} \quad Q_2 = \mathbf{E}_N\Big(|R(f)|^2 L(f, \tfrac{1}{2})\Big),$$

with the resonator polynomial $R(f)$ constructed from the "resonator sequence" $\alpha(\ell)$ and the arithmetic factors such as $\lambda_f(\ell)$ or their variations. Following custom, in this chapter we denote such a resonator polynomial by $R(f) = \sum_{n \leqslant N} r(n)\lambda_f(n)$. (To avoid confusion, we remind the reader that our specific Theorems 7.1 and 7.2 concern the family of twisted forms $f \otimes \chi$ which are naturally indexed by primitive characters χ modulo q, and the above averages are over χ, while f, g are fixed forms; in particular a resonator such as $R(\chi) = \sum_{n \leqslant N} r(n)\lambda_f(n)\chi(n)$ is used.)

In each application, to obtain large values in a family of L-functions (or other arithmetic objects), two steps are required.

(1) The first step is analysis of averages in the family that to some degree isolates the terms contributing to the main term (usually the diagonal terms). Opening the sum in $R(f)$, evaluating the averages in Q_1 and Q_2 asymptotically involves first executing averages of (products of) arithmetic factors and twisted moments of L-functions, or variations of these, in the given family.

(2) The second step is application of the resonator method, with the specific resonator constructed so as to reflect the main term contributions (which typically involve arithmetic factors such as, in the context of Theorems 7.1 and 7.2, Hecke eigenvalues of the fixed form(s)). In this step, the sum over the resonator polynomial is executed, leading in the main terms to sums of the form (7.6) and (7.7) below, and then to a lower bound on the quotient $|Q_2|/Q_1$ as in Lemma 7.5 below. Owing to the multiplicative nature of the resonator, optimizing this lower bound is seen to heavily depend on average information on the arithmetic factors over the primes, such as (7.2) and (7.3) below and their variations.

Step (1) is the key arithmetic input and heavily depends on the family of L-functions considered. In this section, we focus on step (2), the application of the resonator method, and make two points: first, one only needs a fairly limited amount of information about the arithmetic factors, and, second, the machinery of the resonator method can be developed in abstract, with no reference to the specific family and relying only on fairly general assumptions about the arithmetic factors. While probably known to the experts, these facts do not seem to be in the literature in a ready-to-use form and we take the opportunity of this memoir to expose them here.

Soundararajan [79] introduced two variants of the resonator sequence, with each being more efficient depending on whether one is aiming for the highest possible values afforded by the resonator method or for many values of slightly smaller size. We develop both variants abstractly in the two sections below.

7.2.1. Extreme values range. In the extreme values range, we use a resonator polynomial similar to that used by Soundararajan [79] and Hough [34], which involves the multiplicative function $r(n)$ supported on square-free numbers and defined at primes by

$$r(p) = \begin{cases} \dfrac{L}{\sqrt{p}\log p}, & L^2 \leqslant p \leqslant \exp(\log^2 L), \\ 0, & \text{otherwise}, \end{cases} \tag{7.1}$$

where L is a large parameter.

In this section, we prove the two key claims for the application of the resonator method in the extreme values range, Lemmas 7.4 and 7.5.

We consider two non-negative multiplicative arithmetic functions ω, ω' satisfying the following conditions.

There exists $a_\omega, a'_{\omega'} > 0$, $0 < \delta, \delta' \leqslant 1$, such that for all $Y \geqslant 2X \geqslant 4$, we have

$$\sum_{X \leqslant p \leqslant Y} \frac{\omega(p)}{p\log p} \leqslant a_\omega\Big(\frac{1}{\log X} - \frac{1}{\log Y}\Big) + O_\omega\Big(\frac{1}{\log^2 X}\Big), \tag{7.2}$$

$$\sum_{X \leqslant p \leqslant Y} \frac{\omega'(p)}{p\log p} \geqslant a'_{\omega'}\Big(\frac{1}{\log X} - \frac{1}{\log Y}\Big) + O_{\omega'}\Big(\frac{1}{\log^2 X}\Big), \tag{7.3}$$

$$\sum_{p \leqslant X} \omega(p)^\delta \omega'(p) \ll_{\omega,\omega',\delta} X(\log X)^\delta, \quad \sum_{p\leqslant X} \omega'(p)^{1+\delta'} \ll_{\omega',\delta'} X^{1+\delta'/2}. \tag{7.4}$$

For the first lemma, we actually require only the following very generous upper bound:

$$\sum_{p\leqslant X} \frac{\omega'(p)}{p} \ll_{\omega'} \frac{e^{\sqrt{\log X}}}{\log X}, \tag{7.5}$$

which a consequence of (7.4).

In the sequel the implied constant may depend on ω, ω' although we will not always mention explicitly such dependency.

Lemma 7.4. *Let the arithmetic function $r(n)$ be as in (7.1) and let $\omega(n) \geqslant 0$ be a multiplicative arithmetic function satisfying (7.2). Then, for every N such that $L \leqslant \sqrt{a_\omega^{-1}\log N \log\log N}$, we have*

$$\sum_{n\leqslant N} r(n)^2\omega(n) = \big(1 + o^\star(1)\big)\prod_p \big(1 + r(p)^2\omega(p)\big), \tag{7.6}$$

and if, additionally, $\omega'(n) \geqslant 0$ is a multiplicative arithmetic function satisfying (7.5), then

(7.7)
$$\sum_{\substack{nm\leqslant N\\(n,m)=1}} \frac{r(n)^2 r(m)\omega(n)\omega'(m)}{\sqrt{m}} = \big(1+o^\star(1)\big)\prod_p \left(1+r(p)^2\omega(p)+\frac{r(p)}{\sqrt{p}}\omega'(p)\right).$$

Here the notation $o^\star(1)$ is a shortcut to

$$O_C\big(N^{-C/(\log\log N)^3}\big)$$

for any $C>0$.

LEMMA 7.5. *Let the arithmetic function $r(n)$ be as in (7.1) and let $\omega(n),\omega'(n)\geqslant 0$ be multiplicative arithmetic functions satisfying (7.3) and (7.4) (for some $0<\delta,\delta'\leqslant 1$.) Then*

$$\prod_p \left(1+\frac{r(p)\omega'(p)}{\sqrt{p}\big(1+r(p)^2\omega(p)\big)}\right) \gg \exp\left(\big(a'_{\omega'}+o^\star(1)\big)\frac{L}{2\log L}\right),$$

Here the notation $o^\star(1)$ is a shortcut to

$$O\big(1/(\log L)^{\min(\delta,\delta')}\big).$$

The relevance of Lemma 7.5 is clear in the light of Lemma 7.4: it gives a lower bound for the quotient of the right-hand sides of (7.7) and (7.6).

REMARKS. (1) The resonator method as originally formulated is a first moment method, but it can be adapted for applications to products of L-functions such as our Theorem 7.2. For clarity, we prove the corresponding variation of (7.7) separately in Lemma 7.6 below, while (7.6) and Lemma 7.5 are ready to use in their current form.

(2) In the original setup of the resonator method to obtain large values of $\zeta(\frac{1}{2}+it)$ [79], one takes $\omega=\omega'=1$, in which case $a_\omega=a'_{\omega'}=1$. Constant sequences ω and ω' are similarly appropriate for some other families (such as the family of quadratic characters or the family of holomorphic cusp forms of large weight in [79]). As a point of reference, in a situation like Theorems 7.1 and 7.2 where the family consists of twists of a fixed cusp form f, choices that could be of interest include $\omega(n)=1$, $\omega'(n)=|\lambda_f(n)|$ and $\omega(n)=\omega'(n)=|\lambda_f(n)|^2$; we discuss the specific choices for that application in Section 7.4.1. For now we stress that all of our conditions involve only averages of $\omega(p)$, $\omega'(p)$ over at least dyadic intervals (and in fact we only apply them in intervals much longer than dyadic). The conditions (7.2) and (7.3) in particular are only non-empty for $Y\gg X$ with a sufficiently large implied constant.

(3) The error terms in (7.2)– (7.4) are one choice that works, and other choices are possible; for example, any $o_{\omega'}(1/\log X)$ in (7.3) would suffice with an adjustment in the explicit $o^\star$-terms in Lemma 7.5, and (7.2) can similarly be relaxed with a possibly adjusted size of L (compare the critical computation (7.8) below). Often, it is possible to obtain (7.2) and (7.3) with no error term whatsoever by just changing the corresponding constant to $a_\omega+\varepsilon$ and $a'_{\omega'}-\varepsilon$; this need not harm the final extreme value result since one can always take $\varepsilon\to 0$ at the very end.

(4) Finally, the conditions (7.2) and (7.3) can be written simply as

$$\sum_{p\sim X}\omega(p)\ll\frac{X}{\log X}\ll\sum_{p\sim X}\omega'(p)$$

if one is not concerned about the precise values of the constants a_ω and $a'_{\omega'}$; however, these constants have a direct impact on the exponent in the final result (such as our Theorem 7.1), so they can be of significance.

Proof of Lemma 7.4. First, we prove (7.6) by following [34, 79]. Using Rankin's trick with a suitable (soon to be chosen) $\alpha>0$, we have that

$$\sum_{n>N}r(n)^2\omega(n)\leqslant N^{-\alpha}\sum_{n=1}^{\infty}n^\alpha r(n)^2\omega(n)\leqslant N^{-\alpha}\prod_p\left(1+p^\alpha r(p)^2\omega(p)\right).$$

Moreover, for $0\leqslant\alpha\ll 1/\log^2 L$,

$$\begin{aligned}\log\prod_p&\left(1+p^\alpha r(p)^2\omega(p)\right)-\log\prod_p\left(1+r(p)^2\omega(p)\right)\\&=\sum_p\log\left(1+\frac{(p^\alpha-1)r(p)^2\omega(p)}{1+r(p)^2\omega(p)}\right)\\&\leqslant\alpha\sum_p\log p\cdot r(p)^2\omega(p)+O\Bigg(\alpha^2\sum_p\log^2 p\cdot r(p)^2\omega(p)\Bigg)\end{aligned}$$

Using the definition of the resonator sequence $r(p)$, (7.2), and summation by parts, this quantity is seen to be

$$\begin{aligned}&=\alpha L^2\sum_{L^2\leqslant p\leqslant\exp(\log^2 L)}\frac{\omega(p)}{p\log p}+O\Bigg(\alpha^2L^2\sum_{L^2\leqslant p\leqslant\exp(\log^2 L)}\frac{\omega(p)}{p}\Bigg)\\&\leqslant\alpha a_\omega\cdot\frac{L^2}{2\log L}+O_\omega\left(\alpha\frac{L^2}{\log^2 L}+\alpha^2L^2\log\log L\right),\end{aligned}$$

For $0\leqslant\alpha\ll_\omega 1/(\log^2 L\log\log L)$, the second error term may be absorbed in $O_\omega(\alpha L^2/\log^2 L)$. Given that $L\leqslant\sqrt{a_\omega^{-1}\log N\log\log N}$, this estimate is further

$$\begin{aligned}&=\alpha a_\omega\cdot\frac{a_\omega^{-1}\log N\log\log N}{\log\log N+\log\log\log N+O_\omega(1)}+O_\omega\left(\alpha\frac{\log N}{\log\log N}\right)\\&=\alpha\left(\log N-\frac{\log N\log\log\log N}{\log\log N}+O_\omega\left(\frac{\log N}{\log\log N}\right)\right).\end{aligned}\tag{7.8}$$

Combining everything, we have that

$$\begin{aligned}&\sum_{n>N}r(n)^2\omega(n)\\&\quad\leqslant\prod_p\left(1+r(p)^2\omega(p)\right)\exp\left(-\alpha\frac{\log N\log\log\log N}{\log\log N}+O_\omega\left(\alpha\frac{\log N}{\log\log N}\right)\right).\end{aligned}$$

Picking, say, $\alpha=c/(\log^2 L\log\log L)=(4c+o(1))/((\log\log N)^2\log\log\log N)$, we thus have

$$\sum_{n>N}r(n)^2\omega(n)\leqslant\prod_p\left(1+r(p)^2\omega(p)\right)\exp\left(-\frac{C\log N}{(\log\log N)^3}\right),$$

for an arbitrary $C > 0$ (simply by choosing an appropriate $c > 0$). In particular,

$$\begin{aligned}\sum_{n \leqslant N} r(n)^2 \omega(n) &= \prod_p \big(1 + r(p)^2 \omega(p)\big) - \sum_{n > N} r(n)^2 \omega(n) \\ &= \big(1 + O\big(N^{-C/(\log\log N)^3}\big)\big) \prod_p \big(1 + r(p)^2 \omega(p)\big),\end{aligned}$$

completing the proof of (7.6).

The proof of (7.7) is analogous. First of all,

$$\sum_{\substack{n,m \geqslant 1 \\ (n,m)=1}} r(n)^2 \omega(n) \cdot \frac{r(m)\omega'(m)}{\sqrt{m}} = \prod_p \left(1 + r(p)^2 \omega(p) + \frac{r(p)}{\sqrt{p}} \omega'(p)\right).$$

Further, for every $\alpha > 0$,

$$\begin{aligned}&\sum_{\substack{nm > N \\ (n,m)=1}} \frac{r(n)^2 r(m) \omega(n) \omega'(m)}{\sqrt{m}} \\ &\qquad \leqslant N^{-\alpha} \sum_{\substack{n,m \geqslant 1 \\ (n,m)=1}} \big(r(n)^2 \omega(n) n^{\alpha}\big)\big(r(m)\omega'(m) m^{\alpha - 1/2}\big) \\ &\qquad = N^{-\alpha} \prod_p \big(1 + r(p)^2 \omega(p) p^{\alpha} + r(p) \omega'(p) p^{\alpha - 1/2}\big).\end{aligned}$$

Further, for every $0 \leqslant \alpha \ll 1/\log^2 L$,

$$\begin{aligned}&\log \prod_p \big(1 + r(p)^2 \omega(p) p^{\alpha} + r(p)\omega'(p) p^{\alpha-1/2}\big) - \log \prod_p \left(1 + r(p)^2 \omega(p) + \frac{r(p)\omega'(p)}{\sqrt{p}}\right) \\ &\quad = \sum_p \log \left(1 + \frac{(p^{\alpha} - 1)\big(r(p)^2 \omega(p) + r(p)\omega'(p)/\sqrt{p}\big)}{1 + \big(r(p)^2 \omega(p) + r(p)\omega'(p)/\sqrt{p}\big)}\right) \\ &\quad \leqslant \alpha \sum_p \log p \left(r(p)^2 \omega(p) + \frac{r(p)}{\sqrt{p}} \omega'(p)\right) \\ &\qquad + O\left(\alpha^2 \sum_p \log^2 p \left(r(p)^2 \omega(p) + \frac{r(p)}{\sqrt{p}} \omega'(p)\right)\right).\end{aligned}$$

Using the definition of the resonator sequence $r(p)$, (7.2), (7.5), and summation by parts, this quantity is seen to be

$$\begin{aligned}&= \alpha L^2 \sum_{L^2 \leqslant p \leqslant \exp(\log^2 L)} \frac{\omega(p)}{p \log p} + \alpha L \sum_{L^2 \leqslant p \leqslant \exp(\log^2 L)} \frac{\omega'(p)}{p} \\ &\qquad + O\left(\alpha^2 L^2 \sum_{L^2 \leqslant p \leqslant \exp(\log^2 L)} \frac{\omega(p)}{p} + \alpha^2 L \sum_{L \leqslant p \leqslant \exp(\log^2 L)} \frac{\omega'(p)}{p} \log p\right) \\ &\leqslant \alpha a_{\omega} \cdot \frac{L^2}{2 \log L} + O_{\omega,\omega'}\left(\alpha \frac{L^2}{\log^2 L} + \alpha^2 L^2 \log\log L\right),\end{aligned}$$

As before, for $0 \leqslant \alpha \ll_{\omega,\omega'} 1/(\log^2 L \log\log L)$, the second term is absorbed in the first one, and with $L \leqslant \sqrt{a_\omega^{-1} \log N \log\log N}$, the above is

$$= \alpha \left(\log N - \frac{\log N \log\log\log N}{\log\log N} + O_{\omega,\omega'}\left(\frac{\log N}{\log\log N} \right) \right).$$

As above, with $\alpha = c/(\log^2 L \log\log L)$ for a suitable $c > 0$, this leads to the combined estimate

$$\sum_{\substack{nm>N \\ (n,m)=1}} \frac{r(n)^2 r(m)\omega(n)\omega'(m)}{\sqrt{m}} \leqslant \prod_p \left(1 + r(p)^2\omega(p) + \frac{r(p)}{\sqrt{p}}\omega'(p) \right) \exp\left(-\frac{C \log N}{(\log\log N)^3} \right)$$

for an arbitrary $C > 0$. As a consequence,

$$\sum_{\substack{nm\leqslant N \\ (n,m)=1}} \frac{r(n)^2 r(m)\omega(n)\omega'(m)}{\sqrt{m}} = \big(1 + O\big(N^{-1/(\log\log N)^3}\big)\big) \prod_p \left(1 + r(p)^2\omega(p) + \frac{r(p)}{\sqrt{p}}\omega'(p) \right),$$

proving (7.7). □

Proof of Lemma 7.5. Let

$$\mathcal{L} := \prod_p \left(1 + \frac{r(p)\omega'(p)}{\sqrt{p}\big(1 + r(p)^2\omega(p)\big)} \right).$$

Using $1/(1+x) = 1 + O_\delta(x^\delta)$ and $\log(1+x) = x + O_{\delta'}(x^{1+\delta'})$, which hold uniformly for all $x > 0$ (including for trivial reasons possibly large values of x),

$$\log \mathcal{L} = \sum_p \left[\frac{r(p)\omega'(p)}{\sqrt{p}} + O_\delta\left(\frac{r(p)^{1+2\delta}\omega(p)^\delta\omega'(p)}{\sqrt{p}} \right) + O_{\delta'}\left(\frac{r(p)^{1+\delta'}\omega'(p)^{1+\delta'}}{p^{(1+\delta')/2}} \right) \right].$$

Using (7.3), (7.4), and summation by parts, we find that

$$\sum_p \frac{r(p)\omega'(p)}{\sqrt{p}} = L \sum_{L^2 \leqslant p \leqslant \exp(\log^2 L)} \frac{\omega'(p)}{p \log p} \geqslant a'_{\omega'} \frac{L}{2\log L} + O_{\omega'}\left(\frac{L}{\log^2 L} \right),$$

as well as

$$\sum_p \frac{r(p)^{1+2\delta}\omega(p)^\delta\omega'(p)}{\sqrt{p}} = L^{1+2\delta} \sum_{L^2 \leqslant p \leqslant \exp(\log^2 L)} \frac{\omega(p)^\delta\omega'(p)}{p^{1+\delta}\log^{1+2\delta} p} \ll_{\omega,\omega',\delta} \frac{L}{(\log L)^{1+\delta}},$$

$$\sum_p \frac{r(p)^{1+\delta'}\omega'(p)^{1+\delta'}}{p^{(1+\delta')/2}} = L^{1+\delta'} \sum_{L^2 \leqslant p \leqslant \exp(\log^2 L)} \frac{\omega'(p)^{1+\delta'}}{(p\log p)^{1+\delta'}} \ll_{\omega',\delta'} \frac{L}{(\log L)^{1+\delta'}}.$$

Combining everything, we obtain the statement of Lemma 7.5. □

Finally we prove a variation of (7.7) that is useful in applying the method of resonators to products of several L-functions.

LEMMA 7.6. *Let the arithmetic function $r(n)$ be as in (7.1), let $\omega(n) \geqslant 0$ be a multiplicative arithmetic function satisfying (7.2), and let $\omega_1'(n), \ldots, \omega_s'(n) \geqslant 0$ be multiplicative arithmetic functions each satisfying (7.5). Then, for every $N \geqslant 20$ such that $L \leqslant \sqrt{a_\omega^{-1} \log N \log\log N}$,*

$$\sum_{n \leqslant N} r(n)^2 \omega(n) \sum_{\substack{m_1, \ldots, m_s \leqslant N/n \\ (n,m_i)=1,\ (m_i,m_j)=1}} \cdots \sum \prod_{i=1}^{s} \frac{r(m_i)\omega_i'(m_i)}{\sqrt{m_i}} = \big(1 + o^\star(1)\big) \prod_p \left(1 + r(p)^2\omega(p) + \frac{r(p)}{\sqrt{p}} \sum_{i=1}^{s} \omega_i'(p)\right), \tag{7.9}$$

with $o^\star(1) = O_{C,\omega}\big(N^{-C/(\log\log N)^3}\big)$ for an arbitrary $C > 0$.

PROOF. The proof is a straightforward adaptation of the proof of (7.7). Using Rankin's trick, we have that

$$\begin{aligned}
&\sum_{n \leqslant N} r(n)^2 \omega(n) \sum_{\substack{m_1, \ldots, m_s \leqslant N/n \\ (n,m_i)=1,\ (m_i,m_j)=1}} \cdots \sum \prod_{i=1}^{s} \frac{r(m_i)\omega_i'(m_i)}{\sqrt{m_i}} \\
&\quad = \sum_{\substack{n, m_1, \ldots, m_s \geqslant 1 \\ (n,m_i)=1,\ (m_i,m_j)=1}} \sum \cdots \sum r(n)^2 \omega(n) \prod_{i=1}^{s} \frac{r(m_i)\omega_i'(m_i)}{\sqrt{m_i}} \\
&\qquad + O\left(N^{-\alpha} \sum_{\substack{n, m_1, \ldots, n_s \geqslant 1 \\ (n,m_i)=1,\ (m_i,m_j)=1}} \sum \cdots \sum r(n)^2 \omega(n) n^\alpha \prod_{i=1}^{s} r(m_i)\omega_i'(m_i) m_i^{\alpha - 1/2}\right).
\end{aligned}$$

Using multiplicativity, the above expression equals

$$\prod_p \left(1 + r(p)^2\omega(p) + \frac{r(p)\omega'(p)}{\sqrt{p}}\right) + O\left(N^{-\alpha} \prod_p \big(1 + r(p)^2\omega(p)p^\alpha + r(p)\omega'(p)p^{\alpha - 1/2}\big)\right),$$

with $\omega'(p) = \sum_{i=1}^{s} \omega_i'(p)$. From this point on, we proceed as in the proof of (7.7) in Lemma 7.4 and conclude that, with the choice $\alpha = c/(\log^2 L \log\log L)$ for a suitable $c > 0$, the ratio of the error term to the main term is $O\big(N^{-C/(\log\log N)^3}\big)$; this in turn proves the lemma. ☐

7.2.2. Many high values range. Sections 7.2.1 and 7.2.2 prepare ground in general for two different applications of Soundararajan's resonator method (which are demonstrated in the two claims of Theorem 7.1). The first of these, subject of section 7.2.1, is to show the existence of some extremely high values of L-functions in a family. The second is to prove that many L-functions in the family attain high values well beyond the generic size (conjecturally in the sense of any power average) and only slightly below the extreme values range. Such results require a bit different resonator sequence, whose application we develop in abstract here.

Let $X_0 > 0$ be a large parameter (namely sufficiently large so that (7.11)–(7.14) and (7.15) below hold). Let $A > 0$ be arbitrary, and let

$$A_0 = \max(A, X_0).$$

Similarly as in [79], let $r(n)$ be a multiplicative arithmetic function supported on square-free numbers and defined at primes by

$$r(p) = \begin{cases} \dfrac{A}{\sqrt{p}}, & A_0^2 \leqslant p \leqslant N^{c/A_0^2}, \\ 0, & \text{otherwise}, \end{cases} \tag{7.10}$$

where $N > 0$ is a large parameter, and $c > 0$ is a suitable constant (its value will be controlled by (1) in Lemma 7.7). Note that this resonator (which is optimized for the purpose of exhibiting many large values in a family of L-functions) differs somewhat from the one in (7.1) and that it directly depends on N. Also note that the sequence $r(n)$ can only be non-empty for $A_0 \leqslant \sqrt{(c + o^\star(1)) \log N / \log\log N}$, with $o^\star(1) = O(\log\log\log N / \log\log N)$. Although the sequence $r(n)$ is different from the one in (7.1), we keep the same notation since some of the evaluations take literally the same form.

As in Section 7.2.1, the sequence $r(n)$ will be combined with arithmetic factors $\omega(n)$ and $\omega'(n)$ in the particular application of the resonator method. We make the following assumptions on these sequences for all $Y \geqslant 2X$, $X \geqslant X_0$:

$$\sum_{p \leqslant X} \frac{\omega(p) \log p}{p} \leqslant b_\omega \log X + O_\omega(1), \qquad \sum_{p \leqslant X} \frac{\omega'(p) \log p}{p} = O_{\omega'}(\log X), \tag{7.11}$$

$$\sum_{X \leqslant p \leqslant Y} \frac{\omega'(p)}{p} \geqslant b'_{\omega'} \log \frac{\log Y}{\log X} + O_{\omega'}\left(\frac{1}{\log X}\right), \tag{7.12}$$

$$\sum_{p \leqslant X} \omega(p)\omega'(p) = O_{\omega,\omega'}\left(\frac{X}{\log X}\right), \quad \sum_{p \leqslant X} \omega'(p)^2 = O_{\omega'}\left(\frac{X^{3/2}}{\log X}\right), \tag{7.13}$$

$$\sum_{X \leqslant p \leqslant Y} \frac{\omega(p)}{p} \leqslant b_{\omega 2} \log \frac{\log Y}{\log X} + O_\omega\left(\frac{1}{\log X}\right). \tag{7.14}$$

We remark that, of the two upper bounds in (7.11), the first one easily follows from (7.14) but perhaps with a suboptimal value of b_ω, while the second one would follow from a sharpened form of the second condition in (7.13) $O_{\omega'}(X/\log X)$, in which the latter would be typically expected. We keep (7.11) to get the tightest constants and minimal conditions.

Analogously as in Lemmas 7.4 and 7.5, the following statement summarizes the resonator-related inputs into obtaining a large number of high values.

LEMMA 7.7. *Let the arithmetic function $r(n)$ be as in (7.10). Then:*

(1) *If multiplicative arithmetic functions $\omega(n), \omega'(n) \geqslant 0$ satisfy (7.11), then the basic evaluations (7.6) and (7.7) hold for every*

$$c < b_\omega^{-1}, \quad X_0 \gg_{\omega,\omega'} (1 - cb_\omega)^{-1}, \quad 0 < A \ll_{\omega,\omega',c} \sqrt{\log N}, \tag{7.15}$$

with $o^\star(1) = O\big(\exp(-\tilde{\delta} A_0^2)\big)$ for some fixed $\tilde{\delta} > 0$ depending on ω, ω', c only.

(2) *If multiplicative arithmetic functions $\omega(n), \omega'(n) \geqslant 0$ satisfy (7.12) and (7.13), then*

$$\prod_p \left(1 + \frac{r(p)\omega'(p)}{\sqrt{p}\big(1 + r(p)^2 \omega(p)\big)}\right) \gg \exp\left(A b'_{\omega'} \log \frac{c \log N}{2A_0^2 \log A_0} + o^\star(A)\right),$$

with $o^\star(A) = O_{\omega,\omega'}(A/\log A_0)$.

(3) *For every multiplicative function $a(n)$ and $\omega(n) = |a(n)|^2$, we have for every integer $K \geqslant 1$ and $N \leqslant q^{1/K}$*

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \left| \sum_{n \leqslant N} r(n)a(n)\chi(n) \right|^{2K} \leqslant \prod_p \left(1 + r(p)^2\omega(p)\right)^{K^2}.$$

If $\omega(n)$ satisfies (7.14), then

$$\prod_p \left(1 + r(p)^2\omega(p)\right) \ll \exp\left(A^2 b_{\omega 2} \log \frac{c \log N}{2A_0^2 \log A_0} + O_\omega\left(\frac{A^2}{\log A_0}\right)\right).$$

Proof. Claim (1) is proved analogously as Lemma 7.4. We prove the basic evaluation (7.7) by using Rankin's trick as in the proof of Lemma 7.4. Critically, for every $0 \leqslant \alpha \ll A_0^2/\log N$, we estimate using (7.11)

$$\begin{aligned}
&\sum_p \log\left(1 + \frac{(p^\alpha - 1)\left(r(p)^2\omega(p) + r(p)\omega'(p)/\sqrt{p}\right)}{1 + \left(r(p)^2\omega(p) + r(p)\omega'(p)/\sqrt{p}\right)}\right) \\
&\quad \leqslant \alpha \sum_p \log p \left(r(p)^2\omega(p) + \frac{r(p)}{\sqrt{p}}\omega'(p)\right) \\
&\qquad + O\left(\alpha^2 \sum_p \log^2 p \left(r(p)^2\omega(p) + \frac{r(p)}{\sqrt{p}}\omega'(p)\right)\right) \\
&\quad = \alpha A^2 \sum_{A_0^2 \leqslant p \leqslant N^{c/A_0^2}} \frac{\omega(p)\log p}{p} + \alpha A \sum_{A_0^2 \leqslant p \leqslant N^{c/A_0^2}} \frac{\omega'(p)\log p}{p} \\
&\qquad + O\left(\alpha^2 A^2 \sum_{A_0^2 \leqslant p \leqslant N^{c/A_0^2}} \frac{\omega(p)\log^2 p}{p} + \alpha^2 A \sum_{A_0^2 \leqslant p \leqslant N^{c/A_0^2}} \frac{\omega'(p)\log^2 p}{p}\right) \\
&\quad \leqslant c b_\omega \frac{\alpha A^2}{A_0^2} \log N + O_{\omega,\omega'}\left(\alpha A^2 + \frac{\alpha A}{A_0^2}\log N + \frac{\alpha^2 A^2}{A_0^4}\log^2 N\right).
\end{aligned}$$

Recall that $c < b_\omega^{-1}$. Choosing $\alpha = \delta_{\omega,\omega',c} A_0^2/\log N$ for a sufficiently small $\delta_{\omega,\omega',c} > 0$, in light of our conditions (7.15) the above quantity is seen to be $\leqslant (1-\delta')\alpha \log N$ for some (fixed and depending on ω, ω', c only) $\delta' > 0$. Thus, upon application of Rankin's trick,

$$\sum_{\substack{nm > N \\ (n,m)=1}} \frac{r(n)^2 r(m)\omega(n)\omega'(m)}{\sqrt{m}} \leqslant N^{-\delta'\alpha} \prod_p \left(1 + r(p)^2\omega(p) + \frac{r(p)}{\sqrt{p}}\omega'(p)\right),$$

which in turn suffices to prove (7.7) in claim (1). The basic evaluation (7.6) follows analogously simply by omitting the missing terms in the above argument.

In claim (2), we simply compute using (7.12), (7.13), and summation by parts,

$$\sum_p \frac{r(p)\omega'(p)}{\sqrt{p}} = A \sum_{A_0^2 \leqslant p \leqslant N^{c/A_0^2}} \frac{\omega'(p)}{p} \geqslant A b'_{\omega'} \log \frac{c \log N}{2A_0^2 \log A_0} + O_{\omega'}\left(\frac{A}{\log A_0}\right),$$

as well as

$$\sum_p \frac{r(p)^3\omega(p)\omega'(p)}{\sqrt{p}} = A^3 \sum_{A_0^2 \leqslant p \leqslant N^{c/A_0^2}} \frac{\omega(p)\omega'(p)}{p^2} \ll_{\omega,\omega'} \frac{A^3/A_0^2}{\log A_0},$$

$$\sum_p \frac{r(p)^2\omega'(p)^2}{p} = A^2 \sum_{A_0^2 \leqslant p \leqslant N^{c/A_0^2}} \frac{\omega'(p)^2}{p^2} \ll_{\omega'} \frac{A^2/A_0}{\log A_0}.$$

Finally we prove the claim (3). By orthogonality, the condition $N \leqslant q^{1/K}$, multiplicativity, and the fact that $r(n)$ is supported on square-free integers, we have that

$$\begin{aligned}
\frac{1}{\varphi(q)} &\sum_{\chi \bmod q} \Big| \sum_{n \leqslant N} r(n)a(n)\chi(n) \Big|^{2K} \\
&= \sum_{\substack{n_1,\dots,n_{2K} \leqslant N \\ n_1 \cdots n_K = n_{K+1} \cdots n_{2K}}} r(n_1)\cdots r(n_{2K}) a(n_1)\cdots a(n_K)\overline{a(n_{K+1})}\cdots\overline{a(n_{2K})} \\
&\leqslant \prod_p \Big(\sum_{k=0}^{K} \binom{K}{k}^2 r(p)^{2k}\omega(p)^k \Big) \\
&\leqslant \prod_p \Big(\sum_{k=0}^{K^2} \binom{K^2}{k} r(p)^{2k}\omega(p)^k \Big) = \prod_p \big(1 + r(p)^2\omega(p)\big)^{K^2}.
\end{aligned}$$

Using (7.14), we easily find that

$$\begin{aligned}
\log \prod_p \big(1 + r(p)^2\omega(p)\big) \leqslant \sum_p r(p)^2\omega(p) &= A^2 \sum_{A_0^2 \leqslant p \leqslant N^{c/A_0^2}} \frac{\omega(p)}{p} \\
&\leqslant A^2 b_{\omega 2} \log \frac{c \log N}{2A_0^2 \log A_0} + O_\omega\Big(\frac{A^2}{\log A_0}\Big).
\end{aligned}$$

This completes the proof of Lemma 7.7. □

7.3. Evaluation of the moments

In addition to the setup of the resonator method, the crucial input for an application of this method is the evaluation of the first moment twisted by the square of the resonator polynomial. In this section, we complete this and associated steps for the family of twisted L-functions $L(f \otimes \chi, s)$.

7.3.1. Moment evaluations. In this section, we present evaluations of the twisted first and second moments in the form in which they will be used in the application of the resonator method.

LEMMA 7.8. *There is an absolute constant $A \geqslant 0$ such that for any $N < q$, any integers $1 \leqslant n_1, n_2 \leqslant N$ and any $\kappa \in \mathbf{Z}$, the twisted first moment $\mathcal{L}(f; n_1 r^\kappa \bar{n}_2, 2\kappa)$ defined in (4.1) satisfies*

$$\begin{aligned}
\text{(7.16)} \quad \mathcal{L}(f; n_1 r^\kappa \bar{n}_2, 2\kappa) = \delta_{\substack{\kappa=0, \\ n_2 = n_1 m}} \frac{\lambda_f(m)}{\sqrt{m}} + \delta_{\substack{\kappa=-1, \\ n_1 = n_2 m}} \frac{\lambda_f(m)}{\sqrt{m}} \\
+ O_{f,\varepsilon,A}\Big((|\kappa|+1)^A q^\varepsilon \big(q^{-1/8} + (q/N)^{\theta - 1/2}\big)\Big).
\end{aligned}$$

In the first term of the right-hand side the equality $n_2 = n_1 m$ means that the term is zero unless n_1 divides n_2 (and the quotient is defined as m) and similarly for the second term.

PROOF. By Corollary 4.2,

$$\mathcal{L}(f; n_1 r^{\kappa} \bar{n}_2, 2\kappa) = O_{f,\varepsilon,A}\Big((|\kappa| + 1)^A q^{-1/8+\varepsilon}\Big)$$

unless $\kappa = 0$ or -1, in which case the additional terms

$$\delta_{\kappa=0} \frac{\lambda_f(a)}{\sqrt{a}} \quad \text{or} \quad \delta_{\kappa=-1} \varepsilon(f) \frac{\lambda_f(b)}{\sqrt{b}} \tag{7.17}$$

appear, where

$$a = (\overline{n_1 \bar{n}_2})_q$$

is the representative in $[1, q]$ of the congruence class $\overline{n_1 \bar{n}_2}$ modulo q, and

$$b = (n_1 \bar{n}_2)_q$$

is the representative of the congruence class $n_1 \bar{n}_2$ modulo q.

Assume that $\kappa = 0$. Then the congruence $a \equiv \overline{n_1 \bar{n}_2} \pmod q$ implies either that $n_2 = n_1 a$ (so $n_1 \mid n_2$) or that $n_1 a > q$. In the second case, we have $a > q/N$, and the first term of (7.17) is $\ll (q/N)^{-(1/2-\theta+\varepsilon)}$ for any $\varepsilon > 0$.

Assume that $\kappa = -1$. Then the congruence $b \equiv n_1 \bar{n}_2 \pmod q$ implies similarly either that $n_1 = n_2 b$, or that $b > q/N$, in which case the second term of (7.17) is $\ll (q/N)^{-(1/2-\theta+\varepsilon)}$ for any $\varepsilon > 0$. The lemma follows. ☐

Consider now two *distinct* primitive cusp forms f and g of signed level r and r' respectively, with trivial central character. Let q be a prime not dividing rr'. We refer to Section 5.2 for the definition of some of the quantities below. We recall Convention 1.3 concerning the signed level of cusp forms. As in Chapter 5, we write $\delta = (r, r') \geqslant 1$ and $|r| = \varrho\delta$, $|r'| = \varrho'\delta$.

We define arithmetic functions λ_f^* and λ_g^* such that they are supported on squarefree integers and satisfy

$$\lambda_f^*(p) = \lambda_f(p) - \lambda_g(p)/p \text{ and } \lambda_g^*(p) = \lambda_g(p) - \lambda_f(p)/p. \tag{7.18}$$

We note that these functions depend on both f and g, and that λ_f^* and λ_g^* satisfy (2.22), i.e.

$$\lambda_f^*(p) = \lambda_f(p) + O(p^{\theta-1}), \quad \lambda_g^*(p) = \lambda_g(p) + O(p^{\theta-1}).$$

In particular, Corollary 2.17 applies to them.

LEMMA 7.9. *For any integers $1 \leqslant \ell, \ell' \leqslant L \leqslant q^{1/2}$ such that $\ell\ell'$ is squarefree and coprime to rr', the twisted second moment $\mathcal{Q}^{\pm}(f, g; \ell, \ell')$ defined in Sections 5.1 and 5.2 satisfies*

$$\mathcal{Q}^{\pm}(f, g; \ell, \ell') = \mathrm{MT}^{\pm}(f, g; \ell, \ell') + O\big(L^{3/2} q^{-1/144}\big), \tag{7.19}$$

where

$$\mathrm{MT}^{\pm}(f, g; \ell, \ell') = \frac{1}{2} L^*(f \otimes g, 1)\Big(\frac{\lambda_f^*(\ell')\lambda_g^*(\ell)}{(\ell\ell')^{1/2}} + \varepsilon(f)\varepsilon(g) \frac{\lambda_f(\varrho)\lambda_g(\varrho')\lambda_f^*(\ell)\lambda_g^*(\ell')}{(\varrho\varrho'\ell\ell')^{1/2}}\Big) + O\big(q^{-1/2+\varepsilon}\big).$$

PROOF. By the argument in Section 5.2 (see also Proposition 5.2), we obtain the asymptotic formula (7.19) with main term $\mathrm{MT}^{\pm}(f,g;\ell,\ell')$ given by (5.4), (5.6), (5.7), (5.8), namely

$$\mathrm{MT}^{\pm}(f,g;\ell,\ell') = \frac{1}{2}\frac{1}{2i\pi}\int_{(2)} \mathcal{L}^{\pm}_{\infty}(\tfrac{1}{2}+u)\frac{D(1+2u;\ell',\ell)}{(\ell\ell')^{1/2+u}}G(u)(q^2|rr'|)^u\frac{du}{u}$$
$$+\frac{\varepsilon(f)\varepsilon(g)\lambda_f(\varrho)\lambda_g(\varrho')}{2(\varrho\varrho')^{1/2}}\frac{1}{2i\pi}\int_{(2)} \mathcal{L}^{\pm}_{\infty}(\tfrac{1}{2}+u)\frac{D(1+2u;\ell,\ell')}{(\ell\ell')^{1/2+u}}G(u)(q^2|rr'|)^u\frac{du}{u},$$

where

$$\mathcal{L}^{\pm}_{\infty}(s) = \frac{L_\infty(f,\pm,s)}{L_\infty(f,\pm,\frac{1}{2})}\frac{L_\infty(g,\pm,s)}{L_\infty(g,\pm,\frac{1}{2})}$$

and $D(s;\ell,\ell')$ is the Dirichlet series

$$D(s;\ell,\ell') = \sum_{n\geqslant 1}\frac{\lambda_f(\ell n)\lambda_g(\ell' n)}{n^s},$$

which is absolutely convergent for $\mathfrak{Re}(s)>1$.

Since ℓ and ℓ' are squarefree and coprime, we have by multiplicativity the formula

$$D(s;\ell,\ell') = \prod_{p\nmid \ell\ell'} L_p^*(f\otimes g,s)\prod_{p\mid \ell} A_p(f,g;s)\prod_{p\mid\ell'}A_p(g,f;s)$$

where

$$A_p(f,g;s) = \sum_{k\geqslant 0}\frac{\lambda_f(p^{k+1})\lambda_g(p^k)}{p^{ks}}.$$

Using the Hecke relation, we obtain the relation

$$A_p(f,g;s) = \lambda_f(p)L_p^*(f\otimes g,s) - \frac{1}{p^s}A_p(g,f;s).$$

Applying it twice, this leads to the formula

$$A_p(f,g;s) = (1-p^{-2s})^{-1}\Big(\lambda_f(p)-\frac{\lambda_g(p)}{p^s}\Big)L_p^*(f\otimes g,s).$$

It follows that

$$D(s;\ell,\ell') = L^*(f\otimes g,s)\prod_{p\mid\ell\ell'}(1-p^{-2s})^{-1}\prod_{p\mid\ell}\Big(\lambda_f(p)-\frac{\lambda_g(p)}{p^s}\Big)\prod_{p\mid\ell'}\Big(\lambda_g(p)-\frac{\lambda_f(p)}{p^s}\Big).$$

Now, moving the contour of integration to $\mathfrak{Re}\,u = -\frac{1}{4}+\frac{1}{2}\varepsilon$, and recalling that $f\neq g$, so that $L^*(f\otimes g,s)$ is holomorphic inside the contour, we obtain by the residue theorem the formula

$$\mathrm{MT}^{\pm}(f,g;\ell,\ell') = \frac{1}{2}L^*(f\otimes g,1)\Big(\frac{\lambda_f^*(\ell')\lambda_g^*(\ell)}{(\ell\ell')^{1/2}}$$
$$+\varepsilon(f)\varepsilon(g)\frac{\lambda_f(\varrho)\lambda_g(\varrho')\lambda_f^*(\ell)\lambda_g^*(\ell')}{(\varrho\varrho'\ell\ell')^{1/2}}\Big)+O\big(q^{-1/2+\varepsilon}\big)$$

for any $\varepsilon>0$, after picking the simple pole at $u=0$. ∎

7.3.2. Asymptotics involving the resonator polynomial. Let $r(n)$ be one of the following two resonator sequences:

$$\begin{aligned}&\text{Let } L \text{ be a large parameter, and let } r(n) \text{ be as in (7.1), } \textit{or}\\ &\text{Let } N \text{ be a large parameter, let } A, c > 0\text{, and let } r(n) \text{ be as in (7.10).}\end{aligned} \tag{7.20}$$

The values of L, A, c will eventually be restricted by the conditions in Section 7.2, but for now we leave them general. We also set, in each case respectively,

$$\begin{aligned}&o^\star(1) = O_C\big(N^{-C/(\log\log N)^3}\big) \text{ with an arbitrary } C > 0, \textit{ or}\\ &o^\star(1) = O_\delta(e^{-\delta A^2}) \text{ for some suitable fixed } \delta > 0\text{, respectively.}\end{aligned} \tag{7.21}$$

Let $a_f(n)$ be a multiplicative arithmetic function supported on square-free positive integers such that

$$\operatorname{sgn} a_f(n) = \operatorname{sgn} \lambda_f(n) \quad \text{whenever } a_f(n)\lambda_f(n) \neq 0. \tag{7.22}$$

For example, we could pick $a_f(n) = \mu^2(n)\lambda_f(n)$, or $a_f(n) = \mu^2(n)\operatorname{sgn}\lambda_f(n)$. For practical purposes, we only need to be concerned with defining $a_f(n)$ for n such that $r(n) \neq 0$. Define

$$\omega(n) = |a_f(n)|^2, \quad \omega'(n) = \overline{a_f(n)}\lambda_f(n); \tag{7.23}$$

in view of (7.22), ω, ω' are non-negative multiplicative functions.

For every Dirichlet character χ modulo q, we define our resonator polynomial by

$$R(\chi) = \sum_{n \leqslant N} r(n) a_f(n) \chi(n). \tag{7.24}$$

We also recall the definition of the argument (cf. (2.4))

$$e^{i\theta(f\otimes\chi)} := \begin{cases} L\big(f\otimes\chi, \tfrac12\big)/\big|L\big(f\otimes\chi,\tfrac12\big)\big|, & L\big(f\otimes\chi,\tfrac12\big) \neq 0,\\ 1, & \text{else.}\end{cases} \tag{7.25}$$

and the formula

$$e^{2i\theta(f\otimes\chi)} = \varepsilon(f\otimes\chi) = \varepsilon(f)\chi(r)\varepsilon_\chi^2. \tag{7.26}$$

To exhibit the desired large values of $L(f\otimes\chi, \frac12)$ with $\theta(f\otimes\chi)$ in desired angular segments, we will evaluate the following two character averages.

Lemma 7.10. *Let q be a prime modulus, let $N \leqslant q$, let $r(n)$ be as in (7.20), let $a_f(n)$ be an arbitrary multiplicative function supported on square-free integers satisfying (7.22), and, for every primitive character χ of conductor q, let $R(\chi)$ be as in (7.24).*

Assume that $r(n)$ and the multiplicative function $\omega(n) = |a_f(n)|^2$ satisfy the basic evaluation (7.6). Then, with $o^\star(1)$ as in (7.21),

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |R(\chi)|^2 = \big(1 + o^\star(1) + O(N/q)\big) \prod_p \big(1 + r(p)^2\omega(p)\big).$$

PROOF. By orthogonality of characters and the Cauchy–Schwarz inequality,

$$\frac{1}{\varphi^*(q)} \sum_{\chi \bmod q}^{*} |R(\chi)|^2 = \frac{1}{\varphi^*(q)} \sum_{\chi \bmod q} \Big| \sum_{n \leqslant N} r(n) a_f(n) \chi(n) \Big|^2$$
$$- \frac{1}{\varphi^*(q)} \Big| \sum_{n \leqslant N} r(n) a_f(n) \Big|^2 = \big(1 + O(N/q)\big) \sum_{n \leqslant N} r(n)^2 \omega(n),$$

since $N \leqslant q$. Applying the basic evaluation (7.6), we have that

$$\sum_{n \leqslant N} r(n)^2 \omega(n) = \big(1 + o^{\star}(1)\big) \prod_{p} \big(1 + r(p)^2 \omega(p)\big),$$

and this in turn yields Lemma 7.10. $\square$

In the next lemma, we consider functions $\psi \colon \mathbf{R}/2\pi\mathbf{Z} \to \mathbf{C}$ which are π-anti-periodic, namely that satisfy $\psi(\theta + \pi) = -\psi(\theta)$ for all θ. The Fourier expansion of such a function has the form

$$\psi(\theta) = \sum_{\kappa \in \mathbf{Z}} \hat{\psi}(2\kappa + 1) e^{i(2\kappa+1)\theta}.$$

We set

$$I(\psi) := \frac{1}{\pi} \int_{\mathbf{R}/2\pi} \psi(\theta) \cos(\theta) d\theta = \hat{\psi}(1) + \hat{\psi}(-1).$$

If ψ is smooth, then for any integer $B \geqslant 0$, we denote the B-Sobolev norm of ψ by

$$\|\psi\|_B = \sum_{\kappa \in \mathbf{Z}} (|\kappa| + 1)^B \, |\hat{\psi}(1 + 2\kappa)|.$$

LEMMA 7.11. *Let q be a prime modulus, let $N < q$, let $r(n)$ be as in (7.20), and let $a_f(n)$ be an arbitrary multiplicative function supported on square-free integers satisfying (7.22). For every primitive character χ of conductor q, let $R(\chi)$ be as in (7.24).*

Let ψ be a smooth π-anti-periodic function.

Assume that $r(n)$ and the multiplicative functions ω, ω' given in (7.23) satisfy the basic evaluation (7.7). Then, for a sufficiently large absolute $B \geqslant 0$ and with $o^{\star}(1)$ as in (7.21), we have

$$\frac{1}{\varphi^*(q)} \sum_{\chi \bmod q}^{*} |R(\chi)|^2 \big|L\big(f \otimes \chi, \tfrac{1}{2}\big)\big| \psi(\theta(f \otimes \chi))$$
$$= I(\psi)\big(1 + o^{\star}(1)\big) \cdot \prod_{p} \left(1 + r(p)^2 \omega(p) + \frac{r(p)}{\sqrt{p}} \omega'(p)\right)$$
$$+ O_{f,\varepsilon,B}\Big(q^{\varepsilon} N \big(q^{-1/8} + (q/N)^{\theta - 1/2}\big) \|\psi\|_B \cdot \prod_{p} \big(1 + r(p)^2 \omega(p)\big)\Big).$$

Proof. The quantity to evaluate is equal to

$$\begin{aligned}\sum_{n_1,n_2\leqslant N} & r(n_1)r(n_2)a_f(n_1)\overline{a_f(n_2)}\sum_{\kappa\in\mathbf{Z}}\hat{\psi}(2\kappa+1)\\ &\qquad\times\frac{1}{\varphi^*(q)}\sideset{}{^*}\sum_{\chi \bmod q} L\big(f\otimes\chi,\tfrac12\big)e^{i2\kappa\theta(f\otimes\chi)}\chi(n_1\bar{n}_2)\\ &=\sum_{n_1,n_2\leqslant N} r(n_1)r(n_2)a_f(n_1)\overline{a_f(n_2)}\sum_{\kappa\in\mathbf{Z}}\hat{\psi}(2\kappa+1)\varepsilon(f)^\kappa\mathcal{L}(f;n_1r^\kappa\bar{n}_2,2\kappa)\end{aligned}$$

by (7.26).

Using (7.16), (7.23), and keeping in mind that the resonator sequence $r(n)$ is supported on square-free integers, there exists an constant $B\geqslant 0$ such that this is in turn equal to

$$\begin{aligned}(\hat{\psi}(1)+\hat{\psi}(-1))&\sum_{\substack{nm\leqslant N\\(n,m)=1}}\frac{r(n)^2r(m)\omega(n)\omega'(m)}{\sqrt{m}}\\ &+O_{f,\varepsilon,A}\bigg(q^\varepsilon\big(q^{-1/8}+(q/N)^{\theta-1/2}\big)\|\psi\|_B\bigg(\sum_{n\leqslant N}r(n)|a_f(n)|\bigg)^2\bigg).\end{aligned}$$

Applying the basic evaluation (7.7) to the double (n,m)-sum, we have that

$$\sum_{\substack{nm\leqslant N\\(n,m)=1}}\frac{r(n)^2r(m)\omega(n)\omega'(m)}{\sqrt{m}}=\big(1+o^\star(1)\big)\prod_p\bigg(1+r(p)^2\omega(p)+\frac{r(p)}{\sqrt{p}}\omega'(p)\bigg).$$

Finally, by the Cauchy–Schwarz inequality,

$$\bigg(\sum_{n\leqslant N}r(n)|a_f(n)|\bigg)^2\leqslant N\sum_{n\leqslant N}r(n)^2\omega(n)\ll N\prod_p\big(1+r(p)^2\omega(p)\big).\tag{7.27}$$

Lemma 7.11 follows by combining these estimates. ☐

We now turn our attention to large values of the product of twisted L-functions of two *distinct* primitive cusp forms f and g of signed levels r and r'. We use the same notation as before, including ϱ and ϱ'.

We begin with an auxiliary lemma.

Lemma 7.12. *With notation as above, there exists a squarefree integer $u\geqslant 1$ coprime to rr' such that*

$$\lambda_g(u)+\varepsilon(f)\varepsilon(g)\frac{\lambda_f(\varrho)\lambda_g(\varrho')\lambda_f(u)}{(\varrho\varrho')^{1/2}}\neq 0.$$

Proof. If ϱ or ϱ' is not 1, then this holds for $u=1$ (see Proposition 5.2). Otherwise, we need to find $u\geqslant 1$ squarefree and coprime to rr' such that

$$\lambda_g(u)+\varepsilon(f)\varepsilon(g)\lambda_f(u)\neq 0,$$

and the existence of a prime u with this property follows from Rankin-Selberg theory and multiplicity one. ☐

REMARK 7.13. We need to involve u in the resonator method, because otherwise it could be the case that

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |R(\chi)|^2 L\big(f \otimes \chi, \tfrac{1}{2}\big) L\big(g \otimes \bar{\chi}, \tfrac{1}{2}\big)$$

is zero because of the cancellation between a character and its conjugate, although the individual terms have no reason to vanish, or their product to be small (see the last part of Theorem 5.1). In that case, the resonator method would not apply. However, if $u \neq 1$, we consider instead

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |R(\chi)|^2 L\big(f \otimes \chi, \tfrac{1}{2}\big) L\big(g \otimes \bar{\chi}, \tfrac{1}{2}\big) \chi(u)$$

where the symmetry between χ and $\bar{\chi}$ is broken, leading to a non-trivial sum.

We now fix an integer u as given by Lemma 7.12.

We assume given a multiplicative function $\varpi(n)$, supported on squarefree positive integers coprime to urr', such that

$$\operatorname{sgn} \varpi(n) = \operatorname{sgn} \lambda_f(n) = \operatorname{sgn} \lambda_g(n) \quad \text{whenever } \varpi(n)\lambda_f(n)\lambda_g(n) \neq 0.$$

This gives rise to the non-negative multiplicative functions $\omega, \omega_1', \omega_2'$ and the resonator polynomial $R(\chi)$, defined by

$$\omega(n) = |\varpi(n)|^2, \quad \omega_1'(n) = \overline{\varpi(n)} \lambda_f^*(n), \quad \omega_2'(n) = \overline{\varpi(n)} \lambda_g^*(n), \tag{7.28}$$

$$R(\chi) = \sum_{n \leqslant N} r(n) \varpi(n) \chi(n), \tag{7.29}$$

where $\lambda_f^*(n)$ and $\lambda_g^*(n)$ are the multiplicative functions defined before Lemma 7.9.

LEMMA 7.14. *With notation as above, assume that $N \leqslant q^{1/2}$. Assume that $r(n)$ and the multiplicative functions $\omega, \omega_1', \omega_2'$ defined in (7.28) satisfy the basic evaluation (7.9). Then, with $\omega'(p) = \omega_1'(p) + \omega_2'(p)$ and with $o^\star(1)$ as in (7.21), we have*

$$\begin{aligned} \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} & |R(\chi)|^2 L\big(f \otimes \chi, \tfrac{1}{2}\big) L\big(g \otimes \bar{\chi}, \tfrac{1}{2}\big) \chi(u) \\ &= L^*(f \otimes g, 1)\Big(\nu + o^\star(1)\Big) \prod_p \left(1 + r(p)^2 \omega(p) + \frac{r(p)\omega'(p)}{\sqrt{p}}\right) \\ &\qquad + O\bigg(N^{5/2} q^{-1/144} \prod_p \big(1 + r(p)^2 \omega(p)\big)\bigg), \end{aligned}$$

where $\nu \neq 0$.

Proof. Applying the definition or $R(\chi)$ and Lemma 7.9, it follows that

$$\begin{aligned}
&\frac{1}{\varphi^*(q)} \sum_{\chi \bmod q}^{*} |R(\chi)|^2 L\big(f\otimes\chi,\tfrac12\big)L\big(g\otimes\bar\chi,\tfrac12\big)\\
&= \frac{1}{\varphi^*(q)} \sum_{n_1,n_2\leqslant N} r(n_1)\overline{r(n_2)}\varpi(n_1)\overline{\varpi(n_2)} \sum_{\chi \bmod q}^{*} L\big(f\otimes\chi,\tfrac12\big)L\big(g\otimes\bar\chi,\tfrac12\big)\chi(un_1\bar n_2)\\
&= \frac{1}{\sqrt{u}} L^*(f\otimes g,1)\Big(X+\varepsilon(f)\varepsilon(g)Y\Big) + O\Big(N^{3/2}q^{-1/144}\Big(\sum_{n\leqslant N} r(n)|\varpi(n)|\Big)^2\Big)
\end{aligned}$$

where

$$X = \lambda_g(u) \sum_{n_1,n_2\leqslant N} r(n_1)\overline{r(n_2)}\varpi(n_1)\overline{\varpi(n_2)}\frac{\lambda_f^*(n_2)\lambda_g^*(n_1)}{(n_1n_2)^{1/2}}$$

and

$$Y = \lambda_f(u)\frac{\lambda_f(\varrho)\lambda_g(\varrho')}{(\varrho\varrho')^{1/2}} \sum_{n_1,n_2\leqslant N} r(n_1)\overline{r(n_2)}\varpi(n_1)\overline{\varpi(n_2)}\frac{\lambda_f^*(n_1)\lambda_g^*(n_2)}{(n_1n_2)^{1/2}}$$

otherwise. Using the Cauchy–Schwarz inequality and the multiplicativity of $\omega(n)$ to estimate the resulting sum by a product over primes (as in (7.27)), we see that the error term is

$$\ll N^{5/2}q^{-1/144}\prod_p \big(1+r(p)^2\omega(p)\big).$$

We write

$$\begin{aligned}
X = \lambda_g(u) \sum_{n\leqslant N} |r(n)|^2|\varpi(n)|^2 \\
\sum_{\substack{m_1,m_2\leqslant N/n\\ (n,m_1)=(n,m_2)=(m_1,m_2)=1}}\sum \frac{r(m_1)\varpi(m_1)\lambda_g^*(m_1)r(m_2)\overline{\varpi(m_2)}\lambda_f^*(m_2)}{(m_1m_2)^{1/2}}
\end{aligned}$$

and similarly for Y. By the basic evaluation (7.9), we obtain

$$X = \lambda_g(u)\big(1+o^\star(1)\big)\prod_p \Big(1+r(p)^2\omega(p)+\frac{r(p)\omega'(p)}{\sqrt{p}}\Big),$$

and similarly

$$Y = \lambda_f(u)\frac{\lambda_f(\varrho)\lambda_g(\varrho')}{(\varrho\varrho')^{1/2}}\big(1+o^\star(1)\big)\prod_p \Big(1+r(p)^2\omega(p)+\frac{r(p)\omega'(p)}{\sqrt{p}}\Big),$$

(since the function ω' plays the same role in both cases). The result follows, with $\nu\neq 0$ by the defining property of u given by Lemma 7.12. ☐

7.3.3. Asymptotics involving an amplifier. In lower ranges for V in Theorem 7.1, we will be using an amplifier instead of a resonator polynomial. In this section, we prove moment asymptotics that will be useful in this treatment.

We may write

$$\sum_{\ell=1}^{\infty} \mu^2(\ell)\frac{|\lambda_f(\ell)|^2}{\ell^s} = L(f\otimes f,s)G_f(s), \tag{7.30}$$

where $G_f(s)$ is a certain Euler product absolutely convergent for $\mathfrak{Re}(s) > \frac{1}{2}$. The Dirichlet series on the left thus has a simple pole at $s = 1$, and we write

$$c_f = \operatorname*{res}_{s=1} L(f \otimes f, s) \cdot G_f(1) > 0. \tag{7.31}$$

In view of (7.30) and (7.31), we have the asymptotic

$$\sum_{\ell \leqslant L} \mu^2(\ell) \frac{|\lambda_f(\ell)|^2}{\ell} = c_f \log L + O_f(1). \tag{7.32}$$

Let $L \leqslant q$ and

$$A_f(\chi) = \sum_{\ell \leqslant L} \frac{\lambda_f(\ell)}{\sqrt{\ell}} \mu^2(\ell)\chi(\ell). \tag{7.33}$$

We will prove the following two claims.

LEMMA 7.15. *Let q be a prime modulus, let $L \leqslant q$, and, for every primitive character χ of conductor q, let $A_f(\chi)$ be as in (7.33). Then, with $c_f > 0$ as in (7.31),*

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |A_f(\chi)|^2 = c_f \log L + O_f(1), \qquad \text{for } L \leqslant q; \tag{7.34}$$

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |A_f(\chi)|^4 \leqslant c_f^4 \log^4 L + O_f(\log^3 L), \quad \text{for } L \leqslant q^{1/2}. \tag{7.35}$$

PROOF. By orthogonality, asymptotic (7.32), and the Cauchy–Schwarz inequality,

$$\begin{aligned}\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |A_f(\chi)|^2 &= \sum_{\ell \leqslant L} \mu^2(\ell) \frac{|\lambda_f(\ell)|^2}{\ell} - \frac{1}{\varphi^*(q)} \Big| \sum_{\ell \leqslant L} \mu^2(\ell) \frac{\lambda_f(\ell)}{\sqrt{\ell}} \Big|^2 \\ &= c_f \log L + O_f(1).\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |A_f(\chi)|^4 &= M_4(f, L) - \frac{1}{\varphi^*(q)} \Big| \sum_{\ell \leqslant L} \mu^2(\ell) \frac{\lambda_f(\ell)}{\sqrt{\ell}} \Big|^4 \\ &= M_4(f, L) + O_f\big(\log^4 L/q\big),\end{aligned}$$

where

$$\begin{aligned}|M_4(f, L)| &= \Bigg| \sum_{\substack{\ell_1\ell_2 = \ell_3\ell_4 \\ \ell_i \leqslant L}} \mu^2(\ell_1)\mu^2(\ell_2)\mu^2(\ell_3)\mu^2(\ell_4) \frac{\lambda_f(\ell_1)\lambda_f(\ell_2)\overline{\lambda_f(\ell_3)\lambda_f(\ell_4)}}{\sqrt{\ell_1\ell_2\ell_3\ell_4}} \Bigg| \\ &\leqslant \Big(\sum_{\ell \leqslant L} \mu^2(\ell) \frac{|\lambda_f(\ell)|^2}{\ell} \Big)^4 = c_f^4 \log^4 L + O_f(\log^3 L).\end{aligned}$$

□

LEMMA 7.16. *Let $\psi : \mathbf{R}/2\pi\mathbf{Z} \to \mathbf{C}$ be a π-anti-periodic smooth function. Let q be a prime modulus. Let $0 < \theta < \frac{1}{2}$ be an admissible exponent toward the Ramanujan–Petersson conjecture for f, and let $0 < \eta < \frac{1}{4}$ be such that*

$$(1 - \eta)(\theta - \tfrac{1}{2}) + \tfrac{1}{2}\eta < 0. \tag{7.36}$$

Let $L = q^\eta$, and, for every primitive character χ of conductor q, let $A_f(\chi)$ be as in (7.33), and let $\theta(f\otimes\chi)$ be as in (7.25). Then, with $c_f > 0$ as in (7.31),

$$\frac{1}{\varphi^*(q)}\sum_{\chi \bmod q}^{*} \big|L\big(\tfrac12, f\otimes\chi\big)\big|\overline{A_f(\chi)}\psi(\theta(f\otimes\chi)) = c_f\hat\psi(1)\log L + O_{f,\eta}(1).$$

Regarding condition (7.36), we remark that any $\theta < \frac12$ is sufficient to obtain this inequality for some $\eta > 0$, which is all we really need. On the other hand, $\theta < \frac13$ is known, so any $\eta < \frac14$ will be acceptable for this condition.

Proof. Using the evaluation (7.17), we have similarly as in Lemma 7.11

$$\begin{aligned}\frac{1}{\varphi^*(q)}&\sum_{\chi \bmod q}^{*} \big|L\big(\tfrac12, f\otimes\chi\big)\big|\overline{A_f(\chi)}\psi(\theta(f\otimes\chi))\\ &= \sum_{\ell\leqslant L}\frac{\overline{\lambda_f(\ell)}}{\ell^{1/2}}\mu^2(\ell)\sum_{\kappa\in\mathbf{Z}}\hat\psi(1+2\kappa)\varepsilon(f)^\kappa\mathcal{L}(f;\bar\ell r^\kappa, 2\kappa)\\ &= \hat\psi(1)\sum_{\ell\leqslant L}\mu^2(\ell)\frac{|\lambda_f(\ell)|^2}{\ell} + \hat\psi(-1)\varepsilon(f)\\ &\quad + O_{f,\varepsilon,A}\bigg(q^\varepsilon\big(q^{-1/8} + (q/L)^{\theta-1/2}\big)\|\psi\|_A\bigg(\sum_{\ell\leqslant L}\frac{|\lambda_f(\ell)|}{\sqrt{\ell}}\bigg)\bigg).\end{aligned}$$

Using the asymptotic (7.32) and keeping in mind the condition (7.36), this equals

$$\begin{aligned}c_f\hat\psi(1)\log L &+ O_{f,\varepsilon}\Big(1 + q^\varepsilon\big(q^{-1/8} + (q/L)^{\theta-1/2}\big)L^{1/2}\Big)\\ &= c_f\hat\psi(1)\log L + O_{f,\eta}(1). \qquad \square\end{aligned}$$

7.4. Extreme values with angular constraints

By comparing the main terms in Lemmas 7.10 and 7.11, we see that if N is not too large compared to q, we can obtain values of $L(f\otimes\chi,\frac12)$ with $\psi(\theta(f\otimes\chi)) > 0$ as large as the quotient of these main terms. This quotient has a lower bound provided by Lemma 7.5 or Lemma 7.7 (2) (depending on which $r(n)$ is used), which in turn depends on the arithmetic sequence $a_f(n)$ used in the construction of the resonator polynomial $R(\chi)$ in (7.24), subject to the sign condition (7.22).

In this section, we construct an essentially optimal sequence $a_f(n)$ for exhibiting extreme values of $L(f\otimes\chi,\frac12)$, verify that it is allowable for Lemmas 7.4 and 7.5, and then use it to prove the extreme values claim of Theorem 7.1.

7.4.1. Choice of the resonator polynomial. For the purpose of exhibiting extreme values of $L(f\otimes\chi,\frac12)$ in Theorem 7.1, we use the resonator sequence $r(n)$ given by (7.1) that is studied in Section 7.2.1. Construction of the resonator polynomial $R(\chi)$ rests on multiplicative arithmetic factors $a_f(n)$, subject to the sign condition (7.22). From these, we defined (see (7.23))

$$\omega(n) = |a_f(n)|^2, \quad \omega'(n) = \overline{a_f(n)}\lambda_f(n) \tag{7.37}$$

for square-free n.

There are *a priori* many reasonable choices of arithmetic factors $a_f(n)$ satisfying the sign condition (7.22). For a moment, we put aside the issue of actually verifying conditions (7.2)–(7.4), and consider the question of optimizing the choice of $\omega(n)$ and $\omega'(n)$. To get the highest possible lower bound in Lemma 7.5 for a given N (whose allowable size is in turn dictated by computations unrelated to the specific application of the resonator method), one chooses

$$L = \sqrt{a_\omega^{-1} \log N \log\log N}$$

in Lemma 7.4 and thus obtains in Lemma 7.5 a lower bound of the shape

$$\exp\bigg(\Big(\frac{a'_{\omega'}}{\sqrt{a_\omega}} + o(1)\Big)\sqrt{\frac{\log N}{\log\log N}}\bigg).$$

Maximizing the ration $a'_{\omega'}/\sqrt{a_\omega}$ (keeping in mind the conditions (7.2), (7.3), and the definition (7.37)) is tantamount to asymptotically maximizing the ratio

$$\Big|\sum_{X\leqslant p\leqslant Y} \frac{\overline{a_f(p)}\lambda_f(p)}{p\log p}\Big|^2 \Big/ \sum_{X\leqslant p\leqslant Y} \frac{|a_f(p)|^2}{p\log p}. \tag{7.38}$$

By the Cauchy–Schwarz inequality, we see that the choice

$$a_f(n) = \mu^2(n)\lambda_f(n), \quad \omega(n) = \omega'(n) = \mu^2(n)|\lambda_f(n)|^2 \tag{7.39}$$

is actually essentially optimal.

It remains to verify that conditions (7.2)–(7.4) are satisfied for this choice. This is the content of the following lemma, which is a special case of Corollary 2.15.

LEMMA 7.17. *For any primitive cusp form f with trivial central character, we have for $4 \leqslant 2X \leqslant Y$*

$$\sum_{p\leqslant X} \frac{|\lambda_f(p)|^2 \log p}{p} = \log X + O_f(1),$$

$$\sum_{X\leqslant p\leqslant Y} \frac{|\lambda_f(p)|^2}{p\log p} = \left(\frac{1}{\log X} - \frac{1}{\log Y}\right) + O_f\left(\frac{1}{\log^2 X}\right),$$

$$\sum_{X\leqslant p\leqslant Y} \frac{|\lambda_f(p)|^2}{p} = \log\Big(\frac{\log Y}{\log X}\Big) + O_f\left(\frac{1}{\log X}\right),$$

$$\sum_{p\leqslant X} |\lambda_f(p)|^4 \ll_f \frac{X}{\log X}.$$

This lemma shows that the multiplicative arithmetic functions

$$\omega(n) = \omega'(n) = \mu^2(n)|\lambda_f(n)|^2$$

do satisfy the conditions (7.2)–(7.4) and (7.11)–(7.14) with $\delta = \delta' = 1$ and

$$a_\omega = a'_{\omega'} = 1, \quad b_\omega = b_{\omega 2} = b'_\omega = 1.$$

7.4.2. The extreme values claim in Theorem 7.1. In this section, we prove the first part of Theorem 7.1, which is concerned with extreme values of $L(f\otimes\chi,\frac12)$.

We use a resonator polynomial (7.24), with the resonator sequence as in (7.1), and arithmetic factors $a_f(n)$ as in (7.39). According to the previous section, the resulting multiplicative arithmetic functions

$$\omega(n) = \omega'(n) = \mu^2(n)|\lambda_f(n)|^2$$

satisfy the conditions (7.2)–(7.4) with $a_\omega = a'_{\omega'} = 1$. According to Lemma 7.4, ω and ω' satisfy the basic evaluations (7.6) and (7.7).

As in Lemma 7.11, choose an arbitrary smooth π-anti-periodic function $\psi : \mathbf{R}/2\pi \to \mathbf{C}$ such that

$$\operatorname{supp}\psi \cap \left]-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right[\subseteq I, \quad \psi|_I \geqslant 0, \quad \int_I \psi(\theta)\,\mathrm{d}\theta = 1. \tag{7.40}$$

In particular, we have then $I(\psi) > 0$.

Fix an arbitrary $\delta > 0$, and apply Lemmas 7.10 and 7.11 with

$$N = q^{1/8-\delta}$$

and $L = \sqrt{\log N \log\log N}$. Using the available estimate $\theta < \frac{5}{14}$, we have that $N(q/N)^{\theta-1/2} = q^{(\theta-1/2)(7/8+\delta)+1/8-\delta} < q^{-(8/7)\delta} < q^{-\delta}$. Therefore, Lemmas 7.10 and 7.11 give

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |R(\chi)|^2 \sim \prod_p \bigl(1 + r(p)^2|\lambda_f(p)|^2\bigr),$$

and

$$\begin{aligned}
&\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |R(\chi)|^2 \bigl|L\bigl(f\otimes\chi, \tfrac12\bigr)\bigr| \psi(\theta(f\otimes\chi)) \\
&\qquad = I(\psi)\bigl(1+o(1)\bigr)\cdot \prod_p \left(1 + r(p)^2|\lambda_f(p)|^2 + \frac{r(p)}{\sqrt{p}}|\lambda_f(p)|^2\right) \\
&\qquad\qquad + O\Bigl(q^{-\delta+\varepsilon}\cdot \prod_p \bigl(1 + r(p)^2|\lambda_f(p)|^2\bigr)\Bigr) \\
&\qquad \sim I(\psi)\mathcal{L} \prod_p \bigl(1 + r(p)^2|\lambda_f(p)|^2\bigr),
\end{aligned}$$

with implicit constants depending on f, δ, ε, and ψ, and

$$\mathcal{L} = \prod_p \left(1 + \frac{r(p)|\lambda_f(p)|^2}{\sqrt{p}\bigl(1 + r(p)^2|\lambda_f(p)|^2\bigr)}\right).$$

It follows that, for sufficiently large q, there exists at least one primitive character χ of conductor q such that $\psi(\theta(f\otimes\chi)) > 0$ (and so *a fortiori* $\theta(f\otimes\chi) \in I$) and

$$\bigl|L\bigl(f\otimes\chi, \tfrac12\bigr)\bigr| \gg \mathcal{L}.$$

Finally, applying Lemma 7.5, and keeping in mind the present choices

$$L = \sqrt{\log N \log\log N}$$

and $N = q^{1/8-\delta}$ (made below (7.40)) and $a'_{\omega'} = 1$, we obtain a lower bound

$$\mathcal{L} \gg \exp\left(\bigl(1 + o^\star(1)\bigr)\frac{L}{2\log L}\right) = \left(\bigl(\tfrac{1}{\sqrt{8}} - \delta + o^\star(1)\bigr)\sqrt{\frac{\log q}{\log\log q}}\right),$$

with $o^\star(1) = O(\log\log\log q/\log\log q)$. The omega-statement of Theorem 7.1 follows since we may take $\delta > 0$ as small as we please.

7.4.3. Many large values. In this section, we prove the second part of Theorem 7.1 about the number of primitive characters χ of conductor q such that $|L(f \otimes \chi, \frac{1}{2})| \geqslant e^V$ for a sizable V. The argument is an adaptation of that in [79]; here we focus on the specific requirements on the sequences $\omega(n)$ and $\omega'(n)$ and on the few aspects that require some modification (such as the treatment of moderately large V).

Let $c_f > 0$ be as in (7.31), and let $0 < \eta < \frac{1}{4}$ satisfy the condition (7.36). Choose an arbitrary smooth π-anti-periodic function $\psi : \mathbf{R}/2\pi\mathbf{Z} \to \mathbf{C}$ as in (7.40). We have then, $|\hat{\psi}(1)| > 0$. Let $\tilde{c} = \tilde{c}_{f,\psi,\eta} > 0$ be an arbitrary constant such that

$$\tilde{c} < \sqrt{c_f \eta}\, \frac{|\hat{\psi}(1)|}{2\|\psi\|_\infty}. \tag{7.41}$$

We consider two cases, depending on V.

Case 1. *The range* $V \leqslant \frac{1}{2} \log\log q + \log \tilde{c}$. In this range, the second part of Theorem 7.1 states that $|L(f \otimes \chi, \frac{1}{2})|$ achieves moderately high values for a very large number of χ. This is in a sense a complementary range; instead of the method of resonators, we prove Theorem 7.1 by a comparison of moments, including the amplified first moment as follows.

Let $L = q^\eta$, and, for every primitive character χ of conductor q, let the amplifier $A_f(\chi)$ be as in (7.33). Then, according to Lemma 7.16,

$$\begin{aligned} I_{1,f,\psi,L}(q) &= \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} \big|L(\tfrac{1}{2}, f \otimes \chi)\big| \overline{A_f(\chi)} \psi(\theta(f \otimes \chi)) \\ &= c_f \eta \hat{\psi}(1) \log q + O_{f,\psi,\eta}(1). \end{aligned}$$

Note that, in this range, $e^V \leqslant \tilde{c}\sqrt{\log q}$. We split the sum

$$I_{1,f,g,L}(q) = I^0_{1,f,\psi,L}(q) + I^+_{1,f,\psi,L}(q)$$

where $I^0_{1,f,\psi,L}$ restricts to those χ such that $|L(f \otimes \chi, \frac{1}{2})| \leqslant \tilde{c}\sqrt{\log q}$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |I^0_{1,f,\psi,L}(q)| &\leqslant \tilde{c}\sqrt{\log q}\|\psi\|_\infty \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{|L(f\otimes\chi,1/2)|\leqslant \tilde{c}\sqrt{\log q}} |A_f(\chi)| \\ &\leqslant \tilde{c}\sqrt{\log q}\|\psi\|_\infty \Big(\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |A_f(\chi)|^2 \Big)^{1/2}. \end{aligned}$$

Using (7.34) of Lemma 7.15 and recalling that $\tilde{c}$ satisfies (7.41), we deduce that

$$\begin{aligned} |I^0_{1,f,\psi,L}(q)| &\leqslant \tilde{c}\sqrt{c_f \eta}\|\psi\|_\infty \log q + O_{f,\psi}(1) \\ &\leqslant \tfrac{1}{2} c_f |\hat{\psi}(1)| \eta \log q \leqslant \tfrac{1}{2} |I_{1,f,\psi,L}(q)| \end{aligned}$$

for sufficiently large q. This shows that, for sufficiently large q,

$$|I^+_{1,f,\psi,L}(q)| \geqslant \tfrac{1}{2} |I_{1,f,\psi,L}(q)| = \tfrac{1}{2} c_f \eta |\hat{\psi}(1)| \log q + O_{f,\psi,\eta}(1). \tag{7.42}$$

On the other hand, using Hölder's inequality, we estimate

$$
\begin{aligned}
(7.43)\quad |I^{+}_{1,f,\psi,L}(q)| &\leqslant \|\psi\|_\infty \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\substack{|L(f\otimes\chi,1/2)|>\tilde{c}\sqrt{\log q}\\ g(\theta(f\otimes\chi))\neq 0}} \big|L\big(f\otimes\chi,\tfrac12\big)\big||A_f(\chi)| \\
&\leqslant \|\psi\|_\infty \bigg(\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} \big|L\big(f\otimes\chi,\tfrac12\big)\big|^2\bigg)^{1/2} \bigg(\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |A_f(\chi)|^4\bigg)^{1/4} \\
&\quad\times \bigg(\frac{1}{\varphi^*(q)}\Big|\big\{\chi \,(\mathrm{mod}\, q)\ \mid\ \big|L\big(\tfrac12, f\otimes\chi\big)\big| > \tilde{c}\sqrt{\log q},\ \theta(f\otimes\chi)\in I\big\}\Big|\bigg)^{1/4}.
\end{aligned}
$$

Combining (7.42), (7.43), the second moment evaluation of Theorem 5.1, and (7.35) of Lemma 7.15, we conclude that, for sufficiently large q,

$$
\Big|\big\{\chi \,(\mathrm{mod}\, q)\ \mid\ \big|L\big(f\otimes\chi,\tfrac12\big)\big| > \tilde{c}\sqrt{\log q},\ \theta(f\otimes\chi)\in I\big\}\Big| \gg_{f,\psi,I} \frac{\varphi^*(q)}{\log^2 q},
$$

which more than suffices for the second part of Theorem 7.1 for

$$
3 \leqslant V \leqslant \tfrac12 \log\log q + \log\tilde{c}
$$

and any $\eta < \frac14$.

REMARK 7.18. In place of Hölder's inequality above, one could use the Cauchy–Schwarz inequality and then estimate from above the amplified second moment for sufficiently small $\eta > 0$; this would yield a lower bound of the same form save for the numerical values of various constants. We chose the above treatment which is softer and perhaps more universally applicable.

Case 2. *The range* $\frac12\log\log q + \log\tilde{c} \leqslant V \leqslant \frac{3}{14}\sqrt{\log q/\log\log q}$. In this principal range, we use the resonator method to prove Theorem 7.1, proceeding analogously as in [79]. We will be using a resonator sequence $r(n)$ of type (7.10), as studied in Section 7.2.2. For the arithmetic factors $a_f(n)$ in the resonator polynomial $R(\chi)$, we make the same choice as in (7.39), namely

$$
(7.44)\qquad a_f(n) = \mu^2\lambda_f(n), \quad \omega(n) = \omega'(n) = \mu^2(n)|\lambda_f(n)|^2,
$$

which satisfies the sign condition (7.22). As in Section 7.4.1, this choice is essentially optimal: an inspection of (7.48) shows that in generic ranges it allows a choice $A \approx V/(b'_{\omega'}\log Q)$ and eventually to the lower bound

$$
\gg_{f,\psi,c_1} \varphi^*(q)\exp\big(-12(b_{\omega2}/b'^2_{\omega'})\big(V^2/\log Q\big)\big)
$$

in (7.51). Minimizing the constant in this estimate is tantamount to asymptotically maximizing the same ratio (7.38) as in Section 7.4.1, and by the Cauchy–Schwarz inequality leads to the same asymptotically optimal choice (7.44).

We have verified after Lemma 7.17 that the choice (7.44) satisfies conditions (7.11)–(7.14) with $b_\omega = b_{\omega2} = b'_{\omega'} = 1$.

Fix an arbitrary $\delta > 0$ (which will be chosen suitably small under (7.48)), and as in Section 7.4.2 let

$$
N = q^{1/8-\delta}.
$$

Using the available estimate $\theta < \frac{5}{14}$, we have that $N(q/N)^{\theta-1/2} < q^{-\delta}$ as in Section 7.4.2. Further, set

$$
c < 1 \quad\text{and}\quad X_0 \gg_f (1-c)^{-1},\ A \ll_{f,c} \sqrt{\log N} \quad\text{as in (7.15)},
$$

where A will be suitably chosen later, $c < 1$ will be chosen suitably close to 1 under (7.48), and we additionally take X_0 sufficiently large (depending on f, c only) so that the term $o^\star(1) = O\big(e^{-\tilde{\delta}X_0^2}\big)$ in Lemma 7.7(1) is $\leqslant \frac{1}{10}$. With these choices, let the resonator sequence $r(n)$ be as in (7.10) and the arithmetic factors $a_f(n)$ and the resulting multiplicative functions $\omega(n), \omega'(n)$ be as in (7.44), and define the resonator polynomial $R(\chi)$ as in (7.24). Let

$$M_1(q) = \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |R(\chi)|^2,$$
$$M_{2,f,\psi}(q) = \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |R(\chi)|^2 \big|L\big(f\otimes\chi, \tfrac{1}{2}\big)\big| \psi(\theta(f\otimes\chi)).$$

According to Lemma 7.7(1), the basic evaluations (7.6) and (7.7) hold. Lemmas 7.10 and 7.11 then give

$$\begin{aligned} M_1(q) &= \big(1 + o^\star(1) + O(q^{-7/8})\big)\cdot NW, \\ M_{2,f,\psi}(q) &= \big[I(\psi)(1+o^\star(1))\mathcal{A} + O_{f,\psi,\varepsilon}(q^{-\delta+\varepsilon})\big]\cdot NW, \end{aligned} \tag{7.45}$$

where $o^\star(1) = O\big(e^{\tilde{\delta}A_0^2}\big)$ and, according to Lemma 7.7(2) and (3),

$$\begin{aligned} NW &= \prod_p \big(1 + r(p)^2|\lambda_f(p)|^2\big) \\ &\leqslant \exp\left(A^2 \log\frac{c\log N}{2A_0^2\log A_0} + O_f\left(\frac{A^2}{\log A_0}\right)\right), \\ \mathcal{A} &= \prod_p \left(1 + \frac{r(p)|\lambda_f(p)|^2}{\sqrt{p}(1+r(p)^2|\lambda_f(p)|^2)}\right) \\ &\geqslant \exp\left(A\log\frac{c\log N}{2A_0^2\log A_0} + O_f\left(\frac{A}{\log A_0}\right)\right). \end{aligned} \tag{7.46}$$

In particular, since our choice of X_0 ensures that $o^\star(1) \leqslant \frac{1}{10}$ in (7.45), we have that for sufficiently large q,

$$\tfrac{4}{5} I(\psi)\mathcal{A} M_1(q) \leqslant M_{2,f,\psi}(q). \tag{7.47}$$

Let C_f denote an implicit constant sufficient for both asymptotics in (7.46). In the following claim, we now specify our choice of the parameter A.

Claim. With suitable $\delta > 0$ and $0 < c < 1$, for sufficiently large q we can choose $A \ll_{f,c} \sqrt{\log N}$ satisfying

$$\begin{aligned} &A \leqslant \frac{V}{\log Q}, \quad Q = \frac{c\log N}{2V^2\log V} \qquad \text{such that} \\ &A\log\frac{c\log N}{2A_0^2\log A_0} = V + \log\frac{2\|\psi\|_\infty}{I(\psi)} + C_f\frac{A}{\log A_0}. \end{aligned} \tag{7.48}$$

For easier reading, we postpone the proof of this technical claim to the end of this section. With our choice of A satisfying (7.48), we find that

$$\begin{aligned} I(\psi)\mathcal{A} &\geqslant \exp\left(A\log\frac{c\log N}{2A_0^2\log A_0} + \log I(\psi) - C_f\frac{A}{\log A_0}\right) \\ &= \exp\big(V + \log(2\|\psi\|_\infty)\big) = 2\|\psi\|_\infty e^V. \end{aligned}$$

Then, separating the summands in $M_{2,f,\psi}(q)$ according to whether we have $L(f\otimes\chi,\frac12)\leqslant e^V$ or not, we can write

$$M_{2,f,\psi}(q)=M^0_{2,f,\psi}(q)+M^+_{2,f,\psi}(q), \tag{7.49}$$

where

$$|M^0_{2,f,\psi}(q)|\leqslant e^V\|\psi\|_\infty\frac{1}{\varphi^*(q)}\sideset{}{^*}\sum_{\chi:|L(f\otimes\chi,1/2)|\leqslant e^V}|R(\chi)|^2\leqslant\tfrac12 I(\psi)\mathcal{A}M_1(q). \tag{7.50}$$

Combining (7.47), (7.49), (7.50), and Hölder's inequality, we then deduce that

$$\frac{3}{10}I(\psi)\mathcal{A}M_1(q)\leqslant|M^+_{2,f,\psi}(q)|\leqslant\|\psi\|_\infty\frac{1}{\varphi^*(q)}\sideset{}{^*}\sum_{\substack{|L(f\otimes\chi,1/2)|>e^V,\\ \psi(\theta(f\otimes\chi))\neq0}}|R(\chi)|^2\big|L\big(f\otimes\chi,\tfrac12\big)\big|$$

$$\leqslant\|\psi\|_\infty\Big(\frac{1}{\varphi^*(q)}\sideset{}{^*}\sum_{\chi \bmod q}|R(\chi)|^8\Big)^{1/4}\Big(\frac{1}{\varphi^*(q)}\sideset{}{^*}\sum_{\chi \bmod q}\big|L\big(f\otimes\chi,\tfrac12\big)\big|^2\Big)^{1/2}$$

$$\times\Big(\frac{1}{\varphi^*(q)}\Big|\big\{\chi\,(\mathrm{mod}\,q)\;\mid\;\big|L\big(f\otimes\chi,\tfrac12\big)\big|>e^V,\,\theta(f\otimes\chi)\in I\big\}\Big|\Big)^{1/4}.$$

According to part (3) of Lemma 7.7, and using our evaluation of the second moment of twisted L-functions, we conclude that

$$\Big|\big\{\chi\,(\mathrm{mod}\,q)\;\mid\;\big|L\big(f\otimes\chi,\tfrac12\big)\big|>e^V,\,\theta(f\otimes\chi)\in I\big\}\Big|\gg_{f,\psi}\frac{\varphi^*(q)}{\log^2 q}\big(\mathcal{A}M_1(q)\big)^4\frac{1}{NW^{16}}$$

$$\gg_{f,\psi}\frac{\varphi^*(q)}{\log^2 q}\exp\Big((4A-12A^2)\log\frac{c\log N}{2A_0^2\log A_0}-C_f\frac{4A+12A^2}{\log A_0}\Big).$$

With our choice (7.48), the right-hand side of this estimate is

$$\gg_{f,\psi,c}\frac{\varphi^*(q)}{\log^2 q}\exp\big((4-12A)(V+M_\psi)-12C_fA^2/\log A_0\big)$$

$$\gg_\psi\varphi^*(q)\exp\Big(-12(1+o^\star(1))\frac{V^2}{\log Q}\Big), \tag{7.51}$$

with $o^\star(1)=O_{f,\psi}(1/V+1/(\log Q\log A_0))=O_{f,\psi}(1/\log\log q)$. This concludes the proof of Case 2, hence of the theorem. It only remains to prove the technical claim.

Proof of Claim. Let $A_1=V/\log Q$, $M_\psi=\log(2\|\psi\|_\infty/I(\psi))$, and

$$\phi(A):=A\Big(\log\frac{c\log N}{2A_0^2\log A_0}-\frac{C_f}{\log A_0}\Big).$$

We will verify that, for sufficiently large q, $\phi(A_1)>V+M_\psi$; this is tedious but not difficult. Note that $2.72<\frac{196}{72}c(1-8\delta)<Q\leqslant c\log N$ for a sufficiently small $\delta>0$ and a $c<1$ suitably close to 1, and in particular we have, with an absolute $\delta_{\mathrm{ll}}>0$,

$$1<\log Q,\;A_1<V,\;X_0<V,\;0<\delta_{\mathrm{ll}}\leqslant\log\log Q,$$
$$\log Q\leqslant\log\log q+O(1)<(2+\delta)V$$

for sufficiently large q.

Let $X_1=\max(X_0,M_\psi/\delta_{\mathrm{ll}})$. If $A_1\geqslant X_0$, then

$$A_0=A_1,2A_0^2\log A_0\leqslant 2V^2\log V/\log^2 Q,$$

and thus

$$\phi(A_1) \geqslant \frac{V}{\log Q}\left(\log Q + 2\log\log Q - \frac{C_f}{\log A_0}\right) = V + \frac{V}{\log Q}\left(2\log\log Q - \frac{C_f}{\log A_0}\right).$$

This is in particular the case if $V \geqslant (1+\delta)X_1 \log\log q$, when for sufficiently large q, $A_1 \geqslant X_1$. If $\log\log Q \geqslant C_f/\log X_0$, then the second term exceeds $\delta_{ll} V/\log Q = \delta_{ll} A_1 \geqslant M_\psi$, and we are done. If $\log\log Q < C_f/\log X_0$, we have that $V^2 \log V \gg_{C_f, X_0, c} \log N$ and thus $\log A_1 \geqslant (\frac{1}{2} - o_{C_f, X_0, c}(1)) \log\log q$, and for sufficiently large q the second term exceeds $V(\log\log Q/\log Q) \gg_{C_f, X_0} V > M_\psi$.

For $V \leqslant (1+\delta)X_1 \log\log q$, we have that $\log Q = (1+o(1))\log\log q$. If $A_1 \geqslant X_0$, then the above lower bound holds, with the second term $\geqslant (2+\delta)^{-1}((2+o(1))\log\log\log q - C_f/\log X_0) > M_\psi$ for sufficiently large q. Otherwise $A_0 = X_0$ and

$$\phi(A_1) \geqslant V + \frac{V}{\log Q}\left(2\log(V/X_0) - \frac{C_f}{\log X_0}\right),$$

and the second term is again $\geqslant (2+\delta)^{-1}((2+o(1))\log\log\log q - C_f/\log X_0) > M_\psi$ for sufficiently large q.

On the other hand, for $Q_0 = (c\log N)/(2X_0^2 \log X_0)$ and $A_2 = (V + M_\psi)/\log Q_0$, we clearly have that $\phi(A_2) \leqslant V + M_\psi$. Thus the existence of $A \in [A_2, A_1]$ satisfying (7.48) follows simply by continuity. Note that $A \leqslant V/\log Q < V$ guarantees the required condition $A \ll_{f,c} \sqrt{\log N}$ for sufficiently large q. □

7.5. Large values of products

In this section, we prove Theorem 7.2. With the resonator sequence of the form (7.1), Section 7.5.1 is devoted to the construction of arithmetic factors and verification that they satisfy conditions for the application of the resonator method. The proof of Theorem 7.2 then follows in Section 7.5.2.

7.5.1. Choice of the resonator polynomial. Let λ_f^* and λ_g^* be the multiplicative functions supported on squarefree integers and defined by (7.18) and let

$$\mathcal{G} := \{n \geqslant 1 \mid \lambda_f^*(n)\lambda_g^*(n) \neq 0,\ \mathrm{sgn}(\lambda_f^*(n)) = \mathrm{sgn}(\lambda_g^*(n)\}.$$

We construct a multiplicative arithmetic function $\varpi(n)$ supported on squarefree positive integers, subject to the condition

$$\begin{cases} \varpi(n) = 0, & n \notin \mathcal{G}, \\ \varpi(n) = 0 \text{ or } \mathrm{sgn}\,\varpi(n) = \mathrm{sgn}\,\lambda_f^*(n) = \mathrm{sgn}\,\lambda_g^*(n), & n \in \mathcal{G}. \end{cases} \tag{7.52}$$

This multiplicative function $\varpi(n)$ is entirely determined by the sequence of values $\varpi(p)$. We base our construction of the sequence $\varpi(p)$ on the simple observation that, for any two $x, y \in \mathbf{R}$,

$$xy(x+y)^2 = 0 \quad \text{or} \quad \mathrm{sgn}(xy) = \mathrm{sgn}\left(xy(x+y)^2\right).$$

In particular,

$$\lambda_f^*(p)\lambda_g^*(p)\left(\lambda_f^*(p) + \lambda_g^*(p)\right)^2 > 0 \implies p \in \mathcal{G}.$$

We define

$$\varpi(p) = \begin{cases} \mathrm{sgn}(\lambda_f^*(p))\lambda_f^*(p)\lambda_g^*(p)\left(\lambda_f^*(p) + \lambda_g^*(p)\right), & p \in \mathcal{G}, \\ 0, & p \notin \mathcal{G}. \end{cases} \tag{7.53}$$

Then, the multiplicative arithmetic function $\varpi(n)$ satisfies the sign condition (7.52), and hence

$$\omega(n) = |\varpi(n)|^2, \quad \omega'(n) = \varpi(n) \prod_{p \mid n} \left(\lambda_f^*(p) + \lambda_g^*(p)\right) \geqslant 0 \tag{7.54}$$

are non-negative multiplicative functions supported within $\mathcal{G}$. In the following lemma, we verify that ω, ω' satisfy the conditions for application of the resonator method.

Lemma 7.19. *Let f, g be two primitive cusp forms of signed levels r and r' respectively and trivial central character, and let ϖ be the multiplicative arithmetic function supported on square-free integers and defined on primes by (7.53); in particular, ϖ satisfies (7.52).*

Then, for every $\tilde{\delta} > 0$ there exists an $X_0 = X_0(\tilde{\delta}, f, g)$ such that, for every $X \geqslant X_0$ and every $Y \geqslant 2X$, the non-negative multiplicative functions $\omega, \omega' \geqslant 0$ supported on square-free integers and defined by (7.54) satisfy conditions (7.2)–(7.4) with $\delta = \frac{1}{4}$, $\delta' = \frac{1}{3}$, and

$$\begin{aligned} a_\omega &= n_{4,2} + 2n_{3,3} + n_{2,4} + 4\tilde{\delta}, \\ a'_{\omega'} &= n_{3,1} + 2n_{2,2} + n_{1,3} - 4\tilde{\delta}, \end{aligned} \tag{7.55}$$

where $n_{k,k'}$ are the non-negative integers defined in Corollary 2.15. We have

$$n_{3,1} + 2n_{2,2} + n_{1,3} \geqslant 2n_{2,2} \geqslant 2.$$

Proof. We use Corollary 2.17 with various values of the parameters (k, k') and the given $\tilde{\delta} > 0$ to verify that (7.2)–(7.4) are satisfied with the stated values of the parameters a_ω and $a'_{\omega'}$. Choose $X_0 = X_0(\tilde{\delta}, f, g) \geqslant 4$ so that

$$\sum_{x \leqslant p \leqslant y} \frac{\lambda_f(p)^k \lambda_g(p)^{k'}}{p \log p} \geqslant \left(n_{k,k'} - \tilde{\delta}\right) \left(\frac{1}{\log x} - \frac{1}{\log y}\right)$$

and

$$\Big| \sum_{x \leqslant p \leqslant y} \frac{\lambda_f(p)^k \lambda_g(p)^{k'}}{p \log p} \Big| \leqslant \left(n_{k,k'} + \tilde{\delta}\right) \left(\frac{1}{\log x} - \frac{1}{\log y}\right)$$

for $1 \leqslant k$, $k' \leqslant 4$, and for $y \geqslant 2x \geqslant 2X_0 \geqslant 4$, which is possible by (2.21).

First, we find that for $Y \geqslant 2X \geqslant 42$, we have

$$\begin{aligned} \sum_{X \leqslant p \leqslant Y} \frac{\omega(p)}{p \log p} &\leqslant \sum_{X \leqslant p \leqslant Y} \frac{\lambda_f^*(p)^2 \lambda_g^*(p)^2 (\lambda_f^*(p)^2 + 2\lambda_f^*(p)\lambda_g^*(p) + \lambda_g^*(p)^2)}{p \log p} \\ &\leqslant \left(n_{4,2} + 2n_{3,3} + n_{2,4} + 4\tilde{\delta}\right) \left(\frac{1}{\log X} - \frac{1}{\log Y}\right). \end{aligned}$$

This verifies (7.2) with $a_\omega = n_{4,2} + 2n_{3,3} + n_{2,4} + 4\tilde{\delta}$.

We proceed to the proof of (7.3). Keeping in mind that

$$\lambda_f^*(p)\lambda_g^*(p)(\lambda_f^*(p) + \lambda_g^*(p))^2 \leqslant 0$$

if $p \notin \mathcal{G}$, we deduce that

$$\begin{aligned}\sum_{X \leqslant p \leqslant Y} \frac{\omega'(p)}{p \log p} &= \sum_{\substack{X \leqslant p \leqslant Y \\ p \in \mathcal{G}}} \frac{\lambda_f^*(p)\lambda_g^*(p)\big(\lambda_f^*(p) + \lambda_g^*(p)\big)^2}{p \log p} \\ &\geqslant \sum_{X \leqslant p \leqslant Y} \frac{\lambda_f^*(p)\lambda_g^*(p)(\lambda_f^*(p)^2 + 2\lambda_f^*(p)\lambda_g^*(p) + \lambda_g^*(p)^2)}{p \log p} \\ &\geqslant \big(n_{3,1} + 2n_{2,2} + n_{1,3} - 4\tilde{\delta}\big)\left(\frac{1}{\log X} - \frac{1}{\log Y}\right)\end{aligned}$$

for $Y \geqslant 2X \geqslant 4$. This verifies (7.3) with $a'_{\omega'} = n_{3,1} + 2n_{2,2} + n_{1,3} - 4\tilde{\delta} \geqslant 2 - 4\tilde{\delta}$.

Finally, we check that (7.4) holds. Recalling the choice $\delta = \frac{1}{4}$ and using a simple dyadic subdivision, we first find that

$$\begin{aligned}\sum_{p \leqslant X} \omega(p)^{\delta}\omega'(p) &\leqslant \sum_{p \leqslant X} \big|\lambda_f^*(p)\lambda_g^*(p)\big|^{1+2\delta}\big(|\lambda_f^*(p)| + |\lambda_g^*(p)|\big)^{2+2\delta} \\ &\ll \big(n_{4,2} + n_{4,0} + n_{2,4} + n_{0,4} + 4\tilde{\delta}\big)\frac{X}{\log X},\end{aligned}$$

with an absolute implied constant. Similarly recalling that $\delta' = \frac{1}{3}$, we obtain

$$\begin{aligned}\sum_{p \leqslant X} \omega'(p)^{1+\delta'} &\leqslant \sum_{p \leqslant X} \big|\lambda_f^*(p)\lambda_g^*(p)\big|^{1+\delta'}\big(|\lambda_f^*(p)| + |\lambda_g^*(p)|\big)^{2+2\delta'} \\ &\ll \big(n_{4,2} + n_{4,0} + n_{2,4} + n_{0,4} + 4\tilde{\delta}\big)\frac{X}{\log X}.\end{aligned}$$

□

7.5.2. Proof of Theorem 7.2. In this section, we complete the proof of Theorem 7.2.

Proof. We use a resonator polynomial (7.29), with the resonator sequence $r(n)$ as in (7.1), and arithmetic factors $\varpi(n)$ chosen as in (7.53) in Section 7.5.1. Let multiplicative functions ω, ω'_1, ω'_2, and ω' be as in (7.28) and (7.54). Using Lemma 7.19, we have that ω'_1 and ω'_2 satisfy (7.5), while ω and ω' satisfy the conditions (7.2)–(7.4) with a_ω, $a'_{\omega'}$ as in (7.55).

Fix an arbitrary $\delta > 0$, and apply Lemmas 7.10 and 7.14 with

$$N = q^{1/360-\delta}$$

and $L = \sqrt{a_\omega^{-1} \log N \log\log N}$. According to Lemma 7.4, ω satisfies the basic evaluation (7.6); according to Lemma 7.6, ω and ω' satisfy the basic evaluation (7.9). In turn, Lemmas 7.10 and 7.14 give

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |R(\chi)|^2 \sim \prod_p \big(1 + r(p)^2\omega(p)\big)$$

and

$$\begin{aligned}\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} |R(\chi)|^2 L\big(f \otimes \chi, \tfrac12\big) L\big(g \otimes \bar\chi, \tfrac12\big) \\ = L^*(f \otimes g, 1)\big(\nu + o^\star(1)\big) \prod_p \left(1 + r(p)^2\omega(p) + \frac{r(p)\omega'(p)}{\sqrt{p}}\right) \\ + O\bigg(N^{5/2} q^{-1/144} \prod_p \big(1 + r(p)^2\omega(p)\big)\bigg),\end{aligned}$$

where $\nu \neq 0$ depends only on f and g, with implicit constants depending on $(f, g, \delta, \varepsilon)$. According to Lemma 7.5,

$$\begin{aligned}\prod_p \left(1 + r(p)^2\omega(p) + \frac{r(p)\omega'(p)}{\sqrt{p}}\right) \\ \gg \exp\left(\big(a'_{\omega'} + o^\star(1)\big)\frac{L}{2\log L}\right) \prod_p \big(1 + r(p)^2\omega(p)\big).\end{aligned}$$

Since $L^*(f \otimes g, 1) \neq 0$ (Lemma 2.6), it follows that there exists at least one primitive character χ of conductor q such that

$$\big|L\big(f \otimes \chi, \tfrac12\big) L\big(g \otimes \bar\chi, \tfrac12\big)\big| \gg \exp\left(\big(\tfrac{1}{6\sqrt{10}}\sqrt{1 - 360\delta} + o^\star(1)\big)\frac{a'_{\omega'}}{\sqrt{a_\omega}}\sqrt{\frac{\log q}{\log\log q}}\right),$$

with $o^\star(1) = O(\log\log\log q/\log\log q)$. Since we may take δ and $\tilde\delta$ (implicit in a_ω and $a'_{\omega'}$) as small as we wish, this proves the first statement of Theorem 7.2 with the constant

$$C_{f,g} := \frac{1}{6\sqrt{10}} C^0_{f,g}, \quad C^0_{f,g} := \frac{n_{f,g,3,1} + 2n_{f,g,2,2} + n_{f,g,1,3}}{(n_{f,g,4,2} + 2n_{f,g,3,3} + n_{f,g,2,4})^{1/2}} \tag{7.56}$$

in the exponent. It is clear that there is an absolute lower bound $C > 0$ for $n^0_{f,g}$. $\square$

REMARK 7.20. It is clear that, in determining $C > 0$, it suffices to consider the case when f and g are not scalar multiples of each other, for otherwise Theorem 7.2 follows, for example, from Theorem 7.1 (with a better exponent).

In a generic situation, where neither f nor g are of polyhedral type (in particular, $\mathrm{Sym}^k\pi_f$, $\mathrm{Sym}^k\pi_g$ are cuspidal for all $k \leqslant 4$) and if $\mathrm{Sym}^k\pi_f \not\simeq \mathrm{Sym}^k\pi_g$ for every $k \leqslant 4$, then

$$C^0_{f,g} = (0 + 2 + 0)/(2 + 0 + 2)^{1/2} = 1$$

and consequently

$$C_{f,g} = \tfrac{1}{6\sqrt{10}}$$

in (7.56) and Theorem 7.2.

Any of the terms in (7.56), including $n_{f,g,k,k'}$ when k and k' are not both even, can take values larger than the generic ones, for several distinct reasons: first, some of $\mathrm{Sym}^k\pi_f$, $\mathrm{Sym}^k\pi_g$ might not be cuspidal, and the classification of their isobaric components is quoted in Section 2.3.4; second, it is possible to have $\mathrm{Sym}^k\pi_f \simeq \mathrm{Sym}^k\pi_g$ if f and g are character twists of each other (necessarily by a quadratic character due to the trivial central character); and, third, $\mathrm{Sym}^3\pi_f \simeq \mathrm{Sym}^3\pi_g$ can happen even if f and g are not character twists of each other (see Ramakrishnan's paper [68]; for this case, while the known examples arise from icosahedral representations, and are conjectured to be exhaustive, this is not known unconditionally).

Thus, in most cases, cusp forms f and g for which this happens can be explicitly classified, and then the constant $C_{f,g}$ can probably be improved by using custom-made arithmetic factors; however, since such a classification is not available in at least one of the cases, and since getting a tight universal lower bound for our $C^0_{f,g}$ involves an uninspiring case-by-case computation, we are satisfied simply with stating the existence of such a lower bound.

CHAPTER 8

Upper bounds for the analytic rank

8.1. Introduction

In this chapter, we prove Theorem 1.13. We again fix f as in Section 1.2, and we recall the statement.

THEOREM 8.1. *There exist constants* $R \geqslant 0$, $c > 0$ *such that*

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \exp(c\,\mathrm{rk}_{an}(f \otimes \chi)) \leqslant \exp(cR) \tag{8.1}$$

for all primes q.

The proof follows the method of Heath-Brown and Michel, who established a version of Theorem 1.13 for the analytic rank in the family of Hecke L-functions of primitive holomorphic cusp forms of weight 2 and level q ([**31**, Thm 0.1 & Cor 0.2]). This method is robust and general and could be axiomatized (using the definition of families of L-functions as provided in [**47**, **73**]); we will merely indicate where to modify the original argument of [**31**, §2, p. 497].

8.2. Application of the explicit formula

The basic principle is to use the explicit formula of Weil (Proposition 2.5) to bound the analytic rank by a sum over the primes.

Let ϕ be a smooth non-negative function, compactly supported in $[-1, 1]$. We denote by

$$\widehat{\phi}(s) = \int_{\mathbf{R}} \phi(t) e^{st} dt$$

its Fourier-Laplace transform, which is an entire function of $s \in \mathbf{C}$.

In this chapter, we assume that such a function ϕ is chosen once and for all, with the properties that $\widehat{\phi}(0) = 1$ and $\mathfrak{Re}(\widehat{\phi}(s)) \geqslant 0$ for all $s \in \mathbf{C}$ such that $|\,\mathfrak{Re}(s)| \leqslant 1$. (The existence of such functions is standard, see, e.g., [**36**, Prop. 5.55].)

PROPOSITION 8.2. *Let* $\xi > 1/10000$ *be some parameter. We have the inequality*

$$\xi\,\mathrm{rk}_{an}(f \otimes \chi) \leqslant 2\phi(0) \log q - S(f \otimes \chi) - S(f \otimes \overline{\chi}) - 2\,\xi\,\Xi(f \otimes \chi) + O_{\phi,f}(\xi) \tag{8.2}$$

where

$$S(f \otimes \chi) = \sum_p \frac{\chi(p)\lambda_f(p) \log p}{p^{1/2}} \phi\Big(\frac{\log p}{\xi}\Big)$$

and

$$\Xi(f \otimes \chi) = \sum_{\mathfrak{Re}(\varrho - \frac{1}{2}) \geqslant \frac{1}{\xi}} \mathfrak{Re}\Big\{\hat{\phi}\Big(\xi(\varrho - \frac{1}{2})\Big)\Big\}$$

where ϱ *ranges over the non-trivial zeros of* $L(f \otimes \chi, s)$.

Proof. We apply (2.6) to the function

$$\varphi(y)=\frac{1}{\sqrt{y}}\phi\Big(\frac{\log y}{\xi}\Big),$$

with Mellin transform

$$\widetilde{\varphi}(\varrho)=\widehat{\phi}\Big(\xi\Big(\varrho-\frac{1}{2}\Big)\Big).$$

On the side of the sum over powers of primes, we easily get

$$\sum_{l\geqslant 2}\sum_{p}\frac{\chi(p)^l\Lambda_f(p^l)}{p^{l/2}}\phi\Big(\frac{l\log p}{\xi}\Big)\ll\xi$$

by distinguishing the case $l=2$ (for which one applies Corollary 2.15 after noting that $\Lambda_f(p^2)=(\lambda_{\mathrm{Sym}^2 f}(p)-1)\log p$ for $p\nmid r$) and the case $l\geqslant 3$ (when the series can be extended to all primes and converges absolutely). The same bound holds for the corresponding sum with $\bar{\chi}$. Then $S(f\otimes\chi)+S(f\otimes\overline{\chi})$ is the contribution of the primes themselves to the explicit formula.

On the side of the zeros, the assumption on the test function shows that the contribution of any subset of the zeros of $L(f\otimes\chi,s)$ may be dropped by positivity from the explicit formula to obtain an upper bound as in the statement of the proposition. □

Remark 8.3. Note that from (8.2), by taking $\xi=1$ and a suitable ϕ (see [59, p.217 Example]), one obtains the pointwise bound

$$\mathrm{rk}_{an}(f\otimes\chi)\leqslant 2\log q+O_f(1). \tag{8.3}$$

8.2.1. Bounds for moments of the analytic rank. Theoerem 8.1 is a consequence of the following proposition which bound the moments of the analytic ranks:

Proposition 8.4. *There exists an absolute constant C such that for all (sufficiently large) primes q and for all integers $k\geqslant 1$, one has*

$$\frac{1}{\varphi^*(q)}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}\mathrm{rk}_{an}(f\otimes\chi)^{2k}\ll_f(Ck)^{2k}. \tag{8.4}$$

Assuming (8.4), we deduce (8.1): fix $A>0$ such $AC<1/3$; then

$$\sum_{k\geqslant 0}\frac{(ACk)^k}{k!}<\infty,$$

by Stirling's formula, and we therefore obtain

$$\frac{1}{\varphi^*(q)}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}\exp(A\,\mathrm{rk}_{an}(f\otimes\chi))<\infty$$

for all primes q, as desired. □

It remains to prove Proposition 8.4.

8.2.2. Reduction to a mean square estimate. In this section we reduce the proof of Proposition 8.4 to a "second moment" upper bound (Theorem 8.5 below) .

To prove Proposition 8.4 we observe first that because of (8.3), we may assume that

$$k \leqslant \log(q/2).$$

Now for any $k \in [1, \log(q/2)]$, we set

$$\xi = \frac{\log(q/2)}{k} \geqslant 1.$$

By Proposition 8.2, it is enough to prove that there exists a constant $C > 0$, depending only on ϕ, such that

$$\frac{1}{\varphi^*(q)} \sum_{\chi \,(\mathrm{mod}\, q)}^{*} |S(f \otimes \chi)|^{2k} \ll_{f,\phi} (Ck\xi)^{2k} \tag{8.5}$$

$$\frac{1}{\varphi^*(q)} \sum_{\chi \,(\mathrm{mod}\, q)}^{*} |\Xi(f \otimes \chi)|^{2k} \ll_{f,\phi} (Ck)^{2k}. \tag{8.6}$$

We can quickly deal with the first bound as in [**31**, §2.1 (6)]. Since $\exp(\xi) < q^{1/k}$, for any prime numbers $p_i \leqslant \exp(\xi)$ for $1 \leqslant i \leqslant 2k$, we have the equivalence

$$p_1 \cdots p_k \equiv p_{k+1} \cdots p_{2k} \,(\mathrm{mod} q) \Longleftrightarrow p_1 \cdots p_k = p_{k+1} \cdots p_{2k},$$

hence the left-hand side of (8.5) is bounded by

$$\frac{\varphi(q)}{\varphi^*(q)} \sum_{p_1, \cdots, p_{2k}} \prod_{i=1}^{2k} \frac{\lambda_f(p_i) \log p_i}{p_i^{1/2}} \phi\Big(\frac{\log p_i}{\xi}\Big) \delta_{p_1 \cdots p_k = p_{k+1} \cdots p_{2k}},$$

which is

$$\leqslant \frac{\varphi(q)}{\varphi^*(q)} k! \sum_{p_1, \ldots, p_k} \cdots \sum \prod_{i=1}^{k} \frac{\lambda_f(p_i)^2 \log^2 p_i}{p_i} \phi\Big(\frac{\log p_i}{\xi}\Big)^2$$

$$= \frac{\varphi(q)}{\varphi^*(q)} k! \Big(\sum_{p} \frac{\lambda_f(p)^2 \log^2 p}{p} \phi\Big(\frac{\log p}{\xi}\Big)^2 \Big)^k \ll (Ck\xi)^{2k}$$

by Corollary 2.15. This proves (8.5).

The control of the sum over the zeros in (8.6), is achieved by reducting to a second moment estimate. This reduction follows general principles and is explained in [**31**, Th. 0.4, §2.2].

Let $0 < \lambda < 1/360$ be fixed. Define

$$L = q^{\lambda} \tag{8.7}$$

and for $x \geqslant 0$, let

$$P(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant 1/2 \\ 1 & \text{if } 1/2 \leqslant x \leqslant 1. \end{cases}$$

Define furthermore

$$\boldsymbol{x}_L := (x_l)_{l \leqslant L},$$

where the x_l are defined in (6.7). Let then $M(f \otimes \chi, \boldsymbol{x}_L)$ be as in (6.6). The reduction step mentioned above (which relies in particular on an important lemma of Selberg, see [**31**, Lemma 1.1]) shows that (8.6) follows from:

THEOREM 8.5. *For every $0 < \lambda < 1/360$ there exists $\eta = \eta(\lambda) > 0$, such that for any prime $q \geqslant 2$ and any $\sigma - \frac{1}{2} \geqslant -\frac{1}{\log q}$, we have*

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \Big| L(f \otimes \chi, s) M(f \otimes \chi, \boldsymbol{x}_L) - 1 \Big|^2 \ll |s|^{O(1)} q^{-\eta(\sigma - \frac{1}{2})},$$

where $L = q^\lambda$.

8.3. Proof of the mean-square estimate

The rest of this chapter is devoted to the proof of Theorem 8.5.

8.3.1. Application of the twisted second moment formula. By definition of the mollifier $M(f \otimes \chi, \boldsymbol{x}_L)$, we have for $\mathfrak{Re}\, s = \sigma \geqslant 2$ and any $\varepsilon > 0$ the equality

$$L(f \otimes \chi, s) M(f \otimes \chi, \boldsymbol{x}_L) = 1 + \sum_{m > L^{1/2}} \frac{(\lambda_f \cdot \chi * \boldsymbol{x}_L)(m)}{m^s} = 1 + O_\varepsilon(L^{-\frac{1}{2}(\sigma - 1) + \varepsilon}),$$

where $(\lambda_f \cdot \chi * \boldsymbol{x}_L)(m)$ denotes the multiplicative convolution

$$(\lambda_f \cdot \chi * \boldsymbol{x}_L)(m) := \sum_{\substack{n\ell = m \\ \ell \leqslant L}} \lambda_f(n) \chi(n) x_\ell.$$

From the definition of the $(x_\ell)_{\ell \leqslant L}$ (cf. (6.7)) we see that for such s we have

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \Big| L(f \otimes \chi, s) M(f \otimes \chi, \boldsymbol{x}_L) - 1 \Big|^2 \ll q^{-\eta(\sigma - \frac{1}{2})}$$

for some absolute $\eta > 0$. This suffices to establish Theorem 8.5 for $\sigma \geqslant 2$.

By the Phragmen–Lindelöf convexity argument for subharmonic functions, it is then sufficient to show that

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi} \Big| L(f \otimes \chi, s) M(f \otimes \chi, \boldsymbol{x}_L) \Big|^2 \ll |s|^{O(1)} \tag{8.8}$$

for

$$\sigma = \frac{1}{2} - \frac{1}{\log q}.$$

For this we will use the results of Chapter 5.

To establish (8.8), we decompose the sum along even and odd characters. In the sequel, we will evaluate in detail only the contribution of the even characters (multiplied by 2), namely

$$\mathcal{Q}^+(f, s; \boldsymbol{x}_L) = \frac{2}{\varphi^*(q)} \sideset{}{^+}\sum_{\chi \text{ non-trivial}} \Big| L(f \otimes \chi, s) M(f \otimes \chi, \boldsymbol{x}_L) \Big|^2$$

since the treatment of the contribution of odd characters is entirely similar.

We recall that $(x_\ell)_{\ell \leqslant L}$ is supported on integers coprime to r. By Theorem 5.1, we have

$$\mathcal{Q}^+(f, s; \boldsymbol{x}_L) = \mathrm{MT}^+(f, s; \boldsymbol{x}_L) + \mathrm{ET}$$

where, with notations of Chapter 5, the main term is given by (cf. (5.2))

$$\mathrm{MT}^+(f, s; \boldsymbol{x}_L) = \sum_{d \geqslant 1} \sum_{(\ell_1, \ell_2) = 1} \frac{x_{d\ell_1} \overline{x_{d\ell_2}}}{d^{2\sigma} \ell_1^s \ell_2^{\overline{s}}} \mathrm{MT}^+(f, s; \ell_1, \ell_2)$$

and the error term is bounded by

$$\begin{aligned}\mathrm{ET} &\ll_\varepsilon |s|^{O(1)}\Big(q^{-1}+\sum_{d}\sum_{(\ell_1,\ell_2)=1}\frac{|x_{d\ell_1}x_{d\ell_2}|}{d(\ell_1\ell_2)^{1/2}}L^{3/2}q^{-1/144+\varepsilon}\Big)\\ &\ll |s|^{O(1)}L^{5/2}q^{-1/144+2\varepsilon}\end{aligned}$$

(see (6.18) in Section 6.5 for a similar bound).

Before proceeding further we simplify some notations: we set

$$L_\infty(s) := L_\infty(f,s)$$

and

$$R(\ell_1,\ell_2,s) := \sum_{n\geqslant 1}\frac{\lambda_f(\ell_1 n)\lambda_f(\ell_2 n)}{(\ell_1\ell_2 n^2)^s}.$$

Note that we already encountered this function in earlier sections since, with the notation of (5.5), we have the equality

$$R(\ell_1,\ell_2,s) = L\big(f\times f,1,s-\tfrac{1}{2};\ell_1,\ell_2\big).$$

We can then write $\mathrm{MT}^+(f,s;\ell_1,\ell_2)$ in the form

$$\mathrm{MT}^+(f,s;\ell_1,\ell_2) = \frac{1}{2}M(s,\ell_1,\ell_2)+\frac{1}{2}\varepsilon(f,+,s)M(1-s,\ell_1,\ell_2),$$

where

$$M(s,\ell_1,\ell_2) = \frac{1}{2i\pi}\int_{(2)}\frac{L_\infty(s+u)^2}{L_\infty(s)^2}R(\ell_1,\ell_2,s+u)G(u)(q^2|r|)^u\frac{du}{u}. \tag{8.9}$$

We rename s into $s_0=\sigma_0+it_0$. In order to prove Theorem 8.5 it suffices to prove the following estimate:

PROPOSITION 8.6. *There exist two constants C_1 and C_2, such that, for every prime q, for every $s_0=\sigma_0+it_0$ satisfying*

$$\sigma_0=\frac{1}{2}\pm\frac{1}{\log q}\ \textit{ and } t_0\ \textit{ real,} \tag{8.10}$$

we have the inequality

$$\Big|\sum_{d\geqslant 1}\sum_{(\ell_1,\ell_2)=1}\frac{x_{d\ell_1}\overline{x_{d\ell_2}}}{d^{2\sigma_0}\ell_1^{s_0}\ell_2^{\bar{s}_0}}M(s_0,\ell_1,\ell_2)\Big|\leqslant C_1|s_0|^{C_2}. \tag{8.11}$$

We emphasize that σ_0 may be taken to be $<\frac{1}{2}$ in this result.

REMARK 8.7. The upper bound (8.8) for the mollified second moment has many similarities with the evaluation of the mollified second moment at 1/2 discussed Section 6.5. A chief difference, is that in that section, we were looking for a asymptotic formula (see Proposition 6.6) while here, an upper bound of the correct order of magnitude his sufficient for our purpose; another important difference is that the complex variable s, while close to the critical line, is not necessarily located near the central point 1/2, but range along the whole critical line. As we will see below, this significantly complicates the evaluation of the main term (see also Remark 8.12).

8.3.2. Beginning of the proof of Proposition 8.6. For the proof of the result, we consider s_0 fixed and write simply $M(\ell_1, \ell_2) = M(s_0, \ell_1, \ell_2)$.

For $d\ell \leqslant L$, the definition of P and the standard formula

$$\frac{1}{2i\pi} \int_{(2)} y^v \frac{dv}{v^2} = \begin{cases} \log y & \text{if } y \geqslant 1, \\ 0 & \text{if } 0 < y \leqslant 1. \end{cases}$$

show that we have

$$P\Big(\frac{\log(L/d\ell)}{\log L}\Big) = \frac{1}{2i\pi} \int_{(2)} H_L(v)(d\ell)^{-v} \frac{dv}{v}$$

where H_L is the entire function defined by

$$H_L(v) = 2\, \frac{L^{v/2}(L^{v/2} - 1)}{v \log L}$$

for $v \neq 0$ and $H_L(0) = 1$.

We insert this integral in the left-hand side of (8.11), obtaining (see (8.9)) the formula

$$\begin{aligned} (8.12) \quad \mathcal{M} = \mathcal{M}(s_0) &:= \sum_{d \geqslant 1} \sum_{(\ell_1, \ell_2) = 1} \frac{x_{d\ell_1} \overline{x_{d\ell_2}}}{d^{2\sigma_0} \ell_1^{s_0} \ell_2^{\bar{s}_0}} M(s_0, \ell_1, \ell_2) \\ &= \frac{1}{(2\pi i)^3} \int_{(2)} \int_{(2)} \int_{(2)} \frac{L_\infty(s_0 + u)^2}{L_\infty(s_0)^2} G(u) \\ &\qquad \times L(s_0, s_0, \bar{s}_0, u, v, w) H_L(v) H_L(w) (q^2|r|)^u \frac{du}{u} \frac{dv}{v} \frac{dw}{w} \end{aligned}$$

where the auxiliary function L is given by (2.32), namely

$$L(s, z, z', u, v, w) = \sum_{\substack{d, \ell_1, \ell_2, n \\ (\ell_1, \ell_2) = (d\ell_1\ell_2, r) = 1}} \sum \frac{\mu_f(d\ell_1)\lambda_f(\ell_1 n)\mu_f(d\ell_2)\lambda_f(\ell_2 n)}{\ell_1^{s+z+u+v} \ell_2^{s+z'+u+w} d^{z+z'+v+w} n^{2s+2u}}.$$

With this definition, based on integrals, our purpose (see (8.11)) is to prove the inequality

$$(8.13) \qquad \big|\mathcal{M}\big| \leqslant C_1 |s_0|^{C_2},$$

for some absolute C_1 and C_2, uniformly for s_0 satisfying (8.10).

To prove (8.13), we proceed by shifting the three contours slightly to the left of the product of lines

$$\mathfrak{Re}\, u = \mathfrak{Re}\, v = \mathfrak{Re}\, w = 0.$$

In the sequel we decompose the complex variables u, v and w into their real and imaginary parts as

$$u = \sigma_u + it_u, \ v = \sigma_v + it_v, \ w = \sigma_w + it_w.$$

It will also be useful to set

$$\mathcal{L} = (\log q)^{-1}.$$

We will need estimates for the various factors in the integral (8.12). We start with $H_L(v)$.

LEMMA 8.8. *Let $0 < \lambda < 1/360$ and $B > 0$ be two constants. Let L be defined by (8.7). Then there exists a constant C_3 depending only on λ and B, such that uniformly for $\sigma_v \leqslant B\mathcal{L}$, we have the inequality*

$$\big|H_L(v)\big| \leqslant C_3\, L^{\sigma_v/2}\, \min\Big(1, \frac{\mathcal{L}}{|v|}\Big).$$

PROOF. This is an easy combination of the two bounds $L^{v/2} - 1 = O(|v|\log L)$, valid uniformly for $|v|\log q \leqslant 1$, and $L^{v/2} - 1 = O_B(1)$, valid uniformly for $\sigma_v \leqslant B\mathcal{L}$. ☐

Next we provide bounds for the Gamma factors (see (2.2) for the definition of L_∞).

LEMMA 8.9. *Let $G(u)$ be the function defined in (2.23). Then there exists a constant α_f depending only on f, such that, uniformly for s_0 satisfying (8.10) and for $u = \sigma_u + it_u$ with $\sigma_u \in [-1/4, 2]$ and t_u real, we have the bound*

$$\Big(\frac{L_\infty(s_0+u)}{L_\infty(s_0)}\Big)^2 G(u) \ll_f (1+|t_0|)^{\alpha_f} e^{-3\pi|t_u|}.$$

PROOF. In both cases (f holomorphic or not), we have the equalities (see the definition (2.2))

$$L_\infty(s) = \xi_f\, \pi^{-s} \prod_{i=1,2} \Gamma\Big(\frac{s+\mu_{f,i}}{2}\Big)$$

where $\xi_f = 1$ if f is a Hecke–Maaß form, and $\xi_f = \pi^{-1/2}2^{(k-3)/2}$ if f is holomorphic with weight k. Furthermore, the $\mu_{f,i}$ are the archimedean Langlands parameters of the automorphic representation attached to f as in Section 2.2, i.e.

$$\mu_{f,1} = -\frac{k-1}{2},\ \mu_{f,2} = -\frac{k}{2}$$

if f is holomorphic of weight $k \geqslant 2$ and

$$\mu_{f,1} = \frac{1-\kappa_f}{2} + it_f,\ \mu_{f,2} = \frac{1-\kappa_f}{2} - it_f$$

if f is a Maaß form with Laplace eigenvalue $\lambda_f(\infty) = (\frac{1}{2}+it_f)(\frac{1}{2}-it_f)$ and parity $\kappa_f \in \{\pm 1\}$. This implies that, in both cases, we have

$$\frac{1}{16} \leqslant \mathfrak{Re}\Big(\frac{s_0+\mu_{f,i}+u}{2}\Big) \ll_f 1,$$

under the assumptions of Lemma 8.9.

Decompose $\mu_{f,i}$ as $\mu_{f,i} = \sigma_{f,i} + it_{f,i}$. Then by Stirling's formula [27, formula 8.328, page 895] we have for $1/5 \leqslant \sigma \leqslant 3$

$$\begin{aligned} L_\infty(s) &\asymp (1+|t+t_{f,1}|)^{\frac{\sigma}{2}+\frac{\sigma_{f,1}}{2}-\frac{1}{2}} e^{-\frac{\pi}{4}|t+t_{f,1}|} \times (1+|t+t_{f,2}|)^{\frac{\sigma}{2}+\frac{\sigma_{f,2}}{2}-\frac{1}{2}} e^{-\frac{\pi}{4}|t+t_{f,2}|} \\ &\asymp_f (1+|t|)^{\sigma+\frac{\sigma_{f,1}+\sigma_{f,2}}{2}-1} e^{-\frac{\pi}{2}|t|}. \end{aligned}$$

Therefore, since for $\sigma_u \in [-1/4, 2]$ we have $\mathfrak{Re}(s_0+u) \in [1/5, 3]$, we deduce the inequality

$$\frac{L_\infty(s_0+u)^2}{L_\infty(s_0)^2} G(u) \ll_f \frac{(1+|t_0+t_u|)^{2\sigma_0-2+\sigma_{f,1}+\sigma_{f,2}+2\sigma_u}}{(1+|t_0|)^{2\sigma_0-2+\sigma_{f,1}+\sigma_{f,2}}} e^{-\pi(|t_0+t_u|-|t_0|)}\, e^{-4\pi|t_u|}.$$

To control the size of the numerator of the above fraction, we will use either the lower bound $1+|t_u+t_0| \geqslant 1$ or the upper bound $1+|t_u+t_0| \leqslant (1+|t_u|)(1+|t_0|)$ according to the sign of the exponent and we will consider two cases:

Case 1. For $|t_u| \geqslant |t_0|$, using $|t_u + t_0| \geqslant |t_u| - |t_0|$, we have

$$\frac{L_\infty(s_0+u)^2}{L_\infty(s_0)^2} G(u) \ll (1+|t_u|)^{\alpha_f} e^{\pi|t_u|} e^{-4\pi|t_u|} = (1+|t_u|)^{\alpha_f} e^{-3\pi|t_u|}$$

for some absolute $\alpha_f \geqslant 0$.

Case 2. For $|t_u| \leqslant |t_0|$, using the inequality $|t_u + t_0| \geqslant |t_0| - |t_u|$, we get

$$\frac{L_\infty(s_0+u)^2}{L_\infty(s_0)^2} G(u) \ll_f (1+|t_0|)^{\alpha_f}\ e^{\pi|t_u|} e^{-4\pi|t_u|} \ll_f (1+|t_0|)^{\alpha_f} e^{-3\pi|t_u|}$$

again for some absolute constant $\alpha_f \geqslant 0$. □

As in (2.30) we now denote by

$$T(s) = L(f \otimes f, s) = \zeta(s) L(\mathrm{Sym}^2 f, s)$$

the Rankin-Selberg L-function of f, and by

$$T_p(s) = L_p(f \otimes f, s) = \zeta_p(s) L_p(\mathrm{Sym}^2 f, s)$$

its local factor at p.

The analytic properties of $T(s)$ have been reviewed in Section 2.3. Recall in particular that $T(s)$ is holomorphic on $\mathbf{C} - \{1\}$ and has a simple pole at $s = 1$; its residue there is denoted κ_f. Lemma 2.24 implies that there exists $\eta > 0$ and an analytic continuation and factorization of $L(s, z, z', u, v, w)$ of the form

$$L(s,z,z',u,v,w) = \frac{T(2s+2u)T(z+z'+v+w)}{T(s+z+u+v)T(s+z'+u+w)} D(s,z,z',u,v,w).$$

in the region $\mathcal{R}(\eta) \subset \mathbf{C}^6$ defined by the inequalities

$$\begin{aligned} \mathfrak{Re}\, s > \frac{1}{2} - \eta, \quad \mathfrak{Re}\, z > \frac{1}{2} - \eta, \quad \mathfrak{Re}\, z' > \frac{1}{2} - \eta, \\ \mathfrak{Re}\, u > -\eta, \quad \mathfrak{Re}\, v > -\eta, \quad \mathfrak{Re}\, w > -\eta, \end{aligned}$$

where $D(s, z, z', u, v, w)$ is holomorphic and bounded on $\mathcal{R}(\eta)$.

8.3.3. Study of $\mathcal{M}$. We now start the proof of (8.13), which will eventually prove Proposition 8.6.

Let us recall that $\mathcal{M}$ is an integral in three variables (varying along vertical lines) whose integrand contains factors involving translates of Riemann's zeta function $\zeta(s)$ and of the symmetric square L-function $L(\mathrm{Sym}^2 f, s)$. The strategy is similar to that of Section 6.5: we are going to shift contours to the left so that the integrals along the new contours contribute as error terms and to evaluate the residues of the poles that we have met in the process. That would be reasonably easy to do under the *Generalised Riemann Hypothesis*; however to obtain unconditional results, we need to use the standard Hadamard-de la Vallée-Poussin zero free region. The proof is a bit tedious so we give and overview of what is coming up.

(1) We first reduce (up to an admissible error term) to evaluating a truncated version of $\mathcal{M}$ in which the imaginary parts of the variables are bounded by a power of $\log q$ and the real part is such that the arguments at which the various L-functions are evaluated, stay slightly to the right of the critical strip: this is the content of Lemma 8.10.

(2) We shift the contour of one of the variables to the left so that one still remains on the right of the zero free region of any L-function involved in the denominator. We show that the resulting integral contribute a negligible error term and it remains to deal with the contributions of the poles encountered in the process: this is the content of Section 8.3.4 ending with (8.27).
(3) In Section 8.3.5 we describe the contributions of the two poles (these are integrals in two variables) with the aim to bounding them. The target bound is (8.29). We focus on one integral, the treatment of the other being entirely similar.
(4) In Section 8.3.6 we perform a contour shift on one of the remaining variables along the same line as in Step (2). The outcome is the integral along the new contour J_1 and two contributions from poles met in the process J_2, J_3. The target bound for any of these terms is (8.31).
(5) We prove (8.31) for J_1 in Sections 8.3.7 and 8.3.8.
(6) The proofs of (8.31) for J_2 and J_3 (which are integrals in one variable) are spread over Sections 8.3.9, 8.3.10 and 8.3.11. Unlike Section 6.5 we don't need to perform a final contour shift and evaluate the residue: as we only need an upper bound, we simply split the integral into pieces and apply different bounds depending on the position of the piece with respect to the other parameters.

We start with Step (1) which is a reduction to another estimate for the following truncated triple integral: for $C \geqslant 1$ and

$$V_0 := (\log q)^C, \tag{8.14}$$

we define

$$\begin{aligned}\mathcal{M}(V_0) := \frac{1}{(2\pi i)^3} \int_{\substack{(3\mathcal{L})\\ |t_w|\leqslant 2V_0}} \int_{\substack{(3\mathcal{L})\\ |t_v|\leqslant V_0}} \int_{\substack{(3\mathcal{L})\\ |t_u|\leqslant V_0}} &\frac{L_\infty(s_0+u)^2}{L_\infty(s_0)^2} \\ &\times \frac{T(2s_0+2u)T(2\sigma_0+v+w)}{T(2s_0+u+v)T(2\sigma_0+u+w)} \\ &\times E(s_0,u,v,w)G(u)H_L(v)H_L(w)(q^2|r|)^u \frac{du}{u}\frac{dv}{v}\frac{dw}{w}\end{aligned} \tag{8.15}$$

Lemma 8.10. *The bound (8.13) follows from the following bound: for any $C \geqslant 1$ we have uniformly for $|t_0| \leqslant \log q$*

$$|\mathcal{M}(V_0)| \ll |s_0|^{C'}, \tag{8.16}$$

where C' depends on C and the implicit constant on f and C

For the proof of Lemma 8.10 and later arguments will repeatedly use the following Lemma which combines (a special case of) Corollary 2.14 and the Phragmen-Lindelöf principle:

Lemma 8.11. *There exists two constants $c = c_f > 0$ and $A^* = A_f^* \geqslant 0$ such that*

- *For $s = \sigma + it$ in the region*

$$\sigma \geqslant -\frac{c}{\log(2+|t|)}, \tag{8.17}$$

we have $T(1+s)\neq 0$ *and the inequalities*

$$\log^{-A^*}(2+|s|)\ll \frac{s}{1+s}T(1+s)\ll \log^{A^*}(2+|s|). \tag{8.18}$$

- *For* $s=\sigma+it$ *such that* $\sigma\geqslant -1/2$, *we have the inequality*

$$\Big|\frac{s}{s+1}T(1+s)\Big|\ll \max\big(1,(1+|s|)^{\max(0,4(1/2-\sigma))+\varepsilon}\big). \tag{8.19}$$

for any $\varepsilon>0$ *where the constants implied depend only on* f *and* ε.

Proof. (of Lemma 8.10) Recall that $\mathcal{M}$ is defined in (8.12). We use Lemma 2.24 and set

$$E(s,u,v,w)=D(s,s,\bar{s},u,v,w).$$

With these notations, the function L in (8.12) can be written as

$$L(s_0,s_0,\bar{s}_0,u,v,w)=\frac{T(2s_0+2u)T(2\sigma_0+v+w)}{T(2s_0+u+v)T(2\sigma_0+u+w)}E(s_0,u,v,w).$$

We first shift the three lines of integration in (8.12) to

$$\mathfrak{Re}(u)=\mathfrak{Re}(v)=\mathfrak{Re}(w)=3\mathcal{L}.$$

There is no pole encountered in this shift, so that the triple integral $\mathcal{M}$, defined in (8.12), satisfies the equality

$$\mathcal{M}=\frac{1}{(2\pi i)^3}\int_{(3\mathcal{L})}\int_{(3\mathcal{L})}\int_{(3\mathcal{L})}\frac{L_\infty(s_0+u)^2}{L_\infty(s_0)^2}\cdot\frac{T(2s_0+2u)T(2\sigma_0+v+w)}{T(2s_0+u+v)T(2\sigma_0+u+w)}\\ \times E(s_0,u,v,w)G(u)H_L(v)H_L(w)(q^2|r|)^u\frac{du}{u}\frac{dv}{v}\frac{dw}{w}.$$

First, using straightforwardly Lemma 2.24 to bound the E–function, Lemma 8.11 (inequality (8.18)) to bound the T–functions or their inverses, Lemma 8.9 to bound the L_∞ and G-factors, and Lemma 8.8 for the H_L–functions, we can already deduce the rough bound

$$\mathcal{M}\ll |s_0|^{O(1)}(\log q)^{O(1)}. \tag{8.20}$$

In particular, in order to prove (8.13), we may now assume that

$$|t_0|\leqslant \log q. \tag{8.21}$$

This being done we consider the integral truncated in the variable u

$$\mathcal{M}_0(V):=\frac{1}{(2\pi i)^3}\int_{\substack{(3\mathcal{L})\\ |t_w|\leqslant 2V}}\int_{\substack{(3\mathcal{L})\\ |t_v|\leqslant V}}\int_{(3\mathcal{L})}\frac{L_\infty(s_0+u)^2}{L_\infty(s_0)^2}\\ \times\frac{T(2s_0+2u)T(2\sigma_0+v+w)}{T(2s_0+u+v)T(2\sigma_0+u+w)}\\ \times E(s_0,u,v,w)G(u)H_L(v)H_L(w)(q^2|r|)^u\frac{du}{u}\frac{dv}{v}\frac{dw}{w},$$

where $V\geqslant 2$ is some parameter. Using the same lemmas as in the proof of (8.20), we obtain the equality

$$\mathcal{M}=\mathcal{M}_0(V)+O\Big(\frac{(|s_0|\log q)^{C_4}}{V^{1/2}}\Big), \tag{8.22}$$

for some absolute constant $C_4 \geqslant 0$. In view of the inequality (8.13), the error term in (8.22) is admissible if we fix the value of V to be (8.14) for a sufficiently large constant $C \geqslant 2C_4$.

By the same techniques which led to (8.22) (particularly the decay at infinity of the functions $H_L(v)/v$ and $H_L(w)/w$, see Lemma 8.8), we approximate $\mathcal{M}_0(V_0)$ by $\mathcal{M}(V_0)$ with an admissible error. By combining with (8.22), we finally obtain the equality

$$\mathcal{M} = \mathcal{M}(V_0) + O\big(|s_0|^{C_4}\big),$$

where C_4 is some absolute constant, where t_0 satisfies (8.21) and where V_0 is defined by (8.14), with a sufficiently large C. Lemma 8.10 follows from (8.16). □

8.3.4. Shifting the contours of integration. In the u–plane we consider the vertical segment:

$$\gamma_u := \big\{u \in \mathbf{C} \mid \sigma_u = 3\mathcal{L},\ |t_u| \leqslant V_0\big\},$$

and the curve

$$\Gamma_u := \Big\{u \in \mathbf{C} \mid \sigma_u = -\frac{c_f}{\log(V_0^3 + |t_u|)},\ |t_u| \leqslant V_0\Big\},$$

where c_f is the constant appearing in Lemma 8.11. We also introduce two horizontal segments

$$S_u := \Big\{u \in \mathbf{C} \mid -\frac{c_f}{\log(V_0^3 + V_0)} \leqslant \sigma_u \leqslant 3\mathcal{L},\ t_u = V_0\Big\},$$

and its conjugate $\overline{S_u}$. The hypothesis (8.21) and Lemma 8.11 imply that there is no zero of the function

$$u \mapsto T(2s_0 + u + v)T(2\sigma_0 + u + w),$$

in the interior of the curved rectangle $\mathcal{R}_u$ with edges γ_u, S_u, Γ_u and $\overline{S_u}$, when the variables v and w belong to the paths of integration appearing in the definition (8.15) of $\mathcal{M}(V_0)$.

Furthermore, when u belongs to $S_u \cup \Gamma_u \cup \overline{S_u}$ and when v and w are as above, the four numbers

$$2s_0 + 2u - 1,\ 2\sigma_0 + v + w - 1,\ 2s_0 + u + v - 1,\ 2\sigma_0 + u + w - 1$$

all satisfy the lower bound (8.17). Finally, the modulus of these four numbers is also not too small, namely they are $\gg 1/(\log q)$. We then apply (8.18) in the condensed form

$$\frac{T(2s_0 + 2u)T(2\sigma_0 + v + w)}{T(2s_0 + u + v)T(2\sigma_0 + u + w)} \ll (\log q)^{O(1)},$$

uniformly for u, v and w as above and t_0 satisfying (8.21).

To shorten notation we rewrite $\mathcal{M}(V_0)$ into the form

$$\mathcal{M}(V_0) := \frac{1}{(2\pi i)^3} \int_{\substack{(3\mathcal{L})\\ |t_w| \leqslant 2V_0}} \int_{\substack{(3\mathcal{L})\\ |t_v| \leqslant V_0}} \int_{\substack{(3\mathcal{L})\\ |t_u| \leqslant V_0}} \mathcal{T}(s_0, u, v, w) \frac{du}{u}\frac{dv}{v}\frac{dw}{w}, \tag{8.23}$$

From this definition of the $\mathcal{T}$–function and bounding the E–function by Lemma 2.24, we deduce the following bound where the variables are now separated

$$\frac{1}{(2\pi i)^3}\int_{\substack{(3\mathcal{L})\\ |t_w|\leqslant 2V_0}}\int_{\substack{(3\mathcal{L})\\ |t_v|\leqslant V_0}}\int_{u\in S_u\cup\Gamma_u\cup\overline{S_u}}\mathcal{T}(s_0,u,v,w)\frac{du}{u}\frac{dv}{v}\frac{dw}{w}$$
$$\ll(\log q)^{O(1)}\Big(\int_{u\in S_u\cup\Gamma_u\cup\overline{S_u}} q^{\sigma_u}\ \Big|\frac{L_\infty(s_0+u)^2}{L_\infty(s_0)^2}G(u)\Big|\cdot\Big|\frac{du}{u}\Big|\Big)$$
$$\times\Big(\int_{\substack{(3\mathcal{L})\\ |t_v|\leqslant 2V_0}}\Big|\frac{H_L(v)}{v}\Big|\,|dv|\Big)^2. \tag{8.24}$$

To bound the integral $\int_{\Gamma_u}$ we exploit the fact that σ_u is negative and satisfies $|\sigma_u|\gg 1/(\log\log q)$. When combined with Lemma 8.9, we deduce the bound

$$\int_{\Gamma_u}(\cdots)\ll\exp\Big(-c_f'\frac{\log q}{\log\log q}\Big)$$

for some positive constant c_f'. To bound $\int_{S_u}$ and $\int_{\overline{S_u}}$, we use the fact that $|t_u|$ is large, that is $|t_u|=V_0$, to apply Lemma 8.9. These remarks and easy computations lead to the following bound

$$\int_{u\in S_u\cup\Gamma_u\cup\overline{S_u}}(\cdots)\ll(\log q)^{O(1)}\exp\Big(-d_f\frac{\log q}{\log\log q}\Big), \tag{8.25}$$

where d_f is some positive constant. Furthermore, the inequality

$$\int_{\substack{(3\mathcal{L})\\ |t_v|\leqslant V}}\Big|\frac{H_L(v)}{v}\Big|\,|dv|\ll\log q \tag{8.26}$$

is a direct consequence of Lemma 8.8. It remains to combine (8.24), (8.25) and (8.26) to deduce the inequality

$$\frac{1}{(2\pi i)^3}\int_{\substack{(3\mathcal{L})\\ |t_w|\leqslant 2V_0}}\int_{\substack{(3\mathcal{L})\\ |t_v|\leqslant V_0}}\int_{u\in S_u\cup\Gamma_u\cup\overline{S_u}}\mathcal{T}(s_0,u,v,w)\frac{du}{u}\frac{dv}{v}\frac{dw}{w}$$
$$\ll\exp\Big(-\frac{d_f}{2}\cdot\frac{\log q}{\log\log q}\Big). \tag{8.27}$$

This error term is negligible when compared with the right–hand side of (8.16). By the residue formula, we are reduced to proving that the contribution of the residues of the poles which are inside the curved rectangle $\mathcal{R}_u$ are also in modulus less than $C_1|s_0|^{C_2}$.

8.3.5. Description of the residues. During the contour shift from γ_u to $S_u\cup\Gamma_u\cup\overline{S_u}$ we hit exactly two poles. They are both simple and located at $u=0$ (from $1/u$) and at $u=1/2-s_0$ (from the factor $T(2s_0+2u)$). Let us denote by I_0 and $I_{1/2-s_0}$ the contribution of these residues to $\mathcal{M}(V_0)$. More precisely we have the equalities

$$I_0:=\frac{T(2s_0)}{(2\pi i)^2}\int_{\substack{(3\mathcal{L})\\ |t_w|\leqslant 2V_0}}\int_{\substack{(3\mathcal{L})\\ |t_v|\leqslant V_0}}\frac{T(2\sigma_0+v+w)}{T(2s_0+v)T(2\sigma_0+w)}$$
$$\times E(s_0,0,v,w)H_L(v)H_L(w)\frac{dv}{v}\frac{dw}{w} \tag{8.28}$$

and

$$I_{1/2-s_0} := \frac{\kappa_f q^{1-2s_0} G(\frac{1}{2}-s_0)}{(2\pi i)^2(\frac{1}{2}-s_0)} \frac{L_\infty(\frac{1}{2})^2}{L_\infty(s_0)^2} \times \int_{\substack{(3\mathcal{L})\\ |t_w|\leqslant 2V_0}} \int_{\substack{(3\mathcal{L})\\ |t_v|\leqslant V_0}} \frac{T(2\sigma_0+v+w)}{T(s_0+\frac{1}{2}+v)T(\sigma_0-it_0+\frac{1}{2}+w)} \times E(s_0,\tfrac{1}{2}-s_0,v,w)H_L(v)H_L(w)\frac{dv}{v}\frac{dw}{w}.$$

From the above discussions, it remains to prove that, uniformly for (8.21), we have the inequalities

$$|I_0|,\ |I_{1/2-s_0}| \leqslant C_1|s_0|^{C_2}. \tag{8.29}$$

We will concentrate on I_0, since the other bound is similar.

8.3.6. Transformation of I_0. We return to the definitions (8.28) of I_0 and (8.10) of σ_0. We define four paths in the w–plane

$$\begin{aligned}
\gamma_w &= \{w\in\mathbf{C} \mid \sigma_w = 3\mathcal{L},\ |t_w|\leqslant 2V_0\},\\
\Gamma_w &= \{w\in\mathbf{C} \mid \sigma_w = 1-2\sigma_0 - \frac{c_f}{\log(V_0^3+|t_w|)}\ t_w = 2V_0\}\\
&= \{w\in\mathbf{C} \mid \sigma_w = \mp 2\mathcal{L} - \frac{c_f}{\log(V_0^3+|t_w|)},\ |t_w|\leqslant 2V_0\},\\
S_w &= \{w\in\mathbf{C} \mid \mp 2\mathcal{L} - \frac{c_f}{\log(V_0^3+2V_0)} \leqslant \sigma_w \leqslant 3\mathcal{L},\ t_w = 2V_0\}
\end{aligned}$$

and its conjugate $\overline{S_w}$, where c_f is the constant appearing in Lemma 8.11. These four paths define a curved rectangle $\mathcal{R}_w$. Inside $\mathcal{R}_w$, the function

$$w \mapsto \frac{T(2\sigma_0+v+w)}{T(2s_0+v)T(2\sigma_0+w)}E(s_0,0,v,w)$$

has only one pole. It is simple and is located at

$$w_v := 1-2\sigma_0 - v = \mp 2\mathcal{L} - v = (-3\mp 2)\mathcal{L} - it_v.$$

It corresponds to the pole at 1 of the numerator $T(2\sigma_0+v+w)$. Remark that the rectangle $\mathcal{R}_w$ is defined in order to contain no zero of the function $w\mapsto T(2\sigma_0+w)$. The function to integrate with respect to w in (8.28) has another pole at $w=0$

and it is simple. By the residue formula, we have the equality

$$
\begin{aligned}
I_0 \quad &= \frac{T(2s_0)}{(2\pi i)^2}\int_{\substack{\sigma_v=3\mathcal{L}\\|t_v|\leqslant V_0}}\int_{w\in S_w\cup\Gamma_w\cup\overline{S_w}}\frac{T(2\sigma_0+v+w)}{T(2s_0+v)T(2\sigma_0+w)}\\
&\qquad\qquad \times E(s_0,0,v,w)H_L(v)H_L(w)\frac{dw}{w}\frac{dv}{v}\\
&+\frac{T(2s_0)}{2\pi i}\int_{\substack{\sigma_v=3\mathcal{L}\\|t_v|\leqslant V_0}}\frac{L(\mathrm{Sym}^2 f,1)}{T(2s_0+v)T(1-v)}E(s_0,0,v,\mp 2\mathcal{L}-v)\\
&\qquad\qquad \times H_L(v)H_L(\mp 2\mathcal{L}-v)\frac{dv}{v(\mp 2\mathcal{L}-v)} \qquad (8.30)\\
&+\frac{T(2s_0)}{2\pi i}\int_{\substack{\sigma_v=3\mathcal{L}\\|t_v|\leqslant V_0}}\frac{T(2\sigma_0+v)}{T(2s_0+v)T(2\sigma_0)}E(s_0,0,v,0)H_L(v)H_L(0)\frac{dv}{v}\\
&:= \frac{T(2s_0)}{(2\pi i)^2}\Big(J_1+2\pi i J_2+2\pi i J_3\Big).
\end{aligned}
$$

Hence, in order to prove (8.29), it remains to prove the inequalities

$$\big|T(2s_0)J_i\big| \leqslant C_1|s_0|^{C_2}, \tag{8.31}$$

for $i=1$, 2 and 3, for some absolute C_1, C_2 and for any $s_0=\frac{1}{2}\pm\mathcal{L}+it_0$, with t_0 satisfying (8.21).

8.3.7. Dissection of J_1. We decompose J_1 into

$$J_1 = J_{1,1}+J_{1,2}+J_{1,3} \tag{8.32}$$

where $J_{1,1}$ corresponds to the contribution in the double integral defining J_1, of the w in Γ_w, and $J_{1,2}$ (resp. $J_{1,3}$) corresponds to the contribution of the w in S_w (resp. w in $\overline{S_w}$).

For $|t_v|\leqslant V_0$ and $|t_w|=2V_0$, we have $|2\sigma_0+v+w-1|\geqslant 1$. Appealing once again to Lemma 8.11 to bound each of the three T–factors, we deduce the inequality

$$J_{1,2}\ll(\log q)^{O(1)}\Big(\int_{\substack{\sigma_v=3\mathcal{L}\\|t_v|\leqslant V_0}}\Big|\frac{H_L(v)}{v}\Big|\,dv\Big)\Big(\int_{w\in S_w}\Big|\frac{H_L(w)}{w}\Big|dw\Big),$$

uniformly for t_0 satisfying (8.21). We now appeal to Lemma 8.8, which is quite efficient since $|t_w|=2V_0$ is large, to conclude by the inequality

$$J_{1,2}\ll(\log q)^{-10}, \tag{8.33}$$

by choosing C sufficient large in the definition (8.14) of V_0. The same bound holds true for $J_{1,3}$

8.3.8. Study of $J_{1,1}$. To bound $J_{1,1}$ we will benefit from the fact that σ_w is negative and not too small, that is

$$\sigma_w<0 \text{ and } -\sigma_w\gg 1/(\log\log q) \text{ for } w\in\Gamma_w. \tag{8.34}$$

Now remark that, for $w\in\Gamma_w$ and v with $\sigma_v=3\mathcal{L}$, $|t_v|\leqslant V_0$, we have the three lower bounds

$$
\begin{gathered}
|2\sigma_0+v+w-1|\geqslant|\pm 2\mathcal{L}+\sigma_v+\sigma_w|\gg 1/(\log\log q),\\
\mathfrak{Re}(2s_0+v-1)=\pm 2\mathcal{L}+\sigma_v\geqslant\mathcal{L}\geqslant -c_f/\log(2+|2t_0+t_v|),
\end{gathered}
$$

and

$$\mathfrak{Re}(2\sigma_0+w-1)=\pm 2\mathcal{L}-c_f/\log(V_0^3+|t_w|)\geqslant -c_f/\log(2+|t_w|),$$

for sufficiently large q. Appealing one more time to (8.18) and Lemma 2.24 to bound the E–function, we deduce

$$J_{1,1} \ll (\log q)^{O(1)} \Big(\int_{\substack{\sigma_v = 3\mathcal{L} \\ |t_v| \leqslant V_0}} \Big| \frac{H_L(v)}{v} \Big| \, dv \Big) \Big(\int_{w \in \Gamma_w} \Big| \frac{H_L(w)}{w} \Big| dw \Big),$$

and, finally by Lemma 8.8 and the inequality (8.34), we arrive at the inequality

$$J_{1,1} \ll (\log q)^{-10}. \tag{8.35}$$

Gathering (8.32), (8.33) and (8.35), we obtain the bound

$$J_1 \ll (\log q)^{-10}. \tag{8.36}$$

Finally, by the definition of s_0 and the assumption (8.21), we deduce from (8.19) the bound

$$T(2s_0) \ll \log q. \tag{8.37}$$

Combining (8.37) with (8.36) we complete the proof of (8.31) for $i = 1$.

8.3.9. A first bound for J_2 and J_3. Recall that these quantities are defined in (8.30). For v such that $\sigma_v = 3\mathcal{L}$ and $|t_v| \leqslant V_0$, we have the following lower bounds

$$\begin{aligned} \mathfrak{Re}(2s_0 + v - 1) &= \pm 2\mathcal{L} + 3\mathcal{L} \geqslant -c_f / \log(2 + |2t_0 + t_v|), \\ \mathfrak{Re}((1 - v) - 1) &= -\sigma_v = -3\mathcal{L} \geqslant -c_f / \log(2 + |t_v|), \\ \mathfrak{Re}(2\sigma_0 + v - 1) &= \pm 2\mathcal{L} + 3\mathcal{L} \geqslant -c_f / \log(2 + |t_v|). \end{aligned}$$

Furthermore, under the same conditions, we have

$$|v| \asymp | \pm 2\mathcal{L} - v| \asymp \mathcal{L} + |t_v|.$$

These remarks, when inserted in Lemma 8.11 (inequality (8.18)) and Lemma 8.8, give the following bound for J_2

$$\begin{aligned} J_2 \ll \int_{|t_v| \leqslant V_0} & \frac{\mathcal{L} + |2t_0 + t_v|}{1 + \mathcal{L} + |2t_0 + t_v|} \cdot \frac{\mathcal{L} + |t_v|}{1 + \mathcal{L} + |t_v|} \\ & \times \log^{A^*}\big(2 + |2t_0 + t_v|\big) \log^{A^*}\big(2 + |t_v|\big) \min\Big(1, \frac{\mathcal{L}}{\mathcal{L} + |t_v|}\Big)^2 \frac{dt_v}{(\mathcal{L} + |t_v|)^2}, \end{aligned}$$

which is simplified into

$$\begin{aligned} J_2 \ll \log^{A^*}(2 + |t_0|) \int_{|t_v| \leqslant V_0} & \frac{\mathcal{L} + |2t_0 + t_v|}{1 + |2t_0 + t_v|} \\ & \times \frac{\mathcal{L}^2}{1 + |t_v|} \cdot \log^{2A^*}\big(2 + |t_v|\big) \frac{dt_v}{(\mathcal{L} + |t_v|)^3}. \end{aligned} \tag{8.38}$$

Proceeding similarly for J_3, we have

$$\begin{aligned} J_3 \ll \int_{|t_v| \leqslant V_0} & \frac{1 + \mathcal{L} + |t_v|}{\mathcal{L} + |t_v|} \cdot \frac{\mathcal{L} + |2t_0 + t_v|}{1 + \mathcal{L} + |2t_0 + t_v|} \cdot \mathcal{L} \\ & \times \log^{A^*}\big(2 + |t_v|\big) \cdot \log^{A^*}\big(2 + |2t_0 + t_v|\big) \cdot \min\Big(1, \frac{\mathcal{L}}{\mathcal{L} + |t_v|}\Big) \frac{dt_v}{\mathcal{L} + |t_v|}, \end{aligned}$$

which simplifies into

$$J_3 \ll \log^{A^*}(2+|t_0|)\int_{|t_v|\leqslant V_0} \frac{\mathcal{L}+|2t_0+t_v|}{1+|2t_0+t_v|}\cdot \mathcal{L}^2 \times \log^{2A^*}\bigl(2+|t_v|\bigr)\,\frac{1+|t_v|}{(\mathcal{L}+|t_v|)^3}\,dt_v. \tag{8.39}$$

8.3.10. Bound for J_2 and J_3: the case $1\leqslant |t_0|\leqslant \log q$. In that case, the inequality (8.19) asserts the truth of the bound

$$T(2s_0)\ll |s_0|^{O(1)}.$$

Hence, in order to prove (8.31) under the above restriction on t_0, it is sufficient to prove the inequality

$$J_i \ll |s_0|^{O(1)} \text{ for } i=2,\ 3. \tag{8.40}$$

We write $\mathcal{L}_0=\log(2+|t_0|)$.

Estimate of J_2. From (8.38), we deduce that

$$\begin{aligned} J_2 &\ll \mathcal{L}_0^{A^*}\int_{|t_v|\leqslant V_0}\log^{2A^*}(2+|t_v|)\frac{\mathcal{L}^2}{(\mathcal{L}+|t_v|)^3}\,dt_v \\ &= \mathcal{L}_0^{A^*}\Bigl(\int_{|t_v|\leqslant \mathcal{L}}+\int_{\mathcal{L}<|t_v|\leqslant 1}+\int_{1<|t_v|\leqslant V_0}\Bigr)\log^{2A^*}(2+|t_v|)\frac{\mathcal{L}^2}{(\mathcal{L}+|t_v|)^3}\,dt_v \\ &\ll \mathcal{L}_0^{A^*}\bigl(1+1+\mathcal{L}^2\bigr)\ll |s_0|. \end{aligned}$$

This proves (8.40) for J_2.

Estimate of J_3. From (8.39), we deduce that

$$\begin{aligned} J_3 &\ll \mathcal{L}^2\ \mathcal{L}_0^{A^*}\Bigl(\int_{|t_v|\leqslant \mathcal{L}}+\int_{\mathcal{L}<|t_v|\leqslant 1}+\int_{1\leqslant|t_v|\leqslant V_0}\Bigr)\log^{2A^*}\bigl(2+|t_v|\bigr)\cdot\frac{1+|t_v|}{(\mathcal{L}+|t_v|)^3}dt_v \\ &\ll \mathcal{L}^2\ \mathcal{L}_0^{A^*}\ \bigl(\mathcal{L}^{-2}+\mathcal{L}^{-2}+1\bigr)\ll \mathcal{L}_0^{A^*}\ll |s_0|. \end{aligned}$$

This proves (8.40) for J_3.

8.3.11. Bound for J_2 and J_3: the case $|t_0|\leqslant 1$. In that case we have the inequality

$$T(2s_0)\ll \frac{1}{\mathcal{L}+|t_0|}$$

as a direct consequence of (8.19). Hence, in order to prove (8.31) under the above restriction on t_0, it is sufficient to prove the inequality

$$J_i\ll \mathcal{L}+|t_0| \text{ for } i=2,\ 3. \tag{8.41}$$

Estimate of J_2. We start from (8.38), which in that case simplifies into

$$J_2\ll \int_{|t_v|\leqslant V_0}\frac{\mathcal{L}+|2t_0+t_v|}{1+|2t_0+t_v|}\cdot\frac{\mathcal{L}^2}{1+|t_v|}\cdot\log^{2A^*}\bigl(2+|t_v|\bigr)\frac{dt_v}{(\mathcal{L}+|t_v|)^3}.$$

We split this integral in three ranges

$$|t_v|\leqslant\mathcal{L},\quad \mathcal{L}\leqslant|t_v|\leqslant 1,\quad \text{and}\quad 1\leqslant|t_v|\leqslant V_0.$$

We have

$$\int_{|t_v|\leqslant\mathcal{L}}(\cdots)\ll\frac{\mathcal{L}+|t_0|}{1+|t_0|}\int_{|t_v|\leqslant\mathcal{L}}\mathcal{L}^{-1}\,dt_v\ll\mathcal{L}+|t_0|.$$

For the second range, we have

$$\int_{\mathcal{L}\leqslant|t_v|\leqslant 1}(\cdots)\ll \mathcal{L}^2\int_{\mathcal{L}}^{1}\frac{\mathcal{L}+|t_0|+t_v}{t_v^3}dt_v$$
$$\ll \mathcal{L}^2\Big(\frac{\mathcal{L}+|t_0|}{\mathcal{L}^2}+\mathcal{L}^{-1}\Big)\ll \mathcal{L}+|t_0|$$

and for the last one we get

$$\int_{1\leqslant|t_v|\leqslant V_0}(\cdots)\ll \mathcal{L}^2\int_{|t_v|\geqslant 1}\frac{\mathcal{L}+|t_v+2t_0|}{1+|t_v+2t_0|}\log^{2A^*}(2+|t_v|)\frac{dt_v}{|t_v|^4}$$
$$\ll \mathcal{L}^2\ll \mathcal{L}+|t_0|.$$

Gathering the three inequalities above, we complete the proof of (8.41) in the case $i=2$.

Estimate of J_3. In the case of J_3, we first simplify (8.39) into

$$J_3\ll \mathcal{L}^2\int_{|t_v|\leqslant V_0}\frac{\mathcal{L}+|2t_0+t_v|}{1+|2t_0+t_v|}\cdot\log^{2A^*}\big(2+|t_v|\big)\ \frac{1+|t_v|}{(\mathcal{L}+|t_v|)^3}\,dt_v,$$

and we again split this integral in three parts, obtaining

$$\mathcal{L}^2\int_{|t_v|\leqslant\mathcal{L}}(\cdots)\ll \mathcal{L}^2\int_0^{\mathcal{L}}(\mathcal{L}+|t_0|)\,\mathcal{L}^{-3}dt_v\ll \mathcal{L}+|t_0|,$$
$$\mathcal{L}^2\int_{\mathcal{L}\leqslant|t_v|\leqslant 1}(\cdots)\ll \mathcal{L}^2\int_{\mathcal{L}}^{1}(\mathcal{L}+|t_0|+t_v)\frac{dt_v}{t_v^3}\ll \mathcal{L}+|t_0|,$$
$$\mathcal{L}^2\int_{1\leqslant|t_v|\leqslant V_0}(\cdots)\ll \mathcal{L}^2\int_1^{\infty}\log^{2A^*}(2+t_v)\,\frac{dt_v}{t_v^2}\ll \mathcal{L}^2\ll \mathcal{L}+|t_0|.$$

Gathering the three above inequalities, we complete the proof of (8.41) for $i=3$. The proof of (8.41) is now complete. Hence the proof of (8.13) is now complete, and so is the proof of Proposition 8.6.

Remark 8.12. The informed reader will have noticed that the proof presented here is slightly different from similar second moment estimates found in other works (for instance, [**49**, Propositions 4 and 5]). These other arguments (following earlier ideas of Selberg) made key use of the positivity of certain complicated terms to avoid having to evaluate them too precisely (see [**49**, (67), (68)]). In the present work – precisely the proof of Proposition 8.6 – positivity is not so evident so we have to estimate the corresponding sums directly.

CHAPTER 9

A conjecture of Mazur-Rubin concerning modular symbols

9.1. Introduction

In this chapter, we assume that f is a holomorphic primitive cusp form of weight 2 and level r. We recall that for $q \geqslant 1$ and $(a,q)=1$, the modular symbol $\langle a/q\rangle_f$ is defined by

$$\Big\langle \frac{a}{q}\Big\rangle_f = 2\pi i\int_{i\infty}^{a/q} f(z)dz = 2\pi\int_0^{\infty} f\Big(\frac{a}{q}+iy\Big)\,dy$$

and that it only depends on the congruence class $a\,(\mathrm{mod}\,q)$.

In this chapter we investigate some correlation properties of the family

$$\Big\langle \frac{a}{q}\Big\rangle_f, \quad a\in(\mathbf{Z}/q\mathbf{Z})^{\times}$$

when q is a prime number. In particular, we will prove Theorem 1.15 concerning the variance of modular symbols.

Our main ingredient is the Birch-Stevens formula that relates the modular symbols to the central values of the twisted L-functions.

Lemma 9.1. *For any primitive Dirichlet $\chi\,(\mathrm{mod}\,q)$, we have*

$$L\Big(f\otimes\chi,\frac{1}{2}\Big) = \frac{\varepsilon_\chi}{q^{1/2}}\sideset{}{^*}\sum_{a\,(\mathrm{mod}\ q)}\chi(-\overline{a})\Big\langle \frac{a}{q}\Big\rangle_f.$$

Proof. Observe that since f has real Fourier coefficients, we have

$$\overline{f(x+iy)} = f(-x+iy),\ x,y\in\mathbf{R},\ y>0$$

so that

$$\overline{\Big\langle \frac{a}{q}\Big\rangle_f} = 2\pi\int_0^{\infty} f\Big(-\frac{a}{q}+iy\Big)\,dy = \Big\langle \frac{-a}{q}\Big\rangle_f. \tag{9.1}$$

Now denote

$$\Big\langle \frac{a}{q}\Big\rangle_f^{\pm} = \frac{1}{2}\Big(\Big\langle \frac{a}{q}\Big\rangle_f \pm \Big\langle \frac{-a}{q}\Big\rangle_f\Big) \tag{9.2}$$

the even and odd parts of the modular symbols. The Birch-Stevens formula (see [63, (2.2)] or [58, (8.6)]) states that

$$L\Big(f\otimes\chi,\frac{1}{2}\Big) = \frac{1}{\varepsilon_{\overline{\chi}}\,q^{1/2}}\sideset{}{^*}\sum_{a\,(\mathrm{mod}\ q)}\overline{\chi}(a)\Big\langle \frac{a}{q}\Big\rangle_f^{\pm} = \frac{\chi(-1)\,\varepsilon_\chi}{q^{1/2}}\sideset{}{^*}\sum_{a\,(\mathrm{mod}\ q)}\chi(\overline{a})\Big\langle \frac{a}{q}\Big\rangle_f^{\pm}$$

where $a\overline{a}\equiv 1\,(\mathrm{mod}\,q)$, ε_χ is the normalized Gauß sum of χ (cf. (1.9)), and the "exponent" $\pm$ is $\chi(-1)$.

Inserting (9.2) we obtain

$$L\big(f \otimes \chi, \frac{1}{2}\big) = \frac{\varepsilon_\chi}{q^{1/2}} \sideset{}{^*}\sum_{a \,(\mathrm{mod}\ q)} \chi(-\overline{a}) \Big\langle \frac{a}{q} \Big\rangle_f,$$

as claimed. □

By performing discrete Mellin inversion, we will be able to use our results on moments of twisted central values to evaluate asymptotically the first and second moments of the modular symbols, and in fact also correlations between modular symbols for two cusp forms.

We define

$$M_f(q) = \frac{1}{\varphi(q)} \sideset{}{^*}\sum_{a \,(\mathrm{mod}\ q)} \Big\langle \frac{a}{q} \Big\rangle_f.$$

Moreover, we define

$$f_q(z) = f(qz) = \sum_{n \geqslant 1} \lambda_f(n) n^{1/2} e(nqz).$$

If g is a holomorphic primitive cusp form of weight 2 and level r' coprime to q, and u, v are integers coprime to q, we define

$$C_{f,g}(u, v; q) = \frac{1}{\varphi(q)} \sideset{}{^*}\sum_{a \,(\mathrm{mod}\ q)} \Big(\Big\langle \frac{au}{q} \Big\rangle_f - M_f(q)\Big) \overline{\Big(\Big\langle \frac{av}{q} \Big\rangle_g - M_g(q)\Big)}$$

(where here and below, the sum is over invertible residue classes modulo q). In particular, note that the variance in Theorem 1.15 is $V_f(q) = C_{f,f}(1, 1; q)$, so the second part of the next result implies that theorem:

THEOREM 9.2. *Suppose that q is prime. Write the levels r and r' of f and g as $r = \varrho\delta$ and $r' = \varrho'\delta$ where $\delta = (r, r')$ and $(\varrho, \varrho') = 1$.*

(1) *We have*

$$M_f(q) = \Big(\frac{q^{1/2}}{q-1} \cdot \frac{L_q(f_q, 1/2)}{L_q(f, 1/2)} - \frac{1}{q-1}\Big) L(f, 1/2) = O(q^{-1/2}).$$

(2) *We have*

$$C_{f,g}(u, v; q) = \frac{q}{\varphi(q)^2} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} L(f \otimes \chi, 1/2) \overline{L(g \otimes \chi, 1/2)} \chi(u) \overline{\chi(v)}.$$

(3) *In particular, if $r = r'$ and $\varepsilon(f)\varepsilon(g) = -1$, then $C_{f,g}(1, 1; q) = 0$. Otherwise*

$$C_{f,g}(1, 1; q) = \gamma_{f,g} \frac{L^*(f \otimes g, 1)}{\zeta(2)} + O(q^{-1/145}) \text{ if } f \neq g,$$

$$C_{f,f}(1, 1; q) = 2 \prod_{p \mid r} (1 + p^{-1})^{-1} \frac{L^*(\mathrm{Sym}^2 f, 1)}{\zeta(2)} \log q + \beta_f + O(q^{-1/145}),$$

where β_f is a constant, and

$$\gamma_{f,g} = 1 + \varepsilon(f)\varepsilon(g) \frac{\lambda_f(\varrho)\lambda_g(\varrho')}{\sqrt{\varrho\varrho'}}$$

is a non-zero constant.

Part (3) with $f = g$ and $u = v = 1$ confirms a conjecture of Mazur and Rubin, as stated by Petridis and Risager [62, Conj. 1.1], in the case of prime moduli q. Note that their statement of the conjecture involves a quantity which they denote $L(\mathrm{Sym}^2 f, 1)$ and which should be interpreted as our $L^*(\mathrm{Sym}^2 f, 1)$ (although they do not state this formally, it is clear from their proof of [62, Th. 1.6] in Section 8 of loc. cit., especially Section 8.2.1). In fact, Theorem 9.2 and the computations in Section 2.3.3 show that the conjecture would not hold in general if $L(\mathrm{Sym}^2 f, 1)$ is interpreted as the special value of the automorphic (or "motivic") symmetric square.

REMARK 9.3. If $f \neq g$ and $u = v = 1$, and either $r \neq r'$ or $\varepsilon(f)\varepsilon(g) = 1$, then we find by (3) a little bit of correlation between modular symbols as $q \to +\infty$: the modular symbols related to f and g do not become asymptotically independent.

On the other hand, if $r = r'$, $\varepsilon(f)\varepsilon(g) = -1$ and $u = v = 1$, the correlation vanishes exactly, but this fact will not persist in general if u or v is not 1 modulo q, as shown by our evaluation of the twisted second moment (Theorem 5.1). For instance, using Lemma 7.9, we get a formula with non-zero leading term for suitable choices of u and v, even if $r = r'$ and $\varepsilon(f)\varepsilon(g) = -1$.

9.2. Proof of the theorem

We observe first that $M_f(q) \in \mathbf{R}$ because of the relation (9.1). We compute $M_f(q)$ exactly by analytic continuation from a region of absolute convergence, using additive twists of modular forms.

Let a be coprime to q. We have

$$yf\Big(z + \frac{a}{q}\Big) = y \sum_{n \geqslant 1} \lambda_f(n) n^{1/2} e\Big(n\Big(x + \frac{a}{q}\Big)\Big) \exp(-2\pi n y).$$

For any complex number s, we define

$$\Big\langle \frac{a}{q} \Big\rangle_{s,f} = 2\pi \int_0^\infty yf\Big(\frac{a}{q} + iy\Big) y^s \frac{dy}{y}.$$

As a function of s, this expression is holomorphic in the whole complex plane. On the other hand, for $\mathfrak{Re}(s) > 1$, we have

$$\begin{aligned}
\Big\langle \frac{a}{q} \Big\rangle_{s,f} &= (2\pi)^{1/2} \sum_{n \geqslant 1} \lambda_f(n) e\Big(n\frac{a}{q}\Big) \int_0^\infty (2\pi n y)^{1/2} y^{1/2+s} \exp(-2\pi n y) \frac{dy}{y} \\
&= (2\pi)^{-s} \sum_{n \geqslant 1} \frac{\lambda_f(n) e(n\frac{a}{q})}{n^{1/2+s}} \int_0^\infty y^{1+s} e^{-y} \frac{dy}{y} \\
&= (2\pi)^{-s} \Gamma(1+s) L(f, a, \tfrac{1}{2} + s)
\end{aligned}$$

where

$$L(f, a, s) = \sum_{n \geqslant 1} \frac{\lambda_f(n) e(n\frac{a}{q})}{n^s}$$

when the series converges absolutely.

Expressing the additive character in terms of multiplicative characters, it follows that the series $L(f, a, s)$ has analytic continuation to $\mathbf{C}$. Hence the identity above holds for all $s \in \mathbf{C}$. In particular, we obtain

$$\Big\langle \frac{a}{q} \Big\rangle_f = \Big\langle \frac{a}{q} \Big\rangle_{0,f} = L(f, a, 1/2).$$

Since q is prime, we deduce by direct computation that

$$\begin{aligned}\sideset{}{^*}\sum_{a\,(\mathrm{mod}\ q)} \Big\langle \frac{a}{q}\Big\rangle_{s,f} &= (2\pi)^{-s}\Gamma(1+s)\bigg(q\sum_{n\geqslant 1}\frac{\lambda_f(qn)}{(qn)^{s+1/2}} - L(f,s+1/2)\bigg)\\ &= (2\pi)^{-s}\Gamma(1+s)\Big(q^{1/2}\frac{L_q(f_q,s+1/2)}{L_q(f,s+1/2)} - 1\Big)L(f,s+1/2)\end{aligned}$$

in $\mathfrak{Re}\, s > 1/2$, where

$$\begin{aligned}L_q(f_q,1/2) = \sum_{\alpha\geqslant 0}\frac{\lambda_f(q^{\alpha+1})}{q^{\alpha/2}} &= \lambda_f(q) + \frac{L_q(f,1/2)}{q^{1/2}}\Big(\lambda_f(q^2) - \frac{\lambda_f(q)}{q^{1/2}}\Big)\\ &= \lambda_f(q) + O(q^{-1/2}).\end{aligned}$$

Hence

$$M_f(q) = \Big(\frac{q^{1/2}}{q-1}\cdot\frac{L_q(f_q,1/2)}{L_q(f,1/2)} - \frac{1}{q-1}\Big)L(f,1/2) = O(q^{-1/2}).$$

This proves the first part of Theorem 9.2.

Next, let u and v be integers coprime to q. From Lemma 9.1, we derive

$$\begin{aligned}&\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)} L(f\otimes\chi,1/2)\overline{L(g\otimes\chi,1/2)}\chi(u)\overline{\chi(v)}\\ &\qquad = \frac{1}{q}\sideset{}{^*}\sum_{\chi}\chi(u)\overline{\chi(v)}\sideset{}{^*}\sum_{a,a'\,(\mathrm{mod}\ q)}\chi(\bar a)\overline{\chi(\bar a')}\Big\langle\frac{a}{q}\Big\rangle_f\overline{\Big\langle\frac{a'}{q}\Big\rangle_g}\\ &= \frac{1}{q}\sum_{\chi}\chi(u)\overline{\chi(v)}\sideset{}{^*}\sum_{a,a'\,(\mathrm{mod}\ q)}\chi(\bar a)\overline{\chi(\bar a')}\Big\langle\frac{a}{q}\Big\rangle_f\overline{\Big\langle\frac{a'}{q}\Big\rangle_g} - \frac{\varphi(q)^2}{q}M_f(q)\overline{M_g(q)}\\ &\qquad\qquad = \frac{\varphi(q)}{q}\sideset{}{^*}\sum_{\substack{a,a'\,(\mathrm{mod}\ q)\\ u\bar a\equiv v\bar a'\,(\mathrm{mod}\ q)}}\Big\langle\frac{a}{q}\Big\rangle_f\overline{\Big\langle\frac{a'}{q}\Big\rangle_g} - \frac{\varphi(q)^2}{q}M_f(q)\overline{M_g(q)},\end{aligned}$$

hence, putting $b = a\bar u = a'\bar v$, we get

$$\begin{aligned}C_{f,g}(u,v;q) &= \frac{1}{\varphi(q)}\sideset{}{^*}\sum_{b\,(\mathrm{mod}\ q)}\Big(\Big\langle\frac{ub}{q}\Big\rangle_f - M_f(q)\Big)\overline{\Big(\Big\langle\frac{vb}{q}\Big\rangle_g - M_g(q)\Big)}\\ &= \frac{q}{\varphi(q)^2}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}L(f\otimes\chi,1/2)\overline{L(g\otimes\chi,1/2)}\chi(u)\overline{\chi(v)},\end{aligned}$$

which is the formula in Part (2) of the Theorem 9.2.

If $r = r'$, $u = v = 1$ and $\varepsilon(f)\varepsilon(g) = -1$, then the second moment vanishes exactly (see the last part of Theorem 5.1), which proves the first part of Part (3), and otherwise, we obtain the last statement from Theorem 1.18 and Proposition 5.2.

9.3. Modular symbols and trace functions

As we have seen, the modular symbol $\langle a/q\rangle_f$, as a function of a, depends only on the congruence class $a\,(\mathrm{mod}\,q)$ and therefore defines a function on $\mathbf{Z}/q\mathbf{Z}$, where we put $\langle 0/q\rangle_f = 0$.

In the previous sections, we discussed how this function correlates either with the constant function 1, or with itself, or with the modular symbol attached to another modular form. In this section, we will see that we can also evaluate easily the correlations of modular symbols and trace functions $t\colon \mathbf{F}_q \to \mathbf{C}$, as described in Section 3.4.

We consider here the correlation sums

$$C_f(t) = \frac{1}{\varphi(q)} \sideset{}{^*}\sum_{a \,(\mathrm{mod}\ q)} \Big\langle \frac{a}{q} \Big\rangle_f \overline{t(a)}.$$

We will prove that these are small, except in very special cases. This means that trace functions do not correlate with modular symbols.

Proposition 9.4. *Let t be the trace function of a geometrically irreducible ℓ-adic sheaf $\mathcal{F}$. We assume that $\mathcal{F}$ is not geometrically isomorphic to an Artin-Schreier sheaf or to the pull-back of such a sheaf by the map $x \mapsto x^{-1}$. Then we have*

$$C_f(t) \ll q^{-1/8+\varepsilon},$$

for any $\varepsilon > 0$, where the implied constant depends only on ε, f and (polynomially) on the conductor of $\mathcal{F}$.

Remark 9.5. The assumption on the sheaf holds for all the examples in Example 3.9, except for $t(x) = e(f(x)/q)$ if the polynomial f has degree $\leqslant 1$.

Proof. By Lemma 9.1 and Theorem 9.2(1), we have

$$\begin{aligned} C_f(t) &= \frac{1}{\varphi(q)} \sideset{}{^*}\sum_{a \,(\mathrm{mod}\ q)} \Big\langle \frac{a}{q} \Big\rangle_f \overline{t(a)} = \frac{1}{\varphi(q)^2} \sum_{\chi \,(\mathrm{mod}\, q)} \sideset{}{^*}\sum_{a,a' \,(\mathrm{mod}\ q)} \Big\langle \frac{a}{q} \Big\rangle_f \chi(\overline{a})\chi(a')\overline{t(a')} \\ &= \frac{q^{1/2}}{\varphi(q)^{3/2}} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} L(f\otimes\chi, 1/2)\chi(-1)\overline{\varepsilon_\chi \widetilde{t}(\overline{\chi})} + O\left(\frac{1}{q^{3/2}}\right) \\ &= \frac{1}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} L(f\otimes\chi, 1/2)\chi(-1)\overline{\varepsilon_\chi \widetilde{t}(\overline{\chi})} + O\left(\frac{1}{q}\right). \end{aligned}$$

We compute that

$$\begin{aligned} \varepsilon_\chi \widetilde{t}(\overline{\chi}) &= \frac{1}{\sqrt{q}} \sum_x \chi(x) e\Big(\frac{x}{p}\Big) \sum_y \overline{\chi(y)}\, t(y) \\ &= \frac{1}{\sqrt{q}} \sum_a \overline{\chi(a)} \sum_{y/x=a} e\Big(\frac{x}{p}\Big) t(y), \end{aligned}$$

hence

$$\overline{\varepsilon_\chi \widetilde{t}(\overline{\chi})} = \frac{1}{\sqrt{q}} \sum_{a\in\mathbf{F}_q^\times} \chi(a)\tau(a)$$

where

$$\tau(a) = \frac{1}{\sqrt{q}} \sum_{xy=a} e\Big(-\frac{\bar{x}}{p}\Big)\overline{t(y)}$$

is the convolution of $\bar{t}$ and $x \mapsto e(-x^{-1}/p)$. In other words, $\chi \mapsto \overline{\varepsilon_\chi \widetilde{t}(\overline{\chi})}$ is the discrete Mellin transform of this convolution.

We distinguish two cases. If $\mathcal{F}$ is not geometrically isomorphic to a Kummer sheaf, then our assumptions on $\mathcal{F}$ imply that this convolution is the trace function of a Mellin sheaf $\mathcal{G}$ with conductor bounded polynomially in terms of $\mathbf{c}(\mathcal{F})$, and that $\mathcal{G}$ is not geometrically isomorphic to $[x \mapsto a/x]^*\mathcal{K}\ell_2$ for any $a \in \mathbf{F}_q^\times$ (see Lemma 3.13). By Theorem 4.4, we have therefore

$$\frac{1}{\varphi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} L(f \otimes \chi, 1/2)\chi(-1)\overline{\varepsilon_\chi \widetilde{t}(\bar{\chi})} \ll q^{-1/8+\varepsilon}$$

for any $\varepsilon > 0$, and hence $C_f(t) \ll q^{-1/8+\varepsilon}$ for any $\varepsilon > 0$.

In the case of a Kummer sheaf, we have $t(x) = \alpha\chi_0(x)$ for some $\alpha \in \mathbf{C}$ and some non-trivial multiplicative character χ_0 modulo q and

$$\widetilde{\chi}_0(\bar{\chi}) = \begin{cases} \alpha\varphi(q)^{1/2} & \text{if } \chi = \chi_0 \\ 0 & \text{otherwise,} \end{cases}$$

so that we get

$$C_f(\chi_0) = \frac{1}{\varphi(q)^{1/2}}\chi_0(-1)\varepsilon_{\chi_0} L(f \otimes \chi_0, \tfrac{1}{2}) + O(q^{-1}) \ll q^{-1/8+\varepsilon}$$

by the subconvexity estimate of Blomer and Harcos [5, Th. 2]. $\square$

Remark 9.6. For completeness, we consider the correlations in the exceptional cases excluded in the previous proposition. We assume that there exists $l \in \mathbf{F}_q^\times$ such that either

$$t(x) = e\Big(-\frac{\bar{l}x}{q}\Big), \qquad x \in \mathbf{F}_q \tag{9.3}$$

or

$$t(x) = e\Big(\frac{l\bar{x}}{q}\Big), \qquad x \in \mathbf{F}_q^\times, \qquad t(0) = 0. \tag{9.4}$$

(note that if $l = 0$, the correlation sum is just the mean-value $M_f(q)$ that we already investigated).

We follow the steps of the proof of Proposition 9.4 for these specific functions. In both cases, the Mellin transform $\widetilde{t}$ is a multiple of a Gauß sum. More precisely, we obtain

$$\overline{\varepsilon_\chi \widetilde{t}(\bar{\chi})} = \Big(\frac{q}{\varphi(q)}\Big)^{1/2}\chi(l), \qquad \overline{\varepsilon_\chi \widetilde{t}(\bar{\chi})} = \Big(\frac{q}{\varphi(q)}\Big)^{1/2}\chi(l)\varepsilon_\chi^{-2}$$

in the case of (9.3) and of (9.4), respectively. Using the notation of Chapter 4, we therefore have

$$C_f(t) = \Big(\frac{q}{\varphi(q)}\Big)^{1/2}\mathcal{L}(f; l, 0), \qquad C_f(t) = \Big(\frac{q}{\varphi(q)}\Big)^{1/2}\mathcal{L}(f; l, -2),$$

respectively. By Corollary 4.2, we conclude that

$$C_f(t) = \frac{\lambda_f(\bar{l}_q)}{\bar{l}_q^{1/2}} + O_{f,\varepsilon}(q^{-1/8+\varepsilon}), \qquad C_f(t) = \varepsilon(f)\frac{\lambda_f((lr)_q)}{(lr)_q^{1/2}} + O_{f,\varepsilon}(q^{-1/8+\varepsilon}),$$

for any $\varepsilon > 0$, respectively.

Notation index

We list some of the notation used in this book that may not be standard. Further notation and conventions are explained in Section 2.1. Some notation that is local to a single chapter are omitted.

Bibliography

[1] C Aistleitner and Ł. Pańkowski, *Large values of L-functions from the Selberg class*, J. Math. Anal. Appl. **446** (2017), 345-364.

[2] B. Bagchi, *Statistical behaviour and universality properties of the Riemann zeta function and other allied Dirichlet series*, Indian Statistical Institute, Kolkata, 1981, http://library.isical.ac.in:8080/jspui/bitstream/10263/4256/1/TH47.CV01.pdf. PhD thesis.

[3] V. Blomer, *Shifted convolution sums and subconvexity bounds for automorphic L-functions*, Int. Math. Res. Not. (2004), 3905-3926.

[4] V. Blomer, É. Fouvry, E. Kowalski, Ph. Michel, and D. Milićević, *On moments of twisted L–functions*, Amer. J. Math **139** (2017), no. 3, 707–768.

[5] V. Blomer and G. Harcos, *Hybrid bounds for twisted L-functions*, J. reine angew. Math. **621** (2008), 53–79.

[6] V. Blomer and D. Milićević, *The second moment of twisted modular L-functions*, Geom. Funct. Anal. **25** (2015), no. 2, 453–516.

[7] D.A. Burgess, *On character sums and L-series, I*, Proc. London Math. Soc. (3) **12** (1962), 193–206.

[8] C.J. Bushnell and G. Henniart, *An upper bound on conductors for pairs*, J. Number Theory **65** (1997), 183–196.

[9] V. A. Bykovskiĭ, *A trace formula for the scalar product of Hecke series and its applications*, J. Math. Sci. (New York) **89** (1998), 915-932.

[10] G. Chinta, *Analytic ranks of elliptic curves over cyclotomic fields*, J. reine angew. Math. **544** (2002), 13–24.

[11] J. Coates and C.G. Schmidt, *Iwasawa theory for the symmetric square of an elliptic curve*, J. reine angew. Math. **375–376** (1987), 104–156.

[12] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, *Integral moments of L-functions*, Proc. London Math. Soc. (3) **91** (2005), no. 1, 33–104.

[13] J. B. Conrey and K. Soundararajan, *Real zeros of quadratic Dirichlet L-functions*, Invent. math. **150** (2002), no. 1, 1–44.

[14] P. Deligne, *La conjecture de Weil. I*, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273–307.

[15] ———, *La conjecture de Weil, II*, Inst. Hautes Études Sci. Publ. Math. **52** (1980), 137–252.

[16] W. Duke, J.B. Friedlander, and H. Iwaniec, *Bounds for automorphic L-functions*, Invent. math. **112** (1993), 1-8.

[17] É. Fouvry, *Sur le problème des diviseurs de Titchmarsh*, J. reine angew. Math. **357** (1985), 51–76.

[18] É. Fouvry, E. Kowalski, and Ph. Michel, *Algebraic trace functions over the primes*, Duke Math. J. **163** (2014), no. 9, 1683–1736.

[19] É. Fouvry, E. Kowalski, and Ph. Michel, *Trace functions over finite fields and their applications*, Colloquium De Giorgi 2013 and 2014, Colloquia, vol. 5, Ed. Norm., Pisa, 2014, pp. 7–35.

[20] É. Fouvry, E. Kowalski, and Ph. Michel, *Algebraic twists of modular forms and Hecke orbits*, Geom. Funct. Anal. **25** (2015), no. 2, 580–657.

[21] ———, *On the exponent of distribution of the ternary divisor function*, Mathematika **61** (2015), no. 1, 121–144.

[22] J. Fresán, *Équirépartition de sommes exponentielles (travaux de Katz)*, Astérisque (to appear). Bourbaki Seminar, Exposé 1141, http://www.bourbaki.ens.fr/TEXTES/1141.pdf.

[23] P. Gao, R. Khan, and G. Ricotta, *The second moment of Dirichlet twists of Hecke L-functions*, Acta Arith. **140** (2009), no. 1, 57–65.

[24] S. Gelbart, *Automorphic forms on adele groups*, Annals of Mathematics Studies, vol. 83, Princeton University Press, Princeton, NJ, 1975.

[25] S. Gelbart and H. Jacquet, *A relation between automorphic representations of $GL(2)$ and $GL(3)$*, Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 4, 471–542.

[26] D. Goldston, J. Pintz, and C. Y. Yıldırım, *Primes in tuples. I*, Ann. of Math. (2) **170** (2009), no. 2, 819–862.

[27] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Seventh ed., Elsevier/Academic Press, Amsterdam, 2007.

[28] G. Harcos and Ph. Michel, *The subconvexity problem for Rankin-Selberg L-functions and equidistribution of Heegner points. II*, Invent. math. **163** (2006), no. 3, 581–655.

[29] A.J. Harper, *Sharp conditional bounds for moments of the Riemann zeta function*, Preprint (2013). arXiv:1305.4618.

[30] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[31] D.R. Heath-Brown and Ph. Michel, *Exponential decay in the frequency of analytic ranks of automorphic L-functions*, Duke Math. J. **102** (2000), no. 3, 475–484.

[32] G. Henniart, *Correspondance de Langlands et fonctions L des carrés extérieur et symétrique*, Int. Math. Res. Not. IMRN **4** (2010), 633–673 (French).

[33] J. Hoffstein and M. Lee, *Second moments and simultaneous non-vanishing of* GL(2) *automorphic L-series*, Preprint (2013). arXiv:1308.5980.

[34] R. Hough, *The angle of large values of L-functions*, J. Number Theory **167** (2016), 353–393.

[35] H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, RI, 1997.

[36] H. Iwaniec and E. Kowalski, *Analytic number theory*, Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004.

[37] H. Iwaniec and P. Sarnak, *Dirichlet L-functions at the central point*, Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997), de Gruyter, Berlin, 1999, pp. 941–952.

[38] ———, *Perspectives on the analytic theory of L-functions*, Geom. Funct. Anal. **Special Volume** (2000), 705–741. GAFA 2000 (Tel Aviv, 1999).

[39] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), no. 2, 367–464.

[40] N. M. Katz, *Convolution and equidistribution: Sato-Tate theorems for finite-field Mellin transforms*, Annals of Mathematics Studies, vol. 180, Princeton University Press, Princeton, NJ, 2012.

[41] N.M. Katz and P. Sarnak, *Zeroes of zeta functions and symmetry*, Bull. Amer. Math. Soc. (N.S.) **36** (1999), no. 1, 1–26.

[42] J. P. Keating and N. C. Snaith, *Random matrix theory and $\zeta(1/2+it)$*, Comm. Math. Phys. **214** (2000), no. 1, 57–89.

[43] H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2. *With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak*, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183.

[44] H. Kim and F. Shahidi, *Functorial products for* $\mathrm{GL}_2\times\mathrm{GL}_3$ *and the symmetric cube for* GL_2, Ann. of Math. (2) **155** (2002), no. 3, 837–893. With an appendix by Colin J. Bushnell and Guy Henniart.

[45] ———, *Cuspidality of symmetric powers with applications*, Duke Math. J. **112** (2002), no. 1, 177–197.

[46] M. Kim and H.-S. Sun, *Modular symbols and modular L-values with cyclotomic twists* (2017). preprint.

[47] E. Kowalski, *Families of cusp forms*, Actes de la Conférence "Théorie des Nombres et Applications", Publ. Math. Besançon Algèbre Théorie Nr., vol. 2013, Presses Univ. Franche-Comté, Besançon, 2013, pp. 5–40.

[48] ———, *Bagchi's theorem for families of automorphic forms*, Exploring the Riemann zeta function, Springer, Cham, 2017, pp. 181–199.

[49] E. Kowalski and Ph. Michel, *The analytic rank of $J_0(q)$ and zeros of automorphic L-functions*, Duke Math. J. **100** (1999), no. 3, 503–542.

[50] E. Kowalski, Ph. Michel, and J. VanderKam, *Non-vanishing of high derivatives of automorphic L-functions at the center of the critical strip*, J. reine angew. Math. **526** (2000), 1–34.

[51] ———, *Rankin-Selberg L-functions in the level aspect*, Duke Math. J. **114** (2002), no. 1, 123–191.

[52] E. Kowalski, Ph. Michel, and W. Sawin, *Bilinear forms with Kloosterman sums and applications*, Ann. of Math. **186** (2017), no. 2, 413–500.

[53] E. Kowalski and A. Nikeghbali, *Mod-Gaussian convergence and the value distribution of $\zeta(\frac{1}{2}+it)$ and related quantities*, J. Lond. Math. Soc. (2) **86** (2012), no. 1, 291–319.

[54] Wen Ch'ing Winnie Li, *L-series of Rankin type and their functional equation*, Math. Ann. **244** (1979), 135-166.

[55] J. Liu and Y. Ye, *Perron's formula and the prime number theorem for automorphic L-functions*, Pure Appl. Math. Q. **3** (2007), no. 2, 481–497.

[56] J. Matz, *Distribution of Hecke eigenvalues for* $\mathrm{GL}(n)$, Families of automorphic forms and the trace formula, Simons Symp., Springer, [Cham], 2016, pp. 327–350.

[57] B. Mazur and K. Rubin, *Arithmetic conjectures suggested by the statistical behavior of modular symbols* (2019), https://arxiv.org/abs/1910.12798.

[58] B. Mazur, J. Tate, and J. Teitelbaum, *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. math. **84** (1986), no. 1, 1–48.

[59] Jean-François Mestre, *Formules explicites et minorations de conducteurs de variétés algébriques*, Compositio Math. **58** (1986), no. 2, 209–232 (French).

[60] T. Miyake, *Modular forms*, Springer monographs in mathematics, Springer-Verlag, Berlin, 1989.

[61] C. Moreno, *Analytic proof of the strong multiplicity one theorem*, Amer. J. Math. (1985), 163–206.

[62] Y. Petridis and M. Risager, *Arithmetic statistics of modular symbols*, Invent. math. **212** (2018), no. 3, 1–57.

[63] R. Pollack, *Overconvergent modular symbols*, Computations with modular forms, Contrib. Math. Comput. Sci., vol. 6, Springer, Cham, 2014, pp. 69–105.

[64] M. Radziwiłł and K. Soundararajan, *Moments and distribution of central L-values of quadratic twists of elliptic curves*, Invent. math. **202** (2015), no. 3, 1029–1068.

[65] ———, *Selberg's Central Limit Theorem for* $\log|\zeta(1/2+it)|$, L'enseignement math. (to appear).

[66] ———, *Value distribution of L-functions*, Oberwolfach report **40/2017**.

[67] D. Ramakrishnan, *Modularity of the Rankin-Selberg L-series, and multiplicity one for* $\mathrm{SL}(2)$, Ann. of Math. (2) **152** (2000), no. 1, 45–111.

[68] ———, *Recovering cusp forms on* $\mathrm{GL}(2)$ *from symmetric cubes*, SCHOLAR—a scientific celebration highlighting open lines of arithmetic research, 2015, pp. 181–189.

[69] G. Ricotta, *Real zeros and size of Rankin-Selberg L-functions in the level aspect*, Duke Math. J. **131** (2006), no. 2, 291–350.

[70] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Math. J. **81** (1996), no. 2, 269–322.

[71] Z. Rudnick and K. Soundararajan, *Lower bounds for moments of L-functions*, Proc. Natl. Acad. Sci. USA **102** (2005), no. 19, 6837–6838.

[72] P. Sarnak, *Nonvanishing of L-functions on* $\Re(s)=1$, Contributions to automorphic forms, geometry, and number theory, Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 719–732.

[73] P. Sarnak, S.W. Shin, and N. Templier, *Families of L-functions and their Symmetry*, Families of Automorphic Forms and the Trace Formula, Simons Symposia, vol. 12, Springer, 2016, pp. 531-578.

[74] W. Sawin, *Bounds for numerical invariants of sheaves arising from multiplicative convolution* (2018). preprint.

[75] A. Selberg, *On the zeros of Riemann's zeta-function*, Skr. Norske Vid. Akad. Oslo I. **1942** (1942), no. 10, 59.

[76] ———, *Contributions to the theory of the Riemann zeta-function*, Arch. Math. Naturvid. **48** (1946), no. 5, 89–155.

[77] S.W. Shin and N. Templier, *Sato-Tate theorem for families and low-lying zeros of automorphic L-functions*, Invent. math. **203** (2016), no. 1, 1–177. Appendix A by R. Kottwitz, and Appendix B by R. Cluckers, J. Gordon and I. Halupczok.

[78] K. Soundararajan, *Moments of the Riemann zeta function*, Ann. of Math. (2) **170** (2009), no. 2, 981–993.

[79] ———, *Extreme values of zeta and L-functions*, Math. Ann. **342** (2008), no. 2, 467–486.

[80] T. Stefanicki, *Non-vanishing of L-functions attached to automorphic representations of* GL(2) *over* $\mathbf{Q}$, J. reine angew. Math. **474** (1996), 1–24.

[81] S.M. Voronin, *A theorem on the "universality" of the Riemann zeta–function*, Izv. Akad. Nauk SSSR, Ser. Matem. **39** (1975), 475–486; English transl., Math. USSR Izv. **9** (1975), no. 3, 443–453.

[82] P. Xi, *Large sieve inequalities for algebraic trace functions*, Int. Math. Res. Not. IMRN **16** (2017), 4840–4881. With an appendix by Étienne Fouvry, Emmanuel Kowalski, and Philippe Michel.

[83] M. P. Young, *The fourth moment of Dirichlet L-functions*, Ann. of Math. (2) **173** (2011), no. 1, 1–50.

[84] R. Zacharias, *Mollification of the fourth moment of Dirichlet L-functions*, Acta Arith. **191** (2019), no. 4, 201–257.

[85] ———, *Simultaneous non-vanishing for Dirichlet L-functions*, Ann. Inst. Fourier (Grenoble) **69** (2019), no. 4, 1459–1524.