DEPENDENCY ON THE GROUP IN AUTOMORPHIC SOBOLEV INEQUALITIES

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1. Introduction

In [1] and [2], Bernstein and Reznikov have introduced a new way of estimating the coefficients in the spectral expansion of ϕ^2, where ϕ is a Maass cusp of norm 1 on a quotient $Y = \Gamma\backslash\mathbf{H}$ of the Poincaré upper half-plane with finite volume. The question of obtaining the precise exponential decay of those coefficients had been posed by Selberg, and first solved by Good [5] (for holomorphic forms) and Sarnak [13] (for Maass forms). Bernstein and Reznikov obtain in fact the right polynomial growth conjectured by Sarnak: let (φ_i) be an orthonormal basis of the space of cusp forms on Y, eigenfunctions of the Laplace operator with eigenvalue

$$\mu_i = \frac{1-\lambda_i^2}{4} \geqslant 0, \quad 0 < \mu_1 \leqslant \mu_2 \leqslant \cdots$$

and $c_i = \langle \phi^2, \varphi_i \rangle$ the coefficients in question. Then one has

$$\sum_{|\lambda_i|\leqslant T} |c_i|^2 \exp(\tfrac{\pi}{2}|\lambda_i|) \leqslant C(\Gamma,\mu)(\log T)^3 \tag{1}$$

for all $T \geqslant 2$, $C(\Gamma,\mu) \geqslant 0$ being a "constant" depending only on Γ and the eigenvalue μ of ϕ. This is essentially best possible.[1]

For certain arithmetic applications, particularly because of relations between the c_i and special values of triple-product L-functions, it is important to have some control on $C(\Gamma,\mu)$. Especially in the arithmetic case where Γ is a congruence subgroup, one asks for at least a polynomial bound in terms of the level. We indicate two arguments leading to such bounds (indeed, rather better) in this note, following closely the proof of (1) in [2].

Proposition 1. *We have*

$$C(\Gamma,\mu) \ll_\mu 1$$

for all congruence subgroups $\Gamma < SL(2,\mathbf{Z})$, the implied constant depending only on μ.

We deduce from this an extension of (1) which is also arithmetically significant in studying the shifted convolution sums arising in analytic investigations of Rankin-Selberg type L-functions (see e.g. [11], [4], [7], [13]). Here we work (as usual in this situation) with the Hecke groups $\Gamma_0(q)$.

Corollary 2. *Let ϕ be an L^2-normalized Maass cusp form for $\Gamma_0(q)$ with eigenvalue μ, ℓ_1 and $\ell_2 \geqslant 1$ be integers. Let*

$$\Psi(z) = \phi(\ell_1 z)\overline{\phi(\ell_2 z)} \text{ and } c_i = \langle \Psi, \varphi_i\rangle,$$

where (φ_i) is an orthonormal basis of Maass forms of the space of cusp forms on $\Gamma_0(q\ell_1\ell_2)$ with eigenvalues μ_i as above.

We have

$$\sum_{|\lambda_i|\leqslant T} |c_i|^2 \exp(\tfrac{\pi}{2}|\lambda_i|) \ll_\mu [\Gamma_0(q):\Gamma_0(q\ell_1\ell_2)]^2(\log T)^3$$

for any $T \geqslant 2$, the implied constant depending only on μ.

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1. A lower bound for (1) is given in [12]; however Bernstein and Reznikov give a different method in [3], removing the $(\log T)^3$ term, and Krötz and Stanton [8] have also extended and slightly improved the method of [2].

To prove Proposition 1, it will be convenient to summarize the argument leading to (1) as the conjunction of the following two propositions:

Proposition 3. *Let* Γ *be a subgroup of* $SL(2,\mathbf{R})$ *with finite covolume and let* $Y = \Gamma\backslash\mathbf{H}$, $X = \Gamma\backslash SL(2,\mathbf{R})$. *For any Maass cusp form* $\phi \in L^2(Y)$ *with eigenvalue* μ *and norm* 1, *we have*

$$\sum_{|\lambda_i|\leqslant T} |c_i|^2 \exp(\tfrac{\pi}{2}|\lambda_i|) \ll_\mu \|e_\Gamma\|^2 (\log T)^3 \tag{2}$$

for $T \geqslant 2$, *where*

$$e_\Gamma \,:\, H_0^3(X) \rightarrow L^\infty(X)$$

is the possibly unbounded embedding of the third Sobolev space of cuspidal *functions on* X *into* $L^\infty(X)$, *and the implied constant depends only on* μ.

Precisely, our $H_0^3(X)$ is the space of cuspidal functions on X with all partial derivatives (with respect to standard x, y, θ coordinates [9, VI-4], say) of order $\leqslant 3$ in L^2, and the Sobolev norm is denoted $\|\cdot\|_3$ and defined to be

$$\|f\|_3^2 = \sum_{|I|\leqslant 3} \|\partial_I(f)\|_{L^2}^2, \tag{3}$$

where the measure defining the L^2-norm on X is induced by the Haar measure on $G = SL(2,\mathbf{R})$, normalized to give the standard Poincaré measure $y^{-2}dxdy$ on $\mathbf{H} = G/SO(2,\mathbf{R})$.

This proposition is a rephrasing of [2, Prop. 2.2, 2.3] together with [2, Sect. 3 and Prop. 4.1]. To get precisely our statement, note that we need to relate the Sobolev norm $\|\cdot\|_3$ on X to that denoted S_3 in [2, 3.2], which is defined using the Lie-algebra action of $G = SL(2,\mathbf{R})$ on the space V of smooth vectors of a representation π of G. If V is unitarily embedded in $L^2(X)$, say by $\nu : V \hookrightarrow L^2(X)$, the norms are equivalent, i.e. we have

$$c_V S_3(e) \leqslant \|\nu(e)\|_3 \leqslant C_V S_3(e) \text{ for } e \in V, \tag{4}$$

for some constants c_V, $C_V \geqslant 0$ depending only on V. In particular (notation as in [2, Prop. 4.1])

$$N_{sup}(e) = \|\nu(e)\|_\infty \leqslant \|e_\Gamma\|\|\nu(e)\|_3 \leqslant C_V\|e_\Gamma\|S_3(e),$$

i.e. the constant C in (loc. cit.) is $\leqslant C_V\|e_\Gamma\|$. Then the proof of Bernstein-Reznikov gives our Proposition (because $N_{sup} \leqslant C_V\|e_\Gamma\|S_3^G$, see Sect. 3.4, 2.3...)

The inequality (4), on the other hand, holds because of the formulae linking the Lie algebra action and the partial derivatives for functions on G and in induced representations (see e.g. [9, VI-4]). The dependency of this constant on V (i.e. on the eigenvalue of the corresponding Maass form) is of no matter to us because this inequality need only be applied for the representation corresponding to Maass forms of eigenvalue μ (see [2, 2.3]), and we allow a constant depending on μ.

The proof of Proposition 3 uses two beautiful ideas of analytic continuation of representations and invariant semi-norms on representations, which are the crucial points in [2]; note it is important to work with cuspidal forms for the next proposition. When Γ is cocompact, one can replace the third Sobolev space by the second Sobolev space.

Proposition 4. *For* Γ *as above, we have*

$$\|e_\Gamma\| < +\infty,$$

i.e. e_Γ *is a bounded linear operator.*

This is proved in [2, App. B]. Proposition 1 thus follows immediately from the following two facts:

Proposition 5. (1) *The implied constant in Proposition 3 depends only on* μ.

(2) *For any finite index subgroup* $\Gamma < SL(2,\mathbf{Z})$, *we have*

$$\|e_\Gamma\| = \|e_{SL(2,\mathbf{Z})}\|. \tag{5}$$

Of these, (1) is easy, as it follows from reading through the argument of Bernstein-Reznikov, as already partly sketched above. It is mainly due to the fact that the constant in Proposition 3 arises from representation-theoretic computations which only depend on properties of the unitary representation of $SL(2,\mathbf{R})$ corresponding to a Maass form (see [2, Prop. 2.2, (1), (2) and 3.4]).

Thus our much more modest goal is to prove (5), which justifies the title of this note. We will prove a bit more, and give a second argument which applies in more general circumstances, showing that

$$\|e_{\Gamma_q}\| \ll \sqrt{q} \text{ hence } C(\Gamma_q,\mu) \ll_\mu q \tag{6}$$

for any $q \geqslant 3$, where Γ_q is the Hecke triangle group (see e.g [6, 2.3]), taken as a simple concrete example.

2. Sobolev inequalities

Lemma 6. *Let Γ_1 and Γ_2 be commensurable subgroups in $SL(2,\mathbf{R})$ with finite covolume. Then*

$$\|e_{\Gamma_1}\| = \|e_{\Gamma_2}\|.$$

This clearly implies (5), as well as the analogue for the family of congruence subgroups of a cocompact quaternion group (associated to a fixed quaternion algebra).

Proof. It suffices by transitivity to show the result for $\Gamma_1 < \Gamma_2$ of finite index. We denote $e_i = e_{\Gamma_i}$ and let $F_2 \subset G = SL(2,\mathbf{R})$ be an open fundamental domain for the action of Γ_2. Then the open set

$$F_1 = \bigcup_{\gamma\in\Gamma_1\backslash\Gamma_2} \gamma F_2$$

is a (possibly disconnected) fundamental domain for the action of Γ_1. Let $f \in H_0^3(X)$ (where $X = \Gamma_1\backslash G$), which we assume to be smooth and compactly supported (such functions are dense in $H_0^3(X)$). We let g denote the corresponding function on F_1, smooth and compactly supported in F_1. Denoting as above (3) by $\|\cdot\|_3$ the Sobolev norm, we have

$$\|f\|_\infty = \|g\|_\infty = \max_\gamma \|g_\gamma\|_\infty$$

$$\|f\|_3^2 = \|g\|_3^2 = \sum_\gamma \|g_\gamma\|_3^2$$

where we define g_γ for $\gamma \in \Gamma_1\backslash\Gamma_2$ by

$$g_\gamma(x) = g(\gamma x) \text{ for } x \in F_2.$$

The second relation follows from the definition (3) of $\|\cdot\|_3$, and the normalization of Haar measure on G, the norm on the right for g_γ being the sup or Sobolev norm for Γ_2. For any γ, g_γ is smooth and compactly supported (hence cuspidal) for Γ_2, so we have

$$\|g_\gamma\|_\infty \leqslant \|e_2\|\|g_\gamma\|_3$$

by definition. Hence for some γ_0 among the finitely many $\gamma \in \Gamma_1\backslash\Gamma_2$, we have

$$\|f\|_\infty = \|g_{\gamma_0}\|_\infty \leqslant \|e_2\|\|g_{\gamma_0}\|_3 \leqslant \|e_2\|\Bigl(\sum_\gamma \|g_\gamma\|_3^2\Bigr)^{1/2} = \|e_2\|\|f\|_3$$

by positivity, which proves the inequality $\|e_1\| \leqslant \|e_2\|$. The converse is obtained by considering functions (smooth, compactly supported) on F_1 which vanish on all γF_2, $\gamma \neq 1$. ☐

We now come to the second argument, which follows that of [2, App. B], keeping track of the dependency on the group and requires a simple hyperbolic lattice-point counting lemma which we take from [6].

Proposition 7. *For any integer $q \geqslant 3$, let Γ_q be the Hecke triangle group generated by*

$$\pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2\cos(\pi/q) \\ 0 & 1 \end{pmatrix}. \tag{7}$$

We have

$$\|e_{\Gamma_q}\| \ll \sqrt{q} \text{ and } C(\Gamma_q,\mu) \ll_\mu q$$

for all $q \geqslant 3$, the implied constant depending only on μ.

Since Γ_q is a maximal discrete subgroup (see e.g. [14]) and (with few exceptions for $q \leqslant 6$) they are pairwise non-commensurable [10], this statement is essentially disjoint from Lemma 6.

Let $G = SL(2,\mathbf{R})$, $\mathbf{H} = SL(2,\mathbf{R})/SO(2,\mathbf{R})$ (where $SO(2,\mathbf{R})$ is the stabilizer of $i \in \mathbf{H}$), $Y(q) = \Gamma_q\backslash SL(2,\mathbf{R})$, and $X(q) = \Gamma_q\backslash\mathbf{H}$. We have the commutative square

$$\begin{array}{ccc} G & \xrightarrow{\pi} & \mathbf{H} \\ p\downarrow & & \downarrow \\ Y(q) & \longrightarrow & X(q) \end{array}$$

and because of personal habit, we will work mostly on $\mathbf{H}$ and $X(q)$. The upper-horizontal projection map π is simply $g \mapsto gi$.

A fundamental domain for Γ_q in $\mathbf{H}$ is the interior of

$$F_q = \{z \mid |z| \geqslant 1 \text{ and } |x| \leqslant \cos(\pi/q)\}.$$

Let $S_q = \pi^{-1}(\{z \in F_q \mid \mathrm{Im}(z) \geqslant 2\}) \subset G$ be a "Siegel domain" for Γ_q, and let $C_q = \pi^{-1}(F_q) - S_q$, so that C_q is compact.

Now we have the following lemma adapted from [2, App. B.3], where

$$B = \{g \in G \mid d(gi,i) < 1\}$$

is a fixed neighborhood of the identity in G (d denotes the hyperbolic metric on $\mathbf{H}$).

Lemma 8. *Let φ be a smooth cuspidal function on $Y(q)$. We have for $x \in F_q$*

$$|\varphi(x)| \ll v(x)^{1/2}\|\varphi\|_3 \text{ if } g \in C_q$$
$$|\varphi(x)| \ll o(x)v(x)^{1/2}\|\varphi\|_3 \text{ if } g \in S_q,$$

where both implied constants are absolute, $o(x)$ is the length of the shortest horocycle through x and $v(x)$ is the maximal cardinality of a fiber $p_x^{-1}(y)$ of the map

$$p_x \begin{cases} B \to Y(q) \\ g \mapsto p(xg) \end{cases}$$

for $y \in Y(q)$.

Since $SO(2,\mathbf{R}) \subset B$, the function v is $SO(2,\mathbf{R})$-invariant; so is in fact o. Our goal is now to estimate the L^∞-norm of v and o^2v on $Y(q)$, or equivalently on $X(q)$. Let x, $y \in Y(q)$, say $y = p(\tilde{y})$. The fiber $p_x^{-1}(y)$ is given by

$$\begin{aligned} p_x^{-1}(y) &= \{g \in B \mid p(xg) = y\} \\ &= \{g \in B \mid xg = \gamma\tilde{y} \text{ for some } \gamma \in \Gamma_q\} \\ &\simeq \{\gamma \in \Gamma_q \mid x^{-1}\gamma\tilde{y} \in B\} \text{ (where } \simeq \text{ indicates the obvious bijection } g \mapsto \gamma) \\ &= \{\gamma \in \Gamma_q \mid d(\gamma\tilde{y}i, xi) < 1\}. \end{aligned}$$

Thus estimating $v(x)$ is essentially a hyperbolic lattice point problem. Those can be notoriously tricky, but we seek only an upper bound. To apply Lemma 2.11 of [6], we write

$$|p_x^{-1}(y)| = \{\gamma \in \sigma^{-1}\Gamma_q\sigma \mid d(\gamma\sigma^{-1}\tilde{y}i, \sigma^{-1}xi) < 1\}$$

hence by loc. cit. we have

$$v(x) \ll \mathrm{Im}(\sigma^{-1}xi) + c^{-1} + (c\,\mathrm{Im}(\sigma^{-1}xi))^{-1} + 1 \tag{8}$$

with an absolute implied constant, where σ is a scaling matrix for the cusp ∞ of Γ_q, namely σ conjugates a generator of the stabilizer of ∞ in Γ_q to the standard parabolic element $z \mapsto z+1$, and

$$c = \inf\Big\{c \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma^{-1}\Gamma_q\sigma \text{ for some } a,\ b,\ c\Big\}.$$

Clearly one can take

$$\sigma = \begin{pmatrix} \sqrt{2\cos(\pi/q)} & 0 \\ 0 & 1/\sqrt{2\cos(\pi/q)} \end{pmatrix} \text{ acting by } z \mapsto 2z\cos(\pi/q)$$

and since $\sigma^{-1}F_q$ is easily seen to be the Ford polygon for $\sigma^{-1}\Gamma_q\sigma$ (see e.g. [6, p. 44]), we see from the geometric description of c^{-1} as the radius of the largest isometric circle in the Ford polygon (loc. cit., p. 53) that

$$c^{-1} = (2\cos\pi/q)^{-1}, \text{ in particular } c \geqslant 1,$$

and (8) and the value of σ give

$$v(x) \ll (2\cos\pi/q)^{-1}\operatorname{Im}(xi) + \operatorname{Im}(xi)^{-1} + 1 \ll \operatorname{Im}(xi) + \operatorname{Im}(xi)^{-1} + 1. \tag{9}$$

We now apply this to deduce the required estimates:

Lemma 9. *We have*

$$v(x) \ll q \text{ for } x \in C_q \text{ and } o(x)^2 v(x) \ll 1 \text{ for } x \in S_q$$

with absolute implied constants.

Proof. First if $x \in C_q$ we get by (9)

$$v(x) \ll \operatorname{Im}(xi) + \operatorname{Im}(xi)^{-1} + 1 \ll (\sin(\pi/q))^{-1} \ll q$$

with an absolute implied constant, since $\sin(\pi/q) \leqslant \operatorname{Im}(xi) \leqslant 2$ by definition of C_q.

On the other hand, if $x \in S_q$ is in the Siegel domain, we observe that an horocycle through x is given by

$$O = \{u + i\operatorname{Im}(xi) \mid |u| \leqslant 2\cos(\pi/q)\},$$

so that $o(x) \leqslant \operatorname{length}(O) = 2\cos(\pi/q)\operatorname{Im}(xi)^{-1} \leqslant 2\operatorname{Im}(xi)^{-1}$ by definition of the Poincaré metric. We get from (9) that

$$o(x)^2 v(x) \ll \operatorname{Im}(xi)^{-1} + \operatorname{Im}(xi)^{-3} + \operatorname{Im}(xi)^{-2} \ll 1$$

(with an absolute implied constant) since $\operatorname{Im}(xi) \geqslant 2$ from the definition of S_q. □

Putting together Lemma 9 with Lemma 8, we immediately get $\|e_{\Gamma_q}\| \ll \sqrt{q}$ hence Proposition 7.

3. Proof of the corollary

In addition to the notation in the statement of Corollary 2, we let $\psi_1(z) = \phi(\ell_1 z)$ and $\psi_2(z) = \overline{\phi(\ell_2 z)}$. It is well-known that ψ_1 is a Maass form on $\Gamma_0(q\ell_1)$ with the same Laplace-eigenvalue as ϕ, and a fortiori it is a Maass form on $\Gamma_0(q\ell_1\ell_2)$, and so is ψ_2. We consider the L^2-normalizations

$$\tilde{\psi}_i = \frac{\psi_i}{\|\psi_i\|}$$

where $\|\cdot\|$ is the L^2-norm on $\Gamma_0(q\ell_1\ell_2)\backslash\mathbf{H}$. We want to estimate on average

$$c_i = \langle \Psi, \varphi_i\rangle = \langle \psi_1\psi_2, \varphi_i\rangle = \|\psi_1\|\|\psi_2\|\langle\tilde{\psi}_1\tilde{\psi}_2, \varphi_i\rangle.$$

One can either extend (easily) the method of [2] or proceed by simple polarization: let $\tilde{\psi} = \tilde{\psi}_1 \pm \tilde{\psi}_2$, where the sign if chosen so that $\tilde{\psi} \neq 0$. Then $\tilde{\psi}$ is a Maass form on $\Gamma_0(q\ell_1\ell_2)$ with the same eigenvalue as ϕ and

$$\tilde{\psi}^2 = \tilde{\psi}_1^2 + \tilde{\psi}_2^2 + 2\tilde{\psi}_1\tilde{\psi}_2, \quad \text{(taking the + sign for instance)}$$

hence

$$\begin{aligned}\langle\tilde\psi_1\tilde\psi_2,\varphi_i\rangle &= \frac{1}{2}\Big(\langle\tilde\psi^2,\varphi_i\rangle-\langle\tilde\psi_1^2,\varphi_i\rangle-\langle\tilde\psi_2^2,\varphi_i\rangle\Big)\\ &=\frac{1}{2}\Big(\|\tilde\psi\|^2\langle\Big(\frac{\tilde\psi}{\|\tilde\psi\|}\Big)^2,\varphi_i\rangle-\langle\tilde\psi_1^2,\varphi_i\rangle-\langle\tilde\psi_2^2,\varphi_i\rangle\Big)\end{aligned}$$

so (with obvious notation)

$$\frac{|c_i|^2}{\|\psi_1\|^2\|\psi_2\|^2}\leqslant 12\Big\{\Big|c_i\Big(\frac{\tilde\psi}{\|\tilde\psi\|}\Big)\Big|^2+|c_i(\tilde\psi_1)|^2+|c_i(\tilde\psi_2)|^2\Big\}$$

and by Proposition 1 and Proposition 5 (which deals with the passage from $\Gamma_0(q)$ to $\Gamma_0(q\ell_1\ell_2)$) we obtain

$$\sum_{|\lambda_i|\leqslant T}|c_i|^2\exp(\tfrac{\pi}{2}|\lambda_i|)\ll\|\psi_1\|^2\|\psi_2\|^2(\log T)^3$$

the implied constant depending only on μ. Our Corollary 2 is thus a consequence of the following well-known formula:

Lemma 10. *Let ϕ be a Maass form on $\Gamma_0(q)$, $q\geqslant 1$, $\ell\geqslant 1$ and $d\geqslant 1$ two integers, $\psi(z)=\phi(\ell z)$. Then, seeing ψ as a Maass form on $\Gamma_0(qld)\backslash\mathbf{H}$, we have*

$$\|\psi\|=[\Gamma_0(q):\Gamma_0(q\ell d)]^{1/2}\|\phi\|$$

where on the left we have the L^2-norm on $\Gamma_0(q\ell d)\backslash\mathbf{H}$ and on the right the norm on $\Gamma_0(q)\backslash\mathbf{H}$.

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