Twisted - multiplicativity:

\[
V(a; q) \xrightarrow{q \geq 1 \text{ (squarefree)}} 2/\varphi(q)
\]

\[
V(a; q_1, q_2) = V(a \overline{q}_2; q_1) V(a \overline{q}_1; q_2)
\]

\[(q_1, q_2) = 1 \quad \left[ \overline{q}_1 \overline{q}_2 \equiv 1 \mod q_2 \right]
\]

Ex. \[ U(a; q) \]

\[
U(a; q_1, q_2) = U(a; q_1) U(a; q_2)
\]

\[
\left( q_1, q_2 = 1 \right)
\]

Then

\[
V(a; q) = \sum_{x \mod q} U(x; q) e \left( \frac{ax}{q} \right)
\]
is twisted-mult.

Abstractly: \[ \mathbb{Z}/q \mathbb{Z} \cong \mathbb{Z}/q \mathbb{Z} \]
\[
(x \mapsto e \left( \frac{ax}{q} \right)) \quad \text{is not compatible with CRT.}
\]

Ex. 1. \quad U(x; q) = (\text{norm.}) \text{ character function of } A_q \subset \mathbb{Z}/q \mathbb{Z}

with \( A_q = \left\{ x \mod q \mid x \mod \frac{p}{q} \right\} \quad \forall p^n \mid q \}

\[
U(a; q) = \frac{1}{|A_q|} \sum_{x \in A_q} e \left( \frac{ax}{q} \right)
\]
equidistribution of \( \left\{ \frac{x}{q} \right\} \quad x \in A_q \)
\[ \text{Hooray: 60's, } A_q = x \left( f(x) \bmod q \right) \]

\[ f \in \mathbb{Z}[x], \text{ monic} \]

\[ K \text{ - Sound} \]

Ex. 2: \[ f \in \mathbb{Z}[x], \text{ deg } f = d \geq 2 \]

\[ U(x; q) = \frac{1}{\sqrt{q}} \left( \sum_{y \in \mathbb{Z}/q\mathbb{Z}} \frac{1}{1 - \zeta_q} \right) \]

\[ \zeta_q(y) = x \]

\[ \Rightarrow \]

\[ U(a; q) = \frac{1}{\sqrt{q}} \sum_{y \bmod q} e \left( \frac{a f(y)}{q} \right) \]

and

\[ U(0; q) = 0 \]

\[ \text{if } (a, q) = 1 \]

Question: can one estimate (non-trivially)

\[ \sum_{q \leq x} U(1; q) ? \]
\textbf{Theorem} (Fourier–Michel, 2003)

If \( f \) is "suitably generic" then

\[
\sum_{q \leq x} \nu(1; q) \ll \sum_{q \leq x} \nu(1; q) \ll x (\log \log x)^{k_f}
\]

\textbf{Note.} \text{\underline{Weil}}: if \( f \) mod \( q \) is not constant for all \( p/q \) then

\[
\frac{1}{\sqrt{q}} \left| \sum_{\omega(q)} e \left( \frac{a \overline{f}(x)}{q} \right) \right| \leq (d-1)^{\omega(q)}
\]

\[\left[ a \neq 0 \text{ mod } q \right]\]

\[\Rightarrow \sum_{q \leq x} \nu(1; q) \ll x (\log x)^{A_d}\]
Theorem (K.-Sound) \( f \neq g_{0+} \) for \( \delta > 2 \)

\[ \frac{\mathfrak{p}(x) - \mathfrak{p}(y)}{x - y} \in \overline{\Omega} \quad \text{is irreducible} \]

Then \( \exists \quad y = f(d) > 0 \) s.t.

\[ \frac{x}{\log x} \leq \sum_{q \leq x} \mathfrak{U}(1; q) \leq \frac{x}{(\log x)^{\delta}} \]

for \( x \geq 2 \).

In particular

\[ \lim_{x \to \infty} \frac{1}{x} \sum_{x \leq q \leq x} \mathfrak{U}(1; q) = 0. \]

(2) Under the same assumption

\[ \sum_{q \leq x} \frac{\mathfrak{U}(1; q)^2}{q} \leq x \left( \frac{\log \log x}{\log x} \right)^{\delta} \]

Ideas of the proof:

Part 1: (analytic: reduce the problem to \( V(a; p) \) a varying, \( p \) prime)

[Hookey]

Prop. We have (for any \( V(\alpha; q) \))

\[
\sum_{q \leq x \atop q \equiv \alpha \pmod{f}} |V(1; q)| \ll \frac{x}{\log x} \prod_{\varepsilon \leq p \leq x} \left( 1 + \frac{g(p)}{p} \right)
\]

where

\[
\begin{cases} 
|V(a; p)| \leq G(p) \ll 1 \\
\frac{1}{\varphi(\alpha, \varphi)} \sum_{p \mod \varphi} |V(a; p)| \leq g(p)
\end{cases}
\]

and \( \varepsilon = x^{1/\alpha \log \log x} \), \( \alpha \) depends on \( \max G(p) \).
To get our Th. we need: under \( \bigcirc \), we need to find a positive proportion of primes \( p \) s.t.

\[
\frac{1}{\sum_{p \mid q(a,p) = 1} \left| U(a,p) \right|} \leq 1 - \delta
\]

for some \( \delta > 0 \), depending only on \( d \).

(Compare with a paper of Katz in Iwasawa Proceedings)

To get the 1st absolute moment, we use a trick:

if 2nd moment is 1 and the 4th is \( > 1 \) (unif.)

\[ \rightarrow \] the 1st is moment is \( < 1 \) (unif.)
Th. 2. For \( d \geq 2 \) (for \( b \mod p \)) we have a positive proportion of \( p \) with
\[
M_4 = \frac{1}{p} \sum_{(a, p) = 1} |V_p(a; p)|^4 \geq 2 + O\left( \frac{1}{\sqrt{p}} \right).
\]

The proof of Th. 2 is pure algebraic geometry using results of Katz.

1. First we may assume \( \ast \mod p \) (otherwise get “easily” \( M_4 \geq 4 \)) (RHT for curves)

2. Katz: The distribution of \( V(a; p) \) is “controlled” by a subgroup \( G_p \) of \( GL_{d-1} \), which implies their
\[ M_4 = \text{Tr} \left( \delta_{\rho} \left\vert E \right) \right) + O \left( \frac{1}{\sqrt{\rho}} \right) \]

"Frob at $\rho$"

where \( E = \text{End} \left( \text{End} \left( \text{End} \left( \mathbb{K}_{d-1} \right) \right) \right) \)

for some group \( G_\rho \triangleleft G_p \), \( \rho \in G_p \)

**Generic:** \( \text{SL}_{d-1} = \text{SL}_{d-1} \oplus \text{Id} \) \( G_\rho \)

If \( \delta_{\rho} = \text{Id}_E \) then \( \text{Tr} \left( \delta_{\rho} \left\vert E \right) \right) = \dim(E) \)

**Representation theory (Schur's Lemma):**

\[ E = \text{space of } G_\rho \text{-linear maps } \text{End} \left( \mathbb{K}_{d-1} \right) \]

and

\[ \dim E = 1 \iff \text{this action of } G_\rho \]
on $\text{End}(k^{d-1})$ is irreducible.

$G^g = SL_{d-1}$; this space has two stable subspaces

$$\text{C Id } k^{d-1} ; \text{ matrices of Tr. 0}$$

$$\oplus$$

$$\text{PGL}$$

$$\oplus$$

$\exists$ for $d \geq 2$

$$\Rightarrow \dim E \geq 2$$

Our challenge: do this without using knowledge of $G^g, G_p$.

Key ingredients:

1. Show that $G_p$ is also Frobenius for a Galois action over $\mathbb{Q}$

2. So if the Galois action is finite, any totally split
Prime in the kernel has \( p = 1 \).

\[ \rightarrow \text{deduced from a lemma in a previous paper by Michel-Saum} \]

- K.