Remark 2.16. The dual inequality (2.7) can be used to provide an alternate proof of the large sieve inequality of Proposition 2.3. To see this, note that for any $x \in X$ which lies in the sifted set and any $\ell \in \mathcal{L}^*$, we have the analytic "amplification" property

$$\nu_\ell(\Omega_\ell) = \sum_{\varphi \in \mathcal{B}_\ell^*} (-\beta(\ell,\varphi))\varphi(\rho_\ell(F_x))$$

where the $\beta(\ell,\varphi)$ are the coefficients of the characteristic function of $\Omega_\ell$ in the basis $\mathcal{B}_\ell$, as in the first step of the proof of the dual sieve. For any $m \in \mathcal{L}$, taking the product of these identities over $\ell \in m$, we obtain

$$\nu_m(\Omega_m) = \sum_{\varphi \in \mathcal{B}_m^*} \tilde{\beta}(m,\varphi)\varphi(\rho_m(F_x)) \tag{2.18}$$

for some complex coefficients $\tilde{\beta}(m,\varphi)$. Hence, for any $x \in S(X,\Omega;\mathcal{L}^*)$, we have

$$\Bigl|\sum_{m \in \mathcal{L}} \sum_{\varphi \in \mathcal{B}_m^*} \tilde{\beta}(m,\varphi)\varphi(\rho_m(F_x))\Bigr| = \sum_{m \in \mathcal{L}} \nu_m(\Omega_m),$$

and by positivity, we derive from (2.7) that

$$|S(X,\Omega;\mathcal{L})|\Bigl(\sum_{m \in \mathcal{L}} \nu_m(\Omega_m)\Bigr)^2 \leqslant \Delta \sum_{\varphi \in \mathcal{B}_m^*} |\tilde{\beta}(m,\varphi)|^2.$$

Next, by orthonormality (as in the last lines of the proof of Proposition 2.15) and the definition of the product densities, it follows that we have

$$\sum_{\varphi \in \mathcal{B}_m^*} |\tilde{\beta}(m,\varphi)|^2 = \prod_{\ell | m} \sum_{\varphi \in \mathcal{B}_\ell^*} |\tilde{\beta}(\ell,\varphi)|^2 = \prod_{\ell | m} \nu_\ell(\Omega_\ell)(1-\nu_\ell(\Omega_\ell)),$$

and this leads to the inequality

$$|S(X,\Omega;\mathcal{L})| \leqslant \frac{\displaystyle\sum_{m \in \mathcal{L}} \nu_m(\Omega_m) \prod_{\ell | m} (1-\nu_\ell(\Omega_\ell))}{\Bigl(\displaystyle\sum_{m \in \mathcal{L}} \nu_m(\Omega_m)\Bigr)^2}.$$

This is not quite as good as Proposition 2.3, but the following procedure corrects this: if we multiply (2.18) with coefficients

$$\lambda_m = \prod_{\ell | m} \lambda_\ell$$

we obtain

$$\lambda_m \nu_m(\Omega_m) = \sum_{\varphi \in \mathcal{B}_m^*} \lambda_m \tilde{\beta}(m,\varphi)\varphi(\rho_m(F_x)),$$

and the same procedure as before gives the inequality

$$|S(X,\Omega;\mathcal{L})| \leqslant \frac{\displaystyle\sum_{m \in \mathcal{L}} \lambda_m^2 \nu_m(\Omega_m) \prod_{\ell | m} (1-\nu_\ell(\Omega_\ell))}{\Bigl(\displaystyle\sum_{m \in \mathcal{L}} \lambda_m \nu_m(\Omega_m)\Bigr)^2}. \tag{2.19}$$

We now optimize this expression in terms of $\lambda_m$: by Cauchy's inequality, we have

$$\Big(\sum_{m\in\mathcal{L}}\lambda_m\nu_m(\Omega_m)\Big)^2 \leqslant \Big(\sum_{m\in\mathcal{L}}\lambda_m^2\nu_m(\Omega_m)\prod_{\ell\mid m}(1-\nu_\ell(\Omega_\ell))\Big)\times\Big(\sum_{m\in\mathcal{L}}\prod_{\ell\mid m}\frac{\nu_\ell(\Omega)_\ell}{1-\nu_\ell(\Omega_\ell)}\Big),$$

or in other words, the ratio on the left-hand side of (2.19) is always at least as large as $1/H$ (see (2.4)). Moreover, the case of equality of Cauchy's inequality shows that this ratio is precisely equal to $1/H$ (so that (2.19) is then exactly the desired inequality (2.3)), if we select

$$\lambda_\ell = \frac{1}{1-\nu_\ell(\Omega_\ell)}.$$