Sums of three squares: Beyond equidistribution

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Integers as sums of 3 squares

Legendre/Gauss: $$n = x^2 + y^2 + z^2 \iff n \neq 4^a(8b+7)$$

Primitive representation: gcd(x,y,z)=1

n is **primitively** represented as a sum of 3 squares $\leftrightarrow$ $n \neq 0,4,7 \bmod 8$

The number of representations

$$N_n := \#\left\{(x,y,z) : x^2 + y^2 + z^2 = n\right\}$$

Example: n=5 then we have $N_5=24$

$$(\pm 2, \pm 1, 0), (\pm 1, \pm 2, 0), (\pm 2, 0, \pm 1),$$
$$(\pm 1, 0, \pm 2), (0, \pm 2, \pm 1), (0, \pm 1, \pm 2)$$

Exercise: if $n=4^a$ then $N_n=6$

Growth of N_n for large n

For squarefree n, $N_n \approx \sqrt{n} \bullet L(1,\chi_{-n})$

Siegel:

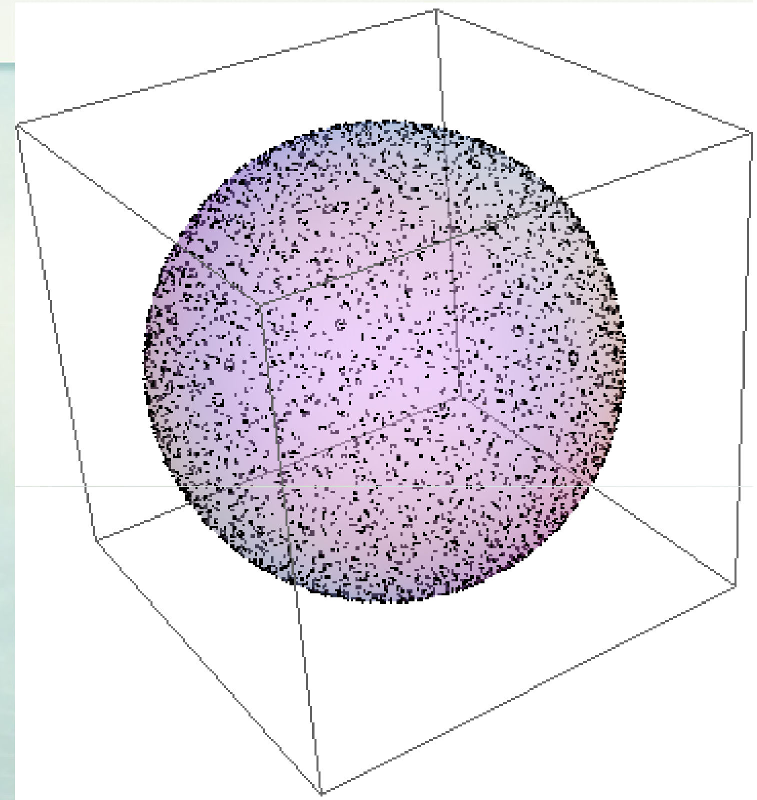
$$\frac{1}{n^{\varepsilon}} << L(1,\chi_{-n}) << n^{\varepsilon}, \quad \forall \varepsilon > 0$$

If n is primitively representable as a sum of three squares then $N_n = n^{1/2 \pm o(1)}$

Spatial distribution of solutions

Project the different representations of n to the unit sphere S^2:

$$(x, y, z) \mapsto \frac{1}{\sqrt{n}}(x, y, z) \in S^2$$



We get a set E(n) of about N_n points on S^2

- call them "Linnik points"

Uniform distribution (Linnik's conjecture)

As $n \to \infty$, $n \neq 0,4,7 \bmod 8$, the set E(n) becomes uniformly distributed on S^2 .

Proved by Linnik assuming GRH, (1940),

Proved unconditionally by Duke, Golubeva-Fomenko (1988), (via Iwaniec).

Uniform distribution on S^2

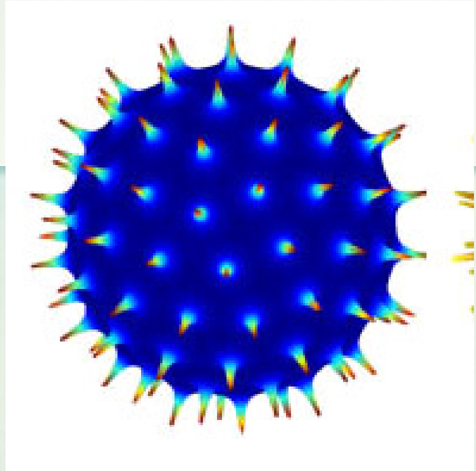
Definition: The sets E(n) become uniformly distributed in S^2 if for any nice set B in S^2

$$\frac{\#(E(n)\cap B)}{\# E(n)} \xrightarrow[n\to\infty]{} \frac{\textbf{area}(B)}{\textbf{area}(S^2)}$$

Equivalently, for any continuous function $f \varepsilon C(S^2)$,

$$\frac{1}{\# E(n)} \sum_{P\in E(n)} f(P) \xrightarrow[n\to\infty]{} \frac{1}{\textbf{area}(S^2)} \int_{S^2} f(x)\, dx$$

Beyond equidistribution I: The electrostatic energy



Visualization: Rob Womersley

The electrostatic energy of N points on the sphere S^2 is

$$Energy(P_1, \dots, P_N) := \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{|P_i - P_j|}$$

Thompson's question: (1904): Find configurations of charges on the sphere which **minimize** E (stable configurations)

Wagner (1992): The energy of stable configurations is

$$\textbf{Energy}_{\min} = N^2 \underbrace{\int_{S^2}\int_{S^2} \frac{dxdy}{|x-y|}}_{=1} - O(N^{3/2})$$

J.J. Thompson, Nobel prize 1903

Question: What is the energy for Linnik points E(n)?

The energy of Linnik points

Theorem (Bourgain, ZR, Sarnak): The energy of the Linnik points E(n) is close to minimal

$$Energy(E(n)) = N^2 + O(N^{2-\delta})$$

Proof: equidistribution + control of # of close neighbours

$$Energy\ (E(n)) := \sum_{P \in E(n)} \left(\sum_{Q \neq P} \frac{1}{|P-Q|} \right)$$

Would like to use uniform distribution to claim that for each P

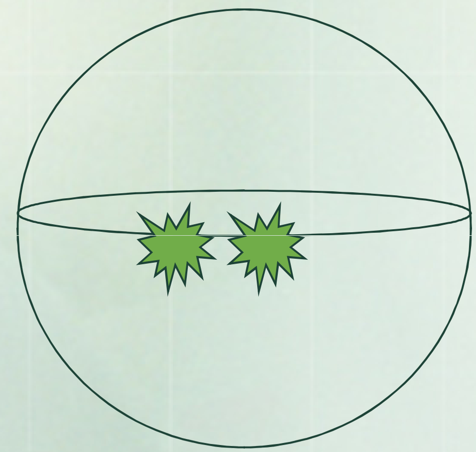
$$\sum_{\substack{Q \in E(n) \\ P \neq Q}} \frac{1}{|P-Q|} \sim N \int_{S^2} \frac{dx}{|P-x|} = N \quad \Longrightarrow \quad Energy \sim \sum_{P \in E(n)} N = N^2$$

Problem: The function Q → 1/|P-Q| is not continuous !

In fact a point Q with $|P-Q|<1/N^{1-o(1)}$, gives a contribution bigger than main term N

Example: A close pair

$$n = k^2 + (k+1)^2, \quad P = \frac{1}{\sqrt{n}}(k+1, k, 0) \quad Q = \frac{1}{\sqrt{n}}(k, k+1, 0)$$



$$\Rightarrow \qquad |P - Q| = \frac{1}{\sqrt{n}} = \frac{L(1, \chi_n)}{N} = \frac{1}{N^{1 \pm o(1)}}$$

Controlling close pairs

Counting pairs of points at a given distance

$$A(n,h) := \#\left\{ x, y \in \mathbf{Z}^3 : |x|^2 = |y|^2 = n, \quad |x-y|^2 = h \right\}$$

Siegel's mass formula: $$A(n,h) = 24 \cdot \alpha_2(n,h) \prod_{\substack{p \mid h(2n-h) \\ p \neq 2}} \alpha_p(n,h)$$

Venkov, 1931, Pall, 1948 : Explicit computation of local factors (crucial).

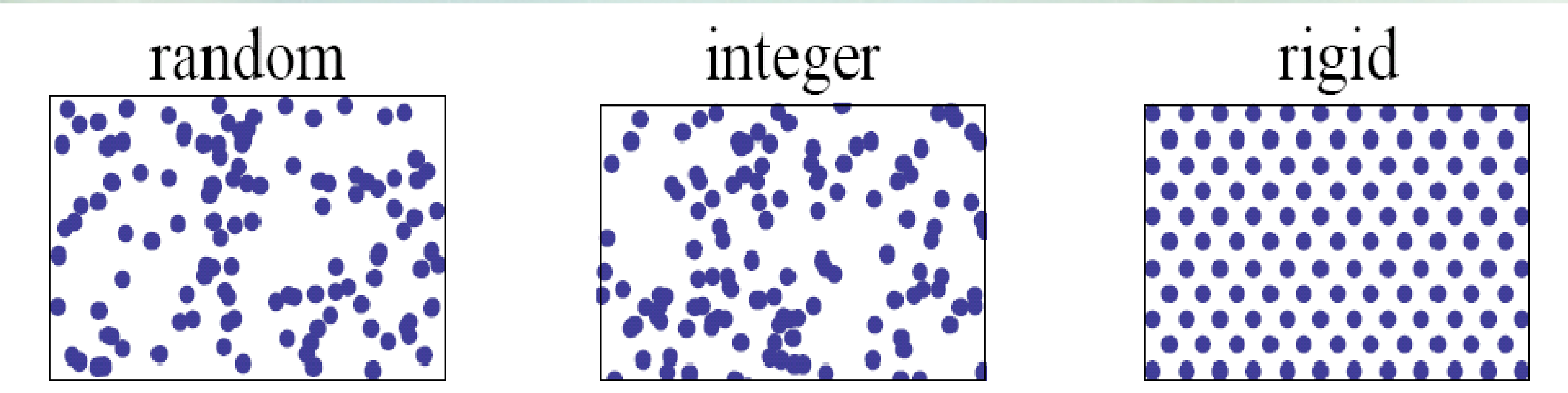
$$\Rightarrow \quad A(n,h) << n^{\varepsilon} \gcd(n,h)^{1/2}$$

- allows to control contribution of "close" pairs and show Energy(E(n))~N^2. QED

Beyond equidistribution II: randomness on smaller scales

Uniform distribution means randomness on scale of O(1) – subsets in S^2 of fixed size.

Question: randomness on smaller scales?



Analyzing point patterns

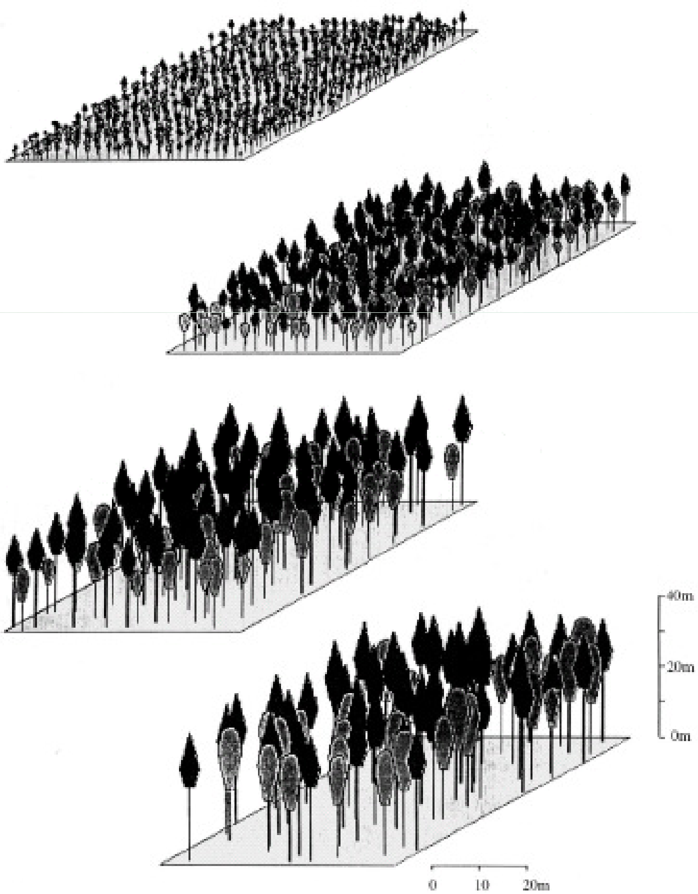


Fig. 7. Some steps of the evolution of a mixed forest with spruces and beeches, obtained by means of the single-tree model SILVA 2.1. Start configuration with spruces and beeches, aged 20 and 25 years, respectively and situations 20, 80 and 145 years later. (Published with the kind permission of H. Pretzsch.)

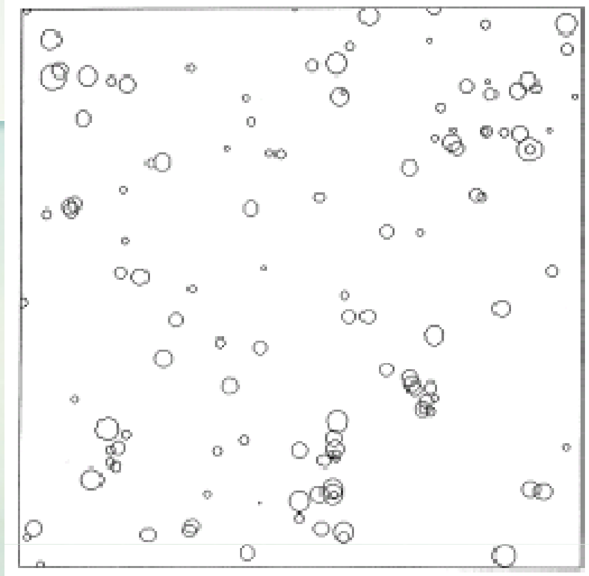


Fig. 1. *126 young pines in a 10 m × 10 m square plot in a Finnish forest. The trees are shown as circles, the diameters of which are proportional to the tree heights. This pattern is highly clustered; some trees are very close together.*

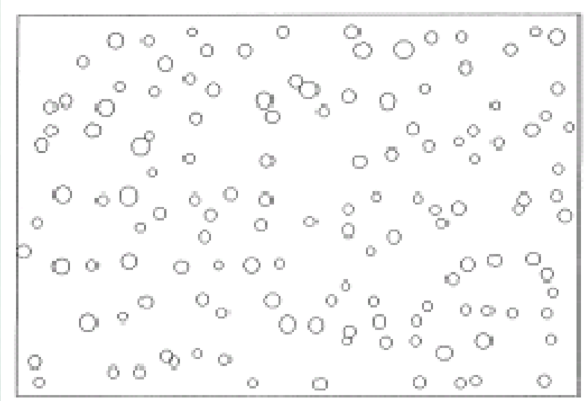


Fig. 2. *134 spruces of age 60 years in a 56 m × 38 m stand of Tharandter Wald (Germany). The trees are shown as circles, where the diameters are proportional to the dbh's.*

Point processes on S^2

Random point process on S^2= random countable subset X in S^2

Intensity of a point process: For any subset A of S^2, E(#A∩X)= μ•σ(A)

σ(A):=area(A)/area(S^2)

Binomial process: N independent points, each uniformly distributed on S^2 :

$$\mathbf{Prob}_{\mathbf{Bin(N)}}(\# A \cap \mathrm{X} = k) = \binom{N}{k} \sigma(A)^k (1-\sigma(A))^{N-k}$$

Poisson process of intensity μ: characterized by

1) #A∩X is a Poisson variable with intensity μ•σ(A)
2) If A,B are **disjoint** then #A∩X , #B∩X are **independent**

$$\mathbf{Prob}_{\mathbf{Pois}(\mu)}(\# A \cap X = k) = e^{-\mu\cdot\sigma(A)} \frac{(\mu\cdot\sigma(A))^k}{k!}$$

Tests for randomness: Minimal distances

Nearest neighbor distance:

$$\delta(x) := \min_{y \neq x} |x - y|^2$$

Average value:

$$\left\langle \delta(x) \right\rangle_n := \frac{1}{N_n} \sum_{x \in E(n)} \delta(x)$$

For random points (Poisson model)

1) $\langle\delta\rangle \sim 4/N$

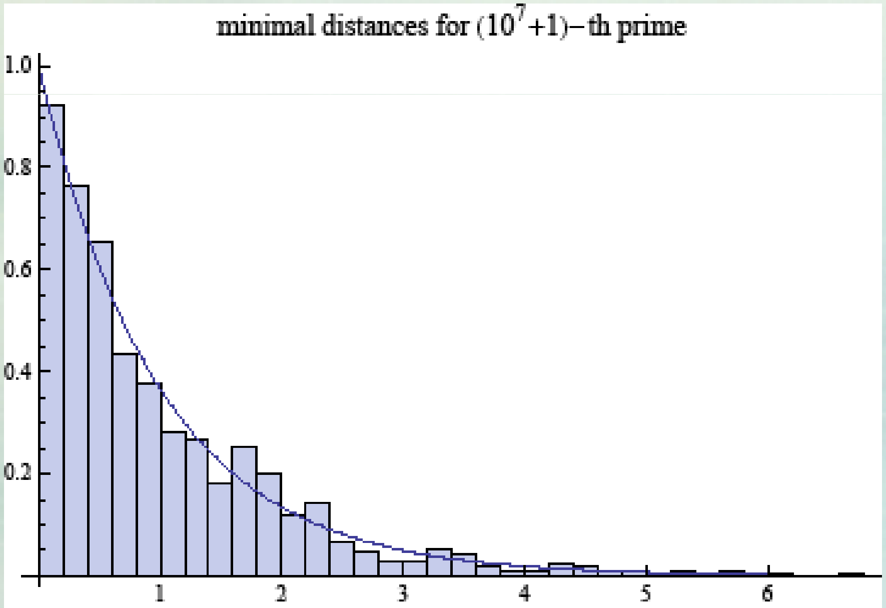
2) Distribution of normalized distances $\delta(x)/\langle\delta\rangle$ is exponential

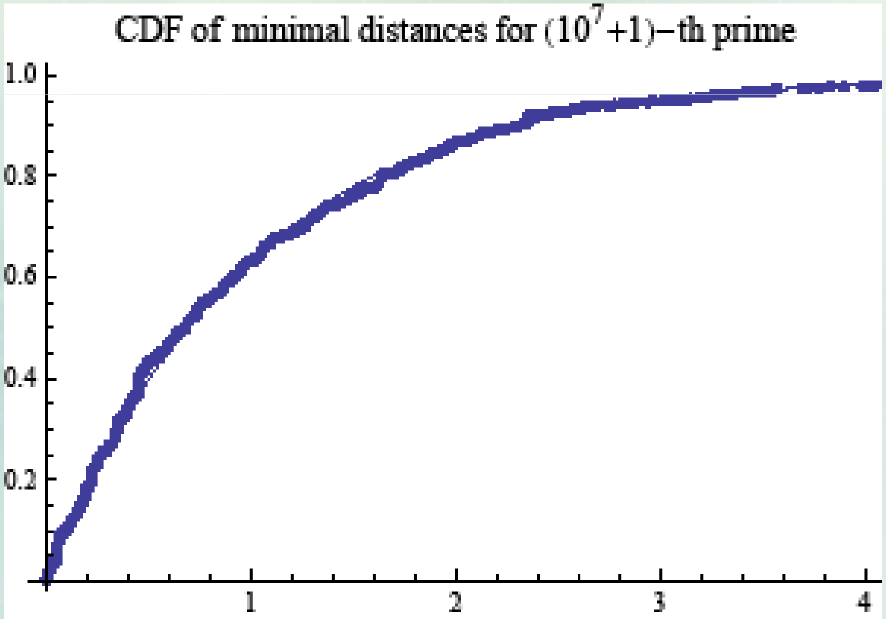
Distribution of minimal distances

Conjecture: For the Linnik points E(n)

1) $\langle\delta\rangle_n \sim 4/N_n$ =Poisson value

2) Distribution P(s) of normalized distances $\delta(x)/\langle\delta\rangle$ is exponential: $P(s)=\exp(-s)$





Proposition: Any possible limit P(s) is absolutely continuous (assuming GRH)

The least spacing statistic

Because the point sets E(n) come from integer points, the least spacing is >1/√n

$$\delta_{\min}(n) := \min_{\substack{x \neq y \\ x, y \in E(n)}} |x-y| > \frac{1}{\sqrt{n}} = N^{-1+o(1)}$$

Is this a non-random feature ?

Compare: For N random points, least spacing is a.s. $N^{-1+o(1)}$ (lower & **upper** bound)
-- "birthday paradox"

Theorem: For almost all n, $\delta_{\min}(n) \approx n^{-1/2+o(1)} = N^{-1+o(1)}$

Implied by: almost all n is a sum of two squares and a "mini-square"

$$n = x^2 + y^2 + z^2, \qquad |z| < n^{\varepsilon}$$

Wooley: can make $|z| < (\log N)^{1+o(1)}$

Second order statistics

Ripley's K-function: for a homogeneous point process with intensity μ,

$\mu K(r)$:=expected number of **other** points at distance r from a typical point of the process

Alternatively,

$$\mu^2 K(r) = \#\text{ distinct pairs } (x,y) \text{ with } |x-y|<r$$

For Poisson process, $\mathbf{E}\{K(r)\} = \pi r^2$ = area

A smaller value of $K(r)$ is interpreted as **repulsion** (e.g. for determinantal processes).

Intensity=# pts
per unit area

The critical scaling regime

we look at values of r where a ball of radius r is expected to contain O(1) points:

Define **critical radius** ρ_n by

$$N_n \cdot \pi\rho_n^2 = area(S^2)$$

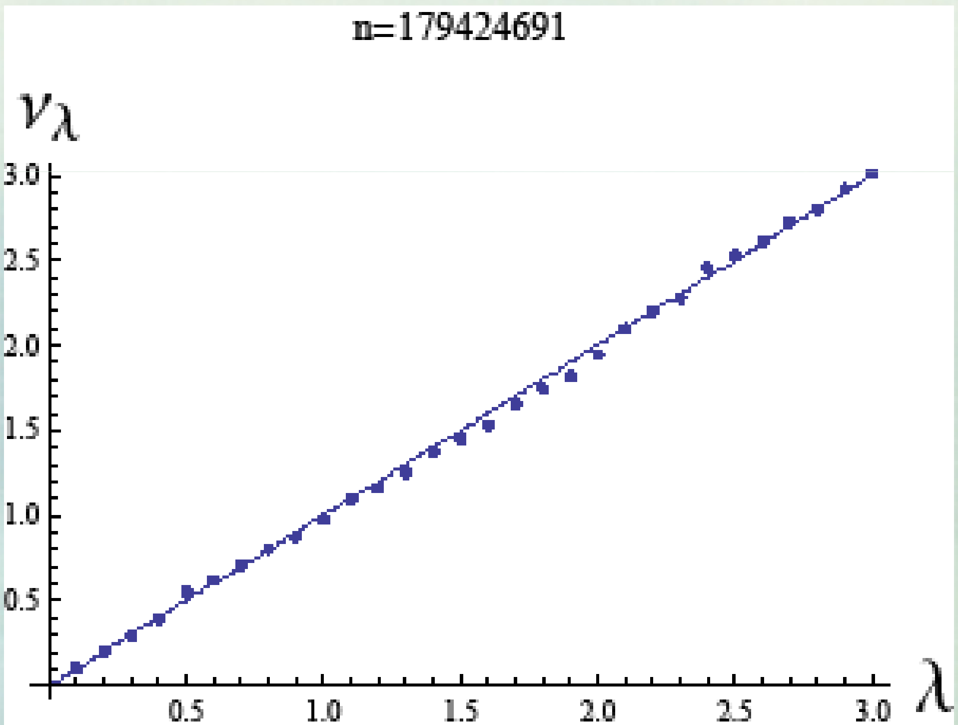
We conjecture that at the critical scale, the two-point statistic K coincides with that of random points

Conjecture: Fix $\lambda>0$. For $r=\sqrt{\lambda}\bullet\rho_n$ then $K_n(r) \sim \pi r^2$ as $n\to\infty$

$$\lim_{n\to\infty} N_n K_n(\sqrt{\lambda}\rho_n) = \lambda$$

K(r) at critical scaling

$$\lim_{n\to\infty} N_n K_n(\sqrt{\lambda}\rho_n) = \lambda$$



n=10^7+1'th prime, N≈100,000 (effectively 2000 points)

bounds at critical scaling

Theorem (on GRH): For $N^{-1+o(1)} < r < 2, \quad K_n(r) < 2010 \bullet r^2$

Theorem: For a **positive density** of square-free n, we have

$$N_n K_n(\sqrt{\lambda}\rho_n) > \frac{1}{2010}\lambda$$

Note: the upper bound holds for **individual** n

Thus we have lower & upper bounds on K(r) consistent with random behavior.

Main tool: Venkov's formula

Arithmetic ingredient: Counting pairs of vectors with length^2 =n and given distance:

$$K_n(r) = \frac{1}{N^2}\#\left\{x, y : |x|^2 = |y|^2 = n, \quad \left|\frac{x-y}{\sqrt{n}}\right|^2 < \lambda\frac{4}{N}\right\} = \frac{1}{N^2}\sum_{h<\frac{4n}{N}\lambda} A(n,h)$$

$$A(n,h) := \#\left\{x, y \in \mathbf{Z}^3 : |x|^2 = |y|^2 = n, \quad |x-y|^2 = h\right\}$$

Siegel's mass formula $$A(n,h) = 24\cdot\alpha_2(n,h)\prod_{\substack{p|h(2n-h)\\ p\neq 2}}\alpha_p(n,h)$$

$$\alpha_p(n,h) = \lim_{k\to\infty}\frac{1}{p^{3k}}\#\left\{x, y \bmod p^k : |x|^2 = |y|^2 = n, |x-y|^2 = h\right\}$$

Venkov, 1931, Pall, 1948 : Explicit computation of local factors (crucial).

Approximating A(n,h)

A(n,h) is essentially a multiplicative function along a quadratic progression

$$A(n,h) \leq 24 \cdot F_n\big(h(2n-h)\big)$$

$$F_n(a) \approx \sum_{\substack{d|a \\ d\ \textbf{odd}}} \left(\frac{-n}{d}\right)$$

This allows us to use results about sums of multiplicative functions along polynomial progressions (Shiu, Nair) to get **upper** bound on K(r)

Nair's theorem (1992)

F=multiplicative function: F(1)=1, F(ab)=F(a)F(b) if a,b coprime,

F non-negative: F≥0, slowly growing: $F(n) \ll n^{\varepsilon}$

$P(t) \varepsilon Z[t]$ polynomial

Then for $x^a < y < x$

$$\sum_{x-y<m<x} f(|P(m)|) \ll_{F,P} y \times \prod_{p \le x} \left(1 - \frac{\rho(p)}{p}\right) \times \exp\left(\sum_{p \le x} \frac{\rho(p) f(p)}{p}\right)$$

$$\rho(m) = \#\left\{x \in \mathbf{Z}/m\mathbf{Z} : \quad P(x) = 0 \bmod m\right\}$$

In our case:

$$P(t) = t(2n-t) \quad , \quad f(a) = F_n(a) \approx \sum_{\substack{d|a \\ d \ \mathbf{odd}}} \left(\frac{-n}{d}\right) \quad , \quad x = n, \quad y = \lambda \frac{4n}{N_n} \approx n^{1/2+o(1)}$$

Why do we need GRH ?

Nair's theorem gives a bound on K(r) of

$$N_n K_n(\rho_n \sqrt{\lambda}) << \lambda \times \left(\frac{1}{L(1, \chi_{-n})} \exp\left(\sum_{p<n} \frac{\chi_{-n}(p)}{p} \right) \right)^2$$

The L-value arises because $N_n \approx \sqrt{n} \bullet L(1,\chi_{-n})$

- Need GRH to guarantee RHS is << λ

summary

We studied properties of the sets E(n) of points on the sphere arising from writing $n=x^2+y^2+z^2$ which go beyond uniform distribution:

- The electrostatic energy is close to minimal .
- Empirical agreement of various statistics with those of a random point process
- Some theoretical results.