SAHAKIAN'S THEOREM AND THE MIHALIK-WIECZOREK PROBLEM

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1. Introduction

The Mihalik–Wieczorek problem, as reported by Pach and Rogers [5, §1], is the question of the existence of a continuous function $f\colon [0,1]\to \mathbf{C}$, with image not contained in a line, such that the image by f of any interval in $[0,1]$ is convex. This remains open, to the author's knowledge at least. The goal of this note is to present a relation between this problem and some results of Sahakian [6], which are in the spirit of the Bohr-Pál Theorem and of the general problem of finding a "reparameterization" of a continuous periodic function f whose Fourier series has better convergence property than f.

Our result does not settle, or even give any definite evidence, for any answer to the Mihalik–Wieczorek Problem. In fact, it is useful here to specifically state that we work in classical logic and set theory, so that the Law of the Excluded Middle holds.

Proposition 1. *One of the following statements holds:*

(1) *There exists a space-filling continuous curve $f\colon [0,1]\to\mathbf{C}$ such that the h-th Fourier coefficient of $g(t)=f(t)-tf(1)$ satisfies*

$$\widehat{g}(h)=O\Bigl(\frac{1}{|h|}\Bigr)$$

for all $h\neq 0$.

(2) *For any continuous function $f\colon [0,1]\to\mathbf{C}$ such that $f([0,1])$ is not contained in an affine line, there exists a segment $[a,b]\subset[0,1]$ such that $f([a,b])$ is not convex.*

In Assertion (1), the meaning of "space-filling curve" is that the interior of $f([0,1])$ is non-empty. This assertion is related to a theorem of Sahakian [6] on improvement of convergence of Fourier series, which is proved for continuous *real-valued* functions. On the other hand, Assertion (2) is a *negative* answer to the question of Mihalik and Wieczorek.

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If we say that a continuous function $f\colon [0,1]\to \mathbf{C}$ is a Mihalik–Wieczorek curve if, for any segment $[a,b]\subset [0,1]$, the image $f([a,b])$ is convex, then what we will really prove is that for any Mihalik–Wieczorek curve, there exists an increasing homeomorphism $\sigma\colon [0,1]\to [0,1]$ such that the function $f\circ\sigma$ satisfies Assertion (1). If the image of f is not contained in an affine line, then the convexity of $f([0,1])$ is convex implies that the image of f has non-empty interior. In other words, the negation of Assertion (2) of Proposition 1 implies that Assertion (1) holds. By the Law of the Excluded Middle, we know that Assertion (2) is either true or false, hence Proposition 1 follows.

Notation

For a set X and functions f, g on X, we use $f \ll g$ and $f = O(g)$ synonymously as meaning that there exists $C\geqslant 0$ such that $|f(x)|\leqslant Cg(x)$ for all $x\in X$.

2. Proof

We denote by $\mathcal{S}_0$ the set of continuous functions $f\colon [0,1]\to\mathbf{C}$ such that the h-th Fourier coefficient of $g(t)=f(t)-f(0)-tf(1)$ satisfies

$$|\widehat{g}(h)|\leqslant \frac{1}{|h|}$$

for $h\neq 0$. We will show, by an adaptation of Sahakian's Theorem, that if f is a Mihalik–Wieczorek curve, then there exists a constant $\delta>0$ and an increasing homeomorphism $\sigma\colon [0,1]\to[0,1]$ such that the function $\delta\,(f\circ\sigma)$ belongs to $\mathcal{S}_0$. By the previous discussion, this implies Proposition 1.

We state formally this key step. We attribute the result to Sahakian, since the reader will see that there is indeed *no change* to his arguments, except that of point of view... Precisely, we will prove:

Proposition 2 (Sahakian)**.** *Let f be a Mihalik-Wieczorek curve. There exists an increasing homeomorphism $\sigma\colon [0,1]\to[0,1]$ such that the function $f\circ\sigma$ has the property that the Fourier coefficients of the function*

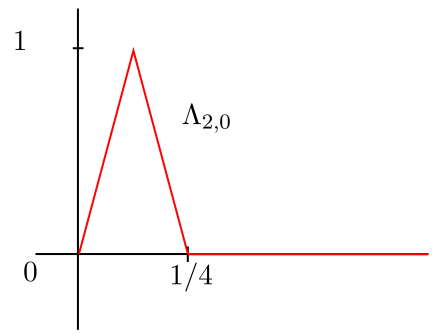
$$g(t)=f(\sigma(t))-f(0)-tf(1)=f(\sigma(t))-f(\sigma(0))-tf(\sigma(1))$$

satisfy

$$\widehat{g}(h)=O\Big(\frac{1}{|h|}\Big)$$

for $h\neq 0$.

Replacing f by δf for $\delta>0$ suitably chosen, this implies the statement above. We will give the full proof, since Sahakian's result is not so well-known, and his main result is a bit stronger, which makes this variant not



as obvious to check. We repeat before doing so that all steps below follow Sahakian's idea.

The construction of the homeomorphism σ will follow from the following elementary lemma:

Lemma 3. *Let $(t_{m,j})$ be a dense family of elements of $[0,1]$ for $m \geqslant 0$ and $1 \leqslant j \leqslant 2^m$ such that the map $j/2^m \mapsto t_{m,j}$ is well-defined, strictly increasing, and maps 0 to 0 and 1 to 1. There exists a unique homeomorphism $\sigma\colon [0,1] \to [0,1]$ such that $\sigma(j/2^m) = t_{m,j}$ for all $m \geqslant 0$ and $1 \leqslant j \leqslant 2^m$.*

Sahakian's construction is based on the Faber-Schauder expansion of a continuous function on $[0,1]$. Recall that the Faber-Schauder functions $\Lambda_{m,j}$ on $[0,1]$ are continuous functions defined for integers $m \geqslant 0$ (which we call the level of $\Lambda_{m,j}$) and $0 \leqslant j < 2^m$ by the following conditions:

(1) The support of $\Lambda_{m,j}$ is the dyadic interval
$$\Big[\frac{j}{2^m}, \frac{j+1}{2^m}\Big],$$
of length 2^{-m},

(2) We have
$$\Lambda_{m,j}\Big(\frac{j}{2^m}\Big) = \Lambda_{m,j}\Big(\frac{j+1}{2^m}\Big) = 0, \qquad \Lambda_{m,j}\Big(\frac{2j+1}{2^{m+1}}\Big) = 1,$$

(3) The function $\Lambda_{m,j}$ is affine on the two intervals
$$\Big[\frac{j-1}{2^m}, \frac{2j-1}{2^{m+1}}\Big], \qquad \Big[\frac{2j-1}{2^{m+1}}, \frac{j}{2^m}\Big].$$

Any continuous function $f\colon [0,1] \to \mathbf{C}$ has a uniformly convergent Faber-Schauder series expansion
$$f(t) = \beta(0) + \beta(1)t + \sum_{m\geqslant 0} \sum_{0\leqslant j<2^m} \beta(m,j)\Lambda_{m,j}(t),$$
with coefficients given by
$$\beta(0) = f(1), \qquad \beta(1) = f(1) - f(0),$$

and

$$\beta(m,j) = f\Big(\frac{2j+1}{2^{m+1}}\Big) - \frac{1}{2}\Big(f\Big(\frac{j}{2^m}\Big) + f\Big(\frac{j+1}{2^m}\Big)\Big) \tag{1}$$

for $m \geqslant 0$ and $0 \leqslant j < 2^m$ (see, e.g., [1, Ch. VI] for these facts).

We will prove:

Proposition 4. *Let f be a Mihalik-Wieczorek function on $[0,1]$. There exists an increasing homeomorphism $\sigma\colon [0,1] \to [0,1]$ such that the function $f \circ \sigma$ has a Faber-Schauder expansion*[1]

$$(f\circ\sigma)(t) = f(0) + f(1)t + \sum_{m\geqslant 0} \gamma_m \Lambda_{m,j(m)}(t)$$

for some integers $0 \leqslant j(m) < 2^m$ and $\gamma_m = \beta(m, j_m) \in \mathbf{C}$.

In other words, a Mihalik-Wieczorek function can be re-parameterized so that its Faber-Schauder expansion has at most one non-zero term per level m.

If we assume this fact, then Proposition 2 follows easily by linearity, using the fact that $|\gamma_m| = |\beta(m, j_m)| \leqslant 2\|f\circ\sigma\|_\infty$, and direct estimates for Fourier coefficients of the Faber-Schauder functions, namely

$$|\widehat{\Lambda}_{m,j}(h)| \ll \min(2^{-m}, 2^m h^{-2}),$$

for $m \geqslant 0$, $0 \leqslant j < 2^m$ and $h \neq 0$, where the implied constant is absolute.

We now begin the proof of Proposition 4, which follows [6, Lemma 1], and the simpler version described by Olevskii in [4, Th. 4.1]. Fix the Mihalik-Wieczorek function f. We may assume that it is not constant.

First, we observe that f satisfies the intermediate value theorem, in the following sense: if $0 \leqslant t_1 < t_2 \leqslant 1$, and $f(t_1) \neq f(t_2)$, then since $f([t_1, t_2])$ is convex, it follows that any point z in the line segment joining t_1 to t_2, which is not one of the two extremities, is of the form $z = f(t)$ for some $t \in]t_1, t_2[$.

Let $m \geqslant 1$. A *decent family* of level m is a family $T_m = (t_j)_{0\leqslant j\leqslant 2^m}$ of elements of $[0,1]$ such that the following conditions are true:

- (α) The map $\sigma_m\colon j/2^m \mapsto t_{m,j}$ is strictly increasing.
- (β) We have $\sigma_m(0) = 0$, $\sigma_m(1) = 1$.
- (γ) For $0 \leqslant j < 2^m$, either
    - (a) we have $f(t_j) \neq f(t_{j+1})$
    - (b) or there exists an open interval I_j contained in $]t_j, t_{j+1}[$ such that $f(t) = f(t_j)$ for all $t \in I_j$.

[1] Note that $f(\sigma(0)) = f(0)$ and $f(\sigma(1)) = f(1)$.

Given a decent family T_m, we denote

$$\delta_m = \delta(T_m) = \max_j |t_j - t_{j-1}|,$$

and we denote by $j(T_m)$, or simply j_m, an integer j such that $\delta_m = |t_j - t_{j-1}|$ (say the smallest j for definitess).

We observe first that there exists a decent family of level 1. Indeed, we must take $t_{0,0} = 0$ and $t_{0,2} = 0$. Since f is non-constant, its image is infinite, and we pick arbitrarily $t_{0,1} \in]0,1[$ such that $f(t_{0,1})$ is different from $f(0)$ and $f(1)$, ensuring Condition (γa).

Given a decent family $T_m = (t_j)_{0 \leqslant j \leqslant 2^m}$ of level m, we say that a decent family $T_{m+1} = (s_k)_{0 \leqslant k \leqslant 2^{m+1}}$ of level $m+1$ is a *good extension* of T_m if $s_{2k} = t_k$ for $0 \leqslant k \leqslant 2^m$, and if, for all integers $k \neq j_m$ such that $0 \leqslant k < 2^m$, we have

$$f(s_{2k+1}) = \frac{1}{2}\Big(f(t_k) + f(t_{k+1})\Big). \tag{2}$$

The following "algorithm" constructs a good extension of a fixed decent family T_m of level m.

The condition $s_{2k} = t_k$ determines all elements of T_{m+1} for even indices $0 \leqslant 2k < 2^{m+1}$. Now consider an odd index $2k+1$, where $0 \leqslant k < 2^m$. We need to insert s_{2k+1} (strictly) between $t_k = s_{2k}$ and $t_{k+1} = s_{2k+2}$. We use the following rules:

(1) If $k \neq j_m$, then
   - (i) If Condition (γa) holds for $[t_k, t_{k+1}]$, then there exists $s_{2k+1} \in]t_k, t_{k+1}[$ satisfying (2), by the Intermediate Value Property for a Mihalik-Wieczorek function.
   - (ii) If Condition (γa) fails for $[t_k, t_{k+1}]$, then (γb) holds, and we define s_{2k+1} to be the mid-point of the interval I_k on which f is constant. Then (2) also holds since f is equal to $f(t_k)$ on I_k.

(2) If $k = j_m$, then let J be the open interval with the same mid-point as $[t_k, t_{k+1}]$ and with half the length;
   - (iii) If f restricted to J is constant, let s_{2k+1} be the mid-point of J.
   - (iv) Otherwise, the set $f(J)$ is infinite; let $s_{2k+1} \in J$ be any element such that $f(s_{2k+1})$ is different from $f(t_k)$ and $f(t_{k+1})$.

Claim. The family T_{m+1} thus defined is a decent family, hence is a good extension of T_m.

First, $\sigma_{m+1}(0) = s_0 = 0$ and $\sigma_{m+1}(1) = s_{2^{m+1}} = 1$. Since the algorithm inserts the values at odd indices always strictly between the adjacent even indices, the family $(s_k)_{0 \leqslant k < 2^{m+1}}$ is strictly increasing, so Conditions (α) and (β) are true.

For Condition (γ), it suffices to check that the intervals $[t_k, s_{2k+1}]$ and $[s_{2k+1}, t_{k+1}]$ satisfy (γ) when $0 \leqslant k < 2^m$.

Consider first a triple $t_k < s_{2k+1} < t_{k+1}$ with $0 \leqslant k < 2^m$ and $k \neq j_m$. Property (2) shows that Condition (γa) holds for the intervals $[t_k, s_{2k+1}]$ and $[s_{2k+1}, t_{k+1}]$ if $f(t_k) \neq f(t_{k+1})$. If this is not the case, then the definition of s_{2k+1} in Rule (1(ii)) implies that f is constant, equal to $f(t_k) = f(t_{k+1})$, on an open neighborhood of s_{2k+1}. This implies Condition (γb) for $[t_k, s_{2k+1}]$ and $[s_{2k+1}, t_{k+1}]$.

We now consider the triple $t_k < s_{2k+1} < t_{k+1}$ with $k = j_m$. Let J be the interval used to define s_{2k+1} in Rule (2). If f is constant on J, then it is constant on an open neighborhood of s_{2k+1}; if the constant value is equal to $f(t_k)$ (resp. $f(t_{k+1})$), then Condition (γb) holds for $[t_k, s_{2k+1}]$ (resp. for $[s_{2k+1}, t_{k+1}]$), and otherwise Condition (γa) holds. If f is not constant on J, we selected s_{2k+1} in Rule (2(iv)) so that Condition (γa) holds.

We now claim that if we construct a sequence $(T_m)_{m\geqslant 1}$ of decent families starting from a decent family of level 1 *and following the algorithm above*, then the numbers $(\sigma_m(j/2^m))$ are dense in $[0, 1]$. We can then apply Lemma 3 to construct a strictly increasing homeomorphism σ such that $\sigma(j/2^m) = \sigma_m(j/2^m)$ for all $m \geqslant 0$ and $0 \leqslant j \leqslant 2^m$. Furthermore, let $m \geqslant 0$ and $0 \leqslant k < 2^m$ with $k \neq j_m$. Property (2) applied to the decent families T_m and T_{m+1} translates to

$$f\Big(\sigma\Big(\frac{2k+1}{2^{m+1}}\Big)\Big) = \frac{1}{2}\Big(f\Big(\sigma\Big(\frac{k}{2^m}\Big)\Big) + f\Big(\sigma\Big(\frac{k+1}{2^m}\Big)\Big),$$

i.e., by (1), the Faber-Schauder coefficient $\beta(m, k)$ of $f \circ \sigma$ vanishes, unless $k = j_m$. To prove that the numbers $(\sigma_m(j/2^m))$ are dense in $[0, 1]$, it suffices to prove that $\delta_m \to 0$ as $m \to +\infty$. By construction, we have $\delta_{m+1} < \delta_m$, so that if this fails, then the sequence (δ_m) converges to a real number $\delta > 0$.

Fix m_0 such that $\delta_{m_0} < 3\delta/2$. There are only finitely many intervals $[t_j, t_{j+1}]$ at level m_0 of length $\geqslant \delta$. For any $m \geqslant m_0$, by construction of the good extensions by the algorithm that was used, the (longest) interval of length δ_m at level m that corresponds to j_m is replaced (at level $m + 1$) by two intervals of length $< \frac{1}{2}\delta_m \leqslant 3\delta/4$. Hence the long intervals at level m_0 will be replaced one by one by intervals of length $\leqslant 3\delta/4$. Since all other intervals are replaced at each step by two intervals of smaller length, this leads to a contradiction since at some level m_1 this would mean that all intervals of level m_1 would have length $< \delta$.

3. Final remarks

This note is partly motivated with another open problem in classical analysis: does Sahakian's Theorem extend to complex-valued continuous functions $f\colon [0,1] \to \mathbf{C}$? As discussed by Lebedev [3, §1], it is unclear whether this is true. Of course, if this is the case, then the existence of space-filling curves would imply that the first Assertion of Proposition 1 is true, independently of the status of the Mihalik-Wieczorek problem. However, note that Lebedev further shows [3, Th. 4] that, for the Sobolev-type space $W_2^{1/2}$ of periodic functions f such that

$$\sum_{h\in\mathbf{Z}} |h||\widehat{f}(h)|^2 < \infty$$

the reparameterization property does not hold for complex-valued functions (i.e., there exist continuous functions with no reparameterization $f \circ \sigma$ in $W_2^{1/2}$).

Finally, we note this question arose in the author's work with W. Sawin and the support of a random Fourier series that appeared in our work [2] on the limiting functional distribution of partial sums of Kloosterman sums.

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