FOURIER EXPANSION OF SCHOENBERG'S SPACE-FILLING CURVE

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This note is motivated by the following question (see [1] for background context):

Question. Does there exist a continuous function $f\colon \mathbf{R}\to\mathbf{C}$ which is periodic of period 1, is *space-filling* (in the sense that the image of f has non-empty interior), and has Fourier coefficients

$$\widehat{f}(h)=\int_0^1 f(x)e(-xh)dx, \qquad e(z)=\exp(2i\pi z),$$

such that

$$\widehat{f}(h)\ll \frac{1}{|h|}$$

for $h\neq 0$?

Most "standard" space-filling curves are described using inductive constructions, from which it is very hard to understand the size of the Fourier coefficients. There is however at least one construction due to Schoenberg [2] (modifying a construction of Lebesgue) that provides a fairly simple analytic expression for a space-filling curve, and our goal is to record the elementary computation of its Fourier coefficients (although we will see that they fail to provide a positive answer to the question).

In fact, Schoenberg's function is 2-periodic, and to simplify references, we use the same convention.

Let f be the continuous function supported on $[0,2]$ such that

$$f(t)=\begin{cases}0 & \text{if } 0\leqslant t\leqslant 1/3 \text{ or} 5/3\leqslant t\leqslant 2,\\ 3t-1 & \text{if } 1/3\leqslant t\leqslant 2/3,\\ 1 & \text{if } 2/3\leqslant t\leqslant 4/3,\\ 5-3t & \text{if } 4/3\leqslant t\leqslant 5/3.\end{cases}$$

Lemma 1. *The Fourier transform*

$$\widehat{f}(h)=\int_{\mathbf{R}} f(t)e(ht)dt$$

for $h\in\mathbf{R}$ is given by

$$\widehat{f}(h)=\frac{3}{4\pi^2h^2}\Big(e\Big(\frac{2h}{3}\Big)-e\Big(\frac{h}{3}\Big)-e\Big(\frac{5h}{3}\Big)+e\Big(\frac{4h}{3}\Big)\Big)$$

if $h\neq 0$.

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Proof. We have

$$\widehat{f}(h) = \int_{1/3}^{2/3} (3t-1)e(ht)dt + \int_{2/3}^{4/3} e(ht)dt + \int_{4/3}^{5/3} (5-3t)e(ht)dt.$$

Using the formulas

$$\int_a^b e(ht)dt = \frac{1}{2i\pi h}(e(hb)-e(ha))$$
$$\int_a^b te(ht)dt = \frac{1}{2i\pi h}(be(hb)-ae(ha)) + \frac{1}{4\pi^2h^2}(e(hb)-e(ha))$$

for $h \neq 0$, simple manipulations give the result. □

Denote by f_{P} the 2-periodic function on $\mathbf{R}$ that coincides with f on $[0,2]$; it is continuous. For an integer $j \geqslant 1$, we define a 2-periodic function by

$$g_j(t) = f_{\mathrm{P}}(3^j t).$$

Lemma 2. *Let*

$$\widehat{g}_j(h) = \frac{1}{2}\int_0^2 f_{\mathrm{P}}(3^j t)e(-ht/2)dt$$

be the 2*-periodic Fourier coefficients of the function* g_j*. For* $h \neq 0$*, they are given by* $\widehat{g}_j(h) = 0$ *unless* $h \equiv 0 \pmod{3^j}$*, and*

$$\widehat{g}_j(3^j m) = \frac{\alpha(m)}{2\pi^2 m^2}$$

for $m \in \mathbf{Z}$*, where*

$$\alpha(m) = e\Big(-\frac{m}{3}\Big) - e\Big(-\frac{m}{6}\Big) - e\Big(-\frac{5m}{6}\Big) + e\Big(-\frac{2m}{3}\Big) = \begin{cases} 0 & \text{if } m \equiv 0 \pmod 2 \\ -2 & \text{if } m \equiv 1,5 \pmod 6 \\ 4 & \text{if } m \equiv 3 \pmod 6. \end{cases}$$

Proof. We have

$$\int_0^2 f_{\mathrm{P}}(3^j t)e(-ht/2)dt = \frac{1}{3^j}\int_0^{2\cdot 3^j} f_{\mathrm{P}}(u)e\Big(-\frac{hu}{2\cdot 3^j}\Big)dt.$$

Using periodicity, this becomes

$$\frac{1}{3^j}\sum_{k=0}^{3^j-1}\int_{2k}^{2k+2} f_{\mathrm{P}}(u)e\Big(-\frac{hu}{2\cdot 3^j}\Big)dt = \frac{1}{3^j}\sum_{k=0}^{3^j-1}\int_0^2 f_{\mathrm{P}}(v)e\Big(-\frac{h(v+2k)}{2\cdot 3^j}\Big)dt.$$

Hence we have

$$\int_0^2 f_{\mathrm{P}}(3^j t)e(-ht/2)dt = \frac{1}{3^j}\widehat{f}\Big(-\frac{h}{2\cdot 3^j}\Big)\Big(\sum_{k=0}^{3^j-1} e\Big(-\frac{hk}{3^j}\Big)\Big).$$

The geometric sum vanishes unless h is divisible by 3^j. When this is the case, it is equal to 3^j; using the previous lemma, we deduce that for $h = 3^j m$ with $m \in \mathbf{Z}$, we have

$$\widehat{g}_j(3^j m) = \frac{1}{2}\times\frac{3}{4\pi^2(m/2)^2}\Big(e\Big(-\frac{2m}{6}\Big) - e\Big(-\frac{m}{6}\Big) - e\Big(-\frac{5m}{6}\Big) + e\Big(-\frac{4m}{6}\Big)\Big)$$

The first formula follows, and the explicit evaluation of $\alpha(m)$ is elementary. □

Finally, the Schoenberg function $s\colon \mathbf{R} \to \mathbf{C}$ is the 2-periodic function defined by

$$s(t) = \sum_{j \geqslant 0} \frac{1}{2^{j+1}} f_{\mathrm{P}}(3^{2j}t) + i \sum_{j \geqslant 0} \frac{1}{2^{j+1}} f_{\mathrm{P}}(3^{2j+1}t)$$

(see [2]). Schoenberg shows that the image of s is the square $[0,1]^2$ (in fact, already the image of the ternay Cantor set in $[0,1]$ is the whole square).

Proposition 3. *Let $h \neq 0$ be an integer and write $h = 3^k m$ where $k \geqslant 0$ and m is coprime to 3. The 2-periodic Fourier coefficient $\widehat{s}(h)$ is given by*

$$\widehat{s}(h) = \frac{1}{2}\int_0^2 s(t)e(-ht/2)dt = \frac{1}{2\pi^2}\Big(\sum_{0 \leqslant j \leqslant k/2} \frac{1}{2^{j+1}} \frac{\alpha(3^{k-2j}m)}{(3^{k-2j}m)^2} + i \sum_{0 \leqslant j \leqslant (k-1)/2} \frac{1}{2^{j+1}} \frac{\alpha(3^{k-2j-1}m)}{(3^{k-2j-1}m)^2} \Big).$$

Proof. Since the series is absolutely and uniformly convergent, we have by the previous lemma the formula

$$\frac{1}{2}\int_0^2 \mathrm{Re}(s(t))e(-ht/2)dt = \frac{1}{2\pi^2} \sum_{\substack{j \geqslant 0 \\ 3^{2j} \mid h}} \frac{1}{2^{j+1}} \frac{\alpha(h/3^{2j})}{(h/3^{2j})^2}.$$

Since k is the 3-adic valuation of h, this becomes

$$\frac{1}{2\pi^2} \sum_{0 \leqslant j \leqslant k/2} \frac{1}{2^{j+1}} \frac{\alpha(3^{k-2j}m)}{(3^{k-2j}m)^2}.$$

Similarly, we obtain

$$\frac{1}{2}\int_0^2 \mathrm{Im}(s(t))e(-ht/2)dt = \frac{1}{2\pi^2} \sum_{0 \leqslant j \leqslant (k-1)/2} \frac{1}{2^{j+1}} \frac{\alpha(3^{k-2j-1}m)}{(3^{k-2j-1}m)^2}.$$

□

Corollary 4. *We have*

$$\limsup_{|h| \to +\infty} |h\widehat{s}(h)| = +\infty.$$

Proof. Let $h = 3^{2k}$ for some integer $k \geqslant 1$. Using the Proposition and the explicit values of α, we see that the real part of $\widehat{s}(h)$ is equal to

$$\frac{1}{2\pi^2} \sum_{0 \leqslant j \leqslant k} \frac{1}{2^{j+1}} \frac{\alpha(3^{2(k-j)})}{3^{4(k-j)}} = \frac{1}{2\pi^2}\Big(-\frac{1}{2^k} + \sum_{0 \leqslant j \leqslant k-1} \frac{1}{2^{j+1}} \frac{4}{3^{4(k-j)}} \Big)$$

since $3^{4(k-j)} \equiv 3 \pmod 6$ unless $k = j$. The second sum is equal to

$$\frac{2}{3^{4k}} \sum_{j=0}^{k-1} \Big(\frac{3^4}{2}\Big)^j = \frac{2}{3^{4k}} \frac{(3^4/2)^k - 1}{(3^4/2) - 1} \leqslant \frac{\varrho}{2^k}$$

where $\varrho = \frac{2}{(3^4/2)-1} = 4/79$. Hence

$$|\operatorname{Re}(\widehat{s}(h))| \geqslant \frac{1-\varrho}{2\pi^2}\frac{1}{2^k} \gg h^{\log(2)/\log(9)}.$$

This implies the result. □

References

[1] E. Kowalski: *Sahakian's theorem and the Mihalik–Wieczorek problem*, note, https://www.math.ethz.ch/~kowalski/sahakian-mihalik-wieczorek.pdf.

[2] I. Schoenberg: *On the Peano curve of Lebesgue*, Bulletin AMS 44 (1938), 519.

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