

# CURVES IN THEIR JACOBIAN ARE SIDON SETS

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ABSTRACT. We report new examples of Sidon sets in abelian groups arising from algebraic geometry.

Let  $A$  be an abelian group. A *Sidon set*  $S$  in  $A$  is a subset such that any solution  $(x_1, x_2, x_3, x_4) \in S^4$  of the equation

$$(1) \quad x_1 + x_2 = x_3 + x_4$$

satisfies  $x_1 \in \{x_3, x_4\}$ , i.e., any  $x \in A$  is in at most one way (up to transposition) the sum of two elements of  $S$ .

We construct some natural examples of Sidon sets in certain abelian groups, arising from algebraic geometry. In some cases, we obtain a slight variant: we say that a set  $S$  in  $A$  is a *symmetric Sidon set* if there exists  $a \in A$  such that  $S = a - S$  and the solutions to the equation above satisfy either  $x_1 \in \{x_3, x_4\}$  or  $x_2 = a - x_1$ .

**Proposition 1** (Diagonal). *Let  $k$  be a field. The diagonal subset*

$$S = \{(x, x) \mid x \in k^\times\}$$

*is a Sidon set in  $k^\times \times k$ .*

*Proof.* For elements  $(x_i, x_i) \in k^\times \times k$  for  $1 \leq i \leq 4$ , the equation (1) becomes

$$\begin{cases} x_1 x_2 = x_3 x_4 \\ x_1 + x_2 = x_3 + x_4. \end{cases}$$

Thus  $x_1$  is a solution of the polynomial equation  $(X - x_3)(X - x_4) = 0$ , and hence  $x_1 \in \{x_3, x_4\}$ . □

**Proposition 2** (Curves in their jacobians). *Let  $k$  be a field and let  $C$  be a smooth projective geometrically connected curve of genus  $g \geq 2$  over  $k$ . Let  $A$  be the jacobian of  $C$ . Assume that there is a  $k$ -rational point  $0_C \in C(k)$ , and let  $\iota: C \rightarrow A$  be the closed immersion induced by the map  $x \mapsto (x) - (0_C)$ .*

- (1) *If  $C$  is not hyperelliptic, then  $\iota(C(k))$  is a Sidon set in  $A(k)$ .*
- (2) *If  $C$  is hyperelliptic, with hyperelliptic involution  $i$ , then  $\iota(C(k))$  is a symmetric Sidon set in  $A(k)$ .*

*Proof.* Let  $x_1, x_2, x_3, x_4$  be points in  $C(k)$  such that

$$\iota(x_1) + \iota(x_2) = \iota(x_3) + \iota(x_4).$$

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If  $x_1 \notin \{x_3, x_4\}$ , this implies the existence of a rational function on  $C$  with zeros  $\{x_1, x_2\}$  and poles  $\{x_3, x_4\}$ , which corresponds to a morphism  $f: C \rightarrow \mathbf{P}^1$  of degree at most 2. This is not possible unless  $C$  is hyperelliptic (see [6, Def. 7.4.7]), proving (1).

On the other hand, if  $C$  is hyperelliptic, then since there exists on  $C$  a unique morphism to  $\mathbf{P}^1$  of degree 2, up to automorphisms (see, e.g., [6, Rem. 7.4.30]), it follows that the hyperelliptic involution exchanges the points on the fibers of  $f$ , which means that we have  $x_2 = i(x_1)$  and  $x_4 = i(x_3)$ . Conversely, for any  $x_1$  and  $x_2$ , there exists a function  $f$  with divisor  $(x_1) + (i(x_1)) - (x_2) - (i(x_2))$ , so that the equation above holds. In particular, the element  $\iota(x) + \iota(i(x))$  in  $A(k)$  is independent of  $x \in C(k)$ . If we denote it by  $a$ , then we have  $a - \iota(x) = \iota(i(x))$  for all  $x$ , hence  $a - \iota(C(k)) = \iota(C(k))$ , and we conclude that  $\iota(C(k))$  is a symmetric Sidon set.  $\square$

**Remark 3.** If  $k$  is algebraically closed, then the group  $A(k)$  can be described concretely as follows: we have  $A(k) = D_0/P$ , where

- denoting by  $D$  the free abelian group generated by formal integral linear combinations of elements of  $C(k)$ , the group  $D_0$  is the subgroup such that the sum of the coefficients is equal to 0;
- $P$  is the subgroup of  $D_0$  formed by looking at non-zero rational functions  $f$  on  $C$  and taking the combination of the sum of the zeros of  $f$ , with multiplicity, minus the sum of the poles, with multiplicity.

If  $k$  is a finite field, with some algebraic closure  $\bar{k}$ , then there is a natural action of the Frobenius automorphism of  $k$  on  $A(\bar{k})$ , and  $A$  is the set of fixed points of this action.

If  $C$  is a hyperelliptic curve, and the characteristic of  $k$  is not 2, then it can be represented by an equation  $y^2 = f(x)$  for some polynomial  $f$  of degree  $2g + 1$  or  $2g + 2$ , together with one or two points at infinity, one of which can be taken as the rational point  $0_C \in C(k)$  if  $\deg(f) = 2g + 1$  or (for instance) if  $f$  is monic (see [6, Prop. 4.24]).

These propositions apply to any field  $k$ . We now specialize to finite fields  $k$ . In the case of Proposition 1, we obtain a Sidon set of size  $|k| - 1$  in the group  $k^\times \times k$ , which is a cyclic group of order  $|k|(|k| - 1)$ . In fact, these finite Sidon sets are “the same” as those described by Ruzsa [7, Th. 4.4] using a primitive root in  $k^\times$ ; S. Eberhard has pointed out to us that they appear previously in a paper of Ganley [3, p. 323], who attributes the example to E. Spence.

In the case of Proposition 2, we obtain a Sidon set (or a symmetric Sidon set)  $S = \iota(C(k))$  of size  $|C(k)|$  that satisfies

$$|k| - 2g\sqrt{|k|} + 1 \leq |S| \leq |k| + 2g\sqrt{|k|} + 1$$

in the group  $A = A(k)$  which satisfies

$$(\sqrt{|k|} - 1)^{2g} \leq |A| \leq (\sqrt{|k|} + 1)^{2g}$$

(all these estimates follow from Weil’s proof of the Riemann Hypothesis for curves over finite fields). Thus,  $S$  has size about  $|A|^{1/g}$ .

Since there is most interest in the literature in large Sidon sets, we consider the case  $g = 2$ . Note that the curve  $C$  is then automatically hyperelliptic (see, e.g., [6, Prop. 4.9]), so that

$\iota(C(k))$  is not a Sidon set, but keeping only one element in any pair  $\{x, i(x)\}$ , we obtain a Sidon set of size about  $\frac{1}{2}|A|^{1/2}$ .

Suppose still that  $g = 2$ . If  $S$  has (close to) maximal size  $|S| = q + (4 - \varepsilon)\sqrt{q} + 1$ , then we get

$$|S| \geq |A|^{1/2} + (2 - \varepsilon)|A|^{1/4} - 2.$$

It may be interesting to note that the right-hand side is of the same shape as the upper bound  $N^{1/2} + N^{1/4} + 1$  (essentially first proved by Erdős–Turán) for the size of a Sidon set in  $\{1, \dots, N\}$ .

**Remark 4.** (1) There has been some speculation (see the blog post [4] of T. Gowers, and the comments there) that “large” Sidon sets in  $\{1, \dots, N\}$  might have some kind of algebraic structure. Since there are many hyperelliptic curves over finite fields (the space of parameters is of dimension 3), and these not infrequently have  $A(k)$  cyclic (see for instance the heuristic in [1]), Proposition 2 shows that such an algebraic structure must be sophisticated enough, in the range of sets of size  $\alpha\sqrt{N}$  for some fixed  $\alpha \geq 1/2$ , to account for jacobians of curves of genus 2 over finite fields.

(2) The fact that the sets in Propositions 1 and 2 are Sidon sets (or symmetric Sidon sets) appears naturally, and plays a key role, in our work [2, §7] in the study of the distribution of exponential sums over finite fields parameterized by characters of the groups  $k^\times \times k$  or  $A(k)$ ; the (symmetric) Sidon property allows us to compute the so-called fourth moment of the relevant (tannakian) monodromy group  $\mathbf{G}$ , and to (almost) determine it by means of the Larsen Alternative [5].

(3) The content of Proposition 2, if not the terminology, was apparently first noticed by N. Katz (unpublished).

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