Let $n \geq 1$. We denote by $\varpi_n$ the random variable on the symmetric group $S_n$ (with the uniform probability measure) that counts the number of cycles in the decomposition of $\sigma \in S_n$ as a product of distinct cycles. A classical computation based on the "Chinese Restaurant" model of a random permutation gives the probability generating function of $\varpi_n$ in the following form:

\[ E(z^{\varpi_n}) = \prod_{j=1}^{n} \left( 1 - \frac{1}{j} + \frac{z}{j} \right). \]

Indeed, this arises directly from the decomposition in law

\[ \varpi_n = \sum_{j=1}^{n} B_j, \]

where $(B_j)$ is a family of independent Bernoulli random variables with

\[ P(B_j = 0) = 1 - \frac{1}{j}, \quad P(B_j = 1) = \frac{1}{j}. \]

We will explain in this note how to derive the probability generating function (hence the identity in law, since equality of probability generating functions implies equality of characteristic functions by taking $z = e^{it}$ with $t \in \mathbb{R}$) from the dimension of symmetric powers of finite-dimensional vector spaces.

First we observe that

\[ E(z^{\varpi_n}) = \frac{1}{n!} \sum_{0 \leq \ell \leq n} \left( \sum_{\varpi_n(g) = \ell} 1 \right) z^\ell \]

is a polynomial function of $z$. It is enough then to show that (1) holds for $z$ restricted to non-negative integers.

Let $m \geq 0$ be an integer and let $V$ be an $m$-dimensional vector space over $\mathbb{C}$. We consider the linear representation of the symmetric group $S_n$ on the tensor power $V^\otimes n$ by permuting the factors of the tensor product.

We claim that the character of this representation, as a function on $S_n$, is $m^{\varpi_n}$. Indeed, if $(v_1, \ldots, v_m)$ is a basis of $V$, then a basis $B$ of $V^\otimes n$ is the family of vectors

\[ w(i_1, \ldots, i_n) = v_{i_1} \otimes \cdots \otimes v_{i_n} \]

where $1 \leq i_j \leq m$ for all $j$. The symmetric group acts on these basis vectors by permuting the factors of the tensor product, which means that

\[ g \cdot w(i_1, \ldots, i_n) = w(i_{g1}, \ldots, i_{gn}). \]
Hence the action of $g$ is described by a permutation matrix in the basis $B$. As such, the character of $V^\otimes n$ at $g$, namely the trace of this permutation matrix, is the number of basis vectors fixed by the action of $g$, i.e., the number of families $(i_1, \ldots, i_n)$ of integers between 1 and $m$ such that

$$i_j = i_{g(j)}$$

for $1 \leq j \leq n$. This means exactly that the function $j \mapsto i_j$ is constant along cycles of the permutation $g$. Since for each cycle of $g$ we can then fix arbitrarily the value of the corresponding indices in $\{1, \ldots, m\}$, we get $m^{m}(g)$ fixed basis vectors, and consequently the character value at $g$ is $m^{m}(g)$, as claimed.

It follows from this computation that

$$E(m^{m}(n)) = \frac{1}{n!} \sum_{g \in S_n} m^{m}(g)$$

and by character theory this coincides with the dimension of the invariant subspace $(V^\otimes n)^{S_n}$. Since we are in characteristic zero, the semisimplicity of the representation shows that this is the dimension of the coinvariant subspace $(V^\otimes n)^{S_n}$. But this last space is by definition the $n$-th symmetric power of $V$. Its dimension is well-known to be

$$\left(\binom{n+m-1}{m-1}\right) = \binom{n+m-1}{n} = \prod_{j=1}^{n} \frac{m+j-1}{j} = \prod_{j=1}^{n} \left(1 - \frac{1}{j} + \frac{m}{j}\right)$$

(see, e.g., Bourbaki, Algèbre, III, p. 75, Algèbre, IV, p. 3 and Ensembles, III, p. 44). Hence we deduce that

$$E(m^{m}(n)) = \prod_{j=1}^{n} \left(1 - \frac{1}{j} + \frac{m}{j}\right).$$

This means that we have established (1) for $m \geq 0$ integer. As we observed at the beginning of the proof, this implies the identity for all $z$.