SYMMETRIC POWERS AND THE CHINESE RESTAURANT PROCESS

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Let $n \geqslant 1$. We denote by ϖ_n the random variable on the symmetric group $\mathfrak{S}_n$ (with the uniform probability measure) that counts the number of cycles in the decomposition of $\sigma \in \mathfrak{S}_n$ as a product of distinct cycles. A classical computation based on the "Chinese Restaurant" model of a random permutation gives the probability generating function of ϖ_n in the following form:

$$\mathbf{E}(z^{\varpi_n}) = \prod_{j=1}^{n} \Bigl(1 - \frac{1}{j} + \frac{z}{j}\Bigr). \tag{1}$$

Indeed, this arises directly from the decomposition in law

$$\varpi_n = \sum_{j=1}^{n} B_j, \tag{2}$$

where (B_j) is a family of independent Bernoulli random variables with

$$\mathbf{P}(B_j = 0) = 1 - \frac{1}{j}, \qquad \mathbf{P}(B_j = 1) = \frac{1}{j}.$$

We will explain in this note how to derive the probability generating function (hence the identity in law, since equality of probability generating functions implies equality of characteristic functions by taking $z = e^{it}$ with $t \in \mathbf{R}$) from the dimension of symmetric powers of finite-dimensional vector spaces.

First we observe that

$$\mathbf{E}(z^{\varpi_n}) = \frac{1}{n!} \sum_{0 \leqslant \ell \leqslant n} \Bigl(\sum_{\varpi_n(g) = \ell} 1 \Bigr) z^{\ell}$$

is a polynomial function of z. It is enough then to show that (1) holds for z restricted to non-negative integers.

Let $m \geqslant 0$ be an integer and let V be an m-dimensional vector space over $\mathbf{C}$. We consider the linear representation of the symmetric group $\mathfrak{S}_n$ on the tensor power $V^{\otimes n}$ by permuting the factors of the tensor product.

We claim that the character of this representation, as a function on $\mathfrak{S}_n$, is m^{ϖ_n}. Indeed, if $(v_1, \ldots, v_m)$ is a basis of V, then a basis B of $V^{\otimes n}$ is the family of vectors

$$w(i_1, \ldots, i_n) = v_{i_1} \otimes \cdots \otimes v_{i_n}$$

where $1 \leqslant i_j \leqslant m$ for all j. The symmetric group acts on these basis vectors by permuting the factors of the tensor product, which means that

$$g \cdot w(i_1, \ldots, i_n) = w(i_{g \cdot 1}, \ldots, i_{g \cdot n}).$$

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Hence the action of g is described by a permutation matrix in the basis B. As such, the character of $V^{\otimes n}$ at g, namely the trace of this permutation matrix, is the number of basis vectors fixed by the action of g, i.e., the number of families $(i_1, \ldots, i_n)$ of integers between 1 and m such that

$$i_j = i_{g\cdot j}$$

for $1 \leqslant j \leqslant n$. This means exactly that the function $j \mapsto i_j$ is constant along cycles of the permutation g. Since for each cycle of g we can then fix arbitrarily the value of the corresponding indices in $\{1, \ldots, m\}$, we get $m^{\varpi_n(g)}$ fixed basis vectors, and consequently the character value at g is $m^{\varpi_n(g)}$, as claimed.

It follows from this computation that

$$\mathbf{E}(m^{\varpi_n}) = \frac{1}{n!} \sum_{g \in \mathfrak{S}_n} m^{\varpi_n(g)}$$

and by character theory this coincides with the dimension of the invariant subspace $(V^{\otimes n})^{\mathfrak{S}_n}$. Since we are in characteristic zero, the semisimplicity of the representation shows that this is the dimension of the coinvariant subspace $(V^{\otimes n})_{\mathfrak{S}_n}$. But this last space is by definition the n-th symmetric power of V. Its dimension is well-known to be

$$\binom{n+m-1}{m-1} = \binom{n+m-1}{n} = \prod_{j=1}^{n} \frac{m+j-1}{j} = \prod_{j=1}^{n} \Bigl(1 - \frac{1}{j} + \frac{m}{j}\Bigr)$$

(see, e.g., Bourbaki, Algèbre, III, p. 75, Algèbre, IV, p. 3 and Ensembles, III, p. 44). Hence we deduce that

$$\mathbf{E}(m^{\varpi_n}) = \prod_{j=1}^{n} \Bigl(1 - \frac{1}{j} + \frac{m}{j}\Bigr).$$

This means that we have established (1) for $m \geqslant 0$ integer. As we observed at the beginning of the proof, this implies the identity for all z.

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