ETH
25th Z. Th. Seminar
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On Zhang's Theorem

This is a survey of the recent result of Y. Zhang:

Th We have

$$(*) \qquad \liminf (p_{n+1} - p_n) < +\infty$$

where p_n is the n-th prime number.

There will be no original work mentionned.

Zhang's strategy is roughly as follows:

(1) An adaptation of the method of Goldston-Pintz-Yildirim which led them to:

Th' Assume $ED(\Lambda) > \frac{1}{2}$. Then $(*)$ is true.

Here $ED(f)$, for an arithmetic function f, refers to a number called the exponent of distribution of f, which measures how well, and especially how uniformly $f(n)$, $n \leq X$, is distributed in arithmetic progressions modulo $q \leq X^\epsilon$ and Λ is the von Mangoldt function.

The point of this result was (1) that it linked $(*)$ to well-studied and widely believed conjectures (namely, one expects $ED(\Lambda) = 1$); (2) that one was not far: the celebrated Bombieri-Vinogradov theorem implies that

Def. $f(n)$ s.t. $|f(n)| \le (\log 2n)^{A_1} \, d(n)^{A_2}$

$ED(f) \geq \theta \iff$

$$\sum_{q \le X^{\theta}} \max_{(a,q)=1} \left| \sum_{\substack{n \le x \\ n \equiv a (q)}} f(n) - \frac{1}{\varphi(q)} \sum_{n \le x} f(n) \right| \ll_A \frac{x}{(\log x)^A}$$

$ED(\Lambda) \geq \frac{1}{2}$ $\boxed{+}$. In fact, using this, GPY deduced the striking result

$$\liminf \frac{p_{n+1}-p_n}{\log n} = 0.$$

(2) Zhang identifies a variant $ED_Z(\Lambda)$ of the exponent of distribution such that

(also observed by Motohashi-Pintz) $\longrightarrow$ (i) $(*)$ follows still from $ED_Z(\Lambda) > \frac{1}{2}$

(ii) ~~one~~ he can prove $ED_Z(\Lambda) > \frac{1}{2}$ ...

(3) ... for the proof of which a number of ideas come into play, inspired by certain well-established methods of analytic number theory, which had been exploited by Fouvry - Iwaniec, Bombieri - Friedlander - Iwaniec, Friedlander - Iwanice to prove that various functions have $ED(\cdot) > \frac{1}{2}$

(4) ... which require crucially some cases of the Riemann Hypothesis over finite fields $\longrightarrow$ the Weil bound for Kloosterman sums

$\longrightarrow$ an estimate of Birch - Bombieri used by Friedlander - Iwaniec to show that $ED(d_3) > \frac{1}{2}$

$$d_3(n) = \sum_{abc=n} 1$$

$\boxed{+}$ $ED(f) \geq \frac{1}{2}$ is a rather general property which can often be established "cheaply" using the large sieve inequality and Siegel-Walfisz theorem. It is also a consequence of GRH.

I want to present just some highlights.

1. <u>The GPY idea</u>

Consider a k-tuple $\underline{h} = (h_1, \ldots, h_k)$ of strictly increasing ~~distinct~~ positive integers s.t.

$$\forall p, \quad |\{h_i \bmod p\}| < p$$

Then no "obvious" reason prevents the existence of n's s.t.

$$n + h_1, \ldots, n + h_k$$

are all primes. Fix <u>one</u> such $\underline{h}$.

Define

$$Q_1(N) = \sum_{\underbrace{N < n \leq 2N}_{n \sim N}} \left(\sum_{i=1}^{k} \Lambda(n + h_i) \right) \underbrace{\mathcal{E}(n)}_{\geq 0 \text{ to be determined}}$$

$$Q_2(N) = \sum_{n \sim N} (\log 3N) \, \mathcal{E}(n)$$

<u>If</u> $Q_1(N) > Q_2(N)$ then

$$\exists\, n \sim N \quad \text{s.t.} \quad \sum \Lambda(n + h_i) > \log 2N$$

which means $\exists\, n \sim N$ and $i \neq j$ with

$$n + h_i, \quad n + h_j \quad \text{primes (powers)}$$

If this holds for all N large enough we get

$$\liminf \, (p_{n+1} - p_n) \leq h_k - h_1 < +\infty.$$

2. <u>Choice of $\mathcal{E}(n)$</u>

<u>GPY</u>:

$$\mathcal{E}(n) = \left(\frac{1}{(k+\ell)!} \sum_{\substack{d \mid P(n) \\ d \leq x}} \mu(d) \left(\log \frac{x}{d} \right)^{k+\ell} \right)^2$$

where $\begin{cases} Q(x) = \prod (x+h_i) \\ x = N^{\alpha} \quad \text{for some (fixed) } \alpha > 0 \\ \ell \geq 1 \quad \text{is some extra parameter} \\ \quad\quad \text{(will be } \simeq \sqrt{k}\text{)} \end{cases}$ (that wa lar

<u>Zhang:</u>

$$\mathcal{E}_z(n) = \Big(\frac{1}{(k+\ell)!} \sum_{\substack{d \mid Q(n) \\ d \leq x \\ d \mid P(z)}} \quad \_\_\_\_\_\_ \Big)^2 \qquad P(z) = \prod_{p<z} p$$

where $z = N^{\beta}$ (β "small"), i.e. we restrict d in the GPY weights to integers <u>without large prime factors</u>. What do we gain? <u>Flexibility</u>, more precisely the possibility to arrange later expressions in <u>bilinear forms</u> (this is very similar to the idea of <u>well-factorable weights</u> of Iwaniec, which are crucial ~~in some~~ in the deepest known results on primes in arithmetic progressions).

The idea to evaluate Q_1, Q_2, is simple: expand the square in $\mathcal{E}_z(n)$, exchange the order of the sum to get (say)

$$Q_1(n) = \sum_{i=1}^{k} \frac{1}{((k+\ell)!)^2} \sum_{\substack{d_1, d_2 \leq x \\ d_i \mid P(z)}} \mu(d_1)\mu(d_2)\Big(\log\frac{x}{d_1}\Big)^{k+\ell}\Big(\log\frac{x}{d_2}\Big)^{k+\ell}$$

$$\underbrace{\sum_{\substack{n \sim N \\ Q(n) \equiv 0\ [d_1, d_2]}} \Lambda(n+h_i)}$$

The inner sum is

$$\sum_{\substack{a \bmod [d_1, d_2] \\ Q(a) \equiv 0}} \underbrace{\sum_{\substack{n \sim N \\ n \equiv a ([d_1, d_2])}} \Lambda(n+h_i)}_{MT + ET}$$

and the contribution of the error term is bounded (essentially) by

fixed

$$\frac{1}{((k+l)!)^2} \sum_i \sum_{\substack{q \leq x^2 \\ (q, \mathcal{P}(\delta)) = 1}} \sum_{\substack{a \bmod q \\ Q(a-h_i) \equiv 0 (q) \\ a \neq h_i}} | E_\Lambda(N, q, a$$

where $E_\Lambda(N, q, a) = \sum_{\substack{n \sim N \\ n \equiv a (q)}} \Lambda(n) - \frac{1}{\varphi(q)} \sum_{n \sim N} \Lambda(n)$

The main term turns out to be positive (for $\begin{cases} N \text{ large enough} \\ l \approx \sqrt{k} \end{cases}$) provided $2\alpha > \frac{1}{2}$

2 - Bilinear tricks

Ideas going back to Vinogradov give decompositions of sums over $\sum_{n \sim N}$ of $\Lambda(n) f(n)$ in terms of simpler expressions, the most important of which are

$$\mathcal{B}(\alpha, \beta) = \sum_{n \sim N_1} \sum_{m \sim M_1} \alpha(n) \beta(m) f(mn)$$

with $N_1, M_1 \approx N$, and α, β relatively arbitrary (bounded) functions which oscillate but rather uncontrollably

One then usually tries to write

$$|B(\alpha,\beta)|^2 \leq \left(\sum_m |\beta(m)|^2\right) \underbrace{\left(\sum_m \left|\sum_n \alpha(n) f(mn)\right|^2\right)}_{\displaystyle \sum_{n_1,n_2} \alpha(n_1)\overline{\alpha(n_2)} \underbrace{\sum_m f(mn_1)\overline{f(mn_2)}}}$$

and one expects that usually [that sum] will be rather small unless $n_1 = n_2$ (diagonal contribution).

Zhang uses such a decomposition to reduce to (**)

(1) such bilinear expression $f = \alpha * \beta$ with M_1, N_1 constrained in specific ways (especially N_1 is not too small)

(2) "type III" expressions ~~[illegible]~~

$$f(n) = \sum_{n_1 \sim N_1} \sum_{\substack{n_2 \\ \sim N_2}} \sum_{\substack{n_3 \\ \sim N_3}} \sum_{\substack{m \sim M_1 \\ m n_1 n_2 n_3 = n}} \alpha(m), \quad M_1 N_1 N_2 N_3 \approx N, \quad M N_1 \leq N^{3/8+\delta}$$

3 - Type II expressions

We basically have to deal with

$$\sum_{q \leq N^{2\alpha}} \sum_{a \in \mathcal{C}(q)} |\Delta(\underset{\substack{\shortparallel \\ \alpha * \beta}}{f}; q, a)| \qquad \alpha = \tfrac{1}{4} + \underset{\substack{| \\ \text{small } \delta}}{\delta}$$

~~[illegible]~~

~~$\sum_q \sum_a \ldots$~~

(6) (**) Proving that $ED_2(f) > \frac{1}{2}$ for f of two types

This requires <u>another</u> bilinear structure on the modulus:

<u>**Lemma**</u> $\forall\ \Delta > 1$, $\left\{\begin{array}{l} q \leq N^{2a} \\ q \mid P(z) \end{array}\right.$ we can write

$$q = rs$$

where $\frac{\Delta}{z} < r < \Delta$.

This leads to expression

$$\sum_{\frac{\Delta}{z} < r < \Delta} \ \sum_{a \in \mathcal{C}(r)} \ \sum_{s} \ \sum_{b \in \mathcal{C}(s)} \varepsilon(r, s, a, b) \ \Delta(\alpha * \beta ; rs, (a, b$$

One interprets this as a bilinear expression in r and s and rearrange the sum, apply Cauchy-Schwarz ... the <u>Dispersion method</u> of Linnik

one has to control averages involving things like

$$\sum_{n} \cdots \sum_{\substack{mn \equiv a\ (r) \\ mn \equiv b\ (s)}} \alpha(m)$$

and one detects congruences $m n \equiv a$ by

$$1_{m \equiv a\bar{n}\ (r)} = \frac{1}{r} \sum_{x \bmod r} e\left(\frac{(m - a\bar{n})x}{r}\right)$$

which lead to Kloosterman sums when summing over $\underline{\underline{n}}$ first ...

4 - <u>Type III expression</u>

We have to deal with sums like

$$\sum_{m\simeq M} \alpha(m) \sum\sum\sum_{\substack{n_1,\, n_2,\, n_3 \\ n_i \simeq N \\ m n_1 n_2 n_3 \equiv a\ (q)}} 1 \;-\; \frac{1}{\varphi(q)} \sum_m \alpha(m) \sum_{n_i} 1$$

with $\begin{cases} q \simeq N^{1/2+\delta}, \ \delta > 0 \ \text{(small)} \\ M N_1 N_2 N_3 \simeq N \end{cases}$, and the crucial case turns out to have

$$\begin{cases} M \simeq q^{1/8} \quad \text{(a tiny bit larger)} \\ N_1, N_2, N_3 \simeq q^{5/8} \end{cases}$$

One (may) begin by applying Poisson summation to handle one variable and this "detects" the main term.

$$\sum_{\substack{n_1 \simeq N_1 \\ n_1 \equiv \overline{a m n_2 n_3}\ (q)}} \varphi(n_1) = \frac{1}{\varphi(q)} \sum_{n_1} \varphi(n_1) + \frac{1}{q} \sum_{1 \le |h| \ll q/N_1} \hat{\varphi}\left(\frac{h}{q}\right) e\left(\frac{-a\overline{m n_2 n_3}}{q}\right)$$

$|\hat{\varphi}| \ll N$

test function C_c^∞

This reduces to the proof of:

<u>Th</u>[4] (Zhang). For some $\delta > 0$ small <u>enough</u> we have

$$\sum_{1 \le |h| \ll q/N_1} \hat{\varphi}\left(\frac{h}{q}\right) \sum_{n_2} \sum_{n_3} e\left(-\frac{a\overline{m n_2 n_3}}{q}\right) \ll \frac{N^{1-\delta}}{N_1}$$

This is highly non-trivial, but note that there are no prime numbers here!

This is close to (but not the same as) expression considered by Friedlander - Iwaniec to prove $EO(d_3) \geq \frac{1}{2} + \frac{1}{230}$.

Zhang manages to adapt some ideas of that result, and ultimately reduces to the proof of:

Th. (Birch - Bombieri) $\exists\, C \geq 1$ s.t.

Let p be a prime, $\psi(a) = e\left(\frac{a}{p}\right) = e^{2i\pi a/p}$ for $a \bmod p$

Then for $\alpha, \beta, \gamma \in \mathbb{F}_p^\times$ we have

$$\left| \sum_{\substack{x, y, z \in \mathbb{F}_p^\times \\ x+\alpha \neq 0}} \psi\left[\frac{y}{x} + \frac{z}{x+\alpha} + \frac{\beta}{y} + \frac{\gamma}{z}\right] \right| \leq C p^{3/2}.$$

Proof (F-K-M)

We interpret this as

$$p \sum_{x} \underbrace{\left(\frac{1}{\sqrt{p}} \sum_{y} \psi\left(\frac{y}{x} + \frac{\beta}{y}\right)\right)}_{Kl_2\left(\frac{\beta}{x}\right)} \overline{\left(\frac{1}{\sqrt{p}} \sum_{z} \psi\left(\frac{\gamma}{z} + \frac{z}{x+\alpha}\right)\right)}$$

$$Kl_2\left(\frac{\beta}{x}\right) \overline{Kl_2\left(\frac{\gamma}{x+\alpha}\right)}$$

$$= p \sum_{v} Kl_2(v) \overline{Kl_2(\sigma \cdot v)}$$

$$\sigma = \begin{pmatrix} \gamma & 0 \\ \alpha & \beta \end{pmatrix}$$

Deligne: $Kl_2(v) = \mathrm{Tr}(\mathrm{Frob}_v \mid \mathcal{K})$

$\mathcal{K}$ sheaf ramified at $0, \infty$

so we have

$$\sum \underbrace{Kl_2(v) \overline{Kl_2(\sigma v)}}_{\text{ramified at } 0 \text{ and } -\beta/\alpha \neq \infty}$$

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and by Deligne

$$|\text{——}| \leq \sqrt{p} \underbrace{\dim H^1_c(\mathbb{A}^1 \setminus \{0, -\beta/\alpha\}, \mathcal{K}\otimes\sigma^*\mathcal{K})}_{\leq 6 \text{ by Euler-Poincaré computation.}}$$

$\square$

<u>Note:</u> Zhang still uses a factorization $q = rs$ with r quite small; he gains an <u>essential</u> factor $r^{-1/2}$ from r, thanks to Ramanujan sums being even <u>smaller</u> than square-root cancellation...