

# Mathematical games

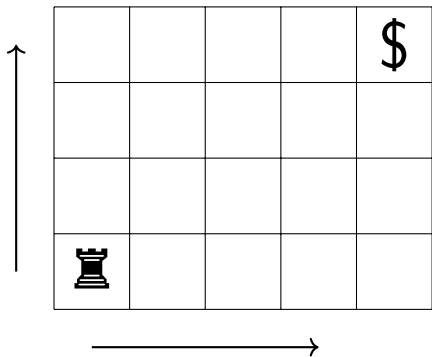
Kaloyan Slavov

Department of Mathematics  
ETH Zürich

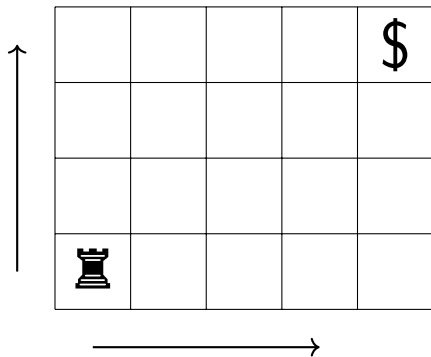
`kaloyan.slavov@math.ethz.ch`

April 7, 2016

# Rook

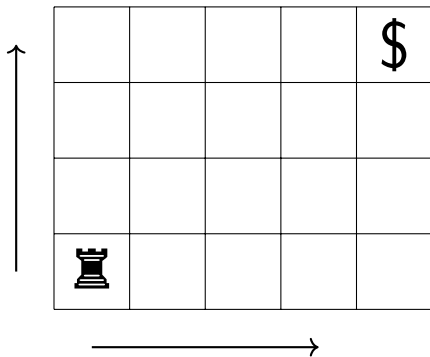


# Rook



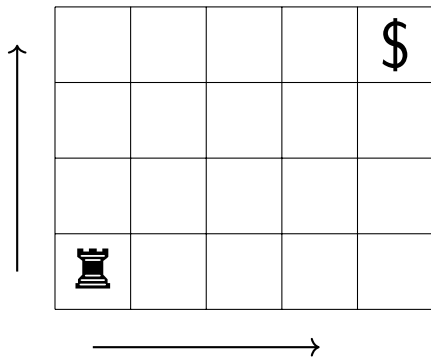
- Alice and Bob alternate turns in moving a rook

# Rook



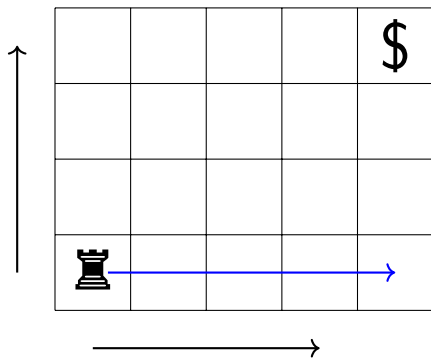
- Alice and Bob alternate turns in moving a rook
- Possible moves:
  - horizontally right, or
  - vertically up

# Rook



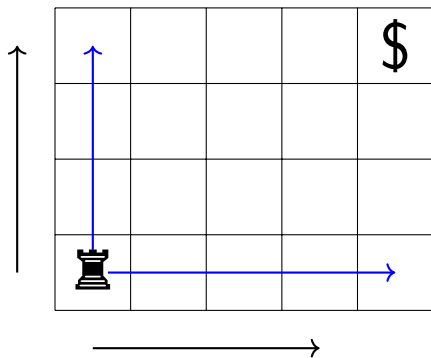
- Alice and Bob alternate turns in moving a rook
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- The player who first reaches the \$, wins.

# Rook



- Alice and Bob alternate turns in moving a rook
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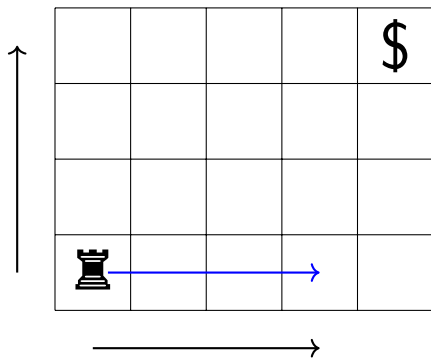
# Rook



bad moves!

- Alice and Bob alternate turns in moving a rook
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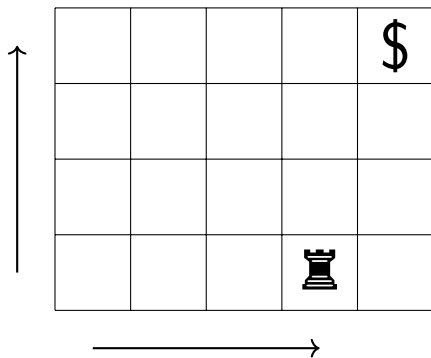
# Rook



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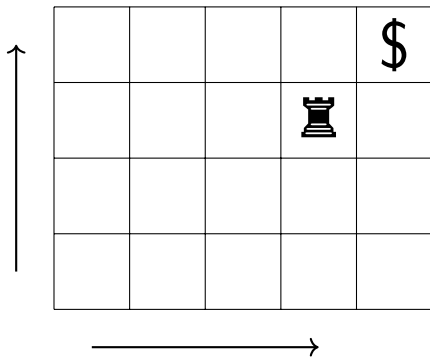


# Rook



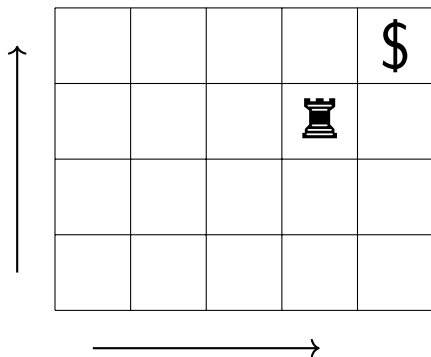
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# Rook



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# Rook



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- Possible moves:
  - horizontally right, or
  - vertically up
- The player who first reaches the \$, wins.

Alice loses this particular game.

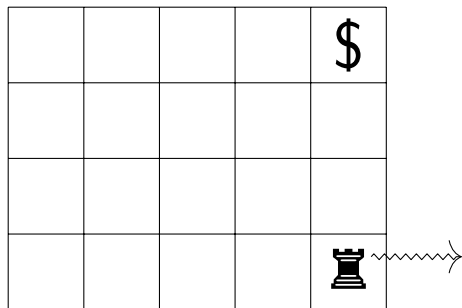
# Backward analysis

				\$

# Backward analysis

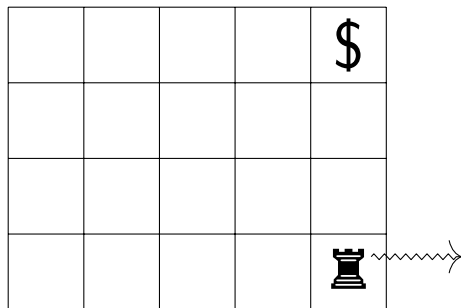
				\$
				

# Backward analysis



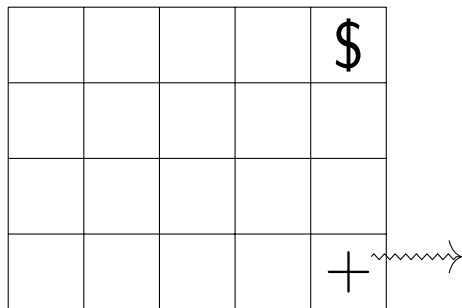
If the rook is here and it is your turn, then you can win, no matter how your opponent plays.

# Backward analysis



If the rook is here and it is your turn, then you can win, no matter how your opponent plays.  
**winning position**

# Backward analysis




If the rook is here and it is your turn, then you can win, no matter how your opponent plays.  
**winning position**



# Backward analysis


+	+	+	+	\$
				+
				+
				+



If the rook is here and it is your turn, then you can win, no matter how your opponent plays.  
**winning position**

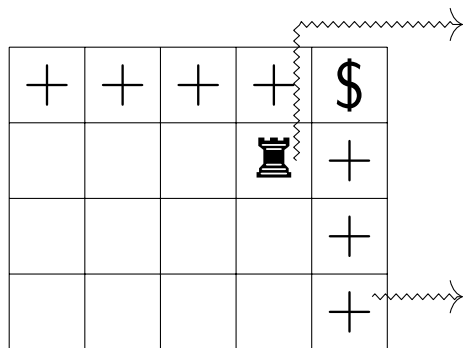
# Backward analysis

+	+	+	+	\$
			♖	+
				+
				+



If the rook is here and it is your turn, then you can win, no matter how your opponent plays.  
**winning position**

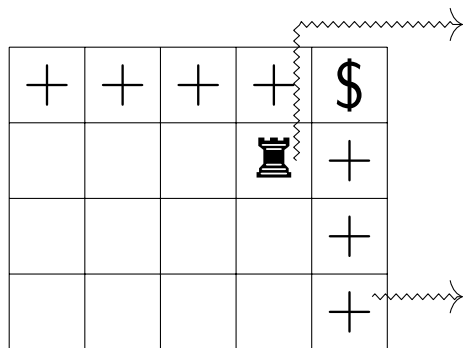
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If the rook is here and it is your turn, then your opponent can win, no matter how you play.

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# Backward analysis



If the rook is here and it is your turn, then your opponent can win, no matter how you play.  
**losing position**

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# Backward analysis


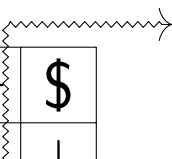
+	+	+	+	\$
			-	+
				+
				+

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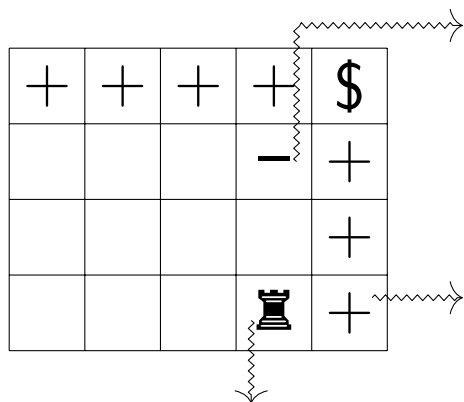
+	+	+	+	\$
			-	+
				+
			♖	+



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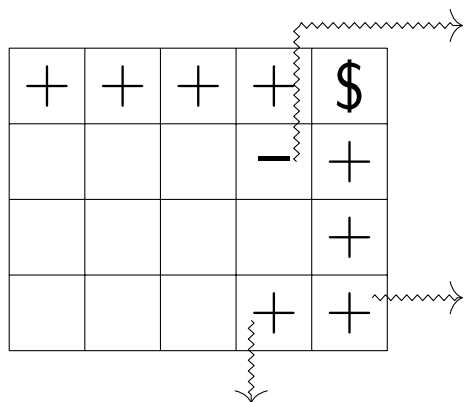


There **exists** a move to a (-)

If the rook is here and it is your turn, then your opponent can win, no matter how you play.  
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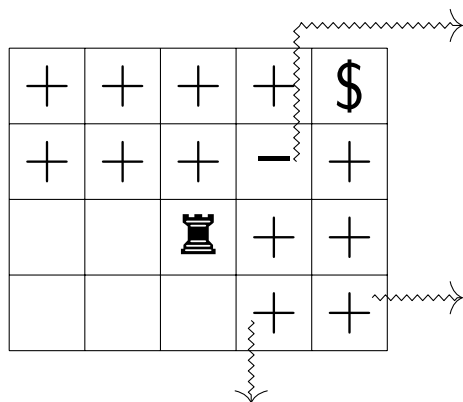
+	+	+	+	\$
+	+	+	-	+
			+	+
			+	+

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# Backward analysis



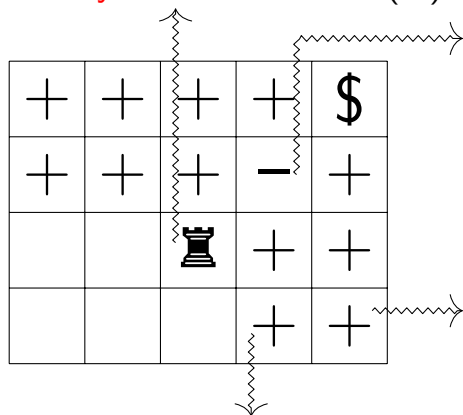
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# Backward analysis

Every move leads to a (+)



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+	+	+	+	\$
+	+	+	-	+
		-	+	+
			+	+

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+	+	+	+	\$
+	+	+	-	+
+	+	-	+	+
		+	+	+

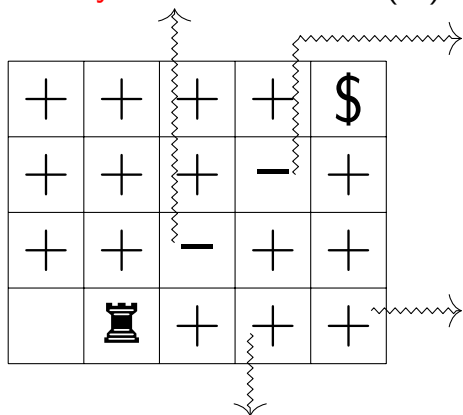
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+	+	+	-	+
+	+	-	+	+
	-	+	+	+

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+	+	+	-	+
+	+	-	+	+
+	-	+	+	+

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Every move leads to a (+)

+	+	+	+	\$
+	+	+	-	+
+	+	-	+	+
+	-	+	+	+

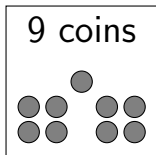
There exists a move to a (-)

⇒ Alice has a winning strategy.

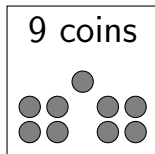
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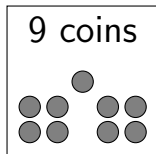
If the rook is here and it is your turn, then you can win, no matter how your opponent plays.  
winning position

# Nim

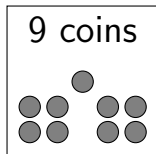


- Alice and Bob alternate turns in taking coins



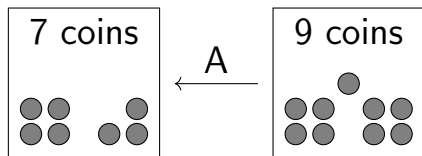


- Alice and Bob alternate turns in taking coins
- On a move, a player can take **1** or **2** coins



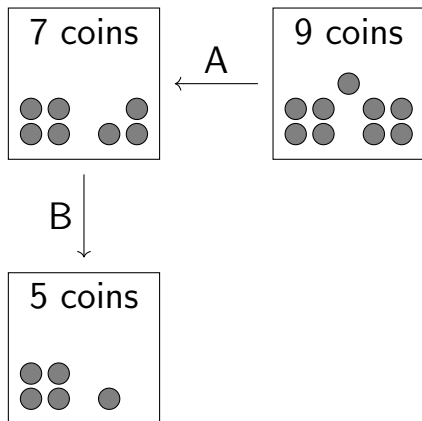
- Alice and Bob alternate turns in taking coins
- On a move, a player can take **1** or **2** coins
- The player who takes the last coin, wins

# Nim



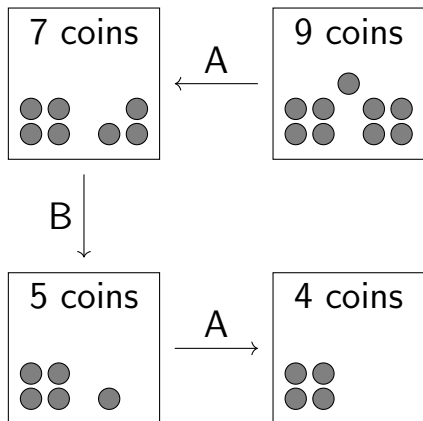
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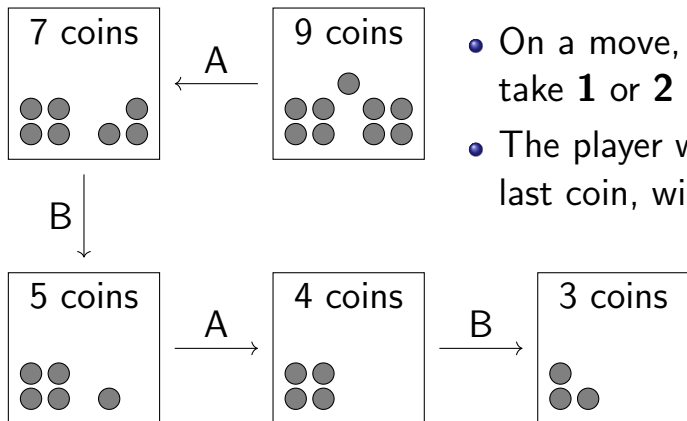
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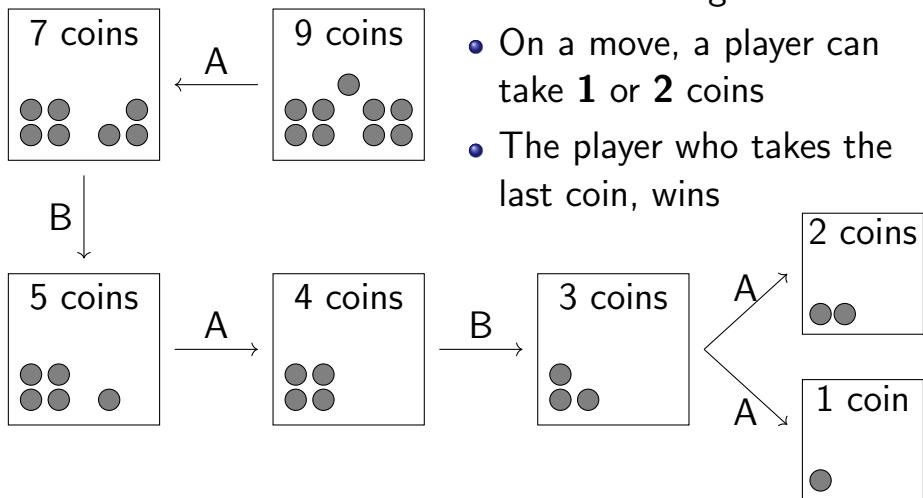


# Nim



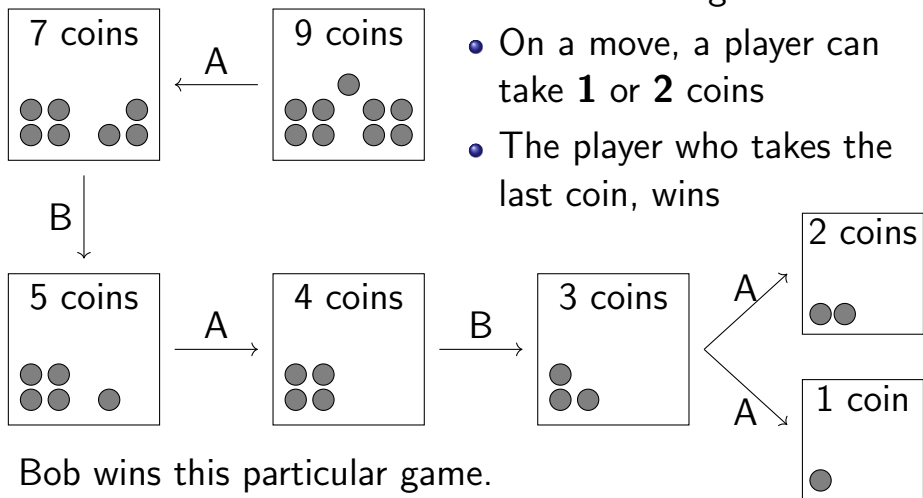
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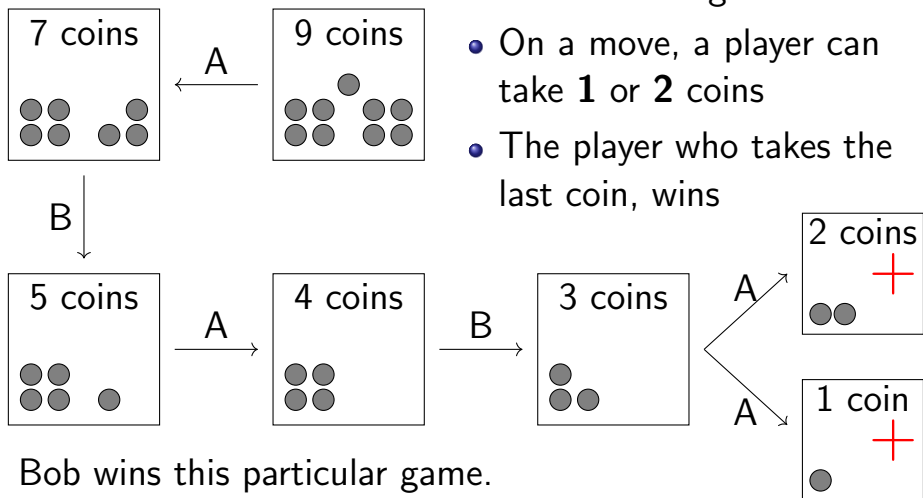
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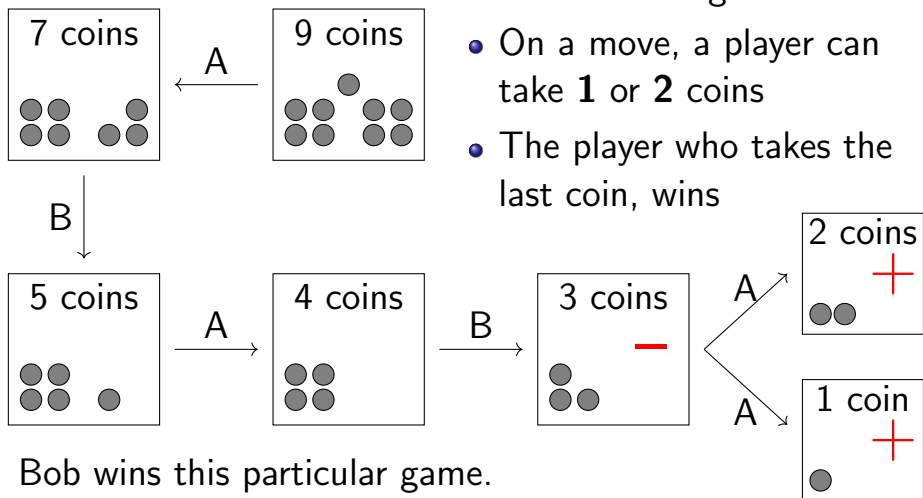
# Nim



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- The player who takes the last coin, wins

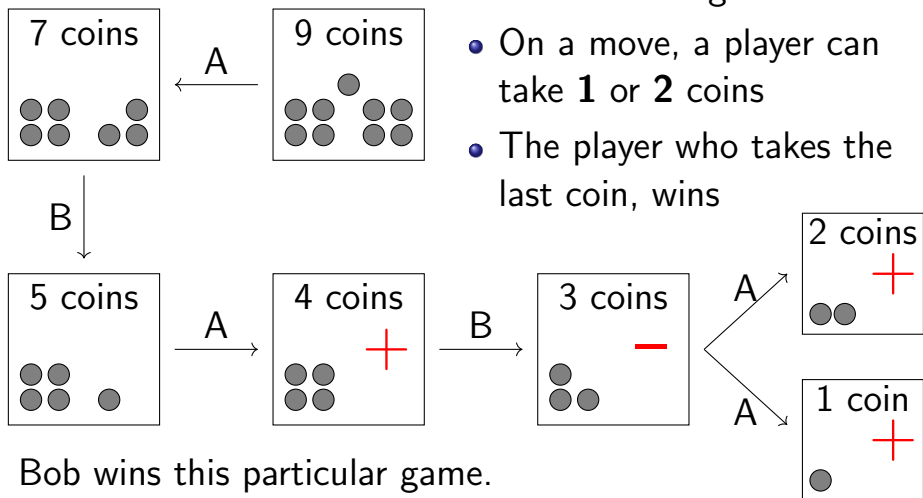
Bob wins this particular game.

# Nim



- Alice and Bob alternate turns in taking coins
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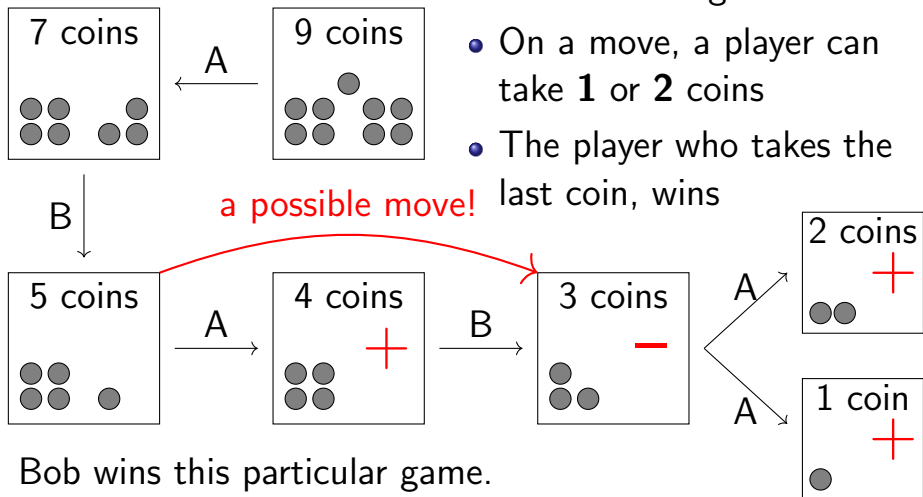
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# Nim

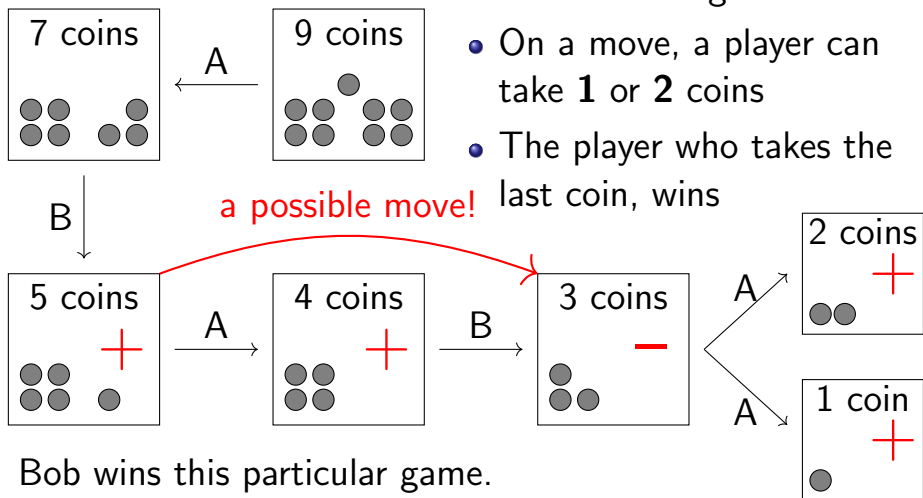
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- The player who takes the last coin, wins



Bob wins this particular game.

# Nim

- Alice and Bob alternate turns in taking coins
- On a move, a player can take **1** or **2** coins
- The player who takes the last coin, wins





# Winning strategy

number of coins	1	2	3	4	5	6	7	8	9
position									

# Winning strategy

number of coins	1	2	3	4	5	6	7	8	9
position	+	+							

# Winning strategy

number of coins	1	2	3	4	5	6	7	8	9
position	+	+	-						

# Winning strategy

number of coins	1	2	3	4	5	6	7	8	9
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There **exists**  
a move to a (-)

# Winning strategy

number of coins	1	2	3	4	5	6	7	8	9
position	+	+	-	+					

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number of coins	1	2	3	4	5	6	7	8	9
position	+	+	-	+	+				

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# Winning strategy

number of coins	1	2	3	4	5	6	7	8	9
position	+	+	-	+	+				

There **exists** a move to a (-) **Every** move leads to a (+)

# Winning strategy

number of coins	1	2	3	4	5	6	7	8	9
position	+	+	-	+	+	-			

There **exists** a move to a (-) **Every** move leads to a (+)



# Winning strategy

number of coins	1	2	3	4	5	6	7	8	9
position	+	+	-	+	+	-	+	+	

There **exists** a move to a (-) **Every** move leads to a (+)

# Winning strategy

number of coins	1	2	3	4	5	6	7	8	9
position	+	+	-	+	+	-	+	+	-

There **exists** a move to a (-) **Every** move leads to a (+)

# Winning strategy

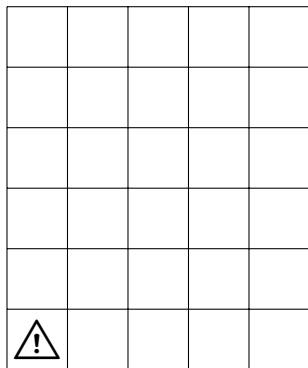
number of coins	1	2	3	4	5	6	7	8	9
position	+	+	-	+	+	-	+	+	-

There **exists** a move to a (-) **Every** move leads to a (+)

$\implies$  Bob has a winning strategy.

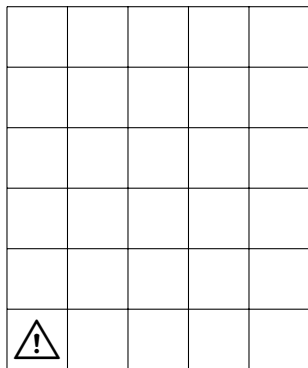
# Chomp

A B



# Chomp

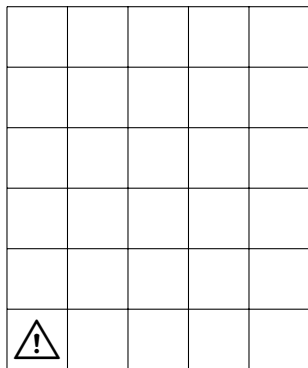
Ⓐ B



- Alice and Bob alternate turns in biting chocolate blocks

# Chomp

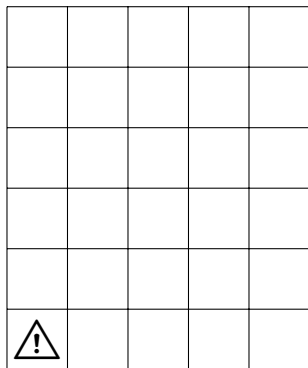
A B




- Alice and Bob alternate turns in biting chocolate blocks
- On a move, a player chooses a block and takes out all blocks north-east of it

# Chomp

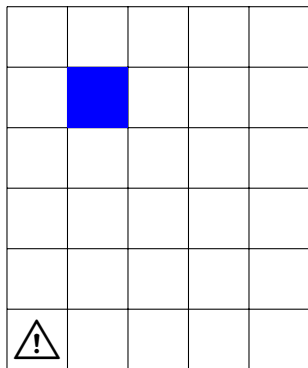
Ⓐ B




- Alice and Bob alternate turns in biting chocolate blocks
- On a move, a player chooses a block and takes out all blocks north-east of it
- The player forced to take the , loses.

# Chomp

Ⓐ B

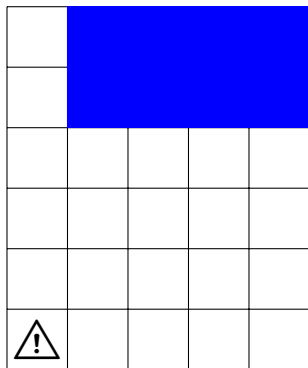


- Alice and Bob alternate turns in biting chocolate blocks
- On a move, a player chooses a block and takes out all blocks north-east of it
- The player forced to take the , loses.



# Chomp

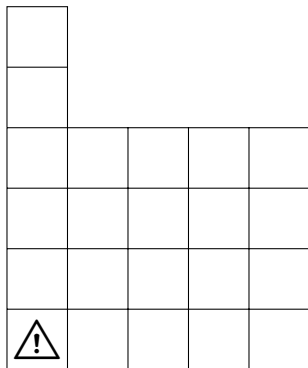
Ⓐ B



- Alice and Bob alternate turns in biting chocolate blocks
- On a move, a player chooses a block and takes out all blocks north-east of it
- The player forced to take the ⚠, loses.

# Chomp

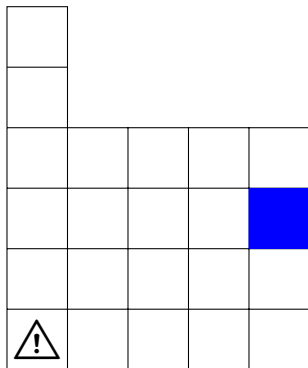
A (B)




- Alice and Bob alternate turns in biting chocolate blocks
- On a move, a player chooses a block and takes out all blocks north-east of it
- The player forced to take the ⚠, loses.

# Chomp

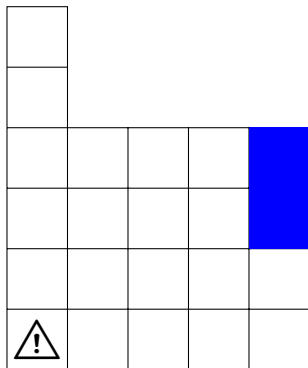
A    **B**



- Alice and Bob alternate turns in biting chocolate blocks
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# Chomp

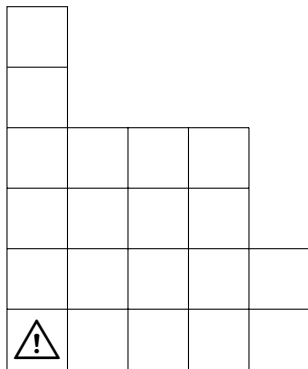
A    **B**



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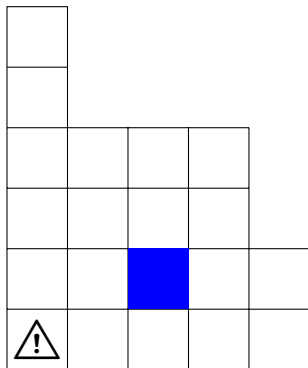
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# Chomp

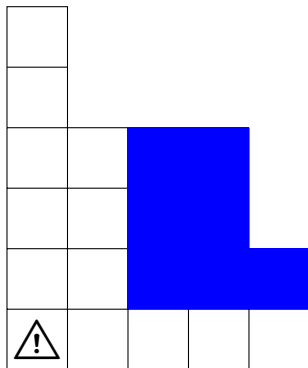
Ⓐ B




- Alice and Bob alternate turns in biting chocolate blocks
- On a move, a player chooses a block and takes out all blocks north-east of it
- The player forced to take the ⚠, loses.

# Chomp

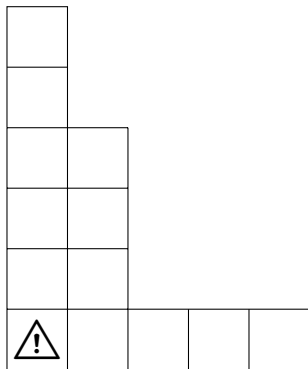
Ⓐ B



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# Chomp

A    **B**

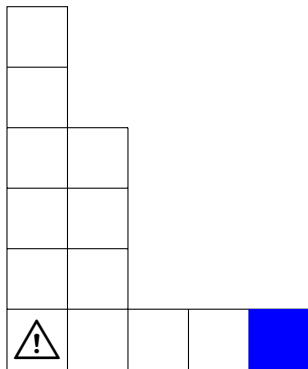



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# Chomp

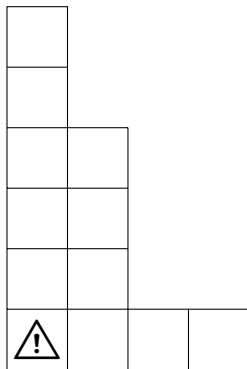
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# Chomp

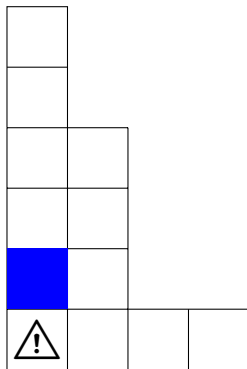
Ⓐ B




- Alice and Bob alternate turns in biting chocolate blocks
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# Chomp

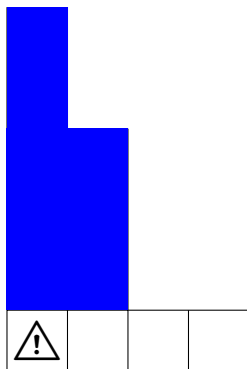
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# Chomp


A    **B**

- Alice and Bob alternate turns in biting chocolate blocks
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# Chomp

Ⓐ B

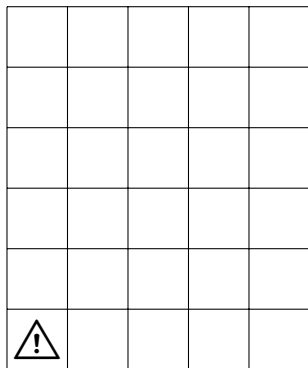
- Alice and Bob alternate turns in biting chocolate blocks
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- The player forced to take the , loses.




Bob wins this particular game.

# Chomp

Ⓐ B



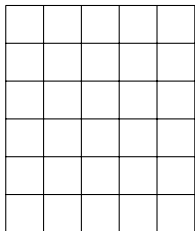
- Alice and Bob alternate turns in biting chocolate blocks
- On a move, a player chooses a block and takes out all blocks north-east of it
- The player forced to take the , loses.

**Claim:** Alice has a winning strategy.



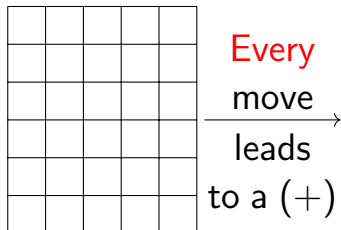
# Proof

Suppose that *Bob* has a winning strategy.



# Proof

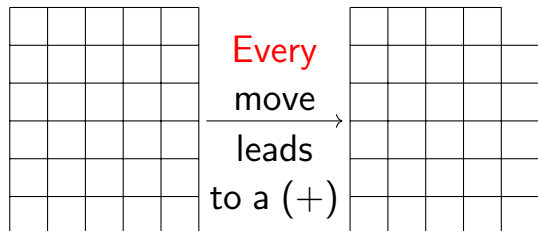
Suppose that *Bob* has a winning strategy.



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# Proof

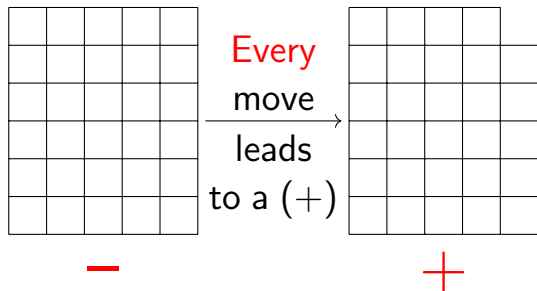
Suppose that *Bob* has a winning strategy.



—

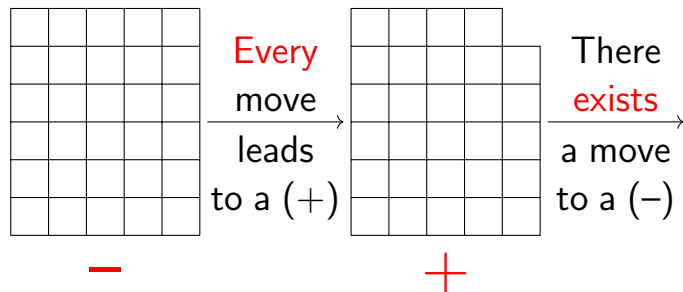
# Proof

Suppose that *Bob* has a winning strategy.



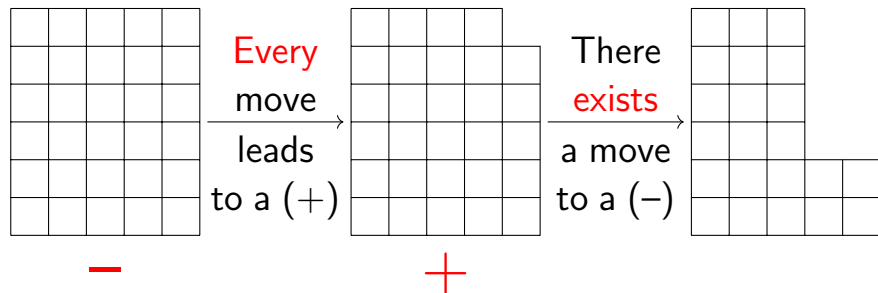
# Proof

Suppose that *Bob* has a winning strategy.



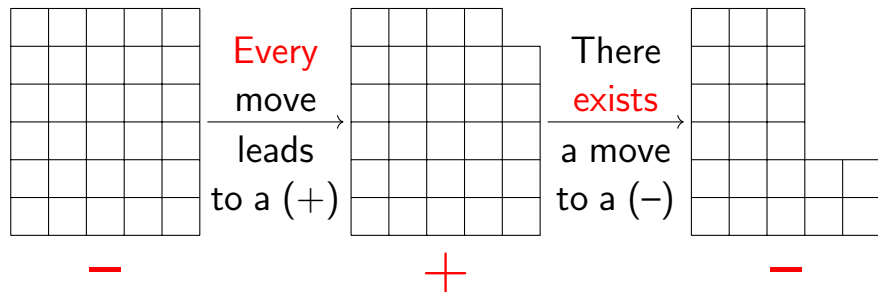
# Proof

Suppose that *Bob* has a winning strategy.



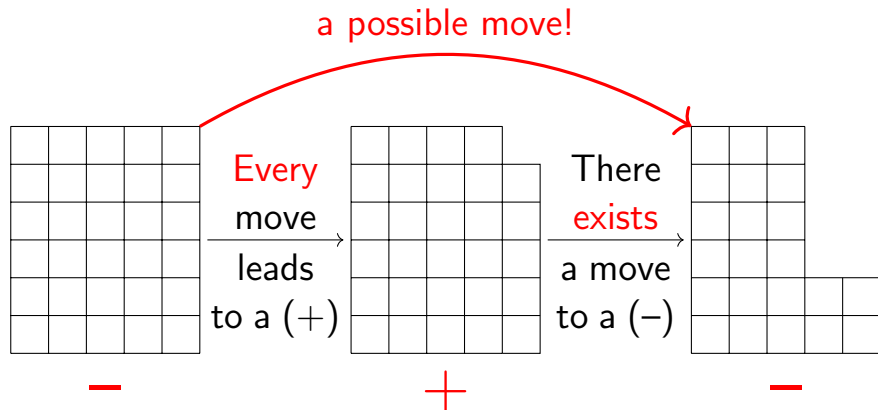
# Proof

Suppose that *Bob* has a winning strategy.



# Proof

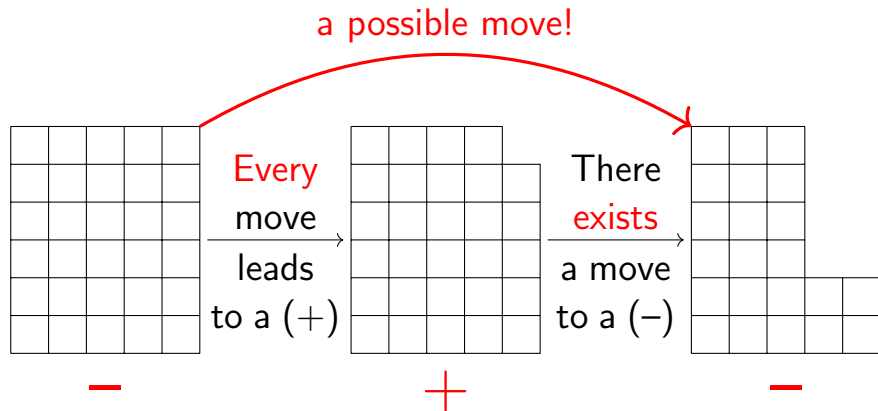
Suppose that *Bob* has a winning strategy.





# Proof

Suppose that *Bob* has a winning strategy.



Contradiction!