Lecture Notes

Differential Geometry I

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Autumn Semester 2024

Preliminary and incomplete version

January 9, 2025

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Differential Geometry in $\mathbb{R}^n$

1 Curves

1.1 Arc length and reparametrization

In the following, the symbol I will always denote an interval, that is, a connected subset of $\mathbb{R}$. A continuous map $c\colon I \to X$ into a topological space X is called a (*parametrized*) *curve* in X. A curve defined on $[0,1]$ is also called a *path*.

Now let $X = (X, d)$ be a metric space. The *length* $L(c) \in [0, \infty]$ of the curve $c\colon I \to X$ is defined as

$$L(c) := \sup \sum_{i=1}^{k} d(c(t_{i-1}), c(t_i)),$$

where the supremum is taken over all finite, non-decreasing sequences $t_0 \le t_1 \le \ldots \le t_k$ in I. The curve c is *rectifiable* if $L(c) < \infty$, and c has *constant speed* or is *parametrized proportionally to arc length* if there exists a constant $\lambda \ge 0$, the *speed* of c, such that for every subinterval $[a,b] \subset I$,

$$L(c|_{[a,b]}) = \lambda(b-a);$$

if $\lambda = 1$, then c has *unit speed* or is *parametrized by arc length*.

The curve $c\colon I \to X$ is a *reparametrization* of another curve $\tilde{c}\colon \tilde{I} \to X$ if there exists a continuous, surjective, non-decreasing or non-increasing map $\varphi\colon I \to \tilde{I}$ (thus $a < b$ implies $\varphi(a) \le \varphi(b)$ or $\varphi(a) \ge \varphi(b)$, respectively) such that $c = \tilde{c} \circ \varphi$. Then clearly $L(c) = L(\tilde{c})$. The following lemma shows that every curve of locally finite length is a reparametrization of a unit speed curve.

1.1 Lemma (reparametrization) *Suppose that $c\colon I \to (X,d)$ is a curve with $L(c|_{[a,b]}) < \infty$ for every subinterval $[a,b] \subset I$. Pick $s \in I$, and define $\varphi\colon I \to \mathbb{R}$ such that $\varphi(t) = L(c|_{[s,t]})$ for $t \ge s$ and $\varphi(t) = -L(c|_{[t,s]})$ for $t < s$. Then φ is continuous and non-decreasing, and there is a well-defined unit speed curve $\tilde{c}\colon \varphi(I) \to X$ such that $\tilde{c}(\varphi(t)) = c(t)$ for all $t \in I$.*

Proof: Whenever $a, b \in I$ and $a < b$, then

$$d(c(a), c(b)) \le L(c|_{[a,b]}) = \varphi(b) - \varphi(a). \qquad (*)$$

Thus φ is non-decreasing. Moreover, given such a, b and $\epsilon > 0$, there exists a sequence $a = t_0 < t_1 < \ldots < t_k = b$ such that

$$L(c|_{[a,b]}) - \epsilon \leq \sum_{i=1}^{k} d(c(t_{i-1}), c(t_i)) \leq d(c(a), c(r)) + L(c|_{[r,b]})$$

for all $r \in (a, t_1]$, and there is a $\delta > 0$ such that $d(c(a), c(r)) < \epsilon$ for all $r \in (a, a + \delta)$; thus $L(c|_{[a,r]}) = L(c|_{[a,b]}) - L(c|_{[r,b]}) < 2\epsilon$ for $r > a$ close enough to a. It follows that φ is right-continuous, and left-continuity is shown analogously.

By $(*)$ there is a well-defined 1-Lipschitz curve $\tilde{c}\colon \varphi(I) \to X$ such that $\tilde{c}(\varphi(t)) = c(t)$ for all $t \in I$. Then $L(\tilde{c}|_{[\varphi(a),\varphi(b)]}) = L(c|_{[a,b]}) = \varphi(b) - \varphi(a)$ for all $[a, b] \subset I$, hence $\tilde{c}$ is parametrized by arc length. □

We now turn to the target space $X = \mathbb{R}^n$, endowed with the canonical inner product

$$\langle x, y \rangle = \big\langle (x^1, \ldots, x^n), (y^1, \ldots, y^n) \big\rangle := \sum_{i=1}^{n} x^i y^i$$

and the Euclidean metric

$$d(x, y) := |x - y| := \sqrt{\langle x - y, x - y \rangle}.$$

In the following we will tacitly assume that the interior of the interval I is non-empty. For $q \in \{0\} \cup \{1, 2, \ldots\} \cup \{\infty\}$ we write as usual $c \in C^q(I, \mathbb{R}^n)$ if c is continuous or q times continuously differentiable or infinitely differentiable, respectively. In the case that $q \geq 1$ and I is not open, this means that c admits an extension $\bar{c} \in C^q(J, \mathbb{R}^n)$ to an open interval $J \supset I$.

Suppose now that $c \in C^q(I, \mathbb{R}^n)$ for some $q \geq 1$. Then

$$L(c|_{[a,b]}) = \int_a^b |c'(t)|\, dt < \infty$$

for every subinterval $[a, b] \subset I$ (exercise), and thus the function φ from Lemma 1.1 satisfies $\varphi(t) = \int_s^t |c'(r)|\, dr$ for all $t \in I$. The curve c is called *regular* if $c'(t) \neq 0$ for all $t \in I$; then $\varphi' = |c'| > 0$ on I, and both $\varphi\colon I \to \varphi(I)$ and the inverse $\varphi^{-1}\colon \varphi(I) \to I$ are also of class C^q, that is, φ is a C^q diffeomorphism. Note also that $c \in C^1(I, \mathbb{R}^n)$ has constant speed $\lambda \geq 0$ if and only if $|c'(t)| = \lambda$ for all $t \in I$.

1.2 Local theory of curves

The following notions go back to Jean Frédéric Frenet (1816–1900).

1.2 Definition (Frenet curve) The curve $c \in C^n(I, \mathbb{R}^n)$ is called a *Frenet curve* if for all $t \in I$ the vectors $c'(t), c''(t), \ldots, c^{(n-1)}(t)$ are linearly independent. The corresponding *Frenet frame* $(e_1, \ldots, e_n)$, $e_i\colon I \to \mathbb{R}^n$, is then characterized by the following conditions:

(1) $(e_1(t), \dots, e_n(t))$ is a positively oriented orthonormal basis of $\mathbb{R}^n$ for $t \in I$;

(2) $\operatorname{span}(e_1(t), \dots, e_i(t)) = \operatorname{span}(c'(t), \dots, c^{(i)}(t))$ and $\langle e_i(t), c^{(i)}(t)\rangle > 0$ for $i = 1, \dots, n-1$ and $t \in I$.

Condition (2) refers to the linear span. The vectors $e_1(t), \dots, e_{n-1}(t)$ are obtained from $c'(t), \dots, c^{(n-1)}(t)$ by means of the Gram–Schmidt process, and $e_n(t)$ is then determined by condition (1). Note that $e_i \in C^{n-i}(I, \mathbb{R}^n)$ for $i = 1, \dots, n-1$, in particular $e_1, \dots, e_n \in C^1(I, \mathbb{R}^n)$.

1.3 Definition (Frenet curvatures) Let $c \in C^n(I, \mathbb{R}^n)$ be a Frenet curve with Frenet frame $(e_1, \dots, e_n)$. For $i = 1, \dots, n-1$, the function $\kappa_i \colon I \to \mathbb{R}$,

$$\kappa_i(t) := \frac{1}{|c'(t)|}\langle e_i'(t), e_{i+1}(t)\rangle,$$

is called the *i-th Frenet curvature* of c.

Note that $\kappa_i \in C^{n-i-1}(I)$; in particular $\kappa_1, \dots, \kappa_{n-1}$ are continuous.

Suppose now that $c = \tilde{c} \circ \varphi$ for some curve $\tilde{c} \in C^n(\tilde{I}, \mathbb{R}^n)$ and a C^n diffeomorphism $\varphi \colon I \to \tilde{I}$ with $\varphi' > 0$. For $i = 1, \dots, n-1$, the i-th derivative $c^{(i)}(t)$ is a linear combination $\sum_{k=1}^{i} a_k(t)\, \tilde{c}^{\,(k)}(\varphi(t))$ with $a_i(t) = (\varphi'(t))^i > 0$, thus

$$\operatorname{span}\bigl(c'(t), \dots, c^{(i)}(t)\bigr) = \operatorname{span}\bigl((\tilde{c}\,' \circ \varphi)(t), \dots, (\tilde{c}^{\,(i)} \circ \varphi)(t)\bigr),$$

c is Frenet if and only if $\tilde{c}$ is Frenet, and the corresponding Frenet vector fields then satisfy the relation $e_i = \tilde{e}_i \circ \varphi$. Likewise, for the Frenet curvatures,

$$\kappa_i = \frac{1}{|c'|}\langle e_i', e_{i+1}\rangle = \frac{1}{|\tilde{c}\,' \circ \varphi||\varphi'|}\bigl\langle (\tilde{e}_i' \circ \varphi)\varphi', \tilde{e}_{i+1} \circ \varphi\bigr\rangle = \tilde{\kappa}_i \circ \varphi.$$

Thus the curvatures are invariant under sense preserving reparametrization.

1.4 Proposition (Frenet equations) *Let $c \in C^n(I, \mathbb{R}^n)$ be a Frenet curve with Frenet frame $(e_1, \dots, e_n)$ and Frenet curvatures $\kappa_1, \dots, \kappa_{n-1}$. Then $\kappa_1, \dots, \kappa_{n-2} > 0$, and*

$$\frac{1}{|c'|}e_i' = \begin{cases} \kappa_1 e_2 & \text{if } i = 1, \\ -\kappa_{i-1}e_{i-1} + \kappa_i e_{i+1} & \text{if } 2 \le i \le n-1, \\ -\kappa_{n-1}e_{n-1} & \text{if } i = n. \end{cases}$$

Proof: Since $(e_1(t), \dots, e_n(t))$ is orthonormal,

$$e_i'(t) = \sum_{j=1}^{n}\langle e_i'(t), e_j(t)\rangle e_j(t)$$

for $i = 1, \ldots, n$, and since $\langle e_i', e_j\rangle + \langle e_i, e_j'\rangle = \langle e_i, e_j\rangle' = 0$, the coefficient matrix $K(t) = (\langle e_i'(t), e_j(t)\rangle)$ is skew-symmetric. For $i = 1, \ldots, n-1$,

$$\langle e_i', e_{i+1}\rangle = |c'|\kappa_i.$$

Now let $i \le n-2$, and recall condition (2) of Definition 1.2. The vector $e_i(t)$ is a linear combination $\sum_{k=1}^{i} a_{ik}(t)\, c^{(k)}(t)$ with $a_{ii}(t) > 0$, so $e_i'(t)$ is of the form $\sum_{k=1}^{i} b_{ik}(t)\, c^{(k)}(t) + a_{ii}(t)\, c^{(i+1)}(t)$, and it follows that

$$\langle e_i', e_{i+2}\rangle = \ldots = \langle e_i', e_n\rangle = 0$$

and $\langle e_i', e_{i+1}\rangle = a_{ii}\langle c^{(i+1)}, e_{i+1}\rangle > 0$. This gives the result. □

In the *case* $n = 2$, a curve $c \in C^2(I, \mathbb{R}^2)$ is Frenet if and only if c is regular. Then the sole Frenet curvature

$$\kappa_{\mathrm{or}} := \kappa_1 = \frac{1}{|c'|}\langle e_1', e_2\rangle$$

is called the *oriented curvature* (or *signed curvature*) of c. Note that $e_1 = c'/|c'|$ and $\langle c', e_2\rangle = 0$, thus

$$\kappa_{\mathrm{or}} = \frac{\langle c'', e_2\rangle}{|c'|^2} = \frac{\det(e_1, c'')}{|c'|^2} = \frac{\det(c', c'')}{|c'|^3}.$$

The Frenet equations may be written in matrix form as

$$\frac{1}{|c'|}\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_{\mathrm{or}} \\ -\kappa_{\mathrm{or}} & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

The *osculating circle* (*Schmiegkreis*) of c at a point t with $\kappa_{\mathrm{or}}(t) \neq 0$ is the circle with center $c(t) + (1/\kappa_{\mathrm{or}}(t))e_2(t)$ and radius $1/|\kappa_{\mathrm{or}}(t)|$, which approximates the curve at t up to second order (exercise).

In the *case* $n = 3$, $c \in C^3(I, \mathbb{R}^3)$ is a Frenet curve if and only if c' and c'' are everywhere linearly independent. The vectors e_2 and $e_3 = e_1 \times e_2$ (vector product) are called the *normal* and the *binormal* of c, respectively. The two Frenet curvatures

$$\kappa := \kappa_1 = \frac{1}{|c'|}\langle e_1', e_2\rangle > 0, \quad \tau := \kappa_2 = \frac{1}{|c'|}\langle e_2', e_3\rangle$$

are called *curvature* and *torsion* of c; the latter measures the rotation of the *osculating plane* (*Schmiegebene*) $\mathrm{span}\{c', c''\} = \mathrm{span}\{e_1, e_2\}$ about e_1. Both κ and τ are also invariant under sense reversing reparametrization, but τ changes sign under orientation reversing isometries of $\mathbb{R}^3$. The Frenet equations for curves in $\mathbb{R}^3$ read

$$\frac{1}{|c'|}\begin{pmatrix} e_1' \\ e_2' \\ e_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

If c is parametrized by arc length, then $2\langle c', c''\rangle = \langle c', c'\rangle' = 0$ and hence $e_2 = c''/|c''|$, thus $\kappa = \langle e_1', e_2\rangle = |c''|$. For a general Frenet curve in $\mathbb{R}^3$, the formulae

$$\kappa = \frac{|c' \times c''|}{|c'|^3}, \quad \tau = \frac{\det(c', c'', c''')}{|c' \times c''|^2}$$

hold (exercise).

1.5 Theorem (fundamental theorem of local curve theory) *If $n - 1$ functions $\kappa_1, \dots, \kappa_{n-1} \in C^\infty(I, \mathbb{R})$ with $\kappa_1, \dots, \kappa_{n-2} > 0$ are given, and if $s_0 \in I$, $x_0 \in \mathbb{R}^n$, and $(b_1, \dots, b_n)$ is a positively oriented orthonormal basis of $\mathbb{R}^n$, then there exists a unique Frenet curve $c \in C^\infty(I, \mathbb{R}^n)$ of constant speed one such that*

(1) $c(s_0) = x_0$;

(2) $(b_1, \dots, b_n)$ *is the Frenet frame of c at s_0;*

(3) $\kappa_1, \dots, \kappa_{n-1}$ *are the Frenet curvatures of c.*

The differentiability assumptions may be weakened.

Proof: Let $K = (k_{i,j}) \in C^\infty(I, \mathbb{R}^{n\times n})$ be the matrix function with

$$k_{i,i+1} = -k_{i+1,i} = \kappa_i \quad \text{for } i = 1, \dots, n-1$$

and all other entries equal to zero, and let $B = (b_i^j) \in \mathbb{R}^{n\times n}$ be the matrix whose ith row is b_i. By the existence and uniqueness theorem for linear ordinary differential equations, there exists a unique solution $E = (e_i^j) \in C^\infty(I, \mathbb{R}^{n\times n})$ of the Frenet matrix equation

$$E' = K\,E$$

satisfying the initial condition $E(s_0) = B$.

To show that the rows of $E(s)$ form a possible Frenet frame for the sought curve, we need to verify that $E(s) \in \mathrm{SO}(n)$ for all $s \in I$. Note that

$$(E\,E^{\mathrm{t}})' = E'E^{\mathrm{t}} + E\,(E')^{\mathrm{t}} = K\,E\,E^{\mathrm{t}} + E\,E^{\mathrm{t}}K^{\mathrm{t}},$$

and $E\,E^{\mathrm{t}} = I_n$ (identity matrix) is the unique solution of this equation with $(E\,E^{\mathrm{t}})(s_0) = I_n$, because $K + K^{\mathrm{t}} = 0$. This shows that $E(s) \in \mathrm{O}(n)$, and since $E(s_0) = B \in \mathrm{SO}(n)$, we have $E(s) \in \mathrm{SO}(n)$ by continuity of the determinant.

Finally, setting

$$c(s) := x_0 + \int_{s_0}^{s} e_1(t)\,dt \quad \text{for all } s \in I,$$

we get a curve $c \in C^\infty(I, \mathbb{R}^n)$ with $c(s_0) = x_0$ and $c' = e_1$. An induction argument using the Frenet equations shows that for $i = 2, \dots, n-1$, the ith derivative is a linear combination $c^{(i)} = \sum_{k=1}^{i} a_{ik}e_k$ with $a_{ii} = \kappa_1\kappa_2 \dots \kappa_{i-1} > 0$. We conclude that c is a Frenet curve with frame $(e_1, \dots, e_n)$ and curvatures $\langle e_i', e_{i+1}\rangle = \langle \kappa_i e_{i+1}, e_{i+1}\rangle = \kappa_i$. □

We now turn to some global results.

1.3 The rotation index of a plane curve

In the following it is assumed that $a < b$. A curve $c\colon [a,b] \to X$ in a topological space X is called *closed* or a *loop* if $c(a) = c(b)$, and c is said to be *simple* if $c|_{[a,b)}$ is injective in addition. Now let again $X = \mathbb{R}^n$. For $q \in \{1, 2, \dots\} \cup \{\infty\}$, a closed curve $c \in C^q([a,b], \mathbb{R}^n)$ will be called C^q*-closed* if c admits a $(b-a)$-periodic extension $\bar{c} \in C^q(\mathbb{R}, \mathbb{R}^n)$, that is, $\bar{c}(t + b - a) = \bar{c}(t)$ for all $t \in \mathbb{R}$.

Suppose now that $c\colon [a,b] \to \mathbb{R}^2$ is a C^1-closed and regular plane curve. Let $S^1 \subset \mathbb{R}^2$ denote the unit circle. The normalized velocity vector $e(t) := c'(t)/|c'(t)| \in S^1$ of c may be represented as

$$e(t) = (\cos\theta(t), \sin\theta(t))$$

for a continuous polar angle function $\theta\colon [a,b] \to \mathbb{R}$, which is uniquely determined up to addition of an integral multiple of 2π. More precisely, θ is a lifting of $e\colon [a,b] \to S^1$ with respect to the canonical covering

$$\sigma\colon \mathbb{R} \to S^1, \quad \sigma(s) := (\cos(s), \sin(s));$$

that is, $\sigma \circ \theta = e$. To show that such a function θ exists, one may use the uniform continuity of e on the compact interval $[a,b]$ to find a subdivision $a = a_0 < a_1 < \ldots < a_k = b$ such that none of the subintervals $[a_{i-1}, a_i]$ is mapped *onto* S^1. Then, for every choice of $\theta(a)$ with $\sigma(\theta(a)) = e(a)$, there are successive unique extensions of θ to the intervals $[a, a_i]$ for $i = 1, \dots, k$.

Since $e(a) = e(b)$, there is a unique integer ϱ_c, independent of the choice of θ, such that

$$\theta(b) - \theta(a) = 2\pi\varrho_c.$$

This number ϱ_c is called the *rotation index* (*Umlaufzahl*) of c. If c is an orientation preserving reparametrization of another C^1-closed regular curve $\tilde{c}$, then $\varrho_c = \varrho_{\tilde{c}}$.

1.6 Theorem (theorem of turning tangents, Umlaufsatz) *The rotation index of a simple C^1-closed, regular curve $c\colon [a,b] \to \mathbb{R}^2$ equals 1 or -1.*

This probably goes back to Riemann. The following elegant argument is due to H. Hopf [Ho1935].

Proof: We assume that c is parametrized by arc length and that $[a,b] = [0,L]$. Furthermore, we suppose that the image of c lies in the upper half-plane $\mathbb{R} \times [0,\infty)$ and that $c(0) = (0,0)$ and $c'(0) = (1,0)$. We will show that $\varrho_c = 1$ under these assumptions.

We consider the triangular domain $D := \{(s,t) \in \mathbb{R}^2 : 0 \le s \le t \le L\}$ and assign to every point in D a unit vector as follows:

$$e(s,t) := \begin{cases} c'(s) & \text{if } s = t, \\ -c'(0) = (-1,0) & \text{if } (s,t) = (0,L), \\ \frac{c(t)-c(s)}{|c(t)-c(s)|} & \text{otherwise.} \end{cases}$$

Note that this definition is possible since c is simple. The resulting map $e\colon D \to S^1$ is easily seen to be continuous.

It then follows from the homotopy lifting property in topology that there is a continuous function $\theta\colon D \to \mathbb{R}$ such that $\sigma \circ \theta = e$, where $\sigma\colon \mathbb{R} \to S^1$ is the canonical covering as above. For an alternative direct argument, note that by the uniform continuity of e on the compact set D there is an integer $k \geq 1$ such that for $\delta := L/(k+1)$, none of the subsets

$$D_{j,i} := D \cap ([i\delta, (i+1)\delta] \times [j\delta, (j+1)\delta]) \quad (j = 0, \dots, k,\ i = 0, \dots, j)$$

is mapped *onto* S^1. Clearly θ may be defined on $D_{0,0}$, and then there exist successive unique extensions to $D_{1,0}, D_{1,1}, D_{2,0}, D_{2,1}, D_{2,2}, \dots$ (lexicographic order).

Now, since $e(0,t)$ lies in the upper half-plane for all $t \in [0,L]$, and $e(0,0) = (1,0)$ and $e(0,L) = (-1,0)$, it follows that

$$\theta(0,L) = \theta(0,0) + \pi.$$

Similarly, $e(s,L)$ is in the lower half-plane for all $s \in [0,L]$, and $e(L,L)$ is again equal to $(1,0)$, hence

$$\theta(L,L) = \theta(0,L) + \pi = \theta(0,0) + 2\pi.$$

Since $s \mapsto \theta(s,s)$ is an angle function for $s \mapsto e(s,s) = c'(s)$, this shows that $\varrho_c = 1$. $\square$

1.4 Total curvature of closed curves

Now let $c\colon [0,L] \to \mathbb{R}^2$ $(L > 0)$ be a C^2 curve of constant speed one with Frenet frame (e_1, e_2). If $\theta\colon [0,L] \to \mathbb{R}$ is continuous and $e_1(s) = (\cos\theta(s), \sin\theta(s))$, then θ is continuously differentiable, and

$$e_1'(s) = \theta'(s)(-\sin\theta(s), \cos\theta(s)) = \theta'(s)e_2(s).$$

On the other hand, $e_1'(s) = \kappa_{\mathrm{or}}(s)e_2(s)$ by the first Frenet equation, thus $\theta' = \kappa_{\mathrm{or}}$. The *total curvature* of c therefore satisfies

$$\int_0^L \kappa_{\mathrm{or}}(s)\, ds = \int_0^L \theta'(s)\, ds = \theta(L) - \theta(0).$$

If c is C^2-closed and simple, then Theorem 1.6 asserts that $|\theta(L) - \theta(0)| = 2\pi$, thus

$$\int_0^L |\kappa_{\mathrm{or}}(s)|\, ds \geq \left| \int_0^L \kappa_{\mathrm{or}}(s)\, ds \right| = 2\pi.$$

Equality holds if and only if κ_{or} does not change sign, that is, $\kappa_{\mathrm{or}} \geq 0$ or $\kappa_{\mathrm{or}} \leq 0$. This in turn holds if and only if c is *convex*, that is, the trace $c([0,L])$ is the boundary of a convex set $C \subset \mathbb{R}^2$ (exercise).

We now turn to curves in $\mathbb{R}^n$ for $n \geq 3$. If $c \in C^n(I, \mathbb{R}^n)$ is a Frenet curve parametrized by arc length, then $\kappa_1 = |c''|$. It is thus consistent to define the *curvature* of an arbitrary unit speed curve $c \in C^2(I, \mathbb{R}^n)$ by

$$\kappa := |c''|.$$

1.7 Theorem (Fenchel–Borsuk) *Suppose that $c\colon [0, L] \to \mathbb{R}^n$ is a C^2-closed unit speed curve whose trace is not contained in a 2-dimensional plane. Then*

$$\int_0^L \kappa(s)\, ds > 2\pi.$$

This is due to Fenchel [Fe1929] for $n = 3$ and to Borsuk [Bo1947] in the general case. The proof below is from [Hor1971].

Proof: It suffices to show the conclusion for $n = 3, 4, \ldots$ under the assumption that the trace of c is not contained in an $(n-1)$-dimensional plane.

The derivative of c, viewed as a (C^1) curve $c'\colon [0, L] \to S^{n-1}$ into the unit sphere, is called the *tangent indicatrix* of c. Clearly

$$\int_0^L \kappa(s)\, ds = \int_0^L |c''(s)|\, ds = L(c').$$

For every fixed unit vector $e \in S^{n-1}$,

$$\int_0^L \langle c'(s), e\rangle\, ds = \langle c(L), e\rangle - \langle c(0), e\rangle = 0,$$

and $\langle c', e\rangle$ cannot be constantly zero, for then $\mathrm{im}(c)$ would be contained in a hyperplane orthogonal to e; thus $\langle c', e\rangle$ must change sign. This shows that no closed hemisphere of S^{n-1} contains the entire trace of the tangent indicatrix. It now follows from the next result that $L(c') > 2\pi$. $\square$

1.8 Proposition *If $c\colon [a, b] \to S^{n-1} \subset \mathbb{R}^n$ is a closed curve whose trace is not contained in a closed hemisphere, then $L(c) > 2\pi$.*

Note that here c is merely continuous. The proof uses a symmetry argument together with the basic fact that the trace of any shortest curve in S^{n-1} between two points is an arc of a great circle of length at most π (exercise).

Proof: We assume that $L(c) < \infty$. Suppose first that there exists a $t \in (a, b)$ such that $c(t) = -c(a)$. Then clearly $L(c) \geq 2\pi$, and equality holds only if c runs on arcs of great circles from $c(a)$ to $-c(a)$ and back, in which case $\mathrm{im}(c)$ would be contained in a closed hemisphere. Thus $L(c) > 2\pi$.

Suppose now that no image point of c is antipodal to $c(a)$. Choose $t \in (a, b)$ such that $l := L(c|_{[a,t]}) = L(c|_{[t,b]})$. Since $c(t) \neq -c(a)$, there exists a unique midpoint $e \in S^{n-1}$ between $c(a)$ and $c(t)$. By the assumption, at least one of the curves $c|_{[a,t]}$ and $c|_{[t,b]}$ leaves the hemisphere $H_e := \{v \in S^{n-1} : \langle e, v\rangle \geq 0\}$. Suppose that $c([a,t]) \not\subset H_e$. Then there exists an $s \in (a,t)$ with $\langle e, c(s)\rangle = 0$. Consider the bigon consisting of the two arcs of great circles from $c(s)$ to $-c(s)$ through $c(a)$ and $c(t)$. By symmetry, $c(a)$ and $c(t)$ subdivide the bigon into two parts of length π. In particular $l \geq \pi$, and equality would imply that $c([a,t]) \subset H_e$. Thus $l > \pi$ and $L(c) = 2l > 2\pi$. □

Fáry [Fa1949] and Milnor [Mi1950] showed independently that the total curvature of a *knotted* curve in $\mathbb{R}^3$ is even $> 4\pi$, thus answering a question raised by Borsuk. We refer to [PeS2024] for a recent survey of various proofs of the Fáry–Milnor theorem.

2 Surfaces

2.1 Submanifolds and immersions

We now consider m-dimensional surfaces in $\mathbb{R}^n$.

2.1 Definition (submanifold) A subset $M \subset \mathbb{R}^n$ is a (smooth) *m-dimensional submanifold* of $\mathbb{R}^n$ if for every point $p \in M$ there exist an open neighborhood $V \subset \mathbb{R}^n$ of p and a C^∞ diffeomorphism $\varphi\colon V \to U$ onto an open set $U \subset \mathbb{R}^n$ such that $\varphi(M \cap V) = (\mathbb{R}^m \times \{0\}) \cap U$.

The number $k := n - m$ is called the *codimension* of M in $\mathbb{R}^n$, and φ is a *submanifold chart* (*Schnittkarte*) of M. Submanifolds of class C^q, $1 \le q \le \infty$, are defined analogously.

Now let $W \subset \mathbb{R}^n$ be an open set, and let $F\colon W \to \mathbb{R}^k$ be a differentiable map. A point $p \in W$ is called a *regular point* of F if the differential dF_p is surjective, otherwise p is called a *singular* or *critical point* of F. A point $x \in \mathbb{R}^k$ is a *regular value* of F if all points $p \in F^{-1}\{x\}$ are regular; otherwise, if $F^{-1}\{x\}$ contains a singular point, x is a *singular* or *critical value* of F. Note that, according to this definition, every $x \in \mathbb{R}^k \setminus F(W)$ is a regular value of F.

2.2 Theorem (regular value theorem) *If $W \subset \mathbb{R}^n$ is open and $F \in C^\infty(W, \mathbb{R}^k)$, and if $x \in F(W)$ is a regular value of F, then $M := F^{-1}\{x\}$ is a submanifold of $\mathbb{R}^n$ of dimension $m := n - k \ge 0$ (thus the codimension of M equals k).*

Proof: We assume that $x = 0$. Let $p \in M = F^{-1}\{0\}$. Since dF_p is surjective, it follows from Theorem A.2 (implicit function theorem, surjective form) that there exist open neighborhoods $U \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$ of $(0,0)$ and $V \subset W$ of p and a C^∞ diffeomorphism $\psi\colon U \to V$ such that $\psi(0,0) = p$ and

$$(F \circ \psi)(x, y) = y \quad \text{for all } (x, y) \in U.$$

Then $\varphi := \psi^{-1}\colon V \to U$ is a submanifold chart of M around p: $\varphi(M \cap V)$ equals the set of all $(x, y) \in U$ such that $\psi(x, y) \in M = F^{-1}\{0\}$ and thus $y = (F \circ \psi)(x, y) = 0$. □

The following alternative notion of surface extends the concept of a regular (parametrized) curve to higher dimensions.

2.3 Definition (immersion) A map $f \in C^\infty(U, \mathbb{R}^n)$ from an open set $U \subset \mathbb{R}^m$ into $\mathbb{R}^n$ is called an *immersion* if for all $x \in U$ the differential $df_x\colon \mathbb{R}^m \to \mathbb{R}^n$ is injective.

2.4 Theorem (immersion theorem) *Let $f \in C^\infty(U, \mathbb{R}^n)$ be an immersion of the open set $U \subset \mathbb{R}^m$. Then, for every point $x \in U$, there exists an open neighborhood $U_x \subset U$ of x such that $f(U_x)$ is an m-dimensional submanifold of $\mathbb{R}^n$.*

Proof: We suppose that $x = 0 \in U$ and $f(0) = p$. Since df_0 is injective, it follows from Theorem A.2 (implicit function theorem, injective form) that there exist open neighborhoods $V \subset \mathbb{R}^n$ of p and $W \subset U \times \mathbb{R}^{n-m}$ of $(0,0)$ and a C^∞ diffeomorphism $\varphi\colon V \to W$ such that $\varphi(p) = (0,0)$ and

$$(\varphi \circ f)(x) = (x, 0) \quad \text{for all } (x,0) \in W.$$

Put $U_0 := \{x \in U : (x,0) \in W\}$ and $M := f(U_0)$. Then φ is a (global) submanifold chart for M, since $\varphi(M \cap V) = \varphi(f(U_0)) = U_0 \times \{0\} = W \cap (\mathbb{R}^m \times \{0\})$. □

In general, even if an immersion is *injective*, its image need not be a submanifold. For example, the trace of the injective regular curve

$$c\colon (0, 2\pi) \to \mathbb{R}^2, \quad c(t) = (\sin(t), \sin(2t)),$$

has the shape of the ∞ symbol. However, the following holds.

2.5 Theorem (local parametrizations) *A subset $M \subset \mathbb{R}^n$ is an m-dimensional submanifold of $\mathbb{R}^n$ if and only if for every point $p \in M$ there exist open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ and an immersion $f\colon U \to \mathbb{R}^n$ such that $p \in f(U) = M \cap V$ and $f\colon U \to M \cap V$ is a homeomorphism.*

Then f is called a *local parametrization*, and $f^{-1}\colon M \cap V \to U$ is a *chart* of M around p.

Proof: Suppose first that $M \subset \mathbb{R}^n$ is a submanifold. Given a point $p \in M$, let $\varphi\colon V \to U' \subset \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ be a submanifold chart around p, and put $U := \{x \in \mathbb{R}^m : (x,0) \in U'\}$ and $f(x) := \varphi^{-1}(x,0)$ for all $x \in U$. Then f is an immersion of U into $\mathbb{R}^n$ and a homeomorphism onto $f(U) = \varphi^{-1}((\mathbb{R}^m \times \{0\}) \cap U') = M \cap V$.

We prove the reverse implication. Let $p \in M$, and suppose that $f\colon U \to \mathbb{R}^n$ is an immersion of an open set $U \subset \mathbb{R}^m$ such that $0 \in U$, $f(0) = p$, and f is a homeomorphism onto $M \cap V$ for some open set $V \subset \mathbb{R}^n$. As in the previous proof, since df_0 is injective, we infer from Theorem A.2 that there exists a C^∞ diffeomorphism $\varphi\colon V' \to W$ between open neighborhoods $V' \subset V$ of p and $W \subset U \times \mathbb{R}^{n-m}$ of $(0,0)$ such that $\varphi(p) = (0,0)$ and

$$(\varphi \circ f)(x) = (x, 0) \quad \text{for all } (x,0) \in W.$$

Furthermore, since $f^{-1}\colon M \cap V \to U$ is continuous, there exists an open neighborhood $V'' \subset V' \subset V$ of p such that

$$U_0 := f^{-1}(M \cap V'') \subset \{x \in U : (x,0) \in W\}.$$

Now $\varphi(M \cap V'') = \varphi(f(U_0)) = U_0 \times \{0\}$, and this is the set of all $(x,0) \in W$ with $f(x) \in V''$ and thus $(x,0) = \varphi(f(x)) \in \varphi(V'')$. Hence, $\varphi|_{V''}\colon V'' \to \varphi(V'')$ is a submanifold chart of M around p. □

2.6 Lemma (parameter transformation) *Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold, and suppose that $f_i \colon U_i \to f(U_i) \subset M$, $i = 1, 2$, are two local parametrizations with $V := f_1(U_1) \cap f_2(U_2) \neq \emptyset$. Then*

$$\psi := f_2^{-1} \circ f_1 \colon f_1^{-1}(V) \to f_2^{-1}(V)$$

is a C^∞ diffeomorphism.

Proof: Suppose that $f_1(0) = p = f_2(0)$. As in the proof of Theorem 2.4, there exists a C^∞ diffeomorphism φ defined on an open neighorhood of p in $\mathbb{R}^n$ such that $\varphi(p) = (0, 0) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ and

$$\varphi(f_2(x)) = (x, 0) \quad \text{for all } (x, 0) \in \operatorname{im}(\varphi).$$

Let $\pi \colon \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ denote the projection $(x, y) \mapsto x$. Then, in a neighborhood of $0 \in \mathbb{R}^m$, we have $\psi = f_2^{-1} \circ f_1 = \pi \circ \varphi \circ f_1$. Thus ψ is locally C^∞ and hence C^∞, and by symmetry the same holds for ψ^{-1}. □

2.2 Tangent spaces and differentials

2.7 Definition (tangent space, normal space) The *tangent space TM_p* of an m-dimensional submanifold $M \subset \mathbb{R}^n$ at the point $p \in M$ is defined as $TM_p := df_x(\mathbb{R}^m) \subset \mathbb{R}^n$ for some (and hence any) local parametrization $f \colon U \to f(U) \subset M$ with $f(x) = p$. The orthogonal complement $TM_p^\perp$ of TM_p in $\mathbb{R}^n$ is the *normal space* of M at p.

The tangent space TM_p is an m-dimensional linear subspace of $\mathbb{R}^n$, whereas the normal space $TM_p^\perp$ is a linear subspace of $\mathbb{R}^n$ of dimension equal to the codimension $k := n - m$ of M.

2.8 Definition (differentiable map, differential) A map $F \colon M \to \mathbb{R}^l$ from a submanifold $M \subset \mathbb{R}^n$ into $\mathbb{R}^l$ is *differentiable* at the point $p \in M$ if for some (and hence any) local parametrization $f \colon U \to f(U) \subset M$ with $f(x) = p$ the composition $F \circ f \colon U \to \mathbb{R}^l$ is differentiable at $x \in U$. The *differential* of $F \colon M \to \mathbb{R}^l$ at p is then defined as the unique linear map $dF_p \colon TM_p \to \mathbb{R}^l$ for which the chain rule

$$d(F \circ f)_x = dF_p \circ df_x$$

holds. For $1 \leq q \leq \infty$, mappings $F \colon M \to \mathbb{R}^l$ of class C^q, $F \in C^q(M, \mathbb{R}^l)$, are defined accordingly.

In order to determine $dF_p(v)$ it is often convenient to represent the vector $v \in TM_p$ as the velocity $c'(0)$ of a differentiable curve $c \colon (-\epsilon, \epsilon) \to M \subset \mathbb{R}^n$ with $c(0) = p$; then

$$dF_p(c'(0)) = (F \circ c)'(0).$$

If $F \colon M \to \mathbb{R}^l$ takes values in a submanifold Q of $\mathbb{R}^l$, then it follows that $dF_p(TM_p) \subset TQ_{F(p)}$.

2.3 Orientability and the separation theorem

2.9 Definition (orientability) A submanifold $M \subset \mathbb{R}^n$ is *orientable* if there exists a system $\{f_\alpha \colon U_\alpha \to f_\alpha(U_\alpha) \subset M\}_{\alpha \in A}$ of local parametrizations of M such that $\bigcup_{\alpha \in A} f_\alpha(U_\alpha) = M$ and every parameter transformation $f_\beta^{-1} \circ f_\alpha$ with $\alpha, \beta \in A$ and $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) \neq \emptyset$ satisfies $\det(d(f_\beta^{-1} \circ f_\alpha)_x) > 0$ everywhere on its domain. A maximal such system is called an *orientation* of M, and every local parametrization belonging to it is then said to be *positively oriented*.

2.10 Proposition (orientable hypersurfaces) *A submanifold $M \subset \mathbb{R}^{m+1}$ of codimension one is orientable if and only if there exists a continuous unit normal vector field on M, that is, a continuous map $N \colon M \to S^m$ with $N(p) \in TM_p^\perp$ for all $p \in M$.*

Such a map N is called a *Gauss map* of M.

Proof: Suppose first that M is orientable, and let $\{f_\alpha \colon U_\alpha \to f_\alpha(U_\alpha) \subset M\}_{\alpha \in A}$ be an oriented system of local parametrizations with $\bigcup_{\alpha \in A} f_\alpha(U_\alpha) = M$. We will briefly write $f_{\alpha,i}$ for the partial derivative $d(f_\alpha)(e_i)$. For every $\alpha \in A$ there exists a unique unit normal vector field $\nu_\alpha \colon U_\alpha \to S^m$ along f_α (thus $\nu_\alpha(x) \in TM^\perp_{f_\alpha(x)}$) such that $(f_{\alpha,1}(x), \ldots, f_{\alpha,m}(x), \nu_\alpha(x))$ is a positively oriented basis of $\mathbb{R}^{m+1}$ for all $x \in U_\alpha$. Since the $f_{\alpha,i}$ are continuous, so is ν_α. In order to define N at $p \in M$, we want to prove that $\nu_\alpha(x) = \nu_\beta(y)$ whenever $f_\alpha(x) = p = f_\beta(y)$. In this case,

$$d(f_\alpha)_x = d(f_\beta)_y \circ d(f_\beta^{-1} \circ f_\alpha)_x,$$

and since $\det(d(f_\beta^{-1} \circ f_\alpha)_x) > 0$, it follows that $(f_{\alpha,1}(x), \ldots, f_{\alpha,m}(x))$ and $(f_{\beta,1}(y), \ldots, f_{\beta,m}(y))$ are equally oriented bases of TM_p. Thus $\nu_\alpha(x) = \nu_\beta(y)$ as desired.

Conversely, suppose that there exists a Gauss map $N \colon M \to S^m$. Choose a system of local parametrizations $\{f_\alpha \colon U_\alpha \to f_\alpha(U_\alpha) \subset M\}_{\alpha \in A}$ such that $\bigcup_{\alpha \in A} f_\alpha(U_\alpha) = M$ and $(f_{\alpha,1}(x), \ldots, f_{\alpha,m}(x), N(f_\alpha(x)))$ is a positively oriented basis of $\mathbb{R}^{m+1}$ for all $\alpha \in A$ and $x \in U_\alpha$. If $f_\alpha(x) = p = f_\beta(y)$, then $(f_{\alpha,1}(x), \ldots, f_{\alpha,m}(x))$ and $(f_{\beta,1}(y), \ldots, f_{\beta,m}(y))$ are equally oriented bases of TM_p, and by the same relation as above it follows that $\det(d(f_\beta^{-1} \circ f_\alpha)_x) > 0$. □

2.11 Theorem (separation theorem) *Suppose that $\emptyset \neq M \subset \mathbb{R}^{m+1}$ is a compact and connected m-dimensional submanifold. Then $\mathbb{R}^{m+1} \setminus M$ has precisely two connected components, a bounded and an unbounded one, M is the boundary of each of them, and M is orientable.*

Proof: Since M is a submanifold of codimension 1, it follows that for every point $p \in M$ there exist an open set $V \subset \mathbb{R}^{m+1}$ and a smooth curve $c \colon [-1, 1] \to V$

with $c(0) = p$ and $c'(0) \notin TM_p$ such that $V \setminus M$ has exactly two connected components containing $c([-1,0))$ and $c((0,1])$, respectively (use a submanifold chart). We claim that $c(-1)$ and $c(1)$ lie in different connected components of $\mathbb{R}^{m+1} \setminus M$. Otherwise, there would exist a C^∞-closed curve $\bar{c}\colon [-1,2] \to \mathbb{R}^{m+1}$ with $\bar{c}(0) = p$, $\bar{c}'(0) \notin TM_p$ and $\bar{c}(t) \notin M$ for $t \neq 0$; this would, however, contradict the homotopy invariance of the intersection number modulo 2, which we will prove later in Theorem 9.12. Hence, every point $p \in M$ is a boundary point of two distinct connected components of $\mathbb{R}^{m+1} \setminus M$.

Now let $p \in M$ be fixed, an let $q \in M$ be any other point. Then $p \in \partial A \cap \partial B$ and $q \in \partial A_q \cap \partial B_q$ for some connected components $A \neq B$ and $A_q \neq B_q$ of $\mathbb{R}^{m+1} \setminus M$. Since M is connected and locally path connected, M is path connected, thus there exists a curve $c_q\colon [0,1] \to M$ from p to q. Let $N_q\colon [0,1] \to \mathbb{R}^{m+1}$ be a continuous unit vector field along c_q normal to M. For a sufficiently small $\epsilon > 0$, the traces of the curves $c_q^\pm\colon t \mapsto c_q(t) \pm \epsilon N_q(t)$ are in $\mathbb{R}^{m+1} \setminus M$. It follows that either $A_q = A$ and $B_q = B$, or $A_q = B$ and $B_q = A$. Since M is bounded, the assertions about the connected components of $\mathbb{R}^{m+1} \setminus M$ are now clear. Furthermore, M admits a Gauss map (pointing everywhere to A, for example), and thus M is orientable by Proposition 2.10. □

Theorem 2.11 holds more generally for the case that $\emptyset \neq M \subset \mathbb{R}^{m+1}$ is the image of a compact and connected m-dimensional topological manifold (Definition 8.1) under a continuous and injective map [Br1911b]. This is the *Jordan–Brouwer separation theorem*, which generalizes the *Jordan curve theorem*. In the latter, M is a *Jordan curve* in $\mathbb{R}^2$, that is, the image of a simple closed curve $c\colon [0,1] \to \mathbb{R}^2$. A first rigorous proof of the Jordan curve theorem was provided by Veblen [Ve1905]. Another generalization of the Jordan curve theorem is *Schönflies' theorem* [Sc1906]:

> Every continuous injective map $f\colon S^1 \to \mathbb{R}^2$ extends to a homeomorphism $\bar{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$, such that $\bar{f}|_{S^1} = f$.

Surprisingly, the analogue for maps $f\colon S^m \to \mathbb{R}^{m+1}$ with $m \geq 2$ fails to be true. *Alexander's horned sphere* in $\mathbb{R}^3$ has the property that the exterior domain is not simply connected [Al1924].

3 Intrinsic geometry of surfaces

3.1 First fundamental form

3.1 Definition (first fundamental form) The *first fundamental form g of a submanifold $M \subset \mathbb{R}^n$* assigns to each point $p \in M$ the inner product g_p on TM_p defined by

$$g_p(X, Y) := \langle X, Y \rangle$$

for $X, Y \in TM_p$ (thus g_p is just the restriction of the standard inner product $\langle \cdot\,, \cdot \rangle$ of $\mathbb{R}^n$ to $TM_p \times TM_p$.) The *first fundamental form g of an immersion $f : U \to \mathbb{R}^n$* of an open set $U \subset \mathbb{R}^m$ assigns to each $x \in U$ the inner product g_x on $\mathbb{R}^m$ defined by

$$g_x(\xi, \eta) := \langle df_x(\xi), df_x(\eta) \rangle$$

for $\xi, \eta \in \mathbb{R}^m$.

The first fundamental form g is also called the (*Riemannian*) *metric* of M or f, respectively. The matrix $(g_{ij}(x))$ of g_x with respect to the canonical basis $(e_1, \ldots, e_m)$ of $\mathbb{R}^m$ is given by

$$g_{ij}(x) = g_x(e_i, e_j) = \langle df_x(e_i), df_x(e_j) \rangle = \left\langle \frac{\partial f}{\partial x^i}(x), \frac{\partial f}{\partial x^j}(x) \right\rangle,$$

where $g_{ij} \in C^\infty(U)$. We will often write this relation briefly as $g_{ij} = \langle f_i, f_j \rangle$.

Now let $M \subset \mathbb{R}^n$ be a submanifold, and suppose that $f : U \to f(U) \subset M$ is a local parametrization (in particular, an immersion). The first fundamental forms of f and M are related as follows: if $x \in U$ and $f(x) = p$, then df_x is an isometry of the Euclidean vector spaces $(\mathbb{R}^m, g_x)$ and (TM_p, g_p). The set $U \subset \mathbb{R}^m$, equipped with the first fundamental form of f, constitutes a "model" for $f(U) \subset M$, in which all quantities belonging to the intrinsic geometry of $f(U) \subset M$ can be computed.

Examples

1. *Norms and angles*: for $X, Y \in TM_p$, $x := f^{-1}(p)$, and the corresponding vectors $\xi := (df_x)^{-1}(X)$ and $\eta := (df_x)^{-1}(Y)$ in $\mathbb{R}^m$,

$$|X| = \sqrt{g_p(X, X)} = \sqrt{g_x(\xi, \xi)} =: |\xi|_{g_x},$$
$$\cos \angle(X, Y) = \frac{g_p(X, Y)}{|X||Y|} = \frac{g_x(\xi, \eta)}{|\xi|_{g_x} |\eta|_{g_x}}.$$

2. *Length* of a C^1 curve $c : I \to f(U) \subset M$: if $\gamma := f^{-1} \circ c : I \to U$ is the corresponding curve in U, then $c'(t) = df_{\gamma(t)}(\gamma'(t))$ and hence

$$L(c) = \int_I |c'(t)|\, dt = \int_I |\gamma'(t)|_{g_{\gamma(t)}}\, dt.$$

3. The *m-dimensional area* of a Borel set $B \subset f(U) \subset M$ is computed as

$$A(B) := \int_{f^{-1}(B)} \sqrt{\det(g_{ij}(x))}\, dx \quad \in [0, \infty];$$

recall that the *Gram determinant*

$$\det(g_{ij}(x)) = \det\big(\langle f_i(x), f_j(x)\rangle\big)$$

equals the square of the volume of the parallelepiped spanned by the vectors $f_i(x) = \frac{\partial f}{\partial x^i}(x)$ for $i = 1, \ldots, m$. The area $A(B)$ is independent of the choice of f and is also denoted by $\int_B dA$.

In order to compute the m-dimensional area of a compact region $C \subset M$, one chooses finitely many local parametrizations $f_\alpha \colon U_\alpha \to f_\alpha(U_\alpha) \subset M$ and Borel sets $B_\alpha \subset f_\alpha(U_\alpha)$ such that $C = \bigcup_\alpha B_\alpha$ is a partition (that is, a decomposition into pairwise disjoint sets). The area

$$A(C) = \sum_\alpha A(B_\alpha) = \sum_\alpha \int_{f_\alpha^{-1}(B_\alpha)} \sqrt{\det(g^\alpha_{ij}(x))}\, dx,$$

where g^α denotes the first fundamental form of f_α, turns out to be independent of the choices made. For a continuous function $b \colon C \to \mathbb{R}$,

$$\int_C b\, dA := \sum_\alpha \int_{f_\alpha^{-1}(B_\alpha)} b \circ f_\alpha(x) \sqrt{\det(g^\alpha_{ij}(x))}\, dx$$

then defines the *surface integral* of b over C.

3.2 Definition (isometries) Two submanifolds $M \subset \mathbb{R}^n$ and $\tilde{M} \subset \mathbb{R}^{\tilde{n}}$ with first fundamental forms g and $\tilde{g}$ are called *isometric* if there exists a diffeomorphism $F \colon M \to \tilde{M}$ such that

$$g_p(X, Y) = \tilde{g}_{F(p)}(dF_p(X), dF_p(Y))$$

for all $p \in M$ and $X, Y \in TM_p$. For open sets $U, \tilde{U} \subset \mathbb{R}^m$, two immersions $f \colon U \to \mathbb{R}^n$ and $\tilde{f} \colon \tilde{U} \to \mathbb{R}^{\tilde{n}}$ with first fundamental forms g and $\tilde{g}$ are called *isometric* if there exists a diffeomorphism $\psi \colon U \to \tilde{U}$ such that

$$g_x(\xi, \eta) = \tilde{g}_{\psi(x)}(d\psi_x(\xi), d\psi_x(\eta))$$

for all $x \in U$ and $\xi, \eta \in \mathbb{R}^m$.

The above relations are briefly expressed as $g = F^*\tilde{g}$ and $g = \psi^*\tilde{g}$, respectively; g equals the *pull-back* of $\tilde{g}$ under the isometry. Note that $\psi^*\tilde{g}$ is just the first fundamental form of the immersion $\tilde{f} \circ \psi$, as

$$\begin{aligned}\tilde{g}(d\psi(\xi), d\psi(\eta)) &= \langle d\tilde{f} \circ d\psi(\xi), d\tilde{f} \circ d\psi(\eta)\rangle \\ &= \langle d(\tilde{f} \circ \psi)(\xi), d(\tilde{f} \circ \psi)(\eta)\rangle.\end{aligned}$$

In particular, if $f = \tilde{f} \circ \psi$ is a reparametrization of $\tilde{f}$, then f and $\tilde{f}$ are isometric.

3.2 Covariant derivative

Let $f\colon U \longrightarrow \mathbb{R}^n$ be an immersion of the open set $U \subset \mathbb{R}^m$. The vectors

$$f_k(x) = \frac{\partial f}{\partial x^k}(x) \quad (k = 1, \ldots, m)$$

form a basis of the tangent space $df_x(\mathbb{R}^m)$ of f at x. We now consider second derivatives

$$f_{ij}(x) := \frac{\partial^2 f}{\partial x^j \partial x^i}(x)$$

of f, which need no longer be tangential. The tangential part has a unique representation

$$\big(f_{ij}(x)\big)^{\mathrm{T}} = \sum_{k=1}^{m} \Gamma_{ij}^k(x) f_k(x).$$

The C^∞ coefficient functions $\Gamma_{ij}^k = \Gamma_{ji}^k\colon U \longrightarrow \mathbb{R}$ defined by this relation are called the *Christoffel symbols* of f.

3.3 Lemma (Christoffel symbols) *Let $f \in C^\infty(U, \mathbb{R}^n)$ be an immersion of the open set $U \subset \mathbb{R}^m$. Then*

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{m} g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right),$$

where (g^{kl}) denotes the matrix inverse to (g_{ij}).

Proof: Since

$$\begin{aligned}
\frac{\partial}{\partial x^i}\langle f_j, f_l\rangle &= \langle f_{ji}, f_l\rangle + \langle f_j, f_{li}\rangle,\\
\frac{\partial}{\partial x^j}\langle f_i, f_l\rangle &= \langle f_{ij}, f_l\rangle + \langle f_i, f_{lj}\rangle,\\
\frac{\partial}{\partial x^l}\langle f_i, f_j\rangle &= \langle f_{il}, f_j\rangle + \langle f_i, f_{jl}\rangle,
\end{aligned}$$

it follows that

$$\frac{1}{2}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right) = \langle f_l, f_{ij}\rangle = \big\langle f_l, (f_{ij})^{\mathrm{T}}\big\rangle = \sum_{k=1}^{m} \Gamma_{ij}^k g_{lk}.$$

By solving this equation for Γ_{ij}^k we get the result. $\square$

In the case $m = 2$ the expression for Γ_{ij}^k has a simpler form, as then always at least two of the indices i, j, l agree. If we use Gauss's notation

$$E := g_{11}, \quad F := g_{12} = g_{21}, \quad G := g_{22}$$

and the abbreviations $D := EG - F^2$ and $E_i := \frac{\partial E}{\partial x^i}$, etc., then

$$\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix} = \frac{1}{2D}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} E_1 & E_2 & 2F_2 - G_1 \\ 2F_1 - E_2 & G_1 & G_2 \end{pmatrix}.$$

3.4 Definition (covariant derivative, parallel vector field) Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold. Suppose that $c\colon I \to M$ is a curve and $X\colon I \to \mathbb{R}^n$ is a C^1 tangent vector field of M along c, that is, $X(t) \in TM_{c(t)}$ for all $t \in I$. The *covariant derivative* $\frac{D}{dt}X$ of X is the vector field along c defined by

$$\frac{D}{dt}X(t) := \left(\frac{d}{dt}X(t)\right)^{\mathrm{T}} = \dot{X}(t)^{\mathrm{T}} \in TM_{c(t)}$$

(tangential part) for $t \in I$. Then X is said to be *parallel* along c if, for all $t \in I$, $\frac{D}{dt}X(t) = 0$, that is, $\dot{X}(t) \in TM_{c(t)}^{\perp}$.

3.5 Theorem (covariant derivative) *Let M be an m-dimensional submanifold of $\mathbb{R}^n$ with first fundamental form g. Suppose that $c\colon I \to M$ is a C^1 curve, $X, Y\colon I \to \mathbb{R}^n$ are two C^1 tangent vector fields of M along c, and $\lambda\colon I \to \mathbb{R}$ is a C^1 function. Then*:

(1)
$$\frac{D}{dt}(X+Y) = \frac{D}{dt}X + \frac{D}{dt}Y, \quad \frac{D}{dt}(\lambda X) = \dot{\lambda}\, X + \lambda\, \frac{D}{dt}X;$$

(2)
$$\frac{d}{dt}g(X,Y) = g\Big(\frac{D}{dt}X, Y\Big) + g\Big(X, \frac{D}{dt}Y\Big);$$

(3) *if $c(I) \subset f(U)$ for some local parametrization $f\colon U \to f(U) \subset M$, and if $\gamma = (\gamma^1, \dots, \gamma^m)\colon I \to U$ and $\xi = (\xi^1, \dots, \xi^m)\colon I \to \mathbb{R}^m$ are the curve and vector field such that $c = f \circ \gamma$ and $X(t) = df_{\gamma(t)}(\xi(t))$, then*
$$\frac{D}{dt}X = \sum_{k=1}^{m}\Big(\dot{\xi}^k + \sum_{i,j=1}^{m} \xi^i\, \dot{\gamma}^j\, \Gamma_{ij}^k \circ \gamma\Big)\frac{\partial f}{\partial x^k} \circ \gamma.$$

Proof: (1) is clear. For (2), note that

$$\frac{d}{dt}\langle X, Y\rangle = \langle \dot{X}, Y\rangle + \langle X, \dot{Y}\rangle = \big\langle \dot{X}^{\mathrm{T}}, Y\big\rangle + \big\langle X, \dot{Y}^{\mathrm{T}}\big\rangle$$

since Y and X are tangential. To prove (3), we use again the shorthand f_i and f_{ij} for the partial derivatives up to second order. We have $X = \sum_{i=1}^m \xi^i\,(f_i \circ \gamma)$, and

$$\frac{d}{dt}\big(\xi^i\,(f_i \circ \gamma)\big) = \dot{\xi}^i\,(f_i \circ \gamma) + \xi^i \sum_{j=1}^{m}(f_{ij} \circ \gamma)\,\dot{\gamma}^j.$$

The tangential part is

$$\frac{D}{dt}(\xi^i\,(f_i \circ \gamma)) = \dot{\xi}^i\,(f_i \circ \gamma) + \xi^i \sum_{j,k=1}^{m} \dot{\gamma}^j\,(\Gamma_{ij}^k\, f_k) \circ \gamma,$$

and the result follows upon summation over i. $\square$

Item (3), together with Lemma 3.3, shows that the covariant derivative can be computed entirely in terms of the first fundamental form and is thus intrinsic. Note also that if X, Y are parallel along c, then $g_{c(t)}(X(t), Y(t))$ is constant, as

$$\frac{d}{dt}g(X,Y) = g\Big(\frac{D}{dt}X, Y\Big) + g\Big(X, \frac{D}{dt}Y\Big) = 0$$

by property (2); in particular $|X| = \sqrt{g(X,X)}$ is constant.

3.6 Theorem (existence and uniqueness of parallel vector fields) *Let $M \subset \mathbb{R}^n$ be a submanifold, and let $c\colon I \to M$ be a C^1 curve with $0 \in I$. Then for every vector $X_0 \in TM_{c(0)}$ there is a unique parallel tangent vector field $X\colon I \to \mathbb{R}^n$ of M along c with $X(0) = X_0$.*

Proof: We may assume that $c = f \circ \gamma$ for some local parametrization $f\colon U \to f(U) \subset M$. By Theorem 3.5, the vector field $X = \sum_{i=1}^m \xi^i\,(f_i \circ \gamma)$ is parallel along c if and only if the function $\xi = (\xi^1, \dots, \xi^m)\colon I \to \mathbb{R}^m$ satisfies the system of linear ordinary differential equations

$$\dot\xi^k + \sum_{i,j=1}^m \xi^i\,\dot\gamma^j\,\Gamma_{ij}^k \circ \gamma = 0 \quad (k = 1, \dots, m).$$

The (given) coefficient functions $\dot\gamma^j\,\Gamma_{ij}^k \circ \gamma$ are continuous and in particular bounded on compact subsets of I. It follows from the existence and uniqueness theorem for linear differential equations that for any vector $X_0 = \sum_{i=1}^m \xi_0^i\, f_i(\gamma(0)) \in TM_{c(0)}$ there exists a unique solution $\xi\colon I \to \mathbb{R}^m$ satisfying the initial condition $\xi(0) = \xi_0$.

□

3.3 Geodesics

3.7 Definition (geodesics) Let $M \subset \mathbb{R}^n$ be a submanifold. A smooth curve $c\colon I \to M$ is a *geodesic* in M if $\dot c$ is parallel along c, that is, $\frac{D}{dt}\dot c = 0$ on I; equivalently, $\ddot c(t) \in TM_{c(t)}^\perp$ for all $t \in I$.

Every geodesic $c\colon I \to M$ has constant speed $|\dot c|$, because

$$\frac{d}{dt}g(\dot c, \dot c) = 2g\Big(\frac{D}{dt}\dot c, \dot c\Big) = 0.$$

If $f\colon U \to f(U) \subset M$ is a local parametrization and $\gamma = (\gamma^1, \dots, \gamma^m)\colon I \to U$ is a smooth curve, then $c := f \circ \gamma\colon U \to M$ is a geodesic if and only if γ satisfies the system of second order ordinary differential equations

$$\ddot\gamma^k + \sum_{i,j=1}^m \dot\gamma^i\,\dot\gamma^j\,\Gamma_{ij}^k \circ \gamma = 0 \quad (k = 1, \dots, m);$$

just put $\xi := \dot{\gamma}$ in Theorem 3.5. Accordingly, we may also speak of a geodesic γ in U with respect to the metric g, or of a geodesic $c = f \circ \gamma$ relative to a general immersion $f : U \to \mathbb{R}^n$.

3.8 Theorem (existence and uniqueness of geodesics) *Let $M \subset \mathbb{R}^n$ be a submanifold, and let $p \in M$ and $X \in TM_p$. Then there exist a unique geodesic $c: I \to M$ with $c(0) = p$ and $\dot{c}(0) = X$ defined on a maximal open interval I around* 0.

Proof: This follows from the existence and uniqueness theorem for solutions of ordinary differential equations. □

We now consider a more concrete instance.

3.9 Theorem (Clairaut's relation) *Let $c: I \to M$ be a non-constant geodesic on a surface of revolution $M \subset \mathbb{R}^3$. For $t \in I$ let $r(t) > 0$ be the distance of $c(t)$ to the axis of rotation, and let $\theta(t) \in [0, \pi]$ denote the angle between $\dot{c}(t)$ and the oriented parallel through $c(t)$ (that is, the circle generated by the rotation). Then $r(t) \cos \theta(t)$ is constant.*

Proof: We assume that M is rotationally symmetric around the third coordinate axis in $\mathbb{R}^3$. Then, for a local parametrization $f : U \to f(U) \subset M$ of the form

$$f(u, v) = (r(v) \cos(u),\ r(v) \sin(u),\ z(v)),$$

where $r(v) > 0$, the first fundamental form is diagonal with $g_{11}(u, v) = r(v)^2$ and $g_{22}(u, v) = r'(v)^2 + z'(v)^2$ (thus $g_{22} = 1$ if the meridians are parametrized by arc length), and the Christoffel symbols are

$$\begin{aligned}
&\Gamma^1_{11} = 0, && \Gamma^1_{12} = \Gamma^1_{21} = \frac{r'}{r}, && \Gamma^1_{22} = 0, \\
&\Gamma^2_{11} = -\frac{rr'}{(r')^2 + (z')^2}, && \Gamma^2_{12} = \Gamma^2_{21} = 0, && \Gamma^1_{22} = \frac{r'r'' + z'z''}{(r')^2 + (z')^2}.
\end{aligned}$$

Hence, every geodesic $c = f \circ \gamma$, where $\gamma : I \to U$, $\gamma(t) = (u(t), v(t))$, satisfies the differential equations

$$\ddot{u} + 2\frac{r'}{r}\, \dot{u}\, \dot{v} = 0 \quad \text{and} \quad \ddot{v} - \frac{rr'}{(r')^2 + (z')^2}\, \dot{u}^2 + \frac{r'r'' + z'z''}{(r')^2 + (z')^2}\, \dot{v}^2 = 0.$$

The first equation can be rewritten as

$$\frac{d}{dt}(r^2\, \dot{u}) = r^2\, \ddot{u} + 2rr'\, \dot{u}\, \dot{v} = 0;$$

note that here r stands briefly for $r \circ v$, thus $\dot{r}(t) = r'(v(t))\,\dot{v}(t)$. This shows that $r^2\,\dot{u}$ is constant. Since $r^2 = g_{11}$, we have

$$\begin{aligned} r(v(t))^2\,\dot{u}(t) &= g_{\gamma(t)}(\dot{\gamma}(t), e_1) = \langle \dot{c}, f_1(\gamma(t)) \rangle \\ &= |\dot{c}|\,|f_1(\gamma(t))|\,\cos(\theta(t)), \end{aligned}$$

where $|\dot{c}|$ is constant and $|f_1(\gamma(t))| = \sqrt{g_{11}(\gamma(t))} = r(t)$. □

3.10 Theorem (first variation of arc length) *Let $M \subset \mathbb{R}^n$ be a submanifold, and let $c_0\colon [a,b] \to M$ be a smooth curve of constant speed $|\dot{c}_0| = \lambda > 0$. If $c\colon (-\epsilon,\epsilon) \times [a,b] \to M$ is a smooth variation of c_0, $c_s(t) := c(s,t)$, with variation vector field $V_s(t) := V(s,t) := \frac{\partial c}{\partial s}(s,t)$, then*

$$\frac{d}{ds}\Big|_{s=0} L(c_s) = \frac{1}{\lambda}\left(g\big(V_0(t), \dot{c}_0(t)\big)\Big|_a^b - \int_a^b g\Big(V_0(t), \frac{D}{dt}\dot{c}_0(t)\Big)\,dt \right).$$

Proof: For all $s \in (-\epsilon,\epsilon)$ and $t \in [a,b]$,

$$\frac{\partial}{\partial s}|\dot{c}_s(t)| = \frac{1}{2|\dot{c}_s(t)|}\,\frac{\partial}{\partial s}\langle \dot{c}_s(t), \dot{c}_s(t)\rangle = \frac{1}{|\dot{c}_s(t)|}\,\Big\langle \frac{\partial}{\partial s}\dot{c}_s(t), \dot{c}_s(t) \Big\rangle,$$

where $\frac{\partial}{\partial s}\dot{c}_s(t) = \frac{\partial^2}{\partial t\,\partial s}c_s(t) = \frac{\partial}{\partial t}V_s(t)$. Hence,

$$\frac{\partial}{\partial s}\Big|_{s=0} L(c_s) = \frac{1}{\lambda}\int_a^b \Big\langle \frac{d}{dt}V_0(t), \dot{c}_0(t) \Big\rangle\,dt.$$

The integrand equals $\frac{d}{dt}\langle V_0(t), \dot{c}_0(t)\rangle - \big\langle V_0(t), \frac{d}{dt}\dot{c}_0(t)\big\rangle$, and the result follows. □

The variation c of c_0 is called a *proper variation* if $c_s(a) = c_0(a)$ and $c_s(b) = c_0(b)$ for all $s \in (-\epsilon,\epsilon)$. It follows from Theorem 3.10 that a non-constant smooth curve $c_0\colon [a,b] \to M$ is a geodesic if and only if c_0 is parametrized proportionally to arc length and $\frac{d}{ds}\big|_{s=0} L(c_s) = 0$ for every proper variation c of c_0. In particular, if a smooth curve $c_0\colon [a,b] \to M$ of constant speed has minimal length among all smooth curves from $p = c_0(a)$ to $q = c_0(b)$, then c_0 is a geodesic.

4 Curvature of hypersurfaces

In this chapter we consider m-dimensional surfaces of codimension 1.

4.1 Second fundamental form

If $M \subset \mathbb{R}^{m+1}$ is an m-dimensional orientable submanifold, then a *Gauss map* N of M is a continuous map $N: M \to S^m$ such that $N(p) \in TM_p^\perp$ for all $p \in M$ (recall Proposition 2.10). If M is connected, then there are precisely two choices for N, and if M is compact in addition, we may speak of the *inner* or *outer* Gauss map according to Theorem 2.11. If $f: U \to \mathbb{R}^{m+1}$ is an immersion of an open set $U \subset \mathbb{R}^m$, then a *Gauss map* ν of f is a continuous map $\nu: U \to S^m$ with $\nu(x) \in df_x(\mathbb{R}^m)^\perp$ for all $x \in U$. For $m = 2$, the standard choice is $\nu = (f_1 \times f_2)/|f_1 \times f_2|$ (vector product). Note that since M and f are smooth, so are the Gauss maps.

In the following, we tacitly assume that for M and f as above a Gauss map is chosen. We now consider the differential

$$dN_p: TM_p \to TS^m_{N(p)} = TM_p \quad \text{or} \quad d\nu_x: \mathbb{R}^m \to TS^m_{\nu(x)} = df_x(\mathbb{R}^m)$$

for $p \in M$ or $x \in U$, respectively.

4.1 Definition (shape operator) For $p \in M$, the linear map

$$L_p: TM_p \to TM_p, \quad L_p := -dN_p,$$

is called the *shape operator* of M at p. For $x \in U$, the linear map

$$L_x: \mathbb{R}^m \to \mathbb{R}^m, \quad L_x := -(df_x)^{-1} \circ d\nu_x,$$

is the *shape operator* of the immersion f at x (here $(df_x)^{-1}: df_x(\mathbb{R}^m) \to \mathbb{R}^m$ is the inverse of the differential viewed as a map $df_x: \mathbb{R}^m \to df_x(\mathbb{R}^m)$ onto its image). In either case, this is also called the *Weingarten map*.

Note that if f is a local parametrization of M with $f(x) = p$ and $\nu = N \circ f$, then the two shape operators are conjugate: $L_x = (df_x)^{-1} \circ L_p \circ df_x$.

4.2 Lemma (self-adjoint) *For $p \in M$, the shape operator L_p is self-adjoint with respect to g_p, thus*

$$g_p(X, L_p(Y)) = g_p(L_p(X), Y)$$

for all $X, Y \in TM_p$. For an immersion $f: U \to \mathbb{R}^n$ and $x \in U$, the shape operator L_x is self-adjoint with respect to g_x, thus

$$g_x(\xi, L_x(\eta)) = g_x(L_x(\xi), \eta)$$

for all $\xi, \eta \in \mathbb{R}^m$.

Proof: For $p \in M$, choose a local parametrization $f: U \to f(U) \subset M$ of M with $f(x) = p$. Put $\nu := N \circ f$. Then $d\nu_x = dN_p \circ df_x$, and the partial derivatives of f and ν satisfy $dN_p(f_j(x)) = \nu_j(x)$, thus

$$g_p\big(f_i(x), L_p(f_j(x))\big) = -\langle f_i(x), \nu_j(x)\rangle.$$

Furthermore, $\langle f_{ij}, \nu\rangle + \langle f_i, \nu_j\rangle = \frac{\partial}{\partial x^j}\langle f_i, \nu\rangle = 0$, hence

$$g_p\big(f_i(x), L_p(f_j(x))\big) = \langle f_{ij}(x), \nu(x)\rangle$$

is symmetric in i and j. Since $f_1(x), \dots, f_m(x)$ is a basis of TM_p, this shows that L_p is self-adjoint with respect to g_p.

Similarly, for an immersion $f: U \to \mathbb{R}^n$ and $x \in U$,

$$g_x(e_i, L_x(e_j)) = -\langle f_i(x), \nu_j(x)\rangle = \langle f_{ij}(x), \nu(x)\rangle$$

is symmetric in i and j. □

4.3 Definition (second fundamental form) The *second fundamental form h of a submanifold $M \subset \mathbb{R}^{m+1}$* assigns to every point $p \in M$ the symmetric bilinear form h_p on TM_p defined by

$$h_p(X, Y) := g_p(X, L_p(Y)) = -\langle X, dN_p(Y)\rangle$$

for $X, Y \in TM_p$. The *second fundamental form h of an immersion $f: U \to \mathbb{R}^{m+1}$* of an open set $U \subset \mathbb{R}^m$ assigns to every point $x \in U$ the symmetric bilinear form h_x on $\mathbb{R}^m$ defined by

$$h_x(\xi, \eta) := g_x(\xi, L_x(\eta)) = -\langle df_x(\xi), d\nu_x(\eta)\rangle$$

for $\xi, \eta \in \mathbb{R}^m$.

The matrix $(h_{ij}(x))$ of h_x with respect to the canonical basis $(e_1, \dots, e_m)$ of $\mathbb{R}^m$ is given by

$$h_{ij}(x) = -\langle f_i(x), \nu_j(x)\rangle = \langle f_{ij}(x), \nu(x)\rangle.$$

We let $(h^i{}_k(x))$ denote the matrix of L_x with respect to $(e_1, \dots, e_m)$; by the definitions, $(g_{ij})(h^j{}_k) = (h_{ik})$ and hence $(h^i{}_k) = (g^{ij})(h_{jk})$, thus

$$h^i{}_k = \sum_{j=1}^m g^{ij} h_{jk}.$$

4.2 Curvature of hypersurfaces

The following lemma yields a geometric interpretation of the second fundamental form.

4.4 Lemma (normal curvature) *Suppose that $M \subset \mathbb{R}^{m+1}$ is an m-dimensional submanifold with Gauss map N, and $X \in TM_p$ is a unit vector. Then*

$$h_p(X, X) = \langle c''(0), N(p)\rangle$$

for every smooth curve $c\colon (-\epsilon, \epsilon) \to M$ with $c(0) = p$ and $c'(0) = X$.

The curve c can be chosen such that it parametrizes the intersection of M with the normal plane $p + \operatorname{span}(X, N(p))$ in a neighborhood of p. Then $h_p(X, X) = \langle c''(0), N(p)\rangle$ equals the oriented curvature $\kappa_{\mathrm{or}}(0)$ of c in this plane with positively oriented basis $(X, N(p))$. For this reason, $h_p(X, X)$ is called the *normal curvature* of M in direction X.

Proof: Note that

$$h_p(X, X) = -\langle X, dN_p(X)\rangle = -\langle c'(0), (N \circ c)'(0)\rangle,$$

furthermore $\langle c', (N \circ c)'\rangle + \langle c'', N \circ c\rangle = \langle c', N \circ c\rangle' = 0$, thus

$$h_p(X, X) = \langle c''(0), (N \circ c)(0)\rangle = \langle c''(0), N(p)\rangle$$

as claimed. □

Since the shape operator L_p is self-adjoint with respect to g_p, it possesses m real eigenvalues $\kappa_1 \le \ldots \le \kappa_m$, and there exists an orthornormal basis $(X_1, \ldots, X_m)$ of TM_p such that $L_p(X_j) = \kappa_j X_j$, thus

$$h_p(X_i, X_j) = g_p(X_i, L_p(X_j)) = \kappa_j \delta_{ij}.$$

In particular, κ_j is the normal curvature of M in direction X_j.

4.5 Definition (principal curvatures) The m real eigenvalues $\kappa_1 \le \ldots \le \kappa_m$ of L_p are called *principal curvatures* of M at p. Every eigenvector X of L_p with $|X| = 1$ is called a *principal curvature direction.*

Analogously, for an immersion $f\colon U \to \mathbb{R}^{m+1}$ and a point $x \in U$, the shape operator L_x has m real eigenvalues $\kappa_1 \le \ldots \le \kappa_m$, the *principal curvatures* of f, and there exists an orthonormal basis $(\xi_1, \ldots, \xi_m)$ of $\mathbb{R}^m$ *with respect to* g_x such that $L_x(\xi_j) = \kappa_j \xi_j$ and $h_x(\xi_i, \xi_j) = \kappa_j \delta_{ij}$.

A point $x \in U$ is called an *umbilical point* of f if $\kappa_1(x) = \ldots = \kappa_m(x) =: \lambda$; equivalently, $L_x = \lambda\, \mathrm{id}_{\mathbb{R}^m}$.

4.6 Theorem (umbilical points) *Let $f: U \to \mathbb{R}^{m+1}$ be an immersion of a connected open set $U \subset \mathbb{R}^m$ for $m \geq 2$. If every point $x \in U$ is an umbilical point of f, then the image $f(U)$ is contained in an m-plane or an m-sphere.*

Proof: We first show that the function $\lambda: U \to \mathbb{R}$ defined by $\lambda(x) := \kappa_1(x) = \ldots = \kappa_m(x)$ is constant. Since

$$-d\nu_x = df_x \circ L_x = \lambda(x)\, df_x,$$

we have $-\nu_i = \lambda f_i$ and $-\nu_{ij} = \lambda_j f_i + \lambda f_{ij}$, hence $0 = \nu_{ji} - \nu_{ij} = \lambda_j f_i - \lambda_i f_j$. Since $f_1, \ldots, f_m$ are everywhere linearly independent, it follows that $f_i = 0$ on U for $i = 1, \ldots, m$. As U is connected, λ is constant as desired.

Now if $\lambda = 0$, then $d\nu_x = 0$ for all $x \in U$, thus ν is constant. We conclude that $\frac{\partial}{\partial x^i}\langle f, \nu\rangle = \langle f_i, \nu\rangle = 0$, so $\langle f, \nu\rangle$ is constant, and $f(U)$ is contained in an m-plane perpendicular to ν.

If $\lambda \neq 0$, put $z(x) := f(x) + \frac{1}{\lambda}\,\nu(x)$ for all $x \in U$. Then $dz_x = df_x + \frac{1}{\lambda}\, d\nu_x = 0$, thus z is constant, and $|f(x) - z| = \left|\frac{1}{\lambda}\right|$. This shows that $f(U)$ is contained in an m-sphere around z. □

4.7 Definition (Gauss curvature, mean curvature) Let $M \subset \mathbb{R}^{m+1}$ be an m-dimensional submanifold. For $p \in M$,

$$K(p) := \det(L_p)$$

is called the *Gauss–Kronecker curvature*, in the case $m = 2$ the *Gauss curvature*, of M at p, and

$$H(p) := \frac{1}{m}\operatorname{trace}(L_p)$$

is the *mean curvature curvature* of M at p.

For an immersion $f: U \to \mathbb{R}^{m+1}$ and a point $x \in U$, one defines analogously $K(x) := \det(L_x)$ and $H(x) := \frac{1}{m}\operatorname{trace}(L_x)$. Then

$$K = \kappa_1 \cdot \ldots \cdot \kappa_m = \det({h^i}_k) = \det((g^{ij})(h_{jk})) = \frac{\det(h_{ij})}{\det(g_{ij})},$$

$$mH = \kappa_1 + \ldots + \kappa_m = \operatorname{trace}({h^i}_k) = \sum_i {h^i}_i = \sum_{i,j} g^{ij} h_{ji}.$$

4.3 Gauss's theorema egregium

In the following we write again f_i for $\frac{\partial f}{\partial x^i}$ and f_{ij} for $\frac{\partial^2 f}{\partial x^j \partial x^i}$, etc.

4.8 Lemma (derivatives of Gauss frame) *Let $U \subset \mathbb{R}^m$ be an open set. For an immersion $f: U \to \mathbb{R}^{m+1}$ with Gauss map $\nu: U \to S^m$, the partial derivatives of f_i and ν satisfy*

(1) (*equations of Gauss*)

$$f_{ij} = \sum_{k=1}^{m} \Gamma_{ij}^{k} f_k + h_{ij}\nu \quad (i, j = 1, \ldots, m),$$

(2) (*equations of Weingarten*)

$$\nu_k = -\sum_{i=1}^{m} h^i{}_k f_i = -\sum_{i,j=1}^{m} g^{ij} h_{jk} f_i \quad (k = 1, \ldots, m).$$

Proof: Since $f_{ij} = (f_{ij})^{\mathrm{T}} + \langle f_{ij}, \nu \rangle \, \nu$, the definitions yield (1). For all $x \in U$,

$$d\nu_x(e_k) = -df_x(L_x(e_k)) = -df_x\left(\sum_{i=1}^{m} h^i{}_k(x) e_i\right) = -\sum_{i=1}^{m} h^i{}_k(x) f_i(x),$$

which shows (2). □

These equations correspond to the Frenet equations of curve theory. For example, when $m = 2$, they can be written in matrix form as

$$\frac{\partial}{\partial x^k} \begin{pmatrix} f_1 \\ f_2 \\ \nu \end{pmatrix} = \begin{pmatrix} \Gamma_{1k}^1 & \Gamma_{1k}^2 & h_{1k} \\ \Gamma_{2k}^1 & \Gamma_{2k}^2 & h_{2k} \\ -h^1{}_k & -h^2{}_k & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \nu \end{pmatrix}.$$

We will now consider second derivatives of the vector fields f_k. The identity $f_{kij} = f_{kji}$ results in the following equations in the coefficients of the first and second fundamental forms.

4.9 Theorem (integrability conditions) *If $f\colon U \to \mathbb{R}^{m+1}$ is an immersion of an open set $U \subset \mathbb{R}^m$, then the following equations hold for all i, j, k:*

(1) (*Gauss equations*)

$$R^s{}_{kij} = h^s{}_i h_{kj} - h^s{}_j h_{ki} = \sum_{l=1}^{m} g^{sl}\big(h_{li} h_{kj} - h_{lj} h_{ki}\big) \quad (s = 1, \ldots, m),$$

where

$$R^s{}_{kij} := \frac{\partial}{\partial x^i}\Gamma_{kj}^s - \frac{\partial}{\partial x^j}\Gamma_{ki}^s + \sum_{r=1}^{m}\big(\Gamma_{kj}^r \Gamma_{ri}^s - \Gamma_{ki}^r \Gamma_{rj}^s\big),$$

(2) (*Codazzi–Mainardi equation*)

$$\frac{\partial}{\partial x^i} h_{kj} - \frac{\partial}{\partial x^j} h_{ki} + \sum_{r=1}^{m}\big(\Gamma_{kj}^r h_{ri} - \Gamma_{ki}^r h_{rj}\big) = 0.$$

For fixed indices i, j, k, the system (1) is equivalent to

$$R_{lkij} := \sum_{s=1}^{m} g_{ls} R^s{}_{kij} = h_{li} h_{kj} - h_{lj} h_{ki} = \det\begin{pmatrix} h_{li} & h_{lj} \\ h_{ki} & h_{kj} \end{pmatrix} \quad (l = 1, \dots, m).$$

Proof: To simplify the notation, we will suppress the sum symbols and use the convention that products containing the same index twice are summed over that index from 1 to m.

Using Lemma 4.8, we get that $f_{kj} = \Gamma^r_{kj}\, f_r + h_{kj}\, \nu$ and

$$\begin{aligned} f_{kji} &= \big(\Gamma^r_{kj}\, f_{ri} + \Gamma^r_{kj,i}\, f_r\big) + \big(h_{kj}\, \nu_i + h_{kj,i}\, \nu\big) \\ &= \Gamma^r_{kj}\big(\Gamma^s_{ri}\, f_s + h_{ri}\, \nu\big) + \Gamma^r_{kj,i}\, f_r + h_{kj}(-h^s{}_i\, f_s) + h_{kj,i}\, \nu. \end{aligned}$$

We split this into the tangential and normal parts:

$$\begin{aligned} (f_{kji})^{\mathrm{T}} &= \big(\Gamma^r_{kj}\, \Gamma^s_{ri} + \Gamma^s_{kj,i} - h_{kj}\, h^s{}_i\big)\, f_s, \\ (f_{kji})^{\perp} &= \big(\Gamma^r_{kj}\, h_{ri} + h_{kj,i}\big)\, \nu. \end{aligned}$$

Now the relation $(f_{kji})^{\mathrm{T}} - (f_{kij})^{\mathrm{T}} = 0$ is equivalent to (1), and $(f_{kji})^{\perp} - (f_{kij})^{\perp} = 0$ is equivalent to (2). □

The coefficients $R^s{}_{kij}$ or R_{lkij} are the components of the *Riemann curvature tensor* of f (see Differential Geometry II). The Gauss equations for $m = 2$ readily imply the following fundamental result.

4.10 Theorem (Gauss's theorema egregium) *Let $f\colon U \to \mathbb{R}^3$ be an immersion of an open set $U \subset \mathbb{R}^2$. Then the Gauss curvature of f is given by*

$$K = \frac{R_{1212}}{\det(g_{ij})},$$

in particular K is intrinsic, that is, computable entirely in terms of the first fundamental form.

Proof: By the definiton of K and the Gauss equations as stated after Theorem 4.9,

$$K = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{R_{1212}}{\det(g_{ij})},$$

and R_{1212} is computable entirely in terms of g. □

In his fundamental investigation [Ga1828], Gauss derived the completely explicit formula

$$\begin{aligned} K = {} & \frac{1}{4D^2}\big(E(G_1^2 - G_2 A) + F(E_1 G_2 - 2E_2 G_1 + AB) + G(E_2^2 - E_1 B)\big) \\ & - \frac{1}{2D}\big(E_{22} - 2F_{12} + G_{11}\big). \end{aligned}$$

Here we are using the same notation as after Lemma 3.3, together with the abbreviations $A := 2F_1 - E_2$ and $B := 2F_2 - G_1$.

4.11 Theorem (g and h determine f) *Suppose that $U \subset \mathbb{R}^m$ is a connected open set and $f, \tilde{f} \colon U \to \mathbb{R}^{m+1}$ are two immersions with Gauss maps $\nu, \tilde{\nu} \colon U \to S^m$ such that $(f_1, \dots, f_m, \nu)$ and $(\tilde{f}_1, \dots, \tilde{f}_m, \tilde{\nu})$ are positively oriented. If $g = \tilde{g}$ and $h = \tilde{h}$ on U, then f and $\tilde{f}$ agree up to an orientation preserving Euclidean isometry $B \colon \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$, that is, $\tilde{f} = B \circ f$.*

Proof: Let $A \colon U \to \mathbb{R}^{(m+1)\times(m+1)}$ be the matrix function such that $A(x)\, f_i(x) = \tilde{f}_i(x)$ $(i = 1, \dots, m)$ and $A(x)\,\nu(x) = \tilde{\nu}(x)$ for all $x \in U$. Note that

$$\begin{aligned} \langle A f_i, A f_j \rangle &= \langle \tilde{f}_i, \tilde{f}_j \rangle = \tilde{g}_{ij} = g_{ij} = \langle f_i, f_j \rangle, \\ \langle A f_i, A\nu \rangle &= \langle \tilde{f}_i, \tilde{\nu} \rangle = 0 = \langle f_i, \nu \rangle, \\ \langle A\nu, A\nu \rangle &= \langle \tilde{\nu}, \tilde{\nu} \rangle = 1 = \langle \nu, \nu \rangle, \end{aligned}$$

and furthermore $\det(A(x)) > 0$; thus $A(x) \in \mathrm{SO}(m+1)$ for all $x \in U$. We want to show that A is constant. Since $g = \tilde{g}$ and $h = \tilde{h}$, it follows from Lemma 4.8 that

$$\begin{aligned} \tilde{f}_{ij} &= \sum_{k=1}^{m} \tilde{\Gamma}_{ij}^{k} \tilde{f}_k + \tilde{h}_{ij}\tilde{\nu} = \sum_{k=1}^{m} \Gamma_{ij}^{k} A f_k + h_{ij} A\nu = A f_{ij}, \\ \tilde{\nu}_k &= -\sum_{i,j=1}^{m} \tilde{g}^{ij} \tilde{h}_{jk} \tilde{f}_i = -\sum_{i,j=1}^{m} g^{ij} h_{jk} A f_i = A\nu_k. \end{aligned}$$

On the other hand, $\tilde{f}_{ij} = (A f_i)_j = A_j f_i + A f_{ij}$ and $\tilde{\nu}_k = (A\nu)_k = A_k \nu + A\nu_k$ for all i, j, k. It follows that $A_i = 0$ on U for $i = 1, \dots, m$, and since U is connnected, A is constant as desired. Furthermore, $\tilde{f} - Af$ is constant as well because $(\tilde{f} - Af)_i = \tilde{f}_i - A f_i = 0$, and so $\tilde{f} = Af + b$ for some $b \in \mathbb{R}^{m+1}$. □

Given symmetric C^∞ matrix functions $(g_{ij}(\cdot))$ and $(h_{ij}(\cdot))$ on an open set $U \subset \mathbb{R}^m$ such that $(g_{ij}(x))$ is positive definite for every $x \in U$, does there exist an immersion with these fundamental forms? By the *fundamental theorem of local surface theory* due to O. Bonnet, the following holds:

> If (g_{ij}) and (h_{ij}) satisfy the integrability conditions of Theorem 4.9, then for all $x_0 \in U$, $p_0 \in \mathbb{R}^{m+1}$, and $b_1, \dots, b_m \in \mathbb{R}^{m+1}$ with $\langle b_i, b_j \rangle = g_{ij}(x_0)$ there exists a connected open neighborhood U' of x_0 in U and precisely one immersion $f \colon U' \to \mathbb{R}^{m+1}$ such that $f(x_0) = p_0$, $f_i(x_0) = b_i$ for $i = 1, \dots, m$, (g_{ij}) is the first fundamental form of f, and (h_{ij}) is the second fundamental form of f with respect to the Gauss map $\nu \colon U' \to S^m$ for which $(b_1, \dots, b_m, \nu(x_0))$ is positively oriented.

(See [Ku] for a sketch of the proof.) Note that the uniqueness assertion follows from Theorem 4.11.

5 Special classes of surfaces

5.1 Geodesic parallel coordinates

In the following we will denote points in $U \subset \mathbb{R}^2$ by (u, v) rather than $x = (x^1, x^2)$, and partial derivatives of functions on U by a respective subscript u or v.

5.1 Proposition (geodesic parallel coordinates, Fermi coordinates) *Let $I, J \subset \mathbb{R}$ be two open intervals, and let f be an immersion of $U := I \times J$ into $\mathbb{R}^3$. Then the following holds.*

(1) *The first fundamental form of f satisfies $g_{12} = g_{21} = 0$ and $g_{22} = 1$ if and only if the curves $v \mapsto f(u_0, v)$ (for fixed u_0) are unit speed geodesics that intersect the curves $u \mapsto f(u, v_0)$ (for fixed v_0) orthogonally.*

(2) *If $E := g_{11}$, $g_{12} = g_{21} = 0$ and $g_{22} = 1$, then the Gauss curvature of f is given by*

$$K = -\frac{(\sqrt{E})_{vv}}{\sqrt{E}} = \frac{E_v^2}{4E^2} - \frac{E_{vv}}{2E}.$$

(3) *If, in addition, $0 \in J$ and $u \mapsto f(u, 0)$ is a unit speed geodesic, then $E(u, 0) = 1$, $E_u(u, 0) = E_v(u, 0) = 0$, and $\Gamma_{ij}^k(u, 0) = 0$ for all i, j, k and $u \in I$.*

Coordinates as in (1) and (2) or as in (3) are called *geodesic parallel coordinates* or *Fermi coordinates*, respectively. For example, if $v \mapsto (r(v), z(v))$ is a unit speed curve in $\mathbb{R}^2$ with $r > 0$, defined on some interval J, then the surface of revolution $f : \mathbb{R} \times J \to \mathbb{R}^3$ defined by

$$f(u, v) := (r(v)\cos(u),\ r(v)\sin(u),\ z(v))$$

is an immersion in geodesic parallel coordinates with $g_{11} = r^2$ and $K = -\frac{r''}{r}$.

Proof: Evidently, $g_{12} = g_{21} = 0$ if and only the parameter lines intersect orthogonally, and $g_{22} = 1$ if and only if the curves $v \mapsto f(u_0, v)$ have unit speed. Thus, for (1), it remains to show that every such curve $c : v \mapsto f(u_0, v)$ is a geodesic. Let $\beta : [a, b] \to U$, $\beta(t) = (u(t), v(t))$, be a smooth curve from $\beta(a) = (u_0, a)$ to $\beta(b) = (u_0, b)$, for any interval $[a, b] \subset J$. Then

$$L_g(\beta) = \int_a^b \sqrt{g_{11}\,\dot{u}^2 + \dot{v}^2}\,dt \geq \int_a^b \dot{v}\,dt = v(b) - v(a) = b - a,$$

and so $L(f \circ \beta) \geq L(c|_{[a,b]})$. Since this holds for all such curves β, it follows from Theorem 3.10 that c is a geodesic.

If $E := g_{11}$, $g_{12} = g_{21} = 0$ and $g_{22} = 1$, then the only non-vanishing Christoffel symbols are

$$\Gamma^1_{11} = \frac{E_u}{2E}, \quad \Gamma^1_{12} = \Gamma^1_{21} = \frac{E_v}{2E}, \quad \Gamma^2_{11} = -\frac{E_v}{2}.$$

Using Theorem 4.10 and the expression for $R^1{}_{212}$ from Theorem 4.9, we get

$$\begin{aligned} K = \frac{R_{1212}}{E} &= \frac{1}{E}\sum_{s=1}^{2} g_{1s}R^s{}_{212} = R^1{}_{212} = -\big(\Gamma^1_{12}\big)_v - \big(\Gamma^1_{12}\big)^2 \\ &= \frac{E_v^{\,2}}{4E^2} - \frac{E_{vv}}{2E} = -\frac{\big(\sqrt{E}\big)_{vv}}{\sqrt{E}}, \end{aligned}$$

showing (2).

As for (3), if $u \mapsto f(u,0)$ is a unit speed geodesic, then $E(u,0) = 1$ and, by the geodesic equation, $\Gamma^k_{11}(u,0) = 0$ for $k = 1,2$. In view of the above list of the Christoffel symbols, the result follows. □

5.2 Theorem (existence of geodesic parallel coordinates) *Suppose that $M \subset \mathbb{R}^3$ is a 2-dimensional submanifold and $z\colon (-\epsilon,\epsilon) \to M$ is a regular smooth curve. Then there exist $\delta \in (0,\epsilon)$ and a local parametrization $f\colon U \to f(U) \subset M$ on $U := (-\delta,\delta)^2$ such that $g_{12} = g_{21} = 0$, $g_{22} = 1$, and $f(u,0) = z(u)$ for all $u \in (-\delta,\delta)$.*

In particular, by choosing the initial curve z to be a geodesic, we obtain local Fermi coordinates.

Proof: Choose a smooth unit vector field $X\colon (-\epsilon,\epsilon) \to \mathbb{R}^3$ such that $X(u) \in TM_{z(u)}$ and $\langle z'(u), X(u)\rangle = 0$. For every $u \in (-\epsilon,\epsilon)$, let $c_u\colon I_u \to M$ be the maximal geodesic with $c_u(0) = z(u)$ and $\dot c_u(0) = X(u)$ (Theorem 3.8). Put $\hat f(u,v) := c_u(v)$ for all (u,v) with $u \in (-\epsilon,\epsilon)$ and $v \in I_u$. Since the geodesics c_u depend smoothly on their initial conditions, $\hat f$ is defined and smooth in an open neighborhood $\hat U$ of $(0,0)$. Furthermore, $d\hat f_{(0,0)}$ is injective, hence there exists $U = (-\delta,\delta)^2 \subset \hat U$ such that $f := \hat f|_U$ is a local parametrization of M (compare Theorem 2.4).

By construction, $g_{22} = 1$ on U and $g_{12}(u,0) = \langle z'(u), X(u)\rangle = 0$ for all $u \in (-\delta,\delta)$. For a fixed $v \in (0,\delta)$, the geodesics $c_u|_{[0,v]}$ have constant length v, so it follows from Theorem 3.10 that $g_{12}(u,v) = \big\langle \frac{\partial f}{\partial u}(u,v), \dot c_u(v)\big\rangle = 0$ for all u, and an analogous argument applies if $v \in (-\delta,0)$. □

5.2 Surfaces with constant Gauss curvature

For $\kappa \in \mathbb{R}$, we define the functions $\mathrm{cs}_\kappa, \mathrm{sn}_\kappa \colon \mathbb{R} \to \mathbb{R}$ by

$$\mathrm{cs}_\kappa(s) := \begin{cases} \cos(\sqrt{\kappa}s) & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ \cosh(\sqrt{|\kappa|}s) & \text{if } \kappa < 0; \end{cases}$$

$$\mathrm{sn}_\kappa(s) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}s) & \text{if } \kappa > 0, \\ s & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{|\kappa|}} \sinh(\sqrt{|\kappa|}s) & \text{if } \kappa < 0. \end{cases}$$

This is a fundamental system of solutions of the equation $f'' + \kappa f = 0$ with $\mathrm{cs}_\kappa(0) = 1$, $\mathrm{cs}_\kappa'(0) = 0$ and $\mathrm{sn}_\kappa(0) = 0$, $\mathrm{sn}_\kappa'(0) = 1$.

5.3 Proposition (constant curvature in Fermi coordinates) *If $f \colon U \to \mathbb{R}^3$ is an immersion of $U = I \times J$ in Fermi coordinates with constant Gauss curvature $K \equiv \kappa \in \mathbb{R}$, then $E(u,v) = g_{11}(u,v) = \mathrm{cs}_\kappa(v)^2$ for all $(u,v) \in U$.*

Proof: By Proposition 5.1,

$$\big(\sqrt{E}\big)_{vv} + \kappa\sqrt{E} = 0,$$

furthermore $\sqrt{E}(u,0) = 1$ and $\big(\sqrt{E}\big)_v(u,0) = E_v(u,0)/\big(2\sqrt{E(u,0)}\big) = 0$. It follows that $\sqrt{E}(u,v) = \mathrm{cs}_\kappa(v)$ for all $(u,v) \in U$. ☐

5.4 Theorem (constant Gauss curvature) *Let $M, \bar{M} \subset \mathbb{R}^3$ be two surfaces with Gauss curvatures $K \colon M \to \mathbb{R}$ and $\bar{K} \colon \bar{M} \to \mathbb{R}$. Then the following are equivalent*:

(1) *$K \equiv k \equiv \bar{K}$ for some constant $k \in \mathbb{R}$;*

(2) *For every pair of points $p \in M$ and $\bar{p} \in \bar{M}$ there exist an open neighborhood $U \subset \mathbb{R}^2$ of 0 and local parametrizations $f \colon U \to f(U) \subset M$ and $\bar{f} \colon U \to \bar{f}(U) \subset \bar{M}$ such that $f(0) = p$, $\bar{f}(0) = \bar{p}$, and $g = \bar{g}$ on U; that is, M and $\bar{M}$ are everywhere locally isometric.*

Proof: For (1) $\Rightarrow$ (2), introduce local Fermi coordinates around p and $\bar{p}$ according to Theorem 5.2. By Proposition 5.3, $g = \bar{g}$.

Conversely, if $g = \bar{g}$ for local parametrizations around p and $\bar{p}$, then $K(p) = \bar{K}(\bar{p})$ by Theorem 4.10. As $p \in M$ and $\bar{p} \in \bar{M}$ are arbitrary, (1) follows. ☐

5.3 Ruled surfaces

Suppose that $c\colon I \to \mathbb{R}^3$ is a C^2 curve and $X\colon I \to \mathbb{R}^3$ is a nowhere vanishing C^2 vector field, where $X(s)$ is viewed as a vector at the point $c(s)$. A map of the form

$$f\colon I \times J \to \mathbb{R}^3, \quad f(s,t) = c(s) + tX(s),$$

for some interval $J \subset \mathbb{R}$, is called a *ruled surface*, regardless of the fact that f is possibly not regular (immersive). The curve c is called a *directrix* of f, and the lines $t \mapsto f(s_0, t)$ (for fixed s_0) are called the *rulings* of f. Intuitively, f is a surface generated by the motion of a line or line segment in $\mathbb{R}^3$. In regions where f is immersive, the Gauss curvature satisfies

$$K = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{-h_{12}^2}{\det(g_{ij})} \le 0,$$

with $K \equiv 0$ if and only if the Gauss map ν is (locally) constant along the rulings: $h_{12} = -\langle f_1, \nu_2 \rangle = 0$ is equivalent to $\nu_2 = 0$, because $\langle \nu, \nu_2 \rangle = 0$ and $\langle f_2, \nu_2 \rangle = -\langle f_{22}, \nu \rangle = 0$.

5.5 Theorem (rulings in flat surfaces) *Suppose that $V \subset \mathbb{R}^2$ is an open set, and $\tilde f\colon V \to \mathbb{R}^3$ is an immersion with vanishing Gauss curvature $\tilde K \equiv 0$ and without planar points* (*that is, points where both principal curvatures are zero*). *Then $\tilde f$ can everywhere locally be reparametrized as a ruled surface.*

The proof uses Lemma A.5.

Proof: Since $\tilde K \equiv 0$ and f has no planar points, there are precisely two orthogonal principal curvature directions at every $y \in V$, up to choice of signs. Thus, given a point $y_0 \in V$, in a neighborhood of y_0 there exist vector fields X_1, X_2 with $\tilde g(X_1, X_2) = 0$ such that $\tilde g(X_i, X_i) = 1$ and $\tilde h(X_i, X_i) = \tilde\kappa_i$ for $i = 1, 2$, where $\tilde\kappa_1 \ne 0 = \tilde\kappa_2$. By Lemma A.5 there exists a diffeomorphism

$$\varphi\colon U = (-\epsilon, \epsilon)^2 \to \varphi(U) \subset V$$

with $\varphi(0,0) = y_0$ such that the immersion $f = \tilde f \circ \varphi$ satisfies $L_{(s,t)} e_i = \kappa_i e_i$ for all $(s,t) \in U$ (a parametrization *by lines of curvature*), where $\kappa_1 \ne 0 = \kappa_2$. Assume in addition that the curve $t \mapsto f(0,t)$ has unit speed (precompose φ with a suitable map, sending horizontal segments to horizontal segments.) We want to show that $f_{22} = 0$ everywhere on U. Notice first that

$$\langle f_{22}, \nu \rangle = h_{22} = g(e_2, \kappa_2 e_2) = 0.$$

Furthermore, by construction, $\nu_1 = -\kappa_1 f_1$ and $\nu_2 = 0$, hence $\nu_{12} = \nu_{21} = 0$ and

$$\kappa_1 \langle f_{22}, f_1 \rangle = \langle f_2, \kappa_1 f_1 \rangle_2 + \langle f_2, \nu_{12} \rangle = 0.$$

In particular, $\langle f_2, f_2\rangle_1 = 2\langle f_2, f_{12}\rangle = 2\langle f_2, f_1\rangle_2 - 2\langle f_{22}, f_1\rangle = 0$. Since $\langle f_2, f_2\rangle = 1$ along the t-axis, it follows that $\langle f_2, f_2\rangle = 1$ on U, and

$$2\langle f_{22}, f_2\rangle = \langle f_2, f_2\rangle_2 = 0.$$

This shows that $f_{22} = 0$ on U as desired. Hence, $f(s,t) = f(s,0) + t f_2(s,0) = c(s) + tX(s)$. $\square$

5.4 Minimal surfaces

An m-dimensional submanifold $M \subset \mathbb{R}^{m+1}$ or an immersion $f\colon U \to \mathbb{R}^{m+1}$ of an open set $U \subset \mathbb{R}^m$ is called *minimal* if its mean curvature H is identically zero.

5.6 Theorem (first variation of area) *Let $U \subset \mathbb{R}^m$ be an open set, and let $f\colon U \to \mathbb{R}^{m+1}$ be an immersion with Gauss map $\nu\colon U \to S^m$ and finite m-dimensional area*

$$A(f) = \int_U dA = \int_U \sqrt{\det(g_{ij}(x))}\, dx < \infty.$$

If $\varphi\colon U \to \mathbb{R}$ is a smooth function with compact support, then

$$\frac{d}{ds}\Big|_{s=0} A(f + s\,\varphi\,\nu) = -m \int_U \varphi\, H\, dA.$$

In particular, f is minimal if and only if $\frac{d}{ds}\big|_{s=0} A(f + s\,\varphi\,\nu) = 0$ for all such functions φ.

Proof: We write $f^s := f + s\,\varphi\,\nu$ and $g^s_{jk} := \langle f^s_j, f^s_k\rangle$. Then

$$\begin{aligned} g^s_{jk} &= \big\langle f_j + s\,\varphi\,\nu_j + s\,\varphi_j\,\nu,\ f_k + s\,\varphi\,\nu_k + s\,\varphi_k\,\nu\big\rangle \\ &= g_{jk} - 2s\,\varphi\,h_{jk} + O(s^2), \end{aligned}$$

in particular f^s is an immersion for s small enough. Furthermore,

$$\sum_{j=1}^m g^{ij} g^s_{jk} = \delta^i{}_k - 2s\,\varphi\,h^i{}_k + O(s^2).$$

Using the identity $\frac{d}{ds}\big|_{s=0} \det(I + sH) = \operatorname{trace}(H)$ for the matrix $H = (h^i{}_k)$, we get

$$\begin{aligned} \det(g^{ij})\,\frac{d}{ds}\Big|_{s=0} \det\big(g^s_{jk}\big) &= \frac{d}{ds}\Big|_{s=0} \det\big(\delta^i{}_k - 2s\,\varphi\,h^i{}_k\big) \\ &= -2\varphi\,\operatorname{trace}(h^i{}_k) = -2m\,\varphi\,H. \end{aligned}$$

Differentiating under the integral, we conclude that

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0} A(f^s) &= \frac{d}{ds}\Big|_{s=0} \int_U \sqrt{\det\big(g^s_{jk}\big)}\, dx \\ &= \int_U \frac{\det(g_{ij})}{2\sqrt{\det(g_{jk})}} (-2m\,\varphi\,H)\, dx = -m \int_U \varphi\, H\, dA \end{aligned}$$

as claimed. $\square$

An immersion $f\colon U \to \mathbb{R}^3$ is called *isothermal* or *conformal* if $(g_{ij}) = \lambda^2(\delta_{ij})$ for some function $\lambda\colon U \to \mathbb{R}$; equivalently, f is angle preserving (exercise).

5.7 Proposition (isothermal minimal surface) *Let $U \subset \mathbb{R}^2$ be an open set, and let $f\colon U \to \mathbb{R}^3$ be an immersion with Gauss map $\nu\colon U \to S^2$. If f is isothermal, $(g_{ij}) = \lambda^2(\delta_{ij})$, then*

$$\Delta f := f_{11} + f_{22} = 2\lambda^2 H\,\nu;$$

thus f is minimal if and only if the coordinate functions f^1, f^2, f^3 are harmonic.

Proof: Differentiating the relations $\langle f_1, f_1\rangle = \langle f_2, f_2\rangle$ and $\langle f_1, f_2\rangle = 0$ we get

$$\langle f_{11}, f_1\rangle = \langle f_{21}, f_2\rangle = \langle f_{12}, f_2\rangle = -\langle f_1, f_{22}\rangle.$$

Thus $\langle f_{11} + f_{22}, f_1\rangle = 0$ and likewise $\langle f_{11} + f_{22}, f_2\rangle = 0$, which shows that $f_{11} + f_{22}$ is normal. Since

$$H = \frac{1}{2}\sum_{i,j=1}^{2} g^{ij} h_{ji} = \frac{1}{2\lambda^2}(h_{11} + h_{22}) = \frac{1}{2\lambda^2}\langle f_{11} + f_{22}, \nu\rangle,$$

it follows that $2\lambda^2 H\,\nu = (f_{11} + f_{22})^\perp = f_{11} + f_{22}$. □

For the next result we use the following notation. Let $U \subset \mathbb{R}^2$ be an open set, and let $f \in C^\infty(U, \mathbb{R}^3)$, $f(u,v) = (f^1(u,v), f^2(u,v), f^3(u,v))$. We view U as a subset of $\mathbb{C}$ and define $\varphi = (\varphi^1, \varphi^2, \varphi^3)\colon U \to \mathbb{C}^3$ by

$$\varphi^k(u + iv) := \frac{\partial f^k}{\partial u}(u,v) - i\frac{\partial f^k}{\partial v}(u,v),$$

$k = 1, 2, 3$. Here f is not assumed to be an immersion, nevertheless we may say that f is conformal or minimal (meaning that $H = 0$ at points where f is immersive).

5.8 Theorem (complexification) *With the above notation, the following holds.*

(1) *The map f is conformal if and only if $\sum_{k=1}^3 (\varphi^k)^2 = 0$ on U.*

(2) *If f is conformal, then f is an immersion if and only if $\sum_{k=1}^3 |\varphi^k|^2 > 0$ on U and f is minimal if and only if $\varphi^1, \varphi^2, \varphi^3$ are holomorphic.*

(3) *If $U \subset \mathbb{C}$ is a simply connected open set, and if $\varphi^1, \varphi^2, \varphi^3\colon U \to \mathbb{C}$ are holomorphic functions such that $\sum_{k=1}^3 (\varphi^k)^2 = 0$ and $\sum_{k=1}^3 |\varphi^k|^2 > 0$ on U, then the map $f = (f^1, f^2, f^3)\colon U \to \mathbb{R}^3$ defined by*

$$f^k(u,v) := \operatorname{Re}\int_{z_0}^{u+iv} \varphi^k(z)\,dz$$

for any $z_0 \in U$ is a conformal and minimal immersion.

Proof: $\square$

Triples of holomorphic functions as above can be found as follows. Suppose that $F\colon U \to \mathbb{C}$ is holomorphic, $G\colon U \to \mathbb{C} \cup \{\infty\}$ is meromorphic, and FG^2 is holomorphic. Put

$$\varphi^1 := \frac{1}{2}F(1-G^2), \quad \varphi^2 := \frac{i}{2}F(1+G^2), \quad \varphi^3 := FG;$$

then it follows that $\sum_{k=1}^3 (\varphi^k)^2 = 0$, and $\varphi^1, \varphi^2, \varphi^3$ are holomorphic. By inserting these functions φ^k into the above definition of f^k one obtains the so-called *Weierstrass representation* of a minimal surface f. Every non-planar minimal surface can locally be written in this form.

5.5 Surfaces of constant mean curvature

5.9 Theorem (Alexandrov–Hopf) *Suppose that $\emptyset \neq M \subset \mathbb{R}^{m+1}$ is a compact and connected m-dimensional submanifold with constant mean curvature H. Then M is a sphere of radius $1/|H|$.*

The theorem is no longer true for *immersed* surfaces in $\mathbb{R}^3$. This was shown by Wente [We1986], who constructed an immersed torus of constant mean curvature.

Proof (sketch): Fix a direction $v \in S^m$. For $r \in \mathbb{R}$, let M_r be the image of M under the reflection of $\mathbb{R}^{m+1}$ with respect to the affine hyperplane $E_r := \{x \in \mathbb{R}^{m+1} : \langle x, v\rangle = r\}$. Let C denote the union of M with its interior domain, and define $M_r^+ := M_r \cap \{\langle x, v\rangle \geq r\}$. Put $s := \max\{r : M_r^+ \subset C\}$. Then there exists a point $p \in M \cap M_s^+$ with $TM_p = T(M_s^+)_p$. In a neighborhood of p, M and M_s^+ may be represented as graphs over $p + TM_p$. This gives functions $f, \bar{f}\colon U \to \mathbb{R}$ on some open neighborhood $U \subset \mathbb{R}^m$ of 0 such that $f(0) = \bar{f}(0)$, $\nabla f(0) = \nabla \bar{f}(0) = 0$, and $\bar{f} \geq f$, as $M_s^+ \subset C$. (It could happen that $p \in E_s$. Then $\bar{f} \geq f$ possibly only on a half-space of $\mathbb{R}^m$, and additional arguments are required.) Notice that

$$\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = mH = \operatorname{div}\left(\frac{\nabla \bar{f}}{\sqrt{1+|\nabla \bar{f}|^2}}\right)$$

on U. Since $\bar{f} \geq f$, it follows from the maximum principle for elliptic PDEs that $f = \bar{f}$. This shows that the set of contact points $M \cap M_s^+$ is open in M_s^+, and obviously also closed. Hence, the connected component of M_s^+ that contains p belongs to M. Since M is connected, $M_s = M$.

This argument applies to every $v \in S^m$ and shows that there exists an $s = s(v) \in \mathbb{R}$ such that M is symmetric with respect to the hyperplane $E_{v,s} := \{x \in \mathbb{R}^{m+1} : \langle x, v\rangle = s\}$. Now it is an exercise to conclude that M is a sphere. $\square$

6 Global surface theory

6.1 The Gauss–Bonnet theorem

6.1 Definition (geodesic curvature) Suppose that $f\colon U \to \mathbb{R}^3$ is an immersion of an open set $U \subset \mathbb{R}^2$ and $\gamma\colon I \to U$ is a C^2 curve such that $c := f \circ \gamma$ is parametrized by arc length. Put $\bar{e}_1(s) := c'(s)$ and choose $\bar{e}_2(s)$ such that $(\bar{e}_1(s), \bar{e}_2(s))$ is a positively oriented orthonormal basis of $df_{\gamma(s)}(\mathbb{R}^2)$ (equivalent to $(f_1 \circ \gamma(s), f_2 \circ \gamma(s))$). Then

$$\kappa_{\mathrm{g}}(s) := \langle \bar{e}_1'(s), \bar{e}_2(s) \rangle = \Big\langle \frac{D}{ds} c'(s), \bar{e}_2(s) \Big\rangle$$

defines the *geodesic curvature* of c at s (relative to f).

If $\nu = (f_1 \times f_2)/|f_1 \times f_2|$ is the Gauss map of f, then there is a decomposition

$$c'' = \langle c'', \bar{e}_1 \rangle\, \bar{e}_1 + \langle c'', \bar{e}_2 \rangle\, \bar{e}_2 + \langle c'', \nu \circ \gamma \rangle\, \nu \circ \gamma,$$

where $\langle c'', \bar{e}_1 \rangle = \langle c'', c' \rangle = 0$ and $\langle c'', \nu \circ \gamma \rangle =: \kappa_{\mathrm{n}}$ is the *normal curvature* of c relative to f (compare Lemma 4.4). Thus $c'' = \kappa_{\mathrm{g}}\, \bar{e}_2 + \kappa_{\mathrm{n}}\, \nu \circ \gamma$ and

$$\kappa^2 = |c''|^2 = \kappa_{\mathrm{g}}^2 + \kappa_{\mathrm{n}}^2,$$

where κ is the curvature of c as a space curve.

6.2 Lemma (geodesic curvature in geodesic parallel coordinates) *Suppose that $f\colon U \to \mathbb{R}^3$ is an immersion with $E := g_{11}$, $g_{12} = g_{21} = 0$ and $g_{22} = 1$, and $\gamma\colon I \to U$ is a C^2 curve such that $c := f \circ \gamma$ is parametrized by arc length. Write $\gamma(s) = (u(s), v(s))$, and let $\varphi\colon I \to \mathbb{R}$ be a continuous function such that*

$$\gamma'(s) = (u'(s), v'(s)) = \left(\frac{\cos(\varphi(s))}{\sqrt{E(\gamma(s))}},\ \sin(\varphi(s)) \right)$$

for all $s \in I$ (note that $|\gamma'(s)|_g = 1$). Then

$$\kappa_{\mathrm{g}}(s) = \varphi'(s) - \big(\sqrt{E}\big)_v(\gamma(s))\, u'(s)$$

for all $s \in I$.

Proof: For every $s \in I$, we introduce the orthonormal basis

$$(X_1(s), X_2(s)) := \left(\frac{f_1(\gamma(s))}{\sqrt{E(\gamma(s))}},\ f_2(\gamma(s)) \right)$$

of the tangent space $df_{\gamma(s)}(\mathbb{R}^2)$. The rotated frame $(\bar{e}_1(s), \bar{e}_2(s))$ with $\bar{e}_1(s) = c'(s)$ is then given by

$$\bar{e}_1 = \cos(\varphi)\, X_1 + \sin(\varphi)\, X_2, \quad \bar{e}_2 = -\sin(\varphi)\, X_1 + \cos(\varphi)\, X_2;$$

furthermore $c'' = \varphi' \bar{e}_2 + \cos(\varphi) X_1' + \sin(\varphi) X_2'$. Using that $\langle X_i, X_i' \rangle = 0$ and $\langle X_1', X_2 \rangle + \langle X_2', X_1 \rangle = \langle X_1, X_2 \rangle' = 0$, we obtain

$$\begin{aligned} \kappa_{\mathrm{g}} = \langle c'', \bar{e}_2 \rangle &= \varphi' + \cos(\varphi)^2 \langle X_1', X_2 \rangle - \sin(\varphi)^2 \langle X_2', X_1 \rangle \\ &= \varphi' - \langle X_1, X_2' \rangle = \varphi' - \Big\langle X_1, \frac{D}{ds} X_2 \Big\rangle. \end{aligned}$$

From the proof of Proposition 5.1 we know that $\Gamma^1_{21} = E_v/(2E)$ and $\Gamma^1_{22} = 0$, so Theorem 3.5 gives

$$\Big\langle X_1, \frac{D}{ds} X_2 \Big\rangle = \Big\langle X_1, u' \,(\Gamma^1_{21} \circ \gamma)\,(f_1 \circ \gamma) \Big\rangle = \Big\langle X_1, u' \Big(\frac{E_v}{2\sqrt{E}} \circ \gamma \Big) X_1 \Big\rangle,$$

and the result follows. □

6.3 Theorem (Gauss–Bonnet, local version) *Let $M \subset \mathbb{R}^3$ be a surface. Suppose that $\bar{D} \subset M$ is a compact set homeomorphic to a disc such that $\partial \bar{D}$ is the trace of a piecewise smooth, simple closed unit speed curve $c\colon [0, L] \to M$, with exterior angles $\alpha_1, \ldots, \alpha_r \in [-\pi, \pi]$ at the vertices of $\bar{D}$. Let $\kappa_{\mathrm{g}}(s) = \langle c''(s), \bar{e}_2(s) \rangle$ denote the geodesic curvature of c (where $c''(s)$ exists) with respect to the normal $\bar{e}_2(s)$ pointing to the interior of $\bar{D}$. Then*

$$\int_{\bar{D}} K \, dA + \int_0^L \kappa_{\mathrm{g}}(s) \, ds + \sum_{i=1}^r \alpha_i = 2\pi.$$

By definition, the *exterior angle* $\alpha_i \in [-\pi, \pi]$ at a vertex of $\bar{D}$ is the complement $\alpha_i = \pi - \beta_i$ of the $[0, 2\pi]$ valued interior angle β_i of $\bar{D}$. If the boundary of $\bar{D}$ is piecewise geodesic, then $\beta_i \in (0, 2\pi)$ and $\alpha_i \in (-\pi, \pi)$.

Proof: Part I. Suppose first that $\bar{D}$ is contained in the image of some local parametrization $f\colon U \to f(U) \subset M$ in geodesic parallel coordinates. Put $D := f^{-1}(\bar{D})$ and $\gamma := f^{-1} \circ c$, and suppose further that γ is positively oriented with respect to D. We write again (u, v) for points in U, and $\gamma(s) = (u(s), v(s))$ for all $s \in [0, L]$. Let $0 < s_1 < \ldots < s_r < L$ be the parameter values of the vertices $c(s_i)$ of $\bar{D}$ (without loss of generality, c is smoothly closed at $c(0) = c(L)$). Since $\det(g_{ij}) = E$, and by Proposition 5.1, we get

$$\int_{\bar{D}} K \, dA = \int_D K \sqrt{E} \, d(u, v) = \int_D -\big(\sqrt{E}\big)_{vv} \, d(u, v).$$

The last integrand equals the rotation of the planar vector field $\big((\sqrt{E})_v, 0\big)$. Thus, using Green's formula and Lemma 6.2, we conclude further that

$$\int_{\bar{D}} K \, dA = \int_0^L \big(\sqrt{E}\big)_v (\gamma(s)) \, u'(s) \, ds = \int_0^L \varphi'(s) - \kappa_{\mathrm{g}}(s) \, ds$$

for any continuous angle function $\varphi\colon [0, L] \setminus \{s_1, \dots, s_r\} \to \mathbb{R}$ for γ' as in the lemma. This function can be chosen such that

$$\alpha_i = \lim_{s \to s_i+} \varphi(s) - \lim_{s \to s_i-} \varphi(s)$$

for $i = 1, \dots, r$. Then it follows that

$$\int_0^L \varphi'(s)\, ds + \sum_{i=1}^r \alpha_i = \varphi(L) - \varphi(0) = 2\pi\, n$$

for some integer n, and it remains to show that $n = 1$. In the case $g_{11} = 1$, when g is the Euclidean metric $\langle \cdot\,, \cdot \rangle$, this is just Theorem 1.6, generalized to piecewise smooth curves. In the general case, consider the interpolating family of metrics $g^t = (1 - t)\langle \cdot\,, \cdot \rangle + tg$ on U for $t \in [0, 1]$. With respect to every g^t, it follows as above that the right side of the desired identity equals $2\pi\, n(t)$ for some integer $n(t)$. Clearly $n(t)$ is continuous in t, and $n(0) = 1$, so $n = n(1) = 1$ as well. This proves the result under the assumption that $\bar{D}$ is contained in the image of some local parametrization in geodesic parallel coordinates.

Part II. For the general case, we write $\bar{D}$ as a union of finitely many sets $\bar{D}_1, \dots, \bar{D}_F$ with pairwise disjoint interiors, such that the result of the first part holds for each $\bar{D}_j$. (Compare the paragraph before Theorem 6.5 below.) Thus

$$\int_{\bar{D}_j} K\, dA + \int_{\partial \bar{D}_j} \kappa_{\mathrm{g}}(s)\, ds = 2\pi - \sum_{i=1}^{V_j} (\pi - \beta_{ji}),$$

where V_j is the number of vertices of $\bar{D}_j$ and the $\beta_{ji} \in [0, 2\pi]$ are the *interior* angles of $\bar{D}_j$. Let V and E be the total number of vertices and edges, respectively, in the decomposition of $\bar{D}$. By Euler's formula, $V - E + F = 1$. Let V' be the number of vertices lying in $\partial \bar{D}$, and let $\gamma_1, \dots, \gamma_{V'} \in [0, 2\pi]$ be the respective interior angles of $\bar{D}$ (possibly V' exceeds the number of original vertices of $\bar{D}$, and then $\gamma_i = \pi$ at the respective subdivision points). Taking the sum of the above identities for $j = 1, \dots, F$, and adding up the β_{ji}, we get

$$\begin{aligned}\int_{\bar{D}} K\, dA + \int_{\partial \bar{D}} \kappa_{\mathrm{g}}(s)\, ds &= 2\pi\, F - \sum_{j=1}^F \sum_{i=1}^{V_j} (\pi - \beta_{ji}) \\ &= 2\pi\, F - \pi \sum_{j=1}^F V_j + 2\pi(V - V') + \sum_{i=1}^{V'} \gamma_i;\end{aligned}$$

on the left, the integrals of κ_{g} over interior edges cancel in pairs. Since every $\bar{D}_j$ has V_j edges, $\sum_{j=1}^F V_j$ equals $2E$ minus the number V' of boundary edges. Hence, the right side simplifies to

$$2\pi(F - E + V) - \pi\, V' + \sum_{i=1}^{V'} \gamma_i = 2\pi - \sum_{i=1}^{V'} (\pi - \gamma_i),$$

where the $\pi - \gamma_i$ (if non-zero) are the exterior angles of $\bar{D}$. □

6.4 Theorem (Gauss, theorema elegantissimum) *For a geodesic triangle $\bar{D} \subset M$ with interior angles $\beta_1, \beta_2, \beta_3 \in (0, 2\pi)$,*

$$\int_{\bar{D}} K\, dA = \beta_1 + \beta_2 + \beta_3 - \pi.$$

Proof: If $\alpha_1, \alpha_2, \alpha_3$ are the corresponding exterior angles, then

$$2\pi - (\alpha_1 + \alpha_2 + \alpha_3) = \beta_1 + \beta_2 + \beta_3 - \pi.$$

Hence, as the integral of the geodesic curvature along $\partial\bar{D}$ is zero, the result follows directly from Theorem 6.3. □

Now let $M \subset \mathbb{R}^3$ be a compact (and hence orientable) surface. A *polygonal decomposition* of M is a cover of M by finitely many compact subsets $\bar{D}_j \subset M$ homeomorphic to a disc, with piecewise smooth boundary $\partial\bar{D}_j$ (like $\bar{D}$ in Theorem 6.3), such that $\bar{D}_j \cap \bar{D}_k$ is either empty, or a singleton corresponding to a common vertex, or a common edge of $\bar{D}_j$ and $\bar{D}_k$ whenever $j \neq k$. If each $\bar{D}_j$ is a (not necessarily geodesic) triangle, then the decomposition is called a *triangulation* of M. If V, E, F are the number of vertices, edges, and faces in a polygonal decomposition of M, then the integer

$$\chi(M) = V - E + F$$

is the *Euler characteristic* of M.

6.5 Theorem (Gauss–Bonnet, global version) *If $M \subset \mathbb{R}^3$ is a compact surface, then*

$$\int_M K\, dA = 2\pi\, \chi(M).$$

Proof: Choose a polygonal decomposition $M = \bigcup_{j=1}^{F} \bar{D}_j$ of M, apply Theorem 6.3 to each $\bar{D}_j$, and take the sum of these identities, similarly as in the second part of the proof of the local result. □

6.2 The Poincaré index theorem

We now discuss another interpretation of $\chi(M)$ in terms of vector fields.

First let $\xi : U \to \mathbb{R}^2$ be a continuous vector field on an open set $U \subset \mathbb{R}^2$. Suppose that x is an isolated zero of ξ, and pick a radius $r > 0$ such that the closed disc $B(x, r) \subset U$ contains no other zeros of ξ. Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ be the parametrization of $\partial B(x, r)$ defined by $\gamma(t) = x + r(\cos(t), \sin(t))$, and let $\varphi : [0, 2\pi] \to \mathbb{R}$ be a continuous function such that $\xi(\gamma(t))/|\xi(\gamma(t))| = (\cos(\varphi(t)), \sin(\varphi(t)))$ for all

$t \in [0, 2\pi]$. Then $\varphi(2\pi) - \varphi(0) = 2\pi I(x)$ for some integer $I(x) = I_\xi(x)$ called the *index* of ξ at x, which is independent of r by continuity. This number agrees with the *mapping degree* $\deg(F)$ (discussed later in Section 9 for the case of smooth maps between manifolds) of the map

$$F\colon S^1 \to S^1, \quad F(e) = \frac{\xi(x+re)}{|\xi(x+re)|}.$$

This second definition of the index generalizes readily to higher dimensions.

If $\psi\colon U \to V$ is C^1 diffeomorphism onto on open set $V \subset \mathbb{R}^2$, and if η is the continuous vector field on V such that $\eta(\psi(x)) = d\psi_x(\xi(x))$ for all $x \in U$, then it can be shown that $I_\eta(\psi(x)) = I_\xi(x)$ for every isolated zero x of ξ (see, for example, [Mi], pp. 33–35). For a surface $M \subset \mathbb{R}^3$ and a continuous (tangent) vector field $X\colon M \to \mathbb{R}^3$ with an isolated zero at $p \in M$, the index $I(p) = I_X(p)$ is then defined via a local parametrization f of M around p such that $I_X(p) := I_\xi(f^{-1}(p))$ for the corresponding vector field ξ with $df_x(\xi(x)) = X(f(x))$.

6.6 Theorem (Poincaré index theorem) *Let $M \subset \mathbb{R}^3$ be a compact C^1 surface, and let X be a continuous vector field on M with only finitely many zeros $p_1, \ldots, p_k$. Then*

$$\sum_{j=1}^{k} I(p_j) = \chi(M).$$

See [Po1885], Chapitre XIII. This was generalized to arbitrary dimensions by H. Hopf [Ho1927b]. The following argument for the two-dimensional case is also due to Hopf.

Proof: We first show that if X_1, X_2 are two vector fields on M with finitely many zeros, then the respective index sums agree. Triangulate M such that there are no zeros on edges, no triangle $\bar{D}_j$ contains two or more zeros of the same vector field, and each $\bar{D}_j$ is contained in the image of some positively oriented local parametrization of M. Fix j for the moment, and let f be such a local parametrization for $\bar{D}_j$. Define $X(p) := f_1(f^{-1}(p))$, and let $c\colon [0, L_j] \to M$ be a positively oriented unit speed parametrization of $\partial\bar{D}_j$ starting and ending at a vertex. Let $I_{i,j}$ denote the index of the only zero of X_i in $\bar{D}_j$, or put $I_{i,j} = 0$ if X_i has no zero in $\bar{D}_j$. There exist continuous functions $\varphi_i\colon [0, L_j] \to \mathbb{R}$ such that $\varphi_i(s)$ is the oriented angle from $X(c(s))$ to $X_i(c(s))$ and $2\pi\, I_{i,j} = \varphi_i(L_j) - \varphi_i(0)$. Let $\varphi(s)$ be the angle from $X_1(s)$ to $X_2(s)$, so that $\varphi(s) = \varphi_2(s) - \varphi_1(s)$. Then

$$2\pi(I_{2,j} - I_{1,j}) = \varphi_2(L_j) - \varphi_2(0) - \varphi_1(L_j) + \varphi_1(0) = \varphi(L_j) - \varphi(0),$$

which is independent of X. Now, for each triangle $\bar{D}_j$, write $\varphi(L_j) - \varphi(0)$ as the sum $(\varphi(L_j) - \varphi(b_j)) + (\varphi(b_j) - \varphi(a_j)) + (\varphi(a_j) - \varphi(0))$ of three differences corresponding to the sides of $\bar{D}_j$. Then it follows that

$$\sum_j 2\pi(I_{2,j} - I_{1,j}) = 0,$$

because the two contributions of a common edge of two adjacent triangles cancel, due to opposite orientation. Thus the two index sums agree, as desired.

By this result, it now suffices to find a specific vector field X with index sum $\sum_j I(p_j) = \chi(M)$. For this, one can take an arbitrary triangulation of M and construct X such that there is a unique zero of index

$$\begin{cases} 1 & \text{at every vertex,} \\ -1 & \text{in the relative interior of every edge,} \\ 1 & \text{in the interior of every face.} \end{cases}$$

This yields $\sum_j I(p_j) = V - E + F = \chi(M)$. □

7 Hyperbolic space

7.1 Spacelike hypersurfaces in Lorentz space

We consider $\mathbb{R}^{m+1}$ together with the non-degenerate symmetric bilinear form

$$\langle x, y\rangle_{\mathrm{L}} := \left(\sum_{i=1}^{m} x^i y^i\right) - x^{m+1}y^{m+1},$$

called *Lorentz product*. The pair

$$\mathbb{R}^{m,1} := (\mathbb{R}^{m+1}, \langle\,\cdot\,,\cdot\,\rangle_{\mathrm{L}})$$

is called *Minkowski space* or *Lorentz space*. A vector $v \in \mathbb{R}^{m,1}$ is *spacelike* if $\langle v, v\rangle_{\mathrm{L}} > 0$ or $v = 0$, *timelike* if $\langle v, v\rangle_{\mathrm{L}} < 0$, and *lightlike* or a *null vector* if $\langle v, v\rangle_{\mathrm{L}} = 0$ and $v \neq 0$. The set of all null vectors is the *nullcone*. A differentiable curve $c\colon I \to \mathbb{R}^{m,1}$ is *spacelike*, *timelike*, or a *null curve* if all tangent vectors $c'(t)$ have the respective character.

A submanifold $M \subset \mathbb{R}^{m,1}$ is *spacelike* if each tangent space TM_p is, that is, all vectors $v \in TM_p$ are spacelike; equivalently, the *first fundamental form* $g_p := \langle\,\cdot\,,\cdot\,\rangle_{\mathrm{L}}|_{TM_p \times TM_p}$ is positive definite.

7.1 Definition (hyperbolic space) The spacelike hypersurface

$$H^m := \{p \in \mathbb{R}^{m,1} : \langle p, p\rangle_{\mathrm{L}} = -1,\ p^{m+1} > 0\},$$

together with its first fundamental form g, is called *hyperbolic m-space*.

The set H^m is the upper half of the two-sheeted hyperboloid given by the equation $(p^{m+1})^2 = 1 + \sum_{i=1}^m (p^i)^2$. For $p \in H^m$, the tangent space TH^m_p equals the m-dimensional linear subspace of $\mathbb{R}^{m,1}$ determined by the equation $\langle p, v\rangle_{\mathrm{L}} = 0$, similarly as for the sphere $S^m \subset \mathbb{R}^{m+1}$.

We now consider an arbitrary spacelike hypersurface $M^m \subset \mathbb{R}^{m,1}$. If $U \subset \mathbb{R}^m$ is an open set and $f\colon U \to f(U) \subset M$ is a local (or global) parametrization of M, then the first fundamental form of f is given by $g_{ij} = \langle f_i, f_j\rangle_{\mathrm{L}}$. All intrinsic concepts and formulae discussed earlier, involving solely the first fundamental form, remain valid and unchanged for M (or f): Christoffel symbols, covariant derivative, parallelism, geodesics, and the formula

$$K = \frac{R_{1212}}{\det(g_{ij})},$$

which is now adopted as a *definition* of the Gauss curvature in the case $m = 2$. Furthermore, there exists a well-defined *Gauss map*

$$N\colon M^m \to H^m$$

such that $\langle v, N(p)\rangle_{\mathrm{L}} = 0$ whenever $v \in TM_p$. For f as above we put again $\nu := N \circ f$. The *shape operator* and the *second fundamental form* h of M or f are then defined as in Section 4. Lemma 4.8 and Theorem 4.9 remain valid as well, except for two sign changes, due to the fact that $\langle \nu, \nu\rangle_{\mathrm{L}} = -1$:

$$f_{ij} = \sum_{k=1}^{m} \Gamma_{ij}^{k} f_k - h_{ij}\nu$$

for $i, j = 1, \dots, m$, and

$$R^s{}_{kij} = -(h^s{}_i h_{kj} - h^s{}_j h_{ki}) = -\sum_{l=1}^{m} g^{sl}\big(h_{li}h_{kj} - h_{lj}h_{ki}\big)$$

for $s = 1, \dots, m$, where the expression of $R^s{}_{kij}$ in terms of the Christoffel symbols remains unchanged. For fixed i, j, k, this system is equivalent to

$$R_{lkij} := \sum_{s=1}^{m} g_{ls}R^s{}_{kij} = -(h_{li}h_{kj} - h_{lj}h_{ki}) = -\det\begin{pmatrix} h_{li} & h_{lj} \\ h_{ki} & h_{kj} \end{pmatrix}$$

for $l = 1, \dots, m$.

7.2 Geometry of hyperbolic space

In the special case that $M = H^2 \subset \mathbb{R}^{2,1}$, the Gauss map is just given by $N(p) = p$, thus $L_p = -dN_p = -\operatorname{id}_{TH^2_p}$ and $\det(L_p) = 1$. It follows that the Gauss curvature of H^2 is

$$K = \frac{R_{1212}}{\det(g_{ij})} = -\frac{\det(h_{ij})}{\det(g_{ij})} = -1.$$

The *Lorentz group* is defined by

$$\mathrm{O}(m,1) := \big\{A \in \mathrm{GL}(m+1,\mathbb{R}) : \langle Ax, Ay\rangle_{\mathrm{L}} = \langle x, y\rangle_{\mathrm{L}}\big\}.$$

For $A \in \mathrm{O}(m,1)$ and $p \in H^m$, $Ap \in \pm H^m$. One puts

$$\mathrm{O}(m,1)_+ := \big\{A \in \mathrm{O}(m,1) : A(H^m) = H^m\big\}.$$

Thus, for $A \in \mathrm{O}(m,1)_+$, the restriction $A|_{H^m} \colon H^m \to H^m$ is an isometry.

7.2 Theorem (homogeneity) *Suppose that $p, q \in H^m$, $(v_1, \dots, v_m)$ is an orthonormal basis of TH^m_p, and $(w_1, \dots, w_m)$ is an orthonormal basis of TH^m_q. Then there exists an $A \in \mathrm{O}(m,1)_+$ such that $Ap = q$ and $Av_i = w_i$ for $i = 1, \dots, m$.*

Proof: It suffices to consider the case $p = e_{m+1}$. Since matrices of the form

$$\left(\begin{array}{c|c} B & 0 \\ \hline 0 & 1 \end{array}\right)$$

with $B \in \mathrm{O}(m)$ are in $\mathrm{O}(m,1)_+$, the claim is easily reduced to showing that for every point in H^m of the form $q = (q^1, 0, \ldots, 0, q^{m+1})$ there exists $A \in \mathrm{O}(m,1)_+$ with $Ae_{m+1} = q$. As $(q^1)^2 - (q^{m+1})^2 = -1$, there exists $s \in \mathbb{R}$ such that $q^1 = \sinh(s)$ and $q^{m+1} = \cosh(s)$, and one can easily check that

$$A = \begin{pmatrix} \cosh(s) & & & & \sinh(s) \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \sinh(s) & & & & \cosh(s) \end{pmatrix}$$

belongs to $\mathrm{O}(m,1)_+$, and $Ae_{m+1} = q$. □

Let $p \in H^m$, and let $v \in TH^m_p$ be such that $\langle v, v\rangle_{\mathrm{L}} = 1$. The unit speed geodesic $c\colon \mathbb{R} \to H^m$ with $c(0) = p$ and $c'(0) = v$ is given by

$$c(s) = \cosh(s)\, p + \sinh(s)\, v;$$

the trace of c is the intersection of H^m with the linear plane spanned by p and v. The distance of two points p, q in H^m satisfies

$$\cosh(d(p,q)) = -\langle p, q\rangle_{\mathrm{L}}.$$

7.3 Models of hyperbolic space

In the following we let $U := \{x \in \mathbb{R}^m : |x| < 1\}$ denote the open unit ball in $\mathbb{R}^m$. The *(Beltrami–)Klein model* $(U, \bar{g})$ of H^m is obtained via the global parametrization

$$\bar{f}\colon U \to H^m, \quad \bar{f}(\bar{x}) := \frac{1}{\sqrt{1 - |\bar{x}|^2}}(\bar{x}, 1);$$

$\bar{f}$ is the inclusion map $U \to U \times \{1\} \subset \mathbb{R}^m \times \mathbb{R}$ followed by the radial projection to H^m. The first fundamental form of $\bar{f}$ is given by

$$\bar{g}_{ij}(\bar{x}) = \left\langle \bar{f}_i(\bar{x}), \bar{f}_j(\bar{x}) \right\rangle_{\mathrm{L}} = \frac{1}{1 - |\bar{x}|^2}\delta_{ij} + \frac{1}{(1 - |\bar{x}|^2)^2}\bar{x}^i \bar{x}^j,$$

and the distance between two points $\bar{x}, \bar{y}$ in $(U, \bar{g})$ satisfies

$$\cosh(d_{\bar{g}}(\bar{x}, \bar{y})) = \frac{1 - \langle \bar{x}, \bar{y}\rangle}{\sqrt{1 - |\bar{x}|^2}\sqrt{1 - |\bar{y}|^2}}.$$

In this model, the trace of any non-constant geodesic $\gamma\colon \mathbb{R} \to (U, \bar{g})$ is simply a chord of U, because inward radial projection maps geodesic lines in H^m to chords in $U \times \{1\}$.

The *Poincaré model* (U, g) of H^m is obtained similarly via the "stereographic projection"

$$f \colon U \to H^m, \quad f(x) := \frac{1}{1 - |x|^2}(2x, 1 + |x|^2);$$

the three points $(0, -1), (x, 0), f(x) \in \mathbb{R}^m \times \mathbb{R}$ are aligned. The first fundamental form of f is given by

$$g_{ij}(x) = \langle f_i(x), f_j(x) \rangle_{\mathrm{L}} = \frac{4}{(1 - |x|^2)^2} \delta_{ij},$$

thus (U, g) is a conformal model. The distance between $x, y \in (U, g)$ satisfies

$$\cosh(d_g(x, y)) = 1 + \frac{2|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}.$$

If $x, \bar{x} \in U$ are two points with the same images $f(x) = \bar{f}(\bar{x})$ in H^m, then a computation shows that the point $\sigma(\bar{x}) := (\bar{x}, \sqrt{1 - |\bar{x}|^2}) \in S^m \subset \mathbb{R}^{m+1}$ lies on the line through $(0, -1)$ and $(x, 0)$. The map σ sends any chord of U to a semicircle orthogonal to ∂S^m_+ in the upper hemisphere $S^m_+ \subset S^m$, and the inward stereographic projection with respect to $(0, -1)$ maps this semicircle to an arc of a circle in $U \times \{0\}$ orthogonal to $\partial U \times \{0\} = \partial S^m_+$. Hence, geodesic lines in (U, g) are represented by arcs of circles orthogonal to ∂U.

Another conformal model of H^m is the *halfspace model* (U^+, g^+), where $U^+ := \{x \in \mathbb{R}^m : x^m > 0\}$. Inversion in the sphere in $\mathbb{R}^m$ with center $-e_m$ and radius $\sqrt{2}$, restricted to U^+, yields the diffeomorphism

$$\psi \colon U^+ \to U, \quad \psi(x) = \frac{2}{|x + e_m|^2}(x + e_m) - e_m.$$

Let g be the Riemannian metric of the Poincaré model as above. Then $g^+ := \psi^* g$ is given by

$$g^+_{ij}(x) = \frac{1}{(x^m)^2} \delta_{ij}.$$

Now let $m = 2$. Then, up to reparametrization, the unit speed geodesics $\gamma \colon \mathbb{R} \to (U^+, g^+)$ are of the form

$$\gamma(s) = \left(a + r \tanh(s), \frac{r}{\cosh(s)} \right) \quad \text{or} \quad \gamma(s) = (a, e^s)$$

for $a \in \mathbb{R}$ and $r > 0$. In the first case, the trace of γ is a semicircle of Euclidean radius r orthogonal to ∂U^+. The group $\mathrm{GL}(2, \mathbb{R})$ acts on $U^+ \subset \mathbb{C}$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{acts as} \quad z \mapsto \frac{az + b}{cz + d} \quad \text{or} \quad z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d}$$

if the determinant $ad - bc$ is positive or negative, respectively. These are precisely the orientation preserving or reversing isometries of (U^+, g), respectively. The kernel of the action is $\{\lambda I : \lambda \neq 0\}$, thus the isometry group of (U^+, g) is isomorphic to $\mathrm{PGL}(2, \mathbb{R}) = \mathrm{GL}(2, \mathbb{R}) / \{\lambda I : \lambda \neq 0\}$ (exercise).

7.4 Hilbert's theorem

We conclude this section with the following famous result [Hi1901].

7.3 Theorem (Hilbert) *There is no isometric C^3 immersion of the hyperbolic plane into $\mathbb{R}^3$, in particular there is no C^3 submanifold in $\mathbb{R}^3$ isometric to H^2.*

By contrast, it follows from a theorem of Nash and Kuiper [Ku1955] that H^m admits an isometric C^1 embedding into $\mathbb{R}^{m+1}$!

Proof (sketch): The proof is indirect. We suppose that there exists an immersion $\tilde{f}\colon V \to \mathbb{R}^3$ of an open set $V \subset \mathbb{R}^2$, with first fundamental form $\tilde{g}$, such that $(V, \tilde{g})$ is isometric to H^2.

Part I. Since the Gauss curvature of $\tilde{f}$ is negative, it follows that for every $y \in V$ there exist exactly four *asymptotic directions* $\pm X_1(y), \pm X_2(y)$ (with $|X_i(y)|_{\tilde{g}} = 1$), that is, $\tilde{h}(\pm X_i(y), \pm X_i(y)) = 0$. As V is diffeomorphic to a disc, one can choose X_1, X_2 continuous on V. Let $y_0 \in V$. By Lemma A.5 there exists a diffeomorphism

$$\psi\colon U = (-\epsilon, \epsilon)^2 \to \psi(U) \subset V$$

such that $\psi(0,0) = y_0$ and the x^i-parameter lines of ψ are everywhere tangent to X_i. Putting $f := \tilde{f} \circ \psi$, we thus get an immersion with $K = -1$ whose parameter lines are *asymptotic curves*, with $h_{ii} = h(e_i, e_i) = 0$.

We first show that the first fundamental form of f satisfies $g_{11,2} = 0$ and $g_{22,1} = 0$. For every $x \in U$,

$$1 = |K(x)| = |\det(L_x)| = \frac{|\nu_1(x) \times \nu_2(x)|}{|f_1(x) \times f_2(x)|},$$

in particular $\nu_i \neq 0$. We have $\langle f_i, \nu \rangle = 0$, $\langle \nu_i, \nu \rangle = 0$, and $\langle f_i, \nu_i \rangle = -h_{ii} = 0$. Thus $f_i = \lambda_i \, \nu \times \nu_i$ for some functions $\lambda_i \neq 0$, for $i = 1, 2$. Now

$$|f_1 \times f_2| = |\lambda_1 \lambda_2| \, |(\nu \times \nu_1) \times (\nu \times \nu_2)|,$$

and $(\nu \times \nu_1) \times (\nu \times \nu_2) = \nu_1 \times \nu_2$, hence $|\lambda_1 \lambda_2| = 1$. Furthermore, since

$$-h_{12} = \langle f_1, \nu_2 \rangle = \lambda_1 \det(\nu, \nu_1, \nu_2)$$

is symmetric in the two indices, whereas the determinant changes the sign, $\lambda_1 = -\lambda_2$. It follows that λ_1, λ_2 are constant (1 or -1), and so

$$\begin{aligned} 2f_{12} = f_{12} + f_{21} &= \lambda_1(\nu_2 \times \nu_1 + \nu \times \nu_{12}) + \lambda_2(\nu_1 \times \nu_2 + \nu \times \nu_{21}) \\ &= 2\lambda_1 \, \nu_2 \times \nu_1. \end{aligned}$$

We conclude that f_{12} is normal, and hence $g_{11,2} = \langle f_1, f_1 \rangle_2 = 2\langle f_1, f_{12} \rangle = 0$ and $g_{22,1} = 0$ as desired.

This shows that g_{11} depends only on x^1 and g_{22} depends only x^2, which allows to simultaneously reparametrize all parameter lines of f by arc length (precompose ψ with a suitable product map, whose first and second components depend only on the respective variable). After this modification, we have $\psi_i(x) = X_i(\psi(x))$, that is, the parameter lines of ψ are integral curves of the vector fields X_1, X_2 with $|X_i|_{\tilde{g}} = 1$. Hence, the first fundamental form of f now satisfies

$$g_{11} = g_{22} = 1, \quad g_{12} = \langle f_1, f_2 \rangle = \cos(\omega),$$

where $\omega = \angle_g(f_1(x), f_2(x)) \in (0, \pi)$. Coordinates of this form are called a *Chebyshev net*. The result is thus that the given immersion $\tilde{f}\colon V \to \mathbb{R}^3$ can everywhere locally be reparametrized as a Chebyshev net.

Part II. Show that there exists a global diffeomorphism $\psi\colon \mathbb{R}^2 \to V$ such that $f := \tilde{f} \circ \psi\colon \mathbb{R}^2 \to \mathbb{R}^3$ has first fundamental form

$$(g_{ij}) = \begin{pmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{pmatrix}$$

with $\omega \in (0, \pi)$ (a global Chebyshev net).

Part III. In such coordinates, $K = -\omega_{12}/\sin(\omega)$, and since $K = -1$,

$$\sqrt{\det(g_{ij})} = \sin(\omega) = \omega_{12}.$$

Hence, for the g-area of a rectangle $R = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$, we have

$$\begin{aligned} A_g(R) &= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \omega_{12}\, dx^2\, dx^1 \\ &= \int_{a_1}^{b_1} \omega_1(x^1, b_2) - \omega_1(x^1, a_2)\, dx^1 \\ &= \omega(b_1, b_2) - \omega(a_1, b_2) - \omega(b_1, a_2) + \omega(a_1, a_2). \end{aligned}$$

As $\omega \in (0, \pi)$, this yields the uniform bound $A_g(R) < 2\pi$ for all rectangles, which contradicts the fact that the hyperbolic plane has infinite area. $\square$

Differential Topology

8 Differentiable manifolds

8.1 Differentiable manifolds and maps

We start with a topological notion.

8.1 Definition (topological manifold) An *m-dimensional topological manifold M* is a Hausdorff topological space with countable basis (that is, M is second countable) and the property that for every point $p \in M$ there exists a homeomorphism $\varphi \colon U \to \varphi(U)$ from an open neighborhood $U \subset M$ of p onto an open set $\varphi(U) \subset \mathbb{R}^m$. Then $\varphi = (\varphi, U)$ is called a *chart* or *coordinate system* of M.

A system of charts $\Phi = \{(\varphi_\alpha, U_\alpha)\}_{\alpha \in A}$ (where A is any index set) forms an *atlas* of the topological manifold M if $\bigcup_{\alpha \in A} U_\alpha = M$. For $\alpha, \beta \in A$, the (possibly empty) homeomorphism

$$\varphi_{\beta\alpha} := \varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is called the *coordinate change* between φ_α and φ_β.

For $1 \le r \le \infty$, the atlas $\{\varphi_\alpha\}_{\alpha \in A}$ is a *C^r atlas* of M if every coordinate change $\varphi_{\beta\alpha}$ is a C^r map. Since $(\varphi_{\beta\alpha})^{-1} = \varphi_{\alpha\beta}$, it then follows that every coordinate change is a C^r diffeomorphism. More generally, we call two charts $(\varphi, U), (\psi, V)$ of a topological manifold *C^r compatible* if $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ is a C^r diffeomorphism.

8.2 Definition (differentiable manifold) For $1 \le r \le \infty$, a *differentiable structure of class C^r* or *C^r structure* on a topological manifold is a maximal C^r atlas, that is, a C^r atlas not contained in a bigger one. A *differentiable manifold of class C^r* or a *C^r manifold* is a topological manifold equipped with a C^r structure.

We use the word "smooth" as a synonym of C^∞. If we speak of a chart of a differentiable manifold M, then we always mean a chart belonging to the differentiable structure of M.

Every C^r atlas Φ of a topological manifold M is contained in a unique C^r structure $\bar{\Phi}$, namely the set of all charts of M that are C^r compatible with all charts in Φ. However, there exist compact topological manifolds that do not admit any C^1 structure [Ke1960]!

Now let $1 \le r < s \le \infty$. Then every C^s structure is a C^r atlas and is thus contained in a unique C^r structure; in this sense, every C^s manifold is also a C^r manifold. Conversely, every C^r structure contains a C^s structure, and this C^s structure is unique up to C^s diffeomorphism (see Definition 8.3 below and Theorem 2.9 in [Hi, Chapter 2] for the proof). In so far there is no essential difference between the classes C^r and C^s for $1 \le r < s \le \infty$.

8.3 Definition (differentiable map, diffeomorphism) Let M, N be two C^r manifolds, $1 \le r \le \infty$. A map $F\colon M \to N$ is *r times continuously differentiable*, briefly C^r, if for every point $p \in M$ there exist a chart (φ, U) of M with $p \in U$ and a chart (ψ, V) of N with $F(U) \subset V$ such that the map

$$\psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$$

is C^r. This composition is called a *local representation* of F around p. The map $F\colon M \to N$ is a C^r *diffeomorphism* if F is bijective and both F, F^{-1} are C^r.

Ist $F\colon M \to N$ is a C^r map, then clearly *every* local representation of F is C^r, because coordinate changes of M and N are C^r.

On $\mathbb{R}^m$, the atlas consisting solely of the identity map $\mathrm{id}_{\mathbb{R}^m}$ determines the usual smooth structure on $\mathbb{R}^m$. On $\mathbb{R}$, the atlases $\Phi = \{\mathrm{id}_{\mathbb{R}}\}$ and $\Psi = \{\psi\}$, where $\psi(x) = x^3$, determine different smooth structures $\bar{\Phi}$ and $\bar{\Psi}$ since $\mathrm{id}_{\mathbb{R}}$ and ψ are not C^1 compatible; however, $F := \psi^{-1}\colon (\mathbb{R}, \bar{\Phi}) \to (\mathbb{R}, \bar{\Psi})$ is a diffeomorphism since the representation $\psi \circ F \circ (\mathrm{id}_{\mathbb{R}})^{-1}$ equals $\mathrm{id}_{\mathbb{R}}$. In fact, any two differentiable structures on $\mathbb{R}$ are diffeomorphic (exercise).

By contrast, there exist topological manifolds that admit different diffeomorphism classes of smooth structures! For example, there are precisely 28 such classes on the 7-dimensional sphere S^7 [Mi1956], [Mi1959]. On $\mathbb{R}^m$, exotic smooth structures exist only for $m = 4$.

8.4 Definition (tangent space) Let M be an m-dimensional C^r manifold, $1 \le r \le \infty$, and let $p \in M$. On the set of all pairs (φ, ξ), where φ is a chart of M around p and $\xi \in \mathbb{R}^m$, we define an equivalence relation such that $(\varphi, \xi) \sim_p (\psi, \eta)$ if and only if

$$d(\psi \circ \varphi^{-1})_{\varphi(p)}(\xi) = \eta.$$

The *tangent space* TM_p of M at p is the set of all equivalence classes. We write $[\varphi, \xi]_p \in TM_p$ for the class of (φ, ξ).

For a fixed chart φ around p we define the map

$$d\varphi_p\colon TM_p \to \mathbb{R}^m, \quad d\varphi_p([\varphi, \xi]_p) := \xi.$$

Since $[\varphi, \xi]_p = [\varphi, \eta]_p$ if and only if $\xi = \eta$, this is a well-defined bijection, which thus induces the structure of an m-dimensional vector space on TM_p, such that $d\varphi_p$ is a linear isomorphism. If ψ is another chart around p and $(\varphi, \xi) \sim_p (\psi, \eta)$, then

$$\begin{aligned} d\psi_p \circ (d\varphi_p)^{-1}(\xi) &= d\psi_p([\varphi, \xi]_p) = d\psi_p([\psi, \eta]_p) = \eta \\ &= d(\psi \circ \varphi^{-1})_{\varphi(p)}(\xi). \end{aligned}$$

Since $d(\psi \circ \varphi^{-1})_{\varphi(p)}$ is an isomorphism of $\mathbb{R}^m$, it follows that the linear structure of TM_p is independent of the choice of the chart φ.

The *tangent bundle* of a C^r manifold M is the (disjoint) union

$$TM := \bigcup_{p \in M} TM_p$$

together with the projection $\pi\colon TM \to M$ that maps every tangent vector $[\varphi, \xi]_p$ to its footpoint p. The set TM has the structure of a $2m$-dimensional C^{r-1} manifold. If (φ, U) is a chart of M, then

$$\begin{aligned} T\varphi\colon TU = \bigcup_{p \in U} TM_p &\to \varphi(U) \times \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^m \\ [\varphi, \xi]_p &\mapsto (\varphi(p), \xi) = (\varphi(p), d\varphi_p([\varphi, \xi]_p)) \end{aligned}$$

is a corresponding *natural chart* of TM. The coordinate change $T\psi \circ (T\varphi)^{-1}$ maps the pair $(x, \xi) \in \mathbb{R}^m \times \mathbb{R}^m$ to $(\psi \circ \varphi^{-1}(x), d(\psi \circ \varphi^{-1})_x(\xi))$.

For a C^1 map $F\colon M \to N$, the *differential* of F at $p \in M$ is the unique linear map

$$dF_p\colon TM_p \to TN_{F(p)}$$

such that for every local representation $H := \psi \circ F \circ \varphi^{-1}$ of F around p the chain rule

$$dF_p = (d\psi_{F(p)})^{-1} \circ dH_{\varphi(p)} \circ d\varphi_p$$

holds, that is, $dF_p([\varphi, \xi]_p) = [\psi, dH_{\varphi(p)}(\xi)]_{F(p)}$ for all $\xi \in \mathbb{R}^m$. Note that for $F = \varphi$ and $\psi = \mathrm{id}_{\mathbb{R}^m}$, this gives $d\varphi_p([\varphi, \xi]_p) = [\mathrm{id}_{\mathbb{R}^m}, \xi]_{\varphi(p)} = \xi$, where the second equality reflects the identification $T\mathbb{R}^m_{\varphi(p)} = \mathbb{R}^m$; thus our notation for the previously defined map $d\varphi_p$ is justified.

8.2 Partition of unity

Let again M be a C^r manifold, $0 \le r \le \infty$. A family of C^r functions $\lambda_\alpha\colon M \to [0, 1]$ indexed by a set A is called a C^r *partition of unity* if every point $p \in M$ has a neighborhood in which all but finitely many λ_α are constantly zero and if

$$\sum_{\alpha \in A} \lambda_\alpha(p) = 1$$

for all $p \in M$. Given a collection of open sets covering M, a partition of unity $\{\lambda_\alpha\}_{\alpha \in A}$ is *subordinate* to this open cover if for every $\alpha \in A$ the support $\operatorname{spt}(\lambda_\alpha) = \overline{\{p \in M : \lambda_\alpha(p) \neq 0\}}$ of λ_α is contained entirely in one of the sets of the cover.

8.5 Theorem (partition of unity) *For every open cover of a C^r manifold M, $0 \le r \le \infty$, there exists a subordinate C^r partition of unity.*

Proof: Among the (open) sets of a countable basis of the topology of M, let $E_1, E_2, \ldots$ be those with compact closure. Every point $p \in M$ has a compact neighborhood N, which is closed since M is Hausdorff, and there is a set E in the above basis such that $p \in E \subset N$; thus the closure of E is compact. This shows that $\bigcup_{j=1}^{\infty} E_j = M$. Now we define recursively a nested sequence of open subsets of M such that $D_{-1} := \emptyset$, $D_0 := \emptyset$, $D_1 := E_1$, and for $i = 1, 2, \ldots$, D_{i+1} is the union of E_{i+1} with finitely many of the sets E_j covering the (compact) closure $\overline{D_i}$. Then $\bigcup_{i=1}^{\infty} C_i = M$, where $C_i := \overline{D_i} \setminus D_{i-1}$ is compact, and $W_i := D_{i+1} \setminus \overline{D_{i-2}}$ is an open neighborhood of C_i intersecting at most two more of these compact sets.

Let now $\{V_\beta\}_{\beta \in B}$ be an open cover of M. For every point $p \in C_i$ there is a chart (φ, U) of M with $\varphi(p) = 0 \in \mathbb{R}^m$ and $\varphi(U) = U(3) = \{x \in \mathbb{R}^m : |x| < 3\}$ such that $U \subset V_\beta \cap W_i$ for some $\beta \in B$. Hence, there is a finite family $\{(\varphi_\alpha, U_\alpha)\}_{\alpha \in A_i}$ of such charts such that $\{\varphi_\alpha^{-1}(U(1))\}_{\alpha \in A_i}$ is an open cover of C_i. Repeating this construction for every index i, and assuming that $A_i \cap A_j = \emptyset$ whenever $i \neq j$, we get an atlas $\{(\varphi_\alpha, U_\alpha)\}_{\alpha \in A}$ of M with $A = \bigcup_{i=1}^{\infty} A_i$ such that $\{U_\alpha\}_{\alpha \in A}$ is a locally finite open refinement of $\{V_\beta\}_{\beta \in B}$.

Finally, choose a C^∞ function $\tau\colon U(3) \to [0, 1]$ such that $\tau|_{U(1)} \equiv 1$ and $\operatorname{spt}(\tau) = \overline{U(2)}$. For every index $\alpha \in A$, define the C^r function $\tilde{\lambda}_\alpha\colon M \to [0, 1]$ such that $\tilde{\lambda}_\alpha = \tau \circ \varphi_\alpha$ on $U_\alpha = \varphi_\alpha^{-1}(U(3))$ and $\tilde{\lambda}_\alpha \equiv 0$ on $M \setminus U_\alpha$. Since $\{\varphi_\alpha^{-1}(U(1))\}_{\alpha \in A}$ covers M and $\{U_\alpha\}_{\alpha \in A}$ is locally finite, it follows that the sum $S := \sum_{\alpha \in A} \tilde{\lambda}_\alpha$ is everywhere greater than or equal to 1 and finite. Now put $\lambda_\alpha := \frac{1}{S} \tilde{\lambda}_\alpha$. □

8.3 Submanifolds and embeddings

8.6 Definition (submanifold) Let N be an n-dimensional C^∞ manifold. A subset $M \subset N$ is an m-dimensional *submanifold* of N if for every point $p \in M$ there is chart $\psi\colon V \to \psi(V) \subset \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ of N such that $p \in V$ and

$$\psi(M \cap V) = \psi(V) \cap (\mathbb{R}^m \times \{0\}).$$

Such charts are called *submanifold charts*, and $k := n - m$ is the *codimension* of M in N.

The restrictions $\psi|_{M \cap V}$ of all submanifold charts (ψ, V) of M form a C^∞ atlas of M, thus M is itself a C^∞ manifold.

Let $F\colon N \to Q$ be a C^1 map between two manifolds. A point $p \in N$ is a *regular point* of F if the differential dF_p is surjective; otherwise p is a *singular* or *critical point* of F. A point $q \in Q$ is a *regular value* of F if all $p \in F^{-1}\{q\}$ are regular points of F, otherwise q is a *singular* or *critical value* of F.

8.7 Theorem (regular value theorem) *Let $F\colon N^n \to Q^k$ be a C^∞ map. If $q \in F(N)$ is a regular value of F, then $M := F^{-1}\{q\}$ is a submanifold of N of dimension* $\dim(M) = n - k \geq 0$.

Proof: □

A C^∞ map $F\colon M \to N$ between two manifolds is an *immersion* or a *submersion* if, for all $p \in M$, the differential dF_p is injective or surjective, respectively. An *embedding* $F\colon M \to N$ is an immersion with the property that $F\colon M \to F(M)$ is a homeomorphism.

8.8 Theorem (image of an embedding) *If $F\colon M \to N$ is an embedding, then the image $F(M)$ is a submanifold, and $F\colon M \to F(M)$ is a diffeomorphism.*

Conversely, if $M \subset N$ is a submanifold, then the inclusion map $i\colon M \to N$ is an embedding.

Proof: □

8.9 Theorem (embedding theorem) *For every compact C^∞ manifold M^m there exist $n \in \mathbb{N}$ and an embedding $F\colon M \to \mathbb{R}^n$.*

This theorem also holds for $n = 2m + 1$, see [Hi], and even for $n = 2m$ and M possibly non-compact [Wh1944].

Proof: Since M is compact, there exists a finite atlas $\{(\varphi_\alpha, U_\alpha)\}_{\alpha=1,\dots,l}$ such that $\varphi_\alpha(U_\alpha) = U(3) = \{x \in \mathbb{R}^m : |x| < 3\}$ and $\bigcup_\alpha \varphi_\alpha^{-1}(U(1)) = M$. Choose C^∞ functions $\lambda_\alpha\colon M \to [0,1]$ with value 1 on $\varphi_\alpha^{-1}(U(1))$ and support $\varphi_\alpha^{-1}(\overline{U(2)})$ (compare the proof of Theorem 8.5). Define $f_\alpha\colon M \to \mathbb{R}^m$ such that $f_\alpha = \lambda_\alpha\varphi_\alpha$ on U_α and $f_\alpha \equiv 0 \in \mathbb{R}^m$ otherwise. Now put $n := lm + l$ and consider the C^∞ map

$$F\colon M \to \mathbb{R}^n, \quad F := (f_1, \dots, f_l, \lambda_1, \dots, \lambda_l).$$

To show that F is an immersion, let $p \in M$. There is an α such that $p \in \varphi_\alpha^{-1}(U(1))$, thus $\lambda_\alpha \equiv 1$ and $f_\alpha \equiv \varphi_\alpha$ in a neighborhood of p. Then the Jacobi matrix of $F \circ \varphi_\alpha^{-1}$ at the point $\varphi_\alpha(p)$, the $n \times m$-matrix

$$\left(\frac{\partial(F^i \circ \varphi_\alpha^{-1})}{\partial x^j}(\varphi_\alpha(p))\right),$$

contains an I_m (identity matrix) block because $F^{(\alpha-1)m+k} = \varphi_\alpha^k$ for $k = 1, \dots, m$. Hence $d(F \circ \varphi_\alpha^{-1})_{\varphi_\alpha(p)}$ has rank m and is therefore injective, and so is dF_p.

To show that $F\colon M \to F(M)$ is a homeomorphism, suppose first that $F(p) = F(q)$ for some $p, q \in M$. Then there is an α such that $\lambda_\alpha(p) = \lambda_\alpha(q) = 1$, in particular $p, q \in U_\alpha$, and

$$\varphi_\alpha(p) = \lambda_\alpha(p)\,\varphi_\alpha(p) = f_\alpha(p) = f_\alpha(q) = \lambda_\alpha(q)\,\varphi_\alpha(q) = \varphi_\alpha(q).$$

Thus $p = q$. Now F is a continuous bijective map from the compact space M onto the Hausdorff space $F(M) \subset \mathbb{R}^m$ and, hence, a homeomorphism. □

8.4 Tangent vectors as derivations

Let M be a C^∞ manifold and $p \in M$. A linear functional $X\colon C^\infty(M) \to \mathbb{R}$ on the algebra of real-valued smooth functions on M is called a *derivation* at p if for all $f, g \in C^\infty(M)$ the product rule (or Leibniz rule)

$$X(fg) = X(f)g(p) + f(p)X(g)$$

holds. It follows from this identity that $X(f) = X(\tilde{f})$ whenever $f \equiv \tilde{f}$ in a neighborhood of p: if $g := f - \tilde{f}$ and $h \in C^\infty(M)$ is such that $h(p) = 1$ and $\mathrm{spt}(h) \subset g^{-1}\{0\}$, then

$$0 = X(0) = X(gh) = X(g)h(p) + g(p)X(h) = X(g) = X(f) - X(\tilde{f}).$$

Hence every derivation X at p has a unique extension, still denoted by X, to the set of functions

$$C^\infty(M)_p := \{f \in C^\infty(U) : U \subset M \text{ an open neighborhood of } p\}$$

such that $X(f) = X(\tilde{f})$ whenever $f, \tilde{f} \in C^\infty(M)_p$ agree in a neighborhood of p. For the constant function on M with value $c \in \mathbb{R}$, $X(c) = c\,X(1) = 0$ since $X(1) = X(1 \cdot 1) = X(1) \cdot 1 + 1 \cdot X(1)$.

For any chart (φ, U) of M^m around p there are canonical derivations $\frac{\partial}{\partial\varphi^1}\big|_p, \dots, \frac{\partial}{\partial\varphi^m}\big|_p$ at p, defined by

$$\frac{\partial}{\partial\varphi^j}\Big|_p(f) := \frac{\partial f}{\partial\varphi^j}(p) := \frac{\partial(f \circ \varphi^{-1})}{\partial x^j}(\varphi(p)).$$

8.10 Theorem (derivations) *The set of all derivations at $p \in M^m$ is an m-dimensional vector space. If φ is a chart around p, then the canonical derivations $\frac{\partial}{\partial\varphi^1}\big|_p, \dots, \frac{\partial}{\partial\varphi^m}\big|_p$ constitute a basis, and every derivation X at p satisfies*

$$X = \sum_{j=1}^m X(\varphi^j)\frac{\partial}{\partial\varphi^j}\Big|_p.$$

Proof: □

For a C^∞ manifold M^m, we now identify the tangent vector $X \in TM_p$ (Definition 8.4) with the derivation X at p defined by

$$X(f) := df_p(X) \in T\mathbb{R}_{f(p)} = \mathbb{R}.$$

It is not difficult to check that then for every chart φ around p and every $\xi = (\xi^1, \ldots, \xi^m) \in \mathbb{R}^m$, the vector $X = [\varphi, \xi]_p$ corresponds to the derivation

$$X = \sum_{j=1}^{m} \xi^j \frac{\partial}{\partial \varphi^j}\Big|_p.$$

9 Transversality

9.1 The Morse–Sard theorem

A *cube* $C \subset \mathbb{R}^m$ of edge length $s > 0$ and volume $|C| = s^m$ is a set isometric to $[0, s]^m$. A set $A \subset \mathbb{R}^m$ has *measure zero* or is a *nullset* if for every $\epsilon > 0$ there exists a sequence of cubes $C_i \subset \mathbb{R}^m$ such that $A \subset \bigcup_i C_i$ and $\sum_i |C_i| < \epsilon$. The union of countably many nullsets is a nullset.

If $V \subset \mathbb{R}^m$ is an open set and $F: V \to \mathbb{R}^m$ a C^1 map, and if $A \subset V$ has measure zero, then $F(A)$ has measure zero. To prove this, note first that V is the union of countably many compact balls B_k. Then each set $A \cap B_k$ lies in the interior of some compact subset of V, on which F is Lipschitz continuous, and it follows easily that $F(A \cap B_k)$ has measure zero.

9.1 Definition (measure zero) A subset A of a differentiable manifold M^m has *measure zero* or is a *nullset* if for every chart (φ, U) of M the set $\varphi(A \cap U) \subset \mathbb{R}^m$ has measure zero.

It follows from the aforementioned properties that $A \subset M$ has measure zero if $\varphi(A \cap U)$ has measure zero for every chart (φ, U) from a fixed countable atlas of M.

9.2 Theorem (Morse–Sard) *If $F: M^m \to N^n$ is a C^r map with $r > \max\{0, m-n\}$, then the set of singular values of F has measure zero in N.*

See [Mo1939] ($n = 1$, $r = m$) and [Sa1942]. For example, the set of singular values of a C^2 function $F: \mathbb{R}^2 \to \mathbb{R}$ has measure zero (and thus $F^{-1}\{t\}$ is a 1-dimensional submanifold for almost every $t \in \mathbb{R}$). The differentiability assumption seems stronger than necessary, but indeed Whitney [Wh1935] constructed an example of a C^1 function $F: \mathbb{R}^2 \to \mathbb{R}$ that is non-constant on a compact connected set of singular points.

Note that if $n = 0$, then there are no singular values in N by definition, whereas if $m = 0$, then $F(M)$ is a countable set. In the general case, the theorem follows easily from the corresponding result for a C^r map F from on open set $U \subset \mathbb{R}^m$ to $\mathbb{R}^n$, because M and N have countable atlases. Then, in the case that $m < n$ and $r = 1$, the proof is simple: $U \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^{n-m}$ is a nullset in $\mathbb{R}^m \times \mathbb{R}^{n-m}$, thus the C^1 map $\tilde{F}: U \times \mathbb{R}^{n-m} \to \mathbb{R}^n$, $\tilde{F}(p, x) := F(p)$, takes it to the nullset $\tilde{F}(U \times \{0\}) = F(U)$ in $\mathbb{R}^n$.

We now prove the result for $m \geq n \geq 1$ and $r = \infty$.

Proof: It suffices to consider a C^∞ map $F = (F^1, \ldots, F^n): U \to \mathbb{R}^n$ on an open set $U \subset \mathbb{R}^m$. Let $\Sigma \subset U$ be the set of singular points of F. Furthermore, for $l = 1, 2, \ldots$, let Z_l denote the set of all points $x \in U$ where all partial derivaties of

F up to order l vanish, that is,

$$F^i_{j_1,\dots,j_k}(x) := \frac{\partial^k F^i}{\partial x^{j_1}\partial x^{j_2}\dots\partial x^{j_k}}(x) = 0$$

for all $k \in \{1,\dots,l\}$, $i \in \{1,\dots,n\}$ and $j_1,\dots,j_k \in \{1,\dots,m\}$. This gives a sequence $\Sigma \supset Z_1 \supset Z_2 \supset \dots$ of closed subsets of U. We now fix $l \geq 1$ as the smallest integer strictly bigger than $\frac{m}{n} - 1$.

We show that $F(Z_l)$ has measure zero. Let $C \subset U$ be a cube of side length s. By virtue of Taylor's formula of order l and the compactness of C,

$$F(y) = F(x) + R(x, y)$$

for all $x \in C \cap Z_l$ and $y \in C$, where $|R(x, y)| \leq c|x - y|^{l+1}$ for some constant c depending only on F and C. Consider a subdivision of C into N^m cubes of side length s/N. If C' is one of these cubes and x is a point in $C' \cap Z_l$, then $F(C')$ lies in the closed ball with center $F(x)$ and radius $c(\sqrt{m}s/N)^{l+1}$. Hence $F(C \cap Z_l)$ can be covered by N^m cubes with total volume $N^m\big(2c(\sqrt{m}s/N)^{l+1}\big)^n$. Since $n(l+1) > m$, this quantity tends to 0 as $N \to \infty$. It follows that $F(Z_l)$ has measure zero.

If $m = n = 1$, then $\Sigma = Z_1 = Z_l$, hence $F(\Sigma)$ has measure zero. We now proceed by induction and complete the argument for $m \geq 2$, $m \geq n \geq 1$ and $r = \infty$ assuming that the set of singular values of every C^∞ map $G\colon M' \to N'$ between manifolds of dimension $\dim(M') = m - 1 \geq \dim(N') \geq 1$ has measure zero.

First we consider $F(Z_k \setminus Z_{k+1})$ for any $k \geq 1$. For every $x \in Z_k \setminus Z_{k+1}$, there exist a k-fold partial derivative $f := F^i_{j_1,\dots,j_k}\colon U \to \mathbb{R}$ and a further index $j \in \{1,\dots,m\}$ such that $f_j(x) := \frac{\partial f}{\partial x^j}(x) \neq 0$. Then $f_j(y) \neq 0$ for all y in an open neighborhood $V \subset U \setminus Z_{k+1}$ of x. Thus the (smooth) function $f|_V$ is everywhere regular, in particular the set $M' := f^{-1}\{0\} \cap V$, which contains $Z_k \cap V$, is an $(m-1)$-dimensional submanifold. Every point $y \in Z_k \cap V \subset \Sigma$ is also a singular point of $F|_{M'}$, hence $F(Z_k \cap V)$ has measure zero in $\mathbb{R}^n$ by the induction hypothesis, or by the remark preceding the proof if $m - 1 < n$. It follows that $F(Z_k \setminus Z_{k+1})$ has measure zero for every $k \geq 1$.

Since $F(Z_1) = F(Z_l) \cup \bigcup_{k=1}^{l-1} F(Z_k \setminus Z_{k+1})$ has measure zero, it remains to consider the set $F(\Sigma \setminus Z_1)$. If $n = 1$, then $\Sigma = Z_1$ and we are done. Now let $n \geq 2$. At every point $x \in \Sigma \setminus Z_1$ at least one partial derivative F^i_j is non-zero. To simplify the notation we assume that $F^i_m(x) \neq 0$. Then x is a regular point of the map

$$\varphi\colon U \to \mathbb{R}^m, \quad \varphi(y) := (y^1,\dots,y^{m-1}, F^i(y)).$$

Hence there exists an open neighborhood $V \subset U \setminus Z_1$ of x such that $\varphi|_V$ is a diffeomorphism onto an open set $W \subset \mathbb{R}^m$, and there is a well-defined map $G\colon W \to \mathbb{R}^n$ such that $F|_V = G \circ \varphi|_V$. For all $y \in V$,

$$G(y^1,\dots,y^{m-1}, F^i(y)) = G(\varphi(y)) = (F^1(y),\dots,F^n(y)),$$

thus G preserves some coordinate. Hence, if $y \in V \cap \Sigma$ is a singular point of F with $F^i(y) = t \in \mathbb{R}$, then $\varphi(y) = (y^1, \dots, y^{m-1}, t)$ is a singular point of G, as well as of the restriction of G to $M_t := W \cap (\mathbb{R}^{m-1} \times \{t\})$, and $F(y) = G(\varphi(y))$ is a singular value of $G|_{M_t}$. Therefore, by the induction hypothesis, the set $F(V \cap \Sigma) \cap \{z \in \mathbb{R}^n : z^i = t\}$ has $(n-1)$-dimensional (Lebesgue) measure zero. By Fubini's theorem, the measurable (in fact, σ-compact) set $F(V \cap \Sigma)$ has n-dimensional measure zero. It follows that also $F(\Sigma \setminus Z_1)$ has measure zero. □

9.2 Manifolds with boundary

Next we introduce manifolds with boundary.

A *halfspace* of $\mathbb{R}^m$ is a set of the form

$$H = \{x \in \mathbb{R}^m : \lambda(x) \geq 0\}$$

for a linear function $\lambda \colon \mathbb{R}^m \to \mathbb{R}$. Note that, according to this definition, also $H = \mathbb{R}^m$ is a halfspace (take $\lambda \equiv 0$). The boundary ∂H of $H = \{\lambda \geq 0\}$ is the kernel of λ if $\lambda \not\equiv 0$ and empty otherwise.

An m-dimensional *topological manifold M with boundary* is a Hausdorff space with countable basis of the topology and the following property: for every point $p \in M$ there exist a homeomorphism $\varphi \colon U \to \varphi(U) \subset H$ from an open neighborhood U of p onto an open subset $\varphi(U)$ of a halfspace $H \subset \mathbb{R}^m$ (with the induced topology). Then $\varphi = (\varphi, U)$ is a *chart* of M. The notions of C^r *atlas*, C^r *structure*, and C^r *manifold with boundary* are then defined in analogy with the boundary-free case. Here, a coordinate change

$$\varphi_{\beta\alpha} := \varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a C^r map between open subsets in halfspaces of $\mathbb{R}^m$; this means that $\varphi_{\beta\alpha}$ admits an extension to a C^r map between open subsets of $\mathbb{R}^m$.

The *boundary* of M is the set

$$\partial M := \{p \in M : \varphi(p) \in \partial H \text{ for some chart } \varphi \colon U \to \varphi(U) \subset H \text{ around } p\}.$$

It follows that if $p \in \partial M$, then $\varphi(p) \in \partial H$ for *every* chart $\varphi \colon U \to \varphi(U) \subset H$ around p. For topological manifolds with boundary this is a consequence of the *theorem on invariance of the domain* [Br1911a]:

> If $V \subset \mathbb{R}^m$ is open and $h \colon V \to \mathbb{R}^m$ is an injective continuous map, then $h(V) \subset \mathbb{R}^m$ is open.

In the C^r case, $r \geq 1$, one may more easily use the inverse function theorem. The boundary ∂M of a C^r manifold M^m with boundary, $r \geq 0$, is in a natural way an $(m-1)$-dimensional C^r manifold (without boundary), and $M \setminus \partial M$ is a manifold as well. According to the above definition, every manifold M is also a manifold with boundary, where $\partial M = \emptyset$.

Example Suppose that N is a manifold, $f\colon N \to \mathbb{R}$ is a smooth function, and $y \in \mathbb{R}$ is a regular value of f. Then $M := f^{-1}([y,\infty))$ is a manifold with boundary $\partial M = f^{-1}\{y\}$: by Theorem 8.7, $f^{-1}\{y\}$ is a submanifold of N of codimension 1, and the restriction of any submanifold chart $\psi\colon V \to \psi(V) \subset \mathbb{R}^n$ to $V \cap M$ is a chart for M around boundary points.

Let now M^m be a C^r manifold with boundary, $1 \le r \le \infty$. For $p \in M$, the *tangent space* TM_p of M at p is defined as in Definition 8.4 (note that the differential $d(\psi \circ \varphi^{-1})_{\varphi(p)}$ is uniquely defined also if $p \in \partial M$). The tangent space $T(\partial M)_p$ of ∂M at a boundary p is in a canonical way an $(m-1)$-dimensional subspace of TM_p. Differentiable maps $F\colon M \to N$ between manifolds with boundary and the differential $dF_p\colon TM_p \to TN_{F(p)}$ are again defined as in the boundary-free case.

The following statement generalizes Theorem 8.7.

9.3 Theorem (regular value theorem, manifolds with boundary) *Let $F\colon N \to Q$ be a C^∞ map, where N^n is a manifold with boundary and Q^k is a manifold. If $q \in F(N)$ is a regular value of $F|_{N\setminus\partial N}$ as well as of $F|_{\partial N}$, then $M := F^{-1}\{q\}$ is a manifold with boundary,* $\dim(M) = n - k \ge 0$, *and $\partial M = M \cap \partial N$.*

Note that the assumption on q is stronger than saying that $q \in F(N)$ is a regular value of F, because ∂N is only $(n-1)$-dimensional. The set $M \cap \partial N$ is non-empty if and only if $q \in F(\partial N)$; in this case, it follows from the assumption that $n - 1 \ge k$ and hence $\dim(M) \ge 1$.

Proof: □

A continuous map $F\colon M \to A$ from a topological space M to a subspace $A \subset M$ such that $F(p) = p$ for all $p \in A$ is called a *retraction* of M onto A.

9.4 Theorem (boundary is not a retract) *Let M be a compact C^∞ manifold with boundary. Then there is no smooth retraction of M onto ∂M.*

In the proof of this result and subsequently we will make use of the *classification of compact* 1*-dimensional manifolds with boundary*:

> Every smooth, compact 1-dimensional manifold with boundary is diffeomorphic to a disjoint union of finitely many circles S^1 and intervals $[0,1]$.

For a proof of this intuitive fact we refer to [Mi, Appendix].

Proof: Suppose to the contrary that there exists a smooth retraction $F\colon M \to \partial M$. By Theorem 9.2 there exists a regular value $q \in \partial M$ of $F|_{M\setminus\partial M}$. Since F is a retraction, q is also a regular value of $F|_{\partial M} = \mathrm{id}_{\partial M}$. It follows from Theorem 9.3 that $F^{-1}\{q\}$ is a compact 1-dimensional manifold with boundary $F^{-1}\{q\} \cap \partial M = \{q\}$.

This contradicts the fact that by the aforementioned classification, such manifolds have an even number of boundary points. □

9.5 Theorem (Brouwer fixed point theorem) *Every continuous map* $G\colon B^m \to B^m = \{x \in \mathbb{R}^m : |x| \leq 1\}$ *has a fixed point.*

Proof: □

9.3 Mapping degree

Let $F, G\colon M \to N$ be two C^∞ maps. A C^∞ map $H\colon M \times [0,1] \to N$ with $H(\cdot, 0) = F$ and $H(\cdot, 1) = G$ is called a *smooth homotopy* from F to G. We write $F \sim G$ and call F and G *smoothly homotopic* if such a map H exists. This defines an equivalence relation on $C^\infty(M, N)$. Transitivity is most easily shown using the following reparametrization trick: if H is a smooth homotopy from F to G, and $\tau\colon [0,1] \to [0,1]$ is a smooth function that is constantly 0 on $[0, \frac{1}{3}]$ and 1 on $[\frac{2}{3}, 1]$, then $\tilde{H}(p,t) := H(p, \tau(t))$ defines a smooth homotopy such that $\tilde{H}(\cdot\,, t) = F$ for $t \in [0, \frac{1}{3}]$ and $\tilde{H}(\cdot\,, t) = G$ for $t \in [\frac{2}{3}, 1]$.

A smooth homotopy $H\colon M \times [0,1] \to N$ from F to G with the additional property that $H(\cdot\,, t)\colon M \to N$ is a C^∞ diffeomorphism for all $t \in [0,1]$ is called a smooth *smooth isotopy* between (the diffeomorphisms) F and G.

9.6 Lemma (isotopies) *If N is a connected manifold, then for every pair of points $q, q' \in N$ there is a smooth isotopy $H\colon N \times [0,1] \to N$ with $H(\cdot, 0) = \mathrm{id}_N$ and $H(q, 1) = q'$.*

Proof: □

Let now $F\colon M \to N$ be a C^∞ map between two manifolds of the same dimension. If $q \in N$ is a regular value of F, then $F^{-1}\{q\}$ is a (possibly empty) 0-dimensional submanifold of M, hence a discrete set. If M is compact, then the number $\#F^{-1}\{q\}$ of points in $F^{-1}\{q\}$ is finite.

9.7 Theorem (mapping degree modulo 2) *Suppose that M, N are two manifolds of the same dimension, M is compact, and N is connected.*

(1) *If $F, G\colon M \to N$ are smoothly homotopic, and if $q \in N$ is a regular value of both F and G, then* $\#F^{-1}\{q\} \equiv \#G^{-1}\{q\} \pmod 2$.

(2) *If $F\colon M \to N$ is a C^∞ map, and if $q, q' \in N$ are two regular values of F, then* $\#F^{-1}\{q\} \equiv \#F^{-1}\{q'\} \pmod 2$.

The *mapping degree modulo* 2 of F is the number

$$\deg_2(F) := (\#F^{-1}\{q\} \bmod 2) \in \{0, 1\};$$

by (2), it does not depend on the choice of the regular value q. Furthermore, by (1), it is invariant under smooth homotopies, that is, $\deg_2(F) = \deg_2(G)$ if $F \sim G$.

Proof: □

If M and N are *oriented* manifolds of the same dimension, M compact and N connected, then the *mapping degree* $\deg(F) \in \mathbb{Z}$ of a smooth map $F\colon M \to N$ is defined as

$$\deg(F) := \sum_{p \in F^{-1}\{q\}} \operatorname{sgn}(dF_p)$$

for any regular value $q \in N$ of F, where

$$\operatorname{sgn}(dF_p) := \begin{cases} +1 & \text{if } dF_p \text{ is orientation preserving,} \\ -1 & \text{otherwise} \end{cases}$$

(note that for every regular point $p \in M$, the differential $dF_p\colon TM_p \to TN_{F(p)}$ is an isomorphism, since $\dim(M) = \dim(N)$). Similarly as for $\deg_2$ one can show that $\deg(F)$ does not depend on the choice of q and that $\deg(F) = \deg(G)$ if $F \sim G$.

9.8 Theorem (hairy ball theorem) *The sphere S^m admits a nowhere vanishing tangent vector field if and only if m is odd.*

Proof: Let $\alpha\colon S^m \to S^m$ be the antipodal map $p \mapsto -p$. We show first that $\deg(\alpha) = (-1)^{m+1}$. If $p \in S^m$ and $(v_1, \dots, v_m)$ is a positively oriented basis of TS^m_p (no matter how S^m is oriented), then $(v_1, \dots, v_m)$ is negatively oriented as a basis of TS^m_{-p}, because $N(-p) = -N(p)$ for any Gauss map. Furthermore, $d\alpha_p(v_i) = -v_i$ (note that α is the restriction of a linear map). Thus $d\alpha_p$ preserves orientation if and only if m is odd. Since α is a diffeomorphism, it follows that $\deg(\alpha) = \operatorname{sgn}(d\alpha_p) = (-1)^{m+1}$.

Suppose now that X is a nowhere zero smooth tangent vector field on S^m. We can assume that $|X| \equiv 1$. Then

$$H(p, s) := \cos(s)\, p + \sin(s)\, X(p)$$

defines a smooth homotopy $H\colon S^m \times [0, \pi] \to S^m$ from id to α. By the homotopy invariance of the degree, $1 = \deg(\mathrm{id}) = \deg(\alpha) = (-1)^{m+1}$, so m is odd. Conversely, if $m = 2k - 1$, then

$$X(p) := (p^2, -p^1, p^4, -p^3, \dots, p^{2k}, p^{2k-1})$$

defines a nowhere vanishing (unit) vector field on $S^m \subset \mathbb{R}^{2k}$. □

An important result about the mapping degree is the following theorem due to Hopf [Ho1927a]:

> For a compact, connected, oriented manifold M of dimension m, two maps $F, G\colon M \to S^m$ are homotopic if and only if $\deg(F) = \deg(G)$.

For a non-orientable manifold M, an analogous result holds with $\deg_2$ instead of deg.

9.4 Transverse maps and intersection number

Let L^l and N^n be two manifolds, and let $M^m \subset N^n$ be a submanifold. A C^∞ map $F\colon L \to N$ is said to be *transverse* to M if

$$TM_q + dF_p(TL_p) = TN_q$$

whenever $p \in L$ and $F(p) =: q \in M$.

Note that if $M = \{q\}$, then F is transverse to M if and only if q is a regular value of F. The following statement generalizes Theorem 9.3 further.

9.9 Theorem (transverse maps) *Suppose that L^l is a manifold with boundary, N^n is a manifold, $M^m \subset N^n$ is a submanifold of codimension $k := n - m$, and $F\colon L \to N$ is a C^∞ map with $F(L) \cap M \neq \emptyset$. If $F|_{L\setminus\partial L}$ and $F|_{\partial L}$ are both transverse to M, then $F^{-1}(M)$ is manifold with boundary $F^{-1}(M) \cap \partial L$, and* $\dim(F^{-1}(M)) = l - k \geq 0$.

Thus $F^{-1}(M)$ has the same codimension in L as M in N. The set $F^{-1}(M) \cap \partial L$ is non-empty if and only if $F(\partial L) \cap M \neq \emptyset$; then $l - 1 \geq k$ by the assumption on $F|_{\partial L}$, and hence $\dim(F^{-1}(M)) \geq 1$.

Proof: □

9.10 Theorem (parametric transversality theorem) *Suppose that L, V, N are manifolds, $M \subset N$ is a submanifold, and $H\colon L \times V \to N$ is a C^∞ map transverse to M. Then, for almost every $v \in V$, the map*

$$H_v := H(\,\cdot\,, v)\colon L \to N$$

is tranverse to M, that is, the set $\{v \in V : H_v \text{ is not transverse to } M\}$ has measure zero in V.

Furthermore, for fixed manifolds L, N and a submanifold $M \subset N$, the set of all C^∞ maps $F\colon L \to N$ transverse to M is dense in $C^\infty(L, N)$ with respect to the compact-open ("weak") C^∞ topology on $C^\infty(L, N)$, see Theorem 2.1, Chapter 3, in [Hi].

Proof: □

9.11 Theorem (homotopy to a transverse map) *If $F\colon L \to N$ is a C^∞ map and $M \subset N$ is a submanifold, then there exists a smooth homotopy $H\colon L \times [0,1] \to N$ from $F = H(\cdot\,, 0)$ to a map $\tilde{F} = H(\cdot\,, 1)$ transverse to M.*

Proof: □

9.12 Theorem (intersection number modulo 2) *Suppose that L^l, N^n are two manifolds, L is compact, and M^m is a submanifold and a closed subset of N such that $l + m = n$. If $F, G\colon L \to N$ are smoothly homotopic and both tranverse to M, then $\#F^{-1}(M) \equiv \#G^{-1}(M) \pmod 2$.*

Note that since $l + m = n$ and $F^{-1}(M)$ is compact, the number $\#F^{-1}(M)$ is finite.

Proof: □

Let again L, N and M be given as in Theorem 9.12, and let $F\colon L \to N$ be an arbitrary C^∞ map. By Theorem 9.11 there exists a map $\tilde{F}\colon L \to N$ that is smoothly homotopic to F and transverse to M. By virtue of Theorem 9.12, the number

$$\#_2(F, M) := (\#\tilde{F}^{-1}(M) \bmod 2) \in \{0, 1\}$$

is independent of the choice of $\tilde{F}$ and invariant under smooth homotopies of F; it is called the *intersection number modulo* 2 of F with M. An application is Theorem 2.11.

10 Vector bundles, vector fields and flows

10.1 Vector bundles

10.1 Definition (smooth vector bundle) A (real, smooth) *vector bundle* with *fiber dimension* k, or briefly a *k-plane bundle*, is a triple (π, E, M) such that $\pi: E \to M$ is a smooth map between manifolds and

(1) for every point $p \in M$, the fiber $E_p := \pi^{-1}\{p\}$ has the structure of a k-dimensional (real) vector space;

(2) for every point $q \in M$ there exist an open neighborhood $U \subset M$ of q and a C^∞ diffeomorphism $\psi: \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that $\psi|_{E_p}: E_p \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism for every $p \in U$.

One calls E the *total space*, M the *base space*, and π the *bundle projection*. Condition (2) is called the *axiom of local triviality*, and a pair (ψ, U) as above is called a *bundle chart* or a *local trivialization* around q.

Topological vector bundles are defined analogously, except that then the projection is merely a continuous map between topological spaces (not necessarily topological manifolds) and bundle charts are homeomorphisms.

A k-plane bundle (π, E, M) is called *trivial* if there exists a global bundle chart $\psi: E \to M \times \mathbb{R}^k$. For every manifold M there is the *trivial $\mathbb{R}^k$-bundle* $\pi: M \times \mathbb{R}^k \to M$ over M with $\pi(p, \xi) = p$ for all $(p, \xi) \in M \times \mathbb{R}^k$ (the identity map on $M \times \mathbb{R}^k$ is a global bundle chart).

A C^∞ map $s: M \to E$ is called a *section* of the vector bundle $\pi: E \to M$ if $\pi \circ s = \mathrm{id}_M$, that is, $s(p) \in E_p$ for all $p \in M$. The set of all sections is denoted by $\Gamma(E)$ or $\Gamma^\infty(E)$, to emphasize that smooth maps are meant. Every vector bundle $\pi: E \to M$ admits the *zero section* with $s(p) = 0 \in E_p$ for all $p \in M$. Note that if (ψ, U) is a bundle chart, then $s|_U = \psi^{-1} \circ i$ for $i: U \to U \times \mathbb{R}^k$, $i(p) = (p, 0)$, thus s is indeed a smooth map.

10.2 Definition (bundle map) Let $\pi: E \to M$ and $\pi': E' \to M'$ be two vector bundles. A C^∞ map $\tilde{F}: E \to E'$ is called a *bundle map* if $\tilde{F}$ maps fibers isomorphically onto fibers, that is, $\tilde{F}$ induces a map $F: M \to M'$ such that $F \circ \pi = \pi' \circ \tilde{F}$ and $\tilde{F}|_{E_p}: E_p \to E'_{F(p)}$ is an isomorphism for all $p \in M$. If F is a diffeomorphism, then $\tilde{F}$ is a *bundle equivalence*. If $M = M'$ and $F = \mathrm{id}_M$, then $\tilde{F}$ is a *bundle isomorphism*.

Note that the map $F: M \to M'$ induced by a bundle map $\tilde{F}: E \to E'$ is smooth as well, because $F = \pi' \circ \tilde{F} \circ s$ for the zero section s of E.

10.3 Proposition (trivial vector bundle) *A k-plane bundle $\pi: E \to M$ is trivial if and only if it admits k everywhere linearly independent sections.*

Proof: Suppose first that there exist sections $s_1, \dots, s_k \in \Gamma(E)$ such that $s_1(p), \dots, s_k(p)$ are linearly independent for every $p \in M$. Let $\psi\colon E \to M \times \mathbb{R}^k$ be the map that sends every linear combination $\sum_{i=1}^k \xi^i s_i(p)$ to (p, ξ). Since the s_i are smooth, it follows that ψ^{-1} is smooth. Furthermore, since ψ^{-1} maps each fiber $\{p\} \times \mathbb{R}^k$ isomorphically onto E_p, all $(p, 0) \in M \times \mathbb{R}^k$ are regular points of ψ^{-1}, thus ψ^{-1} maps an open neighborhood of $M \times \{0\}$ diffeomorphically into E, and it then follows easily that ψ^{-1} and ψ are global diffeomorphisms.

Conversely, given a global bundle chart $\psi\colon E \to M \times \mathbb{R}^k$, the sections $s_1, \dots, s_k$ defined by $s_i(p) := \psi^{-1}(p, e_i)$ are everywhere linearly independent. □

Let $\pi\colon E \to M$ be a k-plane bundle, and let $\{(\psi_\alpha, U_\alpha)\}_{\alpha \in A}$ be a *bundle atlas*, that is, a family of bundle charts such that $\bigcup_{\alpha \in A} U_\alpha = M$. Every chart is of the form $\psi_\alpha = (\pi|_{\pi^{-1}(U_\alpha)}, g_\alpha)$ for a C^∞ map $g_\alpha\colon \pi^{-1}(U_\alpha) \to \mathbb{R}^k$, where $g_\alpha|_{E_p}\colon E_p \to \mathbb{R}^k$ is a linear isomorphism for every $p \in U_\alpha$. Thus, for every pair of indices $\alpha, \beta \in A$, there exists a C^∞ map

$$g_{\beta\alpha}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R}), \quad g_{\beta\alpha}(p) = g_\beta|_{E_p} \circ (g_\alpha|_{E_p})^{-1}.$$

The family $\{g_{\beta\alpha}\}$ satisfies the so-called *cocyle condition*

$$g_{\alpha\alpha}(p) = \mathrm{id}_{\mathbb{R}^k}, \quad g_{\gamma\beta}(p) \circ g_{\beta\alpha}(p) = g_{\gamma\alpha}(p) \quad (p \in U_\alpha \cap U_\beta \cap U_\gamma).$$

If G is a subgroup of $\mathrm{GL}(k, \mathbb{R})$, and if E admits a bundle atlas with transition maps $g_{\beta\alpha}\colon U_\alpha \cap U_\beta \to G$, then E is called a vector bundle with *structure group G*. Conversely, given an open cover $\{U_\alpha\}_{\alpha \in A}$ of M and a family $\{g_{\beta\alpha}\}$ of C^∞ maps $g_{\beta\alpha}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ satisfying the above cocycle condition, one can construct a corresponding k-plane bundle over M from these data.

10.2 The cotangent bundle

Next we discuss the *cotangent bundle TM^** of an m-dimensional manifold M. The total space

$$TM^* = \bigcup_{p \in M} TM_p^*$$

is the (disjoint) union of the dual spaces

$$TM_p^* = \{\lambda\colon TM_p \to \mathbb{R} : \lambda \text{ is linear}\},$$

and $\pi\colon TM^* \to M$ is given by $\pi(\lambda) = p$ for $\lambda \in TM_p^*$. If (φ, U) is a chart of M, then

$$\psi(\lambda) = \Bigl(\pi(\lambda), \sum_{i=1}^m \lambda\Bigl(\frac{\partial}{\partial \varphi^i}\Big|_{\pi(\lambda)}\Bigr) e_i\Bigr)$$

defines a corresponding bundle chart $\psi\colon \pi^{-1}(U) \to U \times \mathbb{R}^m$ of TM^*. For $p \in U$, the differentials $d\varphi^1_p, \dots, d\varphi^m_p \colon TM_p \to \mathbb{R}$ constitute the basis of TM^*_p *dual* to $\frac{\partial}{\partial\varphi^1}\big|_p, \dots, \frac{\partial}{\partial\varphi^m}\big|_p$, as

$$d\varphi^i_p\Big(\frac{\partial}{\partial\varphi^j}\Big|_p\Big) = \frac{\partial\varphi^i}{\partial\varphi^j}(p) = \delta^i_j.$$

The maps $d\varphi^i\colon p \mapsto d\varphi^i_p$ are sections of TU^*. A global section $\omega \in \Gamma(TM^*)$, $p \mapsto \omega_p \in TM^*_p$, is called a *covector field* or a 1-*form* on M. With respect to the chart (φ, U), every such ω has a unique local representation

$$\omega|_U = \sum_{i=1}^m \omega_i \, d\varphi^i$$

for the C^∞ functions $\omega_i\colon U \to \mathbb{R}$ defined by $\omega_i(p) = \omega_p\big(\frac{\partial}{\partial\varphi^i}\big|_p\big)$. In particular, for any $f \in C^\infty(M)$, the differential $df\colon p \mapsto df_p$ is a 1-form with local representation

$$df|_U = \sum_{i=1}^m \frac{\partial f}{\partial\varphi^i}\, d\varphi^i,$$

since $df_p\big(\frac{\partial}{\partial\varphi^i}\big|_p\big) = \frac{\partial f}{\partial\varphi^i}(p)$.

10.3 Constructions with vector bundles

10.4 Definition (pull-back bundle) Suppose that $\pi'\colon E' \to M'$ is a k-plane bundle and $F\colon M \to M'$ is a C^∞ map from another manifold M into M'. The k-plane bundle $\pi\colon F^*E' \to M$ with total space

$$F^*E' := \{(p, v) \in M \times E' : \pi'(v) = F(p)\}$$

and projection $(p, v) \mapsto p$ is called the *pull-back bundle* of π' and F or the *bundle induced by π' and F*.

The map $\tilde{F}\colon F^*E' \to E'$, $\tilde{F}(p, v) = v \in E'_{F(p)}$, is a bundle map over F. If (ψ', U') is a bundle chart for E', $\psi' = (\pi', g')$, then

$$\psi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k, \quad \psi(p, v) = (p, g'(v)),$$

is a corresponding bundle chart for F^*E' over $U := F^{-1}(U')$. If $\{(\psi'_\alpha, U'_\alpha)\}$ is a bundle atlas of E' with transition maps $g'_{\beta\alpha}\colon U'_\alpha \cap U'_\beta \to \mathrm{GL}(k, \mathbb{R})$, then this gives a bundle atlas $\{(\psi_\alpha, U_\alpha)\}$ of E with transitions maps

$$g_{\beta\alpha} = g'_{\beta\alpha} \circ F\colon U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R}).$$

Note that if $E' = TM'$, then a section $s \in \Gamma(F^*TM')$, $s(p) = (p, X(p))$, corresponds to a *vector field along F*, as $X(p) \in TM'_{F(p)}$ for all $p \in M$.

10.5 Definition (Whitney sum) Suppose that $\pi\colon E \to M$ and $\pi'\colon E' \to M$ are vector bundles of rank k and k', respectively, over the same base space M. The *Whitney sum* or *direct sum* of π and π' is the vector bundle $\bar{\pi}\colon E \oplus E' \to M$ of rank $k + k'$ with total space

$$E \oplus E' = \{(v, v') \in E \times E' : \pi(v) = \pi'(v')\}$$

and projection $(v, v') \mapsto \pi(v) = \pi'(v')$; that is, $(E \oplus E')_p = E_p \oplus E'_p$.

If $\psi = (\pi, g)$ and $\psi' = (\pi', g')$ are bundle charts of E and E', respectively, over the same open set $U \subset M$, then

$$\bar{\psi}\colon \bar{\pi}^{-1}(U) \to U \times \mathbb{R}^{k+k'}, \quad \bar{\psi}(v, v') = (\bar{\pi}(v, v'), g(v), g'(v')),$$

is a bundle chart for $E \oplus E'$. Transition maps satisfy

$$\bar{g}_{\beta\alpha}(p) = g_{\beta\alpha}(p) \oplus g'_{\beta\alpha}(p) \in \mathrm{GL}(k + k', \mathbb{R}).$$

The bundles $E \oplus E'$ and $E' \oplus E$ are isomorphic, and

$$(E \oplus E') \oplus E'' = E \oplus (E' \oplus E'').$$

However, $E \oplus E'' \cong E' \oplus E''$ does in general not imply that $E \cong E'$.

If $\pi\colon E \to M$ and $\pi'\colon E' \to M$ are again given as in Definition 10.5, then one may similarly form the *tensor product* $\bar{\pi}\colon E \otimes E' \to M$ of π and π' (of rank kk') with fibers $(E \otimes E')_p = E_p \otimes E'_p$ and transitions maps satisfying

$$\bar{g}_{\beta\alpha}(p) = g_{\beta\alpha}(p) \otimes g'_{\beta\alpha}(p) \in \mathrm{GL}(kk', \mathbb{R})$$

(see Appendix C).

10.6 Definition (tensor bundle, tensor field) Let M be an m-dimensional manifold. The bundle

$$T_{r,s}M := \underbrace{TM \otimes \cdots \otimes TM}_{r} \otimes \underbrace{TM^* \otimes \cdots \otimes TM^*}_{s}$$

of rank m^{r+s} with fibers $T_{r,s}M_p = (TM_p)_{r,s}$ is called the (r, s)-*tensor bundle* over M. An (r, s)-*tensor field* T on M is a section $T \in \Gamma(T_{r,s}M)$.

Note that $T_{1,0}M = TM$ and $T_{0,1}M = TM^*$. By convention, $T_{0,0}M = C^\infty(M)$. In a chart (φ, U) of M, the tensor field $T \in \Gamma(T_{r,s}M)$ has a unique representation

$$T|_U = \sum T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{\partial}{\partial \varphi^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial \varphi^{i_r}} \otimes d\varphi^{j_1} \otimes \cdots \otimes d\varphi^{j_s}$$

for C^∞ functions $T^{i_1 \ldots i_r}_{j_1 \ldots j_s}\colon U \to \mathbb{R}$.

Now let $T\colon (\Gamma(TM))^s \to \Gamma(TM)$ be a multilinear (s-linear) map. We say that T *defines a* $(1,s)$*-tensor field* if for all $p \in M$, the value of the vector field $T(X_1,\dots,X_s)$ at p depends only on $X_1(p),\dots,X_s(p)$; that is, we get an s-linear map $T_p\colon (TM_p)^s \to TM_p$ or, equivalently, an $(1+s)$-linear map

$$T'_p\colon TM_p^* \times (TM_p)^s \to \mathbb{R}, \quad T'_p(\lambda, v_1,\dots,v_s) = \lambda(T_p(v_1,\dots,v_s)),$$

hence a tensor $T'_p \in T_{1,s}M_p$ over TM_p.

10.7 Theorem (tensor fields) *An s-linear map $T\colon (\Gamma(TM))^s \to \Gamma(TM)$ defines a $(1,s)$-tensor field if and only if T is $C^\infty(M)$-homogeneous in every argument, that is,*

$$T(X_1,\dots,X_{i-1}, fX_i, X_{i+1},\dots,X_s) = f\,T(X_1,\dots,X_s)$$

for any $f \in C^\infty(M)$.

The theorem also holds in the following form for (r,s)-tensor fields: *An $(r+s)$-linear map $T\colon (\Gamma(TM^*))^r \times (\Gamma(TM))^s \to C^\infty(M)$ defines an (r,s)-tensor field if and only if T is $C^\infty(M)$-homogeneous in every argument.*

Proof: □

10.4 Vector fields and flows

Let $X \in \Gamma(TM)$ be a vector field on a manifold M. A curve $c\colon (a,b) \to M$ is an *integral curve* of X if

$$\dot c(t) = X_{c(t)}$$

for all $t \in (a,b)$.

10.8 Theorem (local flow) *For all $p \in M$ there exist an open neighborhood U of p and an $\epsilon > 0$ such that for all $q \in U$ there is a unique integral curve $c_q\colon (-\epsilon,\epsilon) \to M$ of X with $c_q(0) = q$. The map $\Phi\colon (-\epsilon,\epsilon) \times U \to M$, $\Phi(t,q) = \Phi^t(q) := c_q(t)$, is C^∞.*

Proof: Choose a chart (ψ, V) of M around p. A curve $c\colon (a,b) \to V$ is an integral curve of X if and only if $\gamma := \psi \circ c$ is an integral curve of the vector field ξ on $\psi(V)$ defined by $\xi_{\psi(p)} := d\psi_p(X_p)$, that is, $\dot\gamma(t) = \xi_{\gamma(t)}$ for all $t \in (a,b)$. Now the result follows from the theorem on existence, uniqueness, and smooth dependence on initial conditions of solutions to ordinary differential equations. □

The map Φ is called a *local flow* of X around p. It follows from the uniqueness assertion in Theorem 10.8 that

$$\Phi^t(\Phi^s(q)) = \Phi^{s+t}(q)$$

whenever $s, t, s+t \in (-\epsilon, \epsilon)$ and $q, \Phi^s(q) \in U$. Then, for any open neighborhood $V \subset U$ of q with $\Phi^s(V) \subset U$, $\Phi^s|_V$ is a C^∞ diffeomorphism from V onto $\Phi^s(V)$, because $\Phi^{-s} \circ \Phi^s|_V = \Phi^0|_V = \mathrm{id}_V$.

A vector field X on M is *completely integrable* if for all $q \in M$ there exists an integral curve $c_q : \mathbb{R} \to M$ of X with $c_q(0) = q$. Then X induces a global *flow* $\Phi : \mathbb{R} \times M \to M$ and a corresponding *1-parameter family of diffeomorphisms* $\{\Phi^t\}_{t \in \mathbb{R}}$.

10.9 Proposition (complete integrability) *Every vector field $X \in \Gamma(TM)$ with compact support is completely integrable.*

Proof: For all $p \in M$ there is a local flow $\Phi : (-\epsilon_p, \epsilon_p) \times U_p \to M$ of X. Then finitely many neighborhoods $U_{p_1}, \ldots, U_{p_k}$ cover the compact support of X. For $\epsilon := \min\{\epsilon_{p_i} : i = 1, \ldots, k\}$, it follows that Φ is defined on $(-\epsilon, \epsilon) \times M$, where $\Phi^t(p) = p$ for all t if $X(p) = 0$. Writing any $t \in \mathbb{R}$ as $t = j \cdot \frac{\epsilon}{2} + r$ with $j \in \mathbb{Z}$ and $r \in [0, \frac{\epsilon}{2})$, we conclude that $\Phi^t = \Phi^r \circ (\Phi^{\epsilon/2})^j$ is the time t flow of X. □

10.10 Lemma (flow-box) *If $X \in \Gamma(TM)$, $p \in M$, and $X_p \neq 0$, then there exists a chart (φ, U) around p such that $X|_U = \frac{\partial}{\partial \varphi^1}$.*

Proof: This follows from the corresponding Euclidean result, Lemma A.4. □

10.5 The Lie bracket

Let $X, Y \in \Gamma(TM)$. For $f \in C^\infty(M)$, the function $Y(f) \in C^\infty(M)$ maps $q \in M$ to $Y_q(f) = df_q(Y_q) \in \mathbb{R}$. For all $p \in M$,

$$[X, Y]_p(f) := X_p(Y(f)) - Y_p(X(f)) \quad (f \in C^\infty(M))$$

defines a derivation at p. This yields a vector field $[X, Y] \in \Gamma(TM)$, called the *Lie bracket* of X and Y. Briefly, $[X, Y] = XY - YX$.

10.11 Theorem (Lie bracket) *For $X, Y, Z \in \Gamma(TM)$ and $f, g \in C^\infty(M)$, the following properties hold*:

(1) *$[X, Y]$ is bilinear, and $[Y, X] = -[X, Y]$;*

(2) *$[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$, in particular $[fX, Y] = f[X, Y] - Y(f)X$ and $[X, gY] = g[X, Y] + X(g)Y$,*

(3) *$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ (Jacobi identity).*

Proof: □

For a chart (φ, U) and $f \in C^\infty(M)$,

$$\frac{\partial}{\partial\varphi^i}\left(\frac{\partial}{\partial\varphi^j}(f)\right) = \frac{\partial}{\partial\varphi^i}\left(\frac{\partial(f\circ\varphi^{-1})}{\partial x^j}\circ\varphi\right) = \frac{\partial^2(f\circ\varphi^{-1})}{\partial x^i\,\partial x^j}\circ\varphi,$$

thus $\left[\frac{\partial}{\partial\varphi^i}, \frac{\partial}{\partial\varphi^j}\right] = 0$. It follows from this fact and properties (1) and (2) above that if $X|_U = \sum_i X^i \frac{\partial}{\partial\varphi^i}$ and $Y|_U = \sum_j Y^j \frac{\partial}{\partial\varphi^j}$, then

$$\begin{aligned}[X,Y]|_U &= \sum_{i,j}\left(X^i \frac{\partial Y^j}{\partial\varphi^i}\frac{\partial}{\partial\varphi^j} - Y^j \frac{\partial X^i}{\partial\varphi^j}\frac{\partial}{\partial\varphi^i}\right)\\ &= \sum_i\left(\sum_j X^j\frac{\partial Y^i}{\partial\varphi^j} - Y^j\frac{\partial X^i}{\partial\varphi^j}\right)\frac{\partial}{\partial\varphi^i}.\end{aligned}$$

The following results relates Lie brackets to flows.

10.12 Theorem (Lie derivative) *If Φ is a local flow of X around p, then*

$$[X,Y]_p = \lim_{t\to 0}\frac{d(\Phi^{-t})(Y_{\Phi^t(p)}) - Y_p}{t} = \frac{d}{dt}\Big|_{t=0} d(\Phi^{-t})(Y_{\Phi^t(p)}).$$

The right side of this identity is called the *Lie derivative* of Y in direction of X at the point p and is denoted by $(L_X Y)_p$; thus $[X,Y] = L_X Y$.

Proof: □

Let N be an n-dimensional manifold. An *m-dimensional C^∞ distribution* Δ on N assigns to each $p \in N$ an m-dimensional linear subspace $\Delta_p \subset TN_p$ such that for every point $p \in N$ there exist an open neighborhood $U \subset N$ of p and vector fields $X_1, \ldots, X_m \in \Gamma(TU)$ with $\Delta_q = \operatorname{span}(X_1(q), \ldots, X_m(q))$ for all $q \in U$. The distribution Δ is called *involutive* or *completely integrable* if for all vector fields $X, Y \in \Gamma(TN)$ with $X_p, Y_p \in \Delta_p$ for all $p \in N$, also $[X,Y]_p \in \Delta_p$ for all $p \in N$. An injective immersion $I\colon M \to N$ of an m-dimensional manifold M is called an *integral manifold* of Δ if $dI_p(TM_p) = \Delta_p$ for all $p \in M$. The *theorem of Frobenius* says:

> For every $p \in N$ there exists an integral manifold of Δ through p if and only if Δ is involutive.

11 Differential forms

11.1 Basic definitions

Let M be a C^∞ manifold of dimension m. For $p \in M$, $\Lambda_s(TM_p^*)$ denotes the vector space of alternating s-linear maps $(TM_p)^s \to \mathbb{R}$ (see Appendix C), and

$$\Lambda_s(TM^*) := \bigcup_{p \in M} \Lambda_s(TM_p^*)$$

denotes the corresponding bundle.

11.1 Definition (differential form) A *differential form of degree s* or an *s-form* on M is a (smooth) section of $\Lambda_s(TM^*)$. We will denote the vector space of s-forms on M more briefly by $\Omega^s(M) := \Gamma(\Lambda_s(TM^*))$.

By convention, $\Lambda_0(TM_p^*) = \mathbb{R}$, hence $\Omega^0(M) = C^\infty(M)$. Recall also that $\Lambda_s(TM_p^*)$ has dimension $\binom{m}{s}$, in particular $\Omega^s(M) = \{0\}$ for $s > m$.

For $\omega \in \Omega^s(M)$ and $\theta \in \Omega^t(M)$, the *exterior product*

$$\omega \wedge \theta \in \Omega^{s+t}(M)$$

is defined by $(\omega \wedge \theta)_p := \omega_p \wedge \theta_p$ for all $p \in M$ (see Definition C.3). Note that

$$\theta \wedge \omega = (-1)^{st} \omega \wedge \theta,$$

in particular $\omega \wedge \omega = 0$ if s is odd. The exterior product is bilinear and associative. For $f \in C^\infty(M) = \Omega^0(M)$ and $\omega \in \Omega^s(M)$, $f \wedge \omega = f\omega$.

In a chart (φ, U), a form $\omega \in \Omega^s(M)$ has the representation

$$\omega|_U = \sum_{1 \le i_1 < \ldots < i_s \le m} \omega_{i_1 \ldots i_s} \, d\varphi^{i_1} \wedge \ldots \wedge d\varphi^{i_s}$$

with components $\omega_{i_1 \ldots i_s} = \omega\big(\frac{\partial}{\partial \varphi^{i_1}}, \ldots, \frac{\partial}{\partial \varphi^{i_s}}\big) \in C^\infty(U)$.

Recall that for $f \in C^\infty(M)$, the pointwise differential $p \mapsto df_p$ is a 1-form $df \in \Gamma(TM^*) = \Gamma(\Lambda_1(TM^*)) = \Omega^1(M)$.

11.2 Theorem (exterior derivative) *There exists a unique sequence of linear operators*

$$d \colon \Omega^s(M) \to \Omega^{s+1}(M), \quad s = 0, 1, \ldots,$$

with the following properties:

(1) *for $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f, thus $df(X) = X(f)$ for $X \in \Gamma(TM)$;*

(2) $d \circ d = 0$;

(3) $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^s \omega \wedge d\theta$ *for* $\omega \in \Omega^s(M)$ *and* $\theta \in \Omega^t(M)$.

Proof: □

The operators d are local, that is, $(d\omega)|_U = d(\omega|_U)$ whenever $\omega \in \Omega^s(M)$ and $U \subset \mathbb{R}^m$ is open. In a chart (φ, U),

$$d\omega|_U = \sum_{1 \le i_1 < \ldots < i_s \le m} d\omega_{i_1 \ldots i_s} \wedge d\varphi^{i_1} \wedge \ldots \wedge d\varphi^{i_s}.$$

11.3 Theorem (exterior derivative, coordinate-free) *For a form* $\omega \in \Omega^s(M)$ *and vector fields* $X_1, \ldots, X_{s+1} \in \Gamma(TM)$*,*

$$\begin{aligned} d\omega(X_1, \ldots, X_{s+1}) &= \sum_{i=1}^{s+1} (-1)^{i+1} X_i\big(\omega(X_1, \ldots, \widehat{X}_i, \ldots, X_{s+1})\big) \\ &+ \sum_{1 \le i < j \le s+1} (-1)^{i+j} \omega\big([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{s+1}\big); \end{aligned}$$

here, $\widehat{X}_i$ *signifies that the entry* X_i *does not occur.*

In particular, if $\omega \in \Omega^1(M)$, then

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

Proof: □

For a C^∞ map $F \colon N \to M$ and $\omega \in \Omega^s(M)$, the *pull-back form* $F^*\omega \in \Omega^s(N)$ is defined by

$$(F^*\omega)_p\big(v_1, \ldots, v_s\big) := \omega_{F(p)}\big(dF_p(v_1), \ldots, dF_p(v_s)\big)$$

for $p \in N$ and $v_1, \ldots, v_s \in TN_p$. If $f \in C^\infty(M) = \Omega^0(M)$, then $F^*f := f \circ F$.

11.4 Proposition (pull-back of forms) *For a* C^∞ *map* $F \colon N \to M$ *and forms* $\omega \in \Omega^s(M)$ *and* $\theta \in \Omega^t(M)$*,*

(1) $F^*(\omega \wedge \theta) = F^*\omega \wedge F^*\theta$*,*

(2) $F^*(d\omega) = d(F^*\omega)$.

Proof: Exercise. □

11.2 Integration of forms

Let M be an oriented manifold of dimension m. A set $M' \subset M$ is *measurable* if $\varphi(M' \cap U) \subset \mathbb{R}^m$ is (Lebesgue) measurable for every chart (φ, U) of M. A *measurable decomposition* of M is a countable family $\{M_\alpha\}_{\alpha\in A}$ of measurable subsets of M such that

(1) $M \setminus \bigcup_{\alpha\in A} M_\alpha$ has measure zero (Definition 9.1), and

(2) $M_\alpha \cap M_\beta$ has measure zero whenever $\alpha \neq \beta$.

For every atlas of M there is a measurable decomposition $\{M_\alpha\}_{\alpha\in A}$ of M such that every set M_α is contained in the domain of some chart of the atlas.

Let now $\omega \in \Omega^m(M)$ be a form of degree $m = \dim(M)$, and let (φ, U) be a positively oriented chart of M. Then

$$\omega|_U = \omega^\varphi \, d\varphi^1 \wedge \ldots \wedge d\varphi^m$$

for $\omega^\varphi = \omega\big(\frac{\partial}{\partial\varphi^1}, \ldots, \frac{\partial}{\partial\varphi^m}\big) \in C^\infty(U)$. If (ψ, V) is another positively oriented chart and $H := \psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ is the change of coordinates, then by applying $\omega|_V = \omega^\psi \, d\psi^1 \wedge \ldots \wedge d\psi^m$ to $\frac{\partial}{\partial\varphi^1}, \ldots, \frac{\partial}{\partial\varphi^m}$ one gets that

$$\omega^\varphi(p) = \omega^\psi(p) \det\left(\frac{\partial\psi^i}{\partial\varphi^j}(p)\right) = \omega^\psi(p) \det J_H(\varphi(p))$$

for all $p \in U \cap V$, where the Jacobi determinant is positive.

Now let $M' \subset U$ be a measurable set. The form ω is *integrable over M'* if the integral of $\left|\omega^\varphi \circ \varphi^{-1}\right|$ over $\varphi(M')$ is finite; then

$$\int_{M'} \omega := \int_{\varphi(M')} \omega^\varphi \circ \varphi^{-1} \, dx$$

defines the *integral of ω over M'*. If (ψ, V) is another positively oriented chart with $M' \subset V$ and H is the change of coordinates, then it follows that

$$\int_{\psi(M')} \omega^\psi \circ \psi^{-1} \, dy = \int_{\varphi(M')} \omega^\psi \circ \varphi^{-1} \, |\det J_H| \, dx = \int_{\varphi(M')} \omega^\varphi \circ \varphi^{-1} \, dx$$

by the change of variables formula and the aforementioned transformation rule for the coefficients of ω.

11.5 Definition (integral of a form) The form $\omega \in \Omega^m(M)$ is *integrable over M* if there exist a measurable decomposition $\{M_\alpha\}_{\alpha\in A}$ and positively oriented charts $(\varphi_\alpha, U_\alpha)$ of M with $M_\alpha \subset U_\alpha$ such that

$$\sum_{\alpha\in A} \int_{\varphi_\alpha(M_\alpha)} \left|\omega^{\varphi_\alpha} \circ \varphi_\alpha^{-1}\right| dx < \infty.$$

In this case,

$$\int_M \omega := \sum_{\alpha\in A} \int_{M_\alpha} \omega = \sum_{\alpha\in A} \int_{\varphi_\alpha(M_\alpha)} \omega^{\varphi_\alpha} \circ \varphi_\alpha^{-1}\, dx$$

defines the *integral of ω over M*.

The integral is independent of the choices of $(\varphi_\alpha, U_\alpha)$ and M_α. Forms with compact support are integrable: this clearly holds if $\mathrm{spt}(\omega)$ lies in the domain of a single chart, and in the general case one may use a partition of unity to write ω as a sum of finitely many forms with this property.

If ω is integrable over M, and N is another oriented m-dimensional manifold and $F\colon N \to M$ is a diffeomorphism, then

$$\int_N F^*\omega = \epsilon \int_M \omega$$

where $\epsilon = 1$ if F is orientation preserving and $\epsilon = -1$ otherwise. Furthermore, if N is compact and M is connected, and $F\colon N \to M$ is an arbitrary C^∞ map, then one can show that $\int_N F^*\omega = \deg(F)\int_M \omega$.

11.6 Theorem (Stokes) *Let M^m be an oriented manifold with (possibly empty) boundary ∂M, and let $\omega \in \Omega^{m-1}(M)$ be an $(m-1)$-form with compact support. Then*

$$\int_M d\omega = \int_{\partial M} \omega$$

(precisely, $\int_M d\omega = \int_{\partial M} i^*\omega$ for the inclusion map $i\colon \partial M \to M$).

Here the boundary ∂M is equipped with the *induced orientation*: a basis $(v_1, \dots, v_{m-1})$ of $T(\partial M)_p \subset TM_p$ is positively oriented if and only if $(v, v_1, \dots, v_{m-1})$ is positively oriented in TM_p for every vector v in the "outer" connected component of $TM_p \setminus T(\partial M)_p$.

Proof: □

A *volume form ω* on M^m is a nowhere vanishing m-form, that is, $\omega_p \neq 0 \in \Lambda_m(TM_p^*)$ for all $p \in M$.

11.7 Theorem (volume form) *There exists a volume form on M if and only if M is orientable.*

Proof: Exercise. □

11.3 Integration without orientation

If V is an m-dimensional (real) vector space and $0 \neq \omega \in \Lambda_m(V^*)$, then

$$|\omega|\colon V \times \cdots \times V \to [0, \infty), \quad |\omega|(v_1, \ldots, v_m) := |\omega(v_1, \ldots, v_m)|,$$

is called a *volume element* on V. Now let M be an m-dimensional manifold. A (C^∞) *volume element* $d\mu$ on M assigns to every point $p \in M$ a volume element $d\mu_p$ on TM_p such that, for every chart (φ, U) of M,

$$d\mu|_U = \varrho^\varphi \left| d\varphi^1 \wedge \ldots \wedge d\varphi^m \right|$$

for some C^∞ density function $\varrho^\varphi\colon U \to (0, \infty)$. (The notation $d\mu$ stems from measure theory and is unrelated to the exterior derivative of differential forms.) If (ψ, V) is another chart and $H = \psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$ is the coordinate change, then

$$\varrho^\varphi(p) = \varrho^\psi(p) \, |\det J_H(\varphi(p))|$$

for all $p \in U \cap V$, similarly as for the coefficients of m-forms.

If $d\mu$ is a volume element on M and M is orientable, then there exists a volume form $\omega \in \Omega^m(M)$ with $d\mu = |\omega|$. For a non-orientable M, such a form exists only locally, due to Theorem 11.7.

From a volume element $d\mu$ on M one obtains a measure μ on (the σ-algebra of measurable subsets of) M as follows: if $\{M_\alpha\}_{\alpha \in A}$ is a measurable decomposition of M such that for every α there is a chart $(\varphi_\alpha, U_\alpha)$ with $M_\alpha \subset U_\alpha$, then

$$\mu(B) := \sum_\alpha \int_{\varphi_\alpha(B \cap M_\alpha)} \varrho^{\varphi_\alpha} \circ \varphi_\alpha^{-1} \, dx$$

for every measurable set $B \subset M$. It follows from the change of variable formula and the above transformation rule for the densities that the measure is well-defined. Now, if $f\colon M \to \mathbb{R}$ is a measurable function, then the meaning of $\int_M f \, d\mu$ results from this measure. However, the integral can also be defined directly in terms of the volume element $d\mu$: f is *integrable* if

$$\int_M |f| \, d\mu := \sum_\alpha \int_{\varphi_\alpha(M_\alpha)} (|f| \, \varrho^{\varphi_\alpha}) \circ \varphi_\alpha^{-1} \, dx < \infty;$$

the same formula with f in place of $|f|$ then defines the integral $\int_M f \, d\mu$.

For a Riemannian manifold (M^m, g), the *volume element $d\mu_g$ induced by g* is given in a chart (φ, U) by

$$d\mu_g|_U := \sqrt{\det(g_{ij}^\varphi)} \, |d\varphi^1 \wedge \ldots \wedge d\varphi^m|,$$

where $g|_U = \sum g_{ij}^\varphi \, d\varphi^i \otimes d\varphi^j$.

11.4 De Rham cohomology

A form $\omega \in \Omega^s(M)$ is *closed* if $d\omega = 0$. The form ω is called *exact* if there exists a $\theta \in \Omega^{s-1}(M)$ such that $\omega = d\theta$; furthermore, by convention, $0 \in C^\infty(M) = \Omega^0(M)$ is the only exact 0-form. Every m-form on an m-dimensional manifold M is closed, because $\Omega^{m+1}(M) = \{0\}$. Since $d \circ d = 0$, every exact form is closed.

11.8 Definition (de Rham cohomology) For $s \geq 0$, the quotient vector space

$$H^s_{\mathrm{dR}}(M) := \frac{\{\omega \in \Omega^s(M) : \omega \text{ is closed}\}}{\{\omega \in \Omega^s(M) : \omega \text{ is exact}\}}$$

is called the *de Rham cohomology* of M in degree s. For a closed form $\omega \in \Omega^s(M)$,

$$[\omega] := \{\omega' \in \Omega^s(M) : \omega' - \omega \text{ is exact}\} \in H^s_{\mathrm{dR}}(M)$$

denotes the *cohomology class* of ω. Two forms $\omega, \omega' \in \Omega^s(M)$ are *cohomologous* if $[\omega] = [\omega']$.

The dimension $b_s(M) := \dim H^s_{\mathrm{dR}}(M)$ is called the *s-th Betti number* of M, and

$$\chi(M) := \sum_{s=0}^{m} (-1)^s\, b_s(M)$$

is the *Euler characteristic* of M. If every closed s-form is exact, then $H^s_{\mathrm{dR}}(M)$ is a trivial (one-point) vector space, which will be denoted by 0. The subscript dR will often be omitted in the following.

Examples

1. $H^0(M) = \{f \in C^\infty(M) : df = 0\}$ is the vector space of the locally constant functions on M. If M has a finite number k of connected components, then $H^0(M) \simeq \mathbb{R}^k$ (isomorphic).

2. On $M = \mathbb{R}^2 \setminus \{(0,0)\}$,

$$\omega = \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$

defines a 1-form that is closed but not exact; in particular, $H^1(M) \neq 0$. Locally, ω agrees with the differential $d\varphi$ of a polar angle φ with respect to the origin $(0,0)$, but φ cannot be defined continuously on all of M.

In the following, M, N are two manifolds and $F \in C^\infty(N, M)$. For $s \geq 0$, the pull-back operator $F^*\colon \Omega^s(M) \to \Omega^s(N)$ induces a well-defined linear map

$$F^*\colon H^s(M) \to H^s(N), \quad F^*[\omega] = [F^*\omega].$$

If L is another manifold and $G \in C^\infty(M, L)$, then

$$F^* \circ G^* = (G \circ F)^* \colon H^s(L) \to H^s(N);$$

in particular, $H^s(M)$ and $H^s(N)$ are isomorphic if F is a diffeomorphism.

11.9 Theorem (Poincaré lemma) *If $F, G \in C^\infty(N, M)$ are smoothly homotopic, $F \sim G$, then the induced maps $F^*, G^* \colon H^s(M) \to H^s(N)$ agree in every degree $s \geq 0$.*

Proof: □

Two manifolds M and $\bar{M}$ are called (smoothly) *homotopy equivalent* if there exist smooth maps $\bar{F} \colon M \to \bar{M}$ and $F \colon \bar{M} \to M$ such that $F \circ \bar{F} \sim \mathrm{id}_M$ and $\bar{F} \circ F \sim \mathrm{id}_{\bar{M}}$; then F and $\bar{F}$ are (smooth) *homotopy equivalences* inverse to each other. The manifold M is (smoothly) *contractible* if id_M is smoothly homotopic to a constant map $M \to \{p_0\} \subset M$; this is the case if and only if M is homotopy equivalent to a one-point space.

11.10 Corollary (1) *If M and $\bar{M}$ are homotopy equivalent, then $H^s(M) \simeq H^s(\bar{M})$ for all $s \geq 0$.*

(2) *If M is contractible, then $H^0(M) \simeq \mathbb{R}$ and $H^s(M) = 0$ for $s \geq 1$.*

Proof: □

If M is a manifold and $U, V \subset M$ are two open sets with $U \cup V = M$, then there exists a long exact sequence

$$\begin{aligned} 0 \to & H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to \dots \\ \dots \to & H^s(M) \to H^s(U) \oplus H^s(V) \to H^s(U \cap V) \\ \to & H^{s+1}(M) \to H^{s+1}(U) \oplus H^{s+1}(V) \to H^{s+1}(U \cap V) \to \dots \end{aligned}$$

(thus the image of each of these linear maps equals the kernel of the following one), the *Mayer–Vietoris sequence*, which constitutes a very useful tool to determine the de Rham cohomology.

Example The sphere $S^m \subset \mathbb{R}^{m+1}$ ($m \geq 1$) is covered by the two open sets $U := S^m \setminus \{-e_{m+1}\}$ and $V := S^m \setminus \{e_{m+1}\}$, both of which are contractible, and $U \cap V$ is homotopy equivalent to S^{m-1}. By Corollary 11.10, for all $s \geq 1$, both $H^s(U) \oplus H^s(V)$ and $H^{s+1}(U) \oplus H^{s+1}(V)$ are trivial, hence the map

$$H^s(S^{m-1}) \simeq H^s(U \cap V) \to H^{s+1}(M) = H^{s+1}(S^m)$$

in the Mayer–Vietoris sequence is injective as well as surjective. Hence, for $m, s \geq 1$, the recursion formula $H^{s+1}(S^m) \simeq H^s(S^{m-1})$ holds. Furthermore, since $H^0(S^m) \simeq \mathbb{R}$ and $H^0(U) \oplus H^0(V) \simeq \mathbb{R}^2$, one obtains the exact sequence

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to H^0(U \cap V) \to H^1(S^m) \to 0.$$

If $m = 1$, then $H^0(U \cap V) \simeq \mathbb{R}^2$ and hence $H^1(S^1) \simeq \mathbb{R}$, and if $m \geq 2$, then $H^0(U \cap V) \simeq \mathbb{R}$ and thus $H^1(S^m) = 0$. It follows that $H^s(S^m) \simeq \mathbb{R}$ for $s \in \{0, m\}$ and $H^s(S^m) = 0$ otherwise.

We mention two other important results, in both of which M is a compact oriented manifold (without boundary) of dimension m, and $s \in \{0, 1, \ldots, m\}$.

The *Poincaré duality theorem* says that the bilinear form

$$(\,\cdot\,,\cdot\,)\colon H^s(M) \times H^{m-s}(M) \to \mathbb{R}, \quad ([\omega], [\theta]) := \int_M \omega \wedge \theta$$

(which is well-defined by the theorem of Stokes), is non-degenerate. This yields an isomorphism $H^s(M) \simeq (H^{m-s}(M))^*$, which assigns to $[\omega]$ the linear form $[\theta] \mapsto ([\omega], [\theta])$. For example, if M is connected, then this implies that $H^m(M) \simeq H^0(M) \simeq \mathbb{R}$.

Now we let $H_s^{(\infty)}(M, \mathbb{R})$ denote the smooth singular homology of M. An element $[\sigma]$ of the vector space $H_s^{(\infty)}(M, \mathbb{R})$ is a homology class $\{\sigma' : \sigma' - \sigma = \partial\tau\}$ of smooth singular s-chains σ' with real coefficients and $\partial\sigma' = 0$. It can be shown that the bilinear form

$$(\,\cdot\,,\cdot\,)\colon H^s_{\mathrm{dR}}(M) \times H_s^{(\infty)}(M, \mathbb{R}) \to \mathbb{R}, \quad ([\omega], [\sigma]) := \int_\sigma \omega,$$

is non-degenerate. (It follows from the generalized theorem of Stokes for smooth singular s-chains that it is well-defined.) This yields a canonical isomorphism

$$H^s_{\mathrm{dR}}(M) \simeq (H_s^{(\infty)}(M, \mathbb{R}))^*,$$

sending $[\omega]$ to the linear form $[\sigma] \mapsto ([\omega], [\sigma])$. Furthermore there are canonical isomorphisms

$$(H_s^{(\infty)}(M, \mathbb{R}))^* \simeq H^s_{(\infty)}(M, \mathbb{R}) \simeq H^s(M, \mathbb{R})$$

to the smooth singular cohomology and the usual singular cohomology, respectively. In particular $H^s_{\mathrm{dR}}(M)$ and $H^s(M, \mathbb{R})$ are isomorphic; this is the *theorem of de Rham*.

12 Lie groups

12.1 Lie groups and Lie algebras

A *topological group* $(G, \cdot)$ is a group endowed with a topology such that the map

$$G \times G \to G, \quad (g, h) \mapsto gh^{-1},$$

is continuous (equivalently, both the group multiplication $G \times G \to G$ and the map $G \to G$ sending each group element to its inverse are continuous).

12.1 Definition (Lie group) A *Lie group* $(G, \cdot)$ is a group with the structure of a C^∞ manifold such that the map $G \times G \to G$, $(g, h) \mapsto gh^{-1}$, is C^∞.

Examples

1. $\mathbb{R}^m$ with vector addition;
2. $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with complex multiplication;
3. $S^1 \subset \mathbb{C}^*$.
4. If G, H are Lie groups, then the product manifold $G \times H$, equipped with the multiplication $(g, h)(g', h') := (gg', hh')$, is a Lie group.
5. $T^m = S^1 \times \ldots \times S^1$ (m factors).
6. $\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n \times n} : \det(A) \neq 0\}$ with matrix multiplication; likewise, $\mathrm{GL}(n, \mathbb{C})$.
7. $\mathrm{GL}(n, \mathbb{R}) \times \mathbb{R}^n$, equipped with the multiplication

   $$(A, v)(B, w) := (AB, Aw + v),$$

   is (isomorphic to) the Lie group of affine transformations $g_{A,v} : x \mapsto Ax + v$ of $\mathbb{R}^n$.

Let G, G' be two Lie groups. A *Lie group homomorphism* $F : G \to G'$ is a C^∞ group homomorphism; a *Lie group isomorphism* is, in addition, a (C^∞) diffeomorphism (and hence also a group isomorphism). A Lie group homomorphism $F : G \to G'$ is also called a *representation* of G in G', in particular when G' is $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$.

In the following, $(G, \cdot)$ denotes a Lie group with neutral element e. For every $g \in G$, the *left multiplication*

$$L_g : G \to G, \quad L_g(h) := gh,$$

is a diffeomorphism of G with inverse $(L_g)^{-1} = L_{g^{-1}}$. Likewise, the *right multiplication* $R_g : G \to G$, $R_g(h) = hg$, is a diffeomorphism.

12.2 Lemma *Let $(G, \cdot)$ be a connected Lie group, and let $U \subset G$ be a neighborhood of e. Then U generates G, that is, every $g \in G$ can be written as a product $g = g_1 \dots g_k$ of finitely many elements of U.*

Proof: We assume that U is open. Then it follows inductively that $U^k = \{g_1 \dots g_k : g_1, \dots, g_k \in U\}$ is open for every $k \geq 1$: if U^k is open, then so is $U^k g = R_g(U^k)$ for all $g \in U$, hence $U^{k+1} = \bigcup_{g \in U} U^k g$ is open. Therefore $V := \bigcup_{k=1}^{\infty} U^{k+1}$ is open. On the other hand, if $g \in G \setminus V$, then $gh \in G \setminus V$ for all $h \in U$, for otherwise $g \in Vh^{-1} = V$; so $gU = L_g(U)$ is an open neighborhood of g disjoint from V. Thus $G \setminus V$ is open as well. Since $e \in V$ and G is connected, it follows that $V = G$, that is, U generates G. □

For a general Lie group G, the connected component containing the neutral element is usually denoted by G_0. For $g \in G$, the diffeomorphisms L_g and R_g map G_0 onto the connected component of G containing g. Thus G_0 is a normal subgroup of G whose cosets are the connected components of G. The quotient G/G_0 is a countable group (and thus a 0-dimensional Lie group with the discrete topology).

12.3 Definition (Lie algebra) A *Lie algebra* V over $\mathbb{R}$ is a vector space over $\mathbb{R}$ together with a bilinear map $[\,\cdot\,,\cdot\,] \colon V \times V \to V$, the *Lie bracket* of V, such that for all $X, Y, Z \in V$,

(1) $[Y, X] = -[X, Y]$ (anti-commutativity);

(2) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity).

Examples

1. Any vector space V (over $\mathbb{R}$) with the trivial bracket $[\,\cdot\,,\cdot\,] \equiv 0$ (*abelian* Lie algebra).
2. The vector space $\Gamma(TM)$ of C^∞ vector fields on a manifold M with the Lie bracket $[X, Y](f) := X(Y(f)) - Y(X(f))$.
3. $\mathbb{R}^{n \times n}$ with $[A, B] := AB - BA$ (matrix multiplication).
4. $\mathbb{R}^3$ with the vector product $[X, Y] := X \times Y$.
5. Any 2-dimensional vector space with basis (X, Y) and the bracket defined by $[X, X] := 0$, $[Y, Y] := 0$, $-[Y, X] = [X, Y] := Y$, and bilinear extension.

Let V, V' be two Lie algebras. A *Lie algebra homomorphism* $L \colon V \to V'$ is a linear map such that $L[X, Y] = [LX, LY]$ for all $X, Y \in V$; a *Lie algebra isomorphism* is, in addition, a linear isomorphism.

A vector field X on a Lie group G is called *left-invariant* if

$$L_{g*}X = X \circ L_g$$

for all $g \in G$, that is, $L_{g*}X_h := d(L_g)_h(X_h) = X_{gh}$ for all $g, h \in G$. For every vector $X_0 \in TG_e$ there exists a unique left-invariant vector field X with $X_e = X_0$, defined by

$$X_g := L_{g*}X_0;$$

then $L_{g*}X_h = L_{g*}L_{h*}X_0 = (L_g \circ L_h)_*X_0 = L_{gh*}X_0 = X_{gh}$ for all $h \in H$. Left-invariant vector fields are C^∞, and if X, Y are left-invariant, then $[X, Y]$ is left-invariant (exercise). Thus the left-invariant vector fields constitute a Lie subalgebra of $(\Gamma(TG), [\,\cdot\,,\cdot\,])$.

12.4 Definition (Lie algebra of a Lie group) The *Lie algebra* $\underline{g}$ *of a Lie group* G is the vector space TG_e with the bracket defined by

$$[X_0, Y_0] := [X, Y]_e$$

for all $X_0, Y_0 \in TG_e$, where X, Y denote the left-invariant vector fields on G such that $X_e = X_0$ and $Y_e = Y_0$.

Examples

1. The Lie algebra of $G = \mathrm{GL}(n, \mathbb{R})$ is the vector space $TG_e = \underline{\mathrm{gl}}(n, \mathbb{R}) = \mathbb{R}^{n\times n}$. If $A \in \underline{\mathrm{gl}}(n, \mathbb{R})$, and if $c\colon (-\epsilon, \epsilon) \to \mathrm{GL}(n, \mathbb{R})$ is a smooth curve with $c(0) = e$ and $c'(0) = A$, then

$$L_{g*}A = L_{g*}(c'(0)) = (L_g \circ c)'(0) = gc'(0) = gA \in TG_g$$

for all $g \in \mathrm{GL}(n, \mathbb{R})$; hence $g \mapsto gA$ is the corresponding left-invariant vector field, viewed as a map from G to $\mathbb{R}^{n\times n}$. For $A, B \in \underline{\mathrm{gl}}(n, \mathbb{R})$ and $X_g := gA$ and $Y_g := gB$, the Lie bracket is given by

$$[A, B] = [X, Y]_e = AB - BA \quad \text{(matrix product)}.$$

To see this, let $\varphi^{ik}\colon\ \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ denote the global coordinate function that assigns to g the matrix entry g_{ik}. The vector $Y_g \in TG_g$, applied as a derivation to φ^{ik}, returns the corresponding matrix entry of $Y_g = gB$, thus

$$Y_g(\varphi^{ik}) = (gB)_{ik} = \sum_{j=1}^{n} g_{ij}b_{jk} = \sum_{j=1}^{n} b_{jk}\,\varphi^{ij}(g).$$

Likewise, $X_e(\varphi^{ij}) = A(\varphi^{ij}) = a_{ij}$ and $(AB)(\varphi^{ik}) = (AB)_{ik}$, hence

$$X_e(Y(\varphi^{ik})) = \sum_{j=1}^{n} b_{jk}\,A(\varphi^{ij}) = \sum_{j=1}^{n} a_{ij}b_{jk} = (AB)(\varphi^{ik}).$$

Since this holds for all $i, k \in \{1, \ldots, n\}$ and also with interchanged roles of A and B, this gives the result.

2. The Lie algebra of $\mathrm{GL}(n, \mathbb{C})$ is the vector space $\underline{\mathrm{gl}}(n, \mathbb{C}) = \mathbb{C}^{n \times n}$ with the bracket given by $[A, B] = AB - BA$ as above.

3. $\mathrm{SL}(n, \mathbb{R}) = \{g \in \mathrm{GL}(n, \mathbb{R}) : \det(g) = 1\}$, dimension $n^2 - 1$,
$$\underline{\mathrm{sl}}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n \times n} : \mathrm{trace}(A) = 0\}.$$

4. $\mathrm{SL}(n, \mathbb{C}) = \{g \in \mathrm{GL}(n, \mathbb{C}) : \det(g) = 1\}$, dimension $2(n^2 - 1)$,
$$\underline{\mathrm{sl}}(n, \mathbb{C}) = \{A \in \mathbb{C}^{n \times n} : \mathrm{trace}(A) = 0\}.$$

5. $\mathrm{O}(n) = \{g \in \mathrm{GL}(n, \mathbb{R}) : gg^{\mathrm{t}} = e\}$, $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$, dimension $\frac{1}{2}n(n-1)$,
$$\underline{\mathrm{o}}(n) = \underline{\mathrm{so}}(n) = \{A \in \mathbb{R}^{n \times n} : A = -A^{\mathrm{t}}\}.$$

6. $\mathrm{U}(n) = \{g \in \mathrm{GL}(n, \mathbb{C}) : g\bar{g}^{\mathrm{t}} = e\}$, dimension n^2,
$$\underline{\mathrm{u}}(n) = \{A \in \mathbb{C}^{n \times n} : A = -\bar{A}^{\mathrm{t}}\}.$$
$\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$, dimension $n^2 - 1$,
$$\underline{\mathrm{su}}(n) = \underline{\mathrm{u}}(n) \cap \underline{\mathrm{sl}}(n, \mathbb{C}).$$

7. Affine group $G = \mathrm{GL}(n, \mathbb{R}) \times \mathbb{R}^n$, $(g, v)(h, w) = (gh, gw + v)$,
$$\underline{g} = \mathbb{R}^{n \times n} \times \mathbb{R}^n, \quad [(A, v), (B, w)] = (AB - BA, Aw - Bv).$$

8. The vector space $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ of quaternions, whose non-commuting imaginary units i, j, k satisfy the relations $i^2 = j^2 = k^2 = ijk = -1$ and hence
$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j,$$
forms a division algebra with norm $\|a+bi+cj+dk\| = (a^2+b^2+c^2+d^2)^{1/2}$. The sphere $S^3 \subset \mathbb{R}^4$ may be viewed as the set
$$\{a + bi + cj + dk \in \mathbb{H} : \|a + bi + cj + dk\| = 1\}$$
of unit quaternions and thus inherits the structure of a Lie group. The corresponding Lie algebra $\underline{s}^3$ is spanned by i, j, k, where
$$[i, j] = ij - ji = 2k, \quad [j, k] = 2i, \quad [k, i] = 2j.$$
The quotient group $S^3/\{1, -1\}$ is a Lie group diffeomorphic to $\mathbb{R}P^3$.

If $F\colon G \to G'$ is a Lie group homomorphism or isomorphism, then the differential $dF_e\colon TG_e \to TG'_e$ is a Lie algebra homomorphism or isomorphism, respectively (exercise).

Example The Lie groups S^3 and $\mathrm{SU}(2)$ are isomorphic, furthermore $S^3/\{1,-1\}$ is isomorphic zu $\mathrm{SO}(3)$. In particular, the Lie algebras $\underline{\mathrm{s}}^3$, $\underline{\mathrm{su}}(2)$, $\underline{\mathrm{so}}(3)$ are mutually isomorphic (exercise).

Let G be a Lie group. A pair (H, i), where H is a Lie group and $i\colon H \to G$ is a Lie group homomorphism and an injective immersion, is called a *Lie subgroup* of G; $i(H)$ is a subgroup of G, but in general i is not a homeomorphism onto $i(H)$ with respect to the topology induced by G.

Example For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the map

$$i\colon (\mathbb{R}, +) \to (T^2 = \mathbb{R}^2/\mathbb{Z}^2, +), \quad t \mapsto (t, \alpha t) \bmod \mathbb{Z}^2,$$

is an injective immersion but not an embedding. In fact, $i(\mathbb{R})$ is dense in T^2.

Using the theorem of Frobenius (see page 70) and Lemma 12.2 one can show that *if $\underline{h}' \subset \underline{g}$ is a Lie subalgebra of the Lie algebra of a Lie group G, then there exists a connected Lie subgroup $i\colon H \to G$ with $di_e(\underline{h}) = \underline{h}'$, and every other connected Lie subgroup $\tilde{i}\colon \tilde{H} \to G$ with $d\tilde{i}_e(\underline{\tilde{h}}) = \underline{h}'$ is of the form $\tilde{i} = i \circ F$ for some Lie group isomorphism $F\colon \tilde{H} \to H$.*

12.2 Exponential map

12.5 Proposition *Left-invariant vector fields are completely integrable. The integral curves $c\colon \mathbb{R} \to G$ with $c(0) = e$ are precisely the Lie group homomorphisms $(\mathbb{R}, +) \to G$.*

Proof: Let X be a left-invariant vector field on G.

There exist an $\epsilon > 0$ and an integral curve $c\colon (-\epsilon, \epsilon) \to G$ of X with $c(0) = e$. Then, for every $g \in G$, the left-translate $gc = L_g \circ c$ is an integral curve of X with $gc(0) = g$, because

$$(gc)'(t) = L_{g*}c'(t) = L_{g*}X_{c(t)} = X_{gc(t)} \quad \text{for all } t \in (-\epsilon, \epsilon)$$

by the product rule and the left-invariance of X. Thus the flow Φ of X is defined on $(-\epsilon, \epsilon) \times G$ by $\Phi^t(g) = gc(t)$, and it then follows as in the proof of Proposition 10.9. that X is completely integrable.

Let now $c\colon \mathbb{R} \to G$ be the integral curve with $c(0) = e$, thus $\Phi^t(e) = c(t)$ for all $t \in \mathbb{R}$. Then, for $s \in \mathbb{R}$ and $g := c(s)$,

$$c(s)c(t) = gc(t) = \Phi^t(g) = \Phi^t(\Phi^s(e)) = \Phi^{s+t}(e) = c(s+t),$$

so c is a homomorphism from $(\mathbb{R}, +)$ into G. Conversely, suppose that $c\colon (\mathbb{R}, +) \to G$ is a Lie group homomorphism with $c'(0) = X_e$. Then $c(s+t) = c(s)c(t) = gc(t)$, and by taking the derivative at $t = 0$ one gets that $c'(s) = L_{g*}c'(0) = X_g = X_{c(s)}$, showing that c is an integral curve. □

12.6 Definition (exponential map) The *exponential map* of G is the map

$$\exp\colon TG_e \to G, \quad \exp(X_e) := c(1),$$

where $c\colon \mathbb{R} \to G$ is the integral curve of the left-invariant vector field X (or, equivalently, the Lie group homomorphism $(\mathbb{R}, +) \to G$) with $c'(0) = X_e$.

Notice that then

$$\exp(tX_e) = c(t) \quad \text{for all } t \in \mathbb{R},$$

since the integral curve through e of the left-invariant vector field $\tilde{X} := tX$ is given by $s \mapsto \tilde{c}(s) := c(ts)$, so that $\exp(tX_e) = \exp(\tilde{X}_e) = \tilde{c}(1) = c(t)$. It follows in particular that

$$\exp(sX_e)\exp(tX_e) = c(s)c(t) = c(s+t) = \exp((s+t)X_e)$$

and $\exp(tX_e)^{-1} = c(t)^{-1} = c(-t) = \exp(-tX_e)$.

Furthermore, exp is smooth. To see this, consider the vector field V on $G \times TG_e$ defined by $V(g, X_e) := (gX_e, 0) \in TG_g \times TG_e$, whose integral curve through (g, X_e) is $t \mapsto (g\exp(tX_e), X_e)$. Thus the flow of V satisfies $\Phi^t(g, X_e) = (g\exp(tX_e), X_e)$ for all $t \in \mathbb{R}$, and if $\pi\colon G \times TG_e \to G$ denotes the canonical projection, then $\exp(X_e) = \pi \circ \Phi^1(e, X_e)$, which depends smoothly on X_e.

The differential $d\exp_0\colon T(TG_e)_0 = TG_e \to TG_e$ is the identity map, as $d\exp_0(X_e) = \frac{d}{dt}\big|_{t=0}\exp(tX_e) = c'(0) = X_e$. In particular, the restriction of exp to a suitable open neighborhood of 0 in TG_e is a diffeomorphism onto an open neighborhood of e in G.

Let now $F\colon G \to G'$ be a Lie group homomorphism. Then, as mentioned earlier, the differential $dF_e\colon TG_e \to TG'_e$ is a Lie algebra homomorphism. Furthermore, the map $t \mapsto F \circ \exp^G(tX_e)$ is a homomorphism $(\mathbb{R}, +) \to G'$ with initial vector $dF_e(X_e)$, hence it agrees with $t \mapsto \exp^{G'}(t\,dF_e(X_e))$. For $t = 1$, this shows that

$$F \circ \exp^G = \exp^{G'} \circ dF_e.$$

Next, consider $\mathrm{GL}(n, \mathbb{C})$ with the matrix exponential function

$$A \mapsto e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

on $\mathbb{C}^{n \times n} = \underline{\mathrm{gl}}(n, \mathbb{C})$. The following properties hold:

(1) $Be^AB^{-1} = e^{BAB^{-1}}$ for all $B \in \mathrm{GL}(n, \mathbb{C})$;

(2) $\det(e^A) = e^{\mathrm{trace}(A)} \neq 0$, in particular $e^A \in \mathrm{GL}(n, \mathbb{C})$;

(3) if $A, B \in \mathbb{C}^{n\times n}$ and $[A, B] = AB - BA = 0$, then $e^{A+B} = e^A e^B$.

Let $A \in \underline{\mathrm{gl}}(n, \mathbb{C})$. Since $[sA, tA] = 0$ for $s, t \in \mathbb{R}$, it follows from (2) and (3) that $c\colon t \mapsto e^{tA}$ is a homomorphism from $(\mathbb{R}, +)$ into G, and $c'(0) = A$. Hence, the Lie group exponential map

$$\exp\colon \underline{\mathrm{gl}}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$$

agrees with the matrix exponential $A \mapsto \exp(A) = e^A$.

Let again G be an arbitrary Lie group. According to the *Campbell–Baker–Hausdorff formula*, for two vectors $X, Y \in TG_e$ in a sufficiently small neighborhood of 0, the identity $\exp(X)\exp(Y) = \exp(S(X, Y))$ holds, where

$$S(X, Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \ldots$$

is a convergent series of nested Lie brackets satisfying $S(Y, X) = -S(-X, -Y)$ (there is an explicit form due to Dynkin (1947)). The formula is particularly useful for *nilpotent* Lie groups, for which S terminates.

Appendix

A Analysis

In the following statements and proofs, all diffeomorphisms are of class C^∞.

A.1 Theorem (inverse function theorem) *Suppose that $W \subset \mathbb{R}^n$ is an open set, $F \in C^\infty(W, \mathbb{R}^n)$, $p \in W$, $F(p) = 0$, and dF_p is bijective. Then there exist open neighborhoods $V \subset W$ of p and $U \subset \mathbb{R}^n$ of 0 such that $F|_V$ is a diffeomorphism from V onto U.*

A.2 Theorem (implicit function theorem, surjective form) *Suppose that $W \subset \mathbb{R}^n$ is an open set, $F \in C^\infty(W, \mathbb{R}^k)$, $p \in W$, $F(p) = 0$, and dF_p is surjective. Then there exist open neighborhoods $U \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$ of $(0, 0)$ and $V \subset W$ of p and a diffeomorphism $\psi\colon U \to V$ such that $\psi(0, 0) = p$ and*

$$(F \circ \psi)(x, y) = y$$

for all $(x, y) \in U$ (canonical projection).

Proof: After a linear change of coordinates on $\mathbb{R}^n$ we can assume that dF_p maps the subspace $\{0\} \times \mathbb{R}^k \subset \mathbb{R}^n$ bijectively onto $\mathbb{R}^k$. Then, for $q = (q^1, \dots, q^n) \in W$ and $q' := (q^1, \dots, q^{n-k})$, put $\tilde{F}(q) := (q', F(q))$. This defines a map $\tilde{F} \in C^\infty(W, \mathbb{R}^{n-k} \times \mathbb{R}^k)$, and $d\tilde{F}_p$ is bijective. By Theorem A.1 there exist open neighborhoods $V \subset W$ of p and $U \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$ of $(0, 0)$ such that $\tilde{F}|_V$ is a diffeomorphism from V onto U. Let $\psi := \big(\tilde{F}|_V\big)^{-1}$. For $(x, y) \in U$ and $\psi(x, y) =: q$, $(q', F(q)) = \tilde{F}(q) = (x, y)$, in particular $(F \circ \psi)(x, y) = F(q) = y$. □

A.3 Theorem (implicit function theorem, injective form) *Suppose that $U \subset \mathbb{R}^m$ is an open set, $f \in C^\infty(U, \mathbb{R}^n)$, $0 \in U$, $f(0) = p$, and df_0 is injective. Then there exist open neighborhoods $V \subset \mathbb{R}^n$ of p and $W \subset U \times \mathbb{R}^{n-m}$ of $(0, 0)$ and a diffeomorphism $\varphi\colon V \to W$ such that $\varphi(p) = (0, 0)$ and*

$$(\varphi \circ f)(x) = (x, 0)$$

for all $(x, 0) \in W$ (canonical inclusion).

Proof: We can assume that the subspace $\{0\} \times \mathbb{R}^{n-m} \subset \mathbb{R}^n$ is complementary to the image of df_0. Define $\tilde{f} \in C^\infty(U \times \mathbb{R}^{n-m}, \mathbb{R}^n)$ by $\tilde{f}(x, y) := f(x) + (0, y)$ for $(x, y) \in U \times \mathbb{R}^{n-m}$. The differential $d\tilde{f}_0$ is bijective. By Theorem A.1 there exist open neighborhoods $W \subset U \times \mathbb{R}^{n-m}$ of $(0, 0)$ and $V \subset \mathbb{R}^n$ of p such that $\tilde{f}|_W$ is a diffeomorphism from W onto V. Let $\varphi := \big(\tilde{f}|_W\big)^{-1}$. For $(x, 0) \in W$, $f(x) = \tilde{f}(x, 0)$, hence $(\varphi \circ f)(x) = (x, 0)$. □

We state two useful facts about smooth vector fields.

A.4 Lemma (flow box) *Suppose that $X\colon V \to \mathbb{R}^m$ is a vector field on a neighborhood V of 0 in $\mathbb{R}^m$, and $X(0) \neq 0$. Then there exist an open neighborhood $W \subset V$ of 0 and a diffeomorphism $\psi\colon W \to \psi(W) \subset \mathbb{R}^m$ such that $d\psi_y(X(y)) = e_1$ for all $y \in W$.*

Proof: We can assume that $X(0) = e_1$. There exist an open set V' in $\{0\} \times \mathbb{R}^{m-1} \subset \mathbb{R}^m$ with $0 \in V' \subset V$ and an $\epsilon > 0$ such that for every $x \in V'$ there is an integral curve $c_x\colon (-\epsilon, \epsilon) \to \mathbb{R}^m$ of X with $c_x(0) = x$, and the map $(t, x) \mapsto c_x(t)$ on $(\epsilon, \epsilon) \times V'$ is C^∞ (compare Theorem 10.8). Then the map sending $x + te_1$ to $c_x(t)$ for every $(t, x) \in (\epsilon, \epsilon) \times V'$ is also C^∞ and furthermore regular at 0, because $\dot{c}_0(0) = X(0) = e_1$ and $c_x(0) = x$ for all $x \in V'$. Hence the restriction of this map to a suitable neighborhood of 0 is a diffeomorphism whose inverse $\psi\colon W \to \psi(W)$ satisfies $\psi(y) = x + te_1$ and $d\psi_y(X(y)) = d\psi_y(\dot{c}_x(t)) = e_1$ for all $y = c_x(t) \in W$. □

A.5 Lemma (parametrization by flow lines) *Suppose that $X_1, X_2\colon V \to \mathbb{R}^2$ are two vector fields on a neighborhood V of 0 in $\mathbb{R}^2$, and $X_1(0), X_2(0)$ are linearly independent. Then there exist an open set $U \subset \mathbb{R}^2$ and a diffeomorphism $\varphi\colon U \to \varphi(U) \subset V$ with $0 \in \varphi(U)$ such that*

$$\frac{\partial \varphi}{\partial x^i}(x) = \lambda_i(x)\, X_i(\varphi(x))$$

for all $x \in U$ and some functions $\lambda_i\colon U \to \mathbb{R}$, $i = 1, 2$.

Proof: Since $X_i(0) \neq 0$ for $i = 1, 2$, by Lemma A.4 there exist an open neighborhood $W \subset V$ of 0 and diffeomorphisms $\psi_i = (\psi_i^1, \psi_i^2)\colon W \to \psi_i(W) \subset \mathbb{R}^2$ such that $d(\psi_i)_y(X_i(y)) = e_i$ for all $y \in W$. Then $h^1 := \psi_2^1$ and $h^2 := \psi_1^2$ are regular functions on W whose level curves are flow lines of X_2 and X_1, respectively. Define $h := (h^1, h^2)\colon W \to \mathbb{R}^2$. Since $X_1(0), X_2(0)$ are linearly independent and h^1, h^2 are regular at 0, whereas $d(h^1)_0(X_2(0)) = 0$ and $d(h^2)_0(X_1(0)) = 0$, it follows that $d(h^i)_0(X_i(0)) \neq 0$ for $i = 1, 2$, thus h is regular at 0. Hence, the restriction of h to a suitable neighborhood of 0 has an inverse φ as claimed, mapping horizontal and vertical lines to flow lines of X_1 and X_2, respectively. □

B General topology

B.1 Definition (topology, topological space) Let M be a set. A *topology* on M is a collection of subsets of M, called *open sets*, with the following properties:

(1) $\emptyset$ and M are open;

(2) the union of arbitrarily many open sets is open;

(3) the intersection of finitely many open sets is open.

A *topological space* is a set equipped with a topology.

Examples

1. Let (M, d) be a metric space. With respect to the *topology induced by* d, a set $U \subset M$ is open if and only if for all $p \in U$ there is an $r > 0$ such that $B(p, r) = \{q \in M : d(p, q) < r\} \subset U$.

2. The usual topology on $\mathbb{R}^m$ is induced by the standard metric $d(x, y) = |x - y|$.

3. The *trivial topology* on a set M consists only of $\emptyset$ and M, whereas the *discrete topology* on M is the entire power set.

A subset A of a topological space M is called *closed* if the complement $M \setminus A$ is open; thus $\emptyset$ and M are both open and closed.

A map $f : M \to N$ between two topological spaces is *continuous* if $f^{-1}(V) \subset M$ is open for every open set $V \subset N$. The map f is a *homeomorphism* if f is bijective and both f and f^{-1} are continuous.

B.2 Definition (induced topology) Let N be a topological space, and let $M \subset N$ be a subset. The *induced topology* or *subspace topology* on M consists of all sets $U \subset M$ of the form $U = M \cap V$ where V is open in N.

B.3 Definition (compactness) A topological space M is *compact* if every open cover of M has a finite subcover; that is, whenever $\bigcup_{\alpha \in A} U_\alpha = M$ for open sets $U_\alpha \subset M$ and an index set A, there exists a finite set $B \subset A$ such that $\bigcup_{\beta \in B} U_\beta = M$.

If M is compact and $f : M \to N$ is continuous, then $f(M)$ is a compact subspace of N. If M is compact and A is closed in M, then A is a compact subspace of M.

A set $U \subset M$ is called a *neighborhood* of a point $p \in M$ if there exists an open set V with $p \in V \subset U$.

B.4 Definition (Hausdorff space) A topological space M is called a *Hausdorff space* if for every pair of distinct points $p, q \in M$ there exist disjoint neighborhoods U of p and V of q.

Every metric space is a Hausdorff space.

B.5 Lemma *If M is a Hausdorff space and $A \subset M$ is a compact subspace, then A is closed in M.*

It follows easily that every continuous bijective map $f : M \to N$ from a compact space M onto a Hausdorff space N is a homeomorphism.

B.6 Definition (basis, subbasis) Let M be a topological space. A collection $\mathcal{B}$ of open sets is called a *basis of the topology* if every open set can be written as a union of sets in $\mathcal{B}$. A collection $\mathcal{S}$ of open sets is a *subbasis of the topology* if every open set is a union of sets that are intersections of finitely many sets in $\mathcal{S}$.

Examples

1. The set of all open balls forms a basis of the topology of a metric space.
2. The set of all open balls $B(x, r)$ with $x \in \mathbb{Q}^m$ and $r \in \mathbb{Q}$, $r > 0$, is a countable basis of the usual topology on $\mathbb{R}^m$.

B.7 Definition (product topology) Let M, N be two topological spaces. The *product topology* on $M \times N$ is the topology for which the sets of the form $U \times V$ where U is open in M and V is open in N constitute a basis.

B.8 Definition (quotient topology) Suppose that M is a topological space, $\sim$ is an equivalence relation on M, and $\pi : M \to M/\!\sim$ is the projection onto the set of equivalence classes. The *quotient topology* on $M/\!\sim$ consists of all sets $V \subset M/\!\sim$ for which $\pi^{-1}(V)$ is open in M.

A topological space M is called *connected* if $\emptyset$ and M are the only open and closed subsets of M. A topological space M is *path connected* if for every pair of points $p, q \in M$ there is a path from p to q (that is, a continuous map $c : [0, 1] \to M$ with $c(0) = p$ and $c(1) = q$), and M is *locally path connected* if every point $p \in M$ has a neighborhood that is path connected in the induced topology. Every path connected space is connected. The subspace

$$\{(x, \sin(1/x)) : x \in \mathbb{R},\ x > 0\} \cup \{(0, y) : y \in [-1, 1]\}$$

of $\mathbb{R}^2$ is connected but not path connected. Every connected and locally path connected space is (globally) path connected.

C Multilinear algebra

Let $V, V_1, \ldots, V_n$ and W be vector spaces (over $\mathbb{R}$). We denote by $L(V; W)$ the vector space of linear maps from V to W. A map

$$f : V_1 \times \ldots \times V_n \to W$$

is *multilinear* or *n-linear* if for every index $i \in \{1, \ldots, n\}$ and for fixed vectors $v_j \in V_j$, $j \neq i$, the map

$$v \mapsto f(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_n)$$

from V_i to W is linear. We let $L(V_1, \ldots, V_n; W)$ denote the vector space of all such n-linear maps.

C.1 Theorem (tensor product) *Given vector spaces $V_1, \ldots, V_n$, there exist a vector space $\mathcal{T}$ and an n-linear map $\tau \in L(V_1, \ldots, V_n; \mathcal{T})$ with the following property: for every n-linear map $f \in L(V_1, \ldots, V_n; W)$ into any vector space W there is a unique linear map $g \in L(\mathcal{T}; W)$ such that $f = g \circ \tau$.*

This property characterizes the pair $(\tau, \mathcal{T})$ uniquely up to a linear isomorphism; $(\tau, \mathcal{T})$ is called the *tensor product* of $V_1, \ldots, V_n$, and one writes

$$V_1 \otimes \ldots \otimes V_n := \mathcal{T}, \quad v_1 \otimes \ldots \otimes v_n := \tau(v_1, \ldots, v_n).$$

The unique assignment $f \mapsto g$ given by the theorem is a linear isomorphism

$$L(V_1, \ldots, V_n; W) \cong L(V_1 \otimes \ldots \otimes V_n; W).$$

For every permutation σ of $\{1, \ldots, n\}$ there exists a linear isomorphism

$$V_1 \otimes \ldots \otimes V_n \cong V_{\sigma(1)} \otimes \ldots \otimes V_{\sigma(n)}$$

mapping $v_1 \otimes \ldots \otimes v_n$ to $v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}$. For $m < n$,

$$(V_1 \otimes \ldots \otimes V_m) \otimes (V_{m+1} \otimes \ldots \otimes V_n) \cong V_1 \otimes \ldots \otimes V_n.$$

For every vector space V the scalar multiplication is a bilinear map $\mathbb{R} \times V \to V$; this induces an isomorphism

$$\mathbb{R} \otimes V \cong V$$

mapping $a \otimes v$ to av. If $V \cong V_1 \oplus V_2$ (direct sum), then

$$V \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W).$$

The construction of the tensor product is natural in the following sense: if linear maps $f_j : V_j \to V'_j$ are given, $j = 1, \ldots, n$, then there exists a unique linear map $f_1 \otimes \ldots \otimes f_n : V_1 \otimes \ldots \otimes V_n \to V'_1 \otimes \ldots \otimes V'_n$ such that

$$(f_1 \otimes \ldots \otimes f_n)(v_1 \otimes \ldots \otimes v_n) = f_1(v_1) \otimes \ldots \otimes f_n(v_n)$$

whenever $v_j \in V_j$ for $j = 1, \ldots, n$.

We now assume that the vector spaces $V, V_1, \ldots, V_n$ are finite dimensional. If B_j is a basis of V_j for $j = 1, \ldots, n$, then the products $b_1 \otimes \ldots \otimes b_n$ with $b_j \in B_j$ constitute a basis of $V_1 \otimes \ldots \otimes V_n$. In particular,

$$\dim(V_1 \otimes \ldots \otimes V_n) = \dim(V_1) \cdots \dim(V_n).$$

We let $V^* := L(V; \mathbb{R})$ denote the dual space of V. The map $v \mapsto \tilde{v} \in (V^*)^*$, $\tilde{v}(\lambda) := \lambda(v)$, is a canonical isomorphism $V \cong V^{**}$. If $\lambda_j \in V_j^*$, $j = 1, \ldots, n$, then $\lambda_1 \otimes \ldots \otimes \lambda_n \in V_1^* \otimes \ldots \otimes V_n^*$ may also be viewed as the tensor product

$$\lambda_1 \otimes \ldots \otimes \lambda_n \colon V_1 \otimes \ldots \otimes V_n \longrightarrow \mathbb{R} \otimes \ldots \otimes \mathbb{R} \cong \mathbb{R}$$

of the linear maps $\lambda_j \colon V_j \longrightarrow \mathbb{R}$ described above; this yields an isomorphism

$$V_1^* \otimes \ldots \otimes V_n^* \cong (V_1 \otimes \ldots \otimes V_n)^*.$$

Note that

$$(\lambda_1 \otimes \ldots \otimes \lambda_n)(v_1 \otimes \ldots \otimes v_n) = \lambda_1(v_1) \cdots \lambda_n(v_n).$$

An (r, s)-*tensor* over V is an element of

$$\begin{aligned} V_{r,s} &:= \underbrace{V \otimes \ldots \otimes V}_{r} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s} \\ &\cong (\underbrace{V^* \otimes \ldots \otimes V^*}_{r} \otimes \underbrace{V \otimes \ldots \otimes V}_{s})^* \\ &\cong \{T \colon \underbrace{V^* \times \ldots \times V^*}_{r} \times \underbrace{V \times \ldots \times V}_{s} \to \mathbb{R} : T \text{ ist } (r+s)\text{-linear}\}. \end{aligned}$$

Note that $\dim(V_{r,s}) = \dim(V)^{r+s}$, $V_{1,0} = V$, $V_{0,1} = V^*$, and one puts $V_{0,0} := \mathbb{R}$. If $(e_1, \ldots, e_m)$ is a basis of V and $(\epsilon^1, \ldots, \epsilon^m)$ is the dual basis of V^*, $\epsilon^i(e_j) = \delta^i_j$, then $T \in V_{r,s}$ possesses the representation

$$T = \sum_{j_1, \ldots, j_r, i_1, \ldots, i_s = 1}^{m} T^{j_1 \ldots j_r}_{i_1 \ldots i_s} e_{j_1} \otimes \ldots \otimes e_{j_r} \otimes \epsilon^{i_1} \otimes \ldots \otimes \epsilon^{i_s}$$

with components $T^{j_1 \ldots j_r}_{i_1 \ldots i_s} \in \mathbb{R}$.

In the following, $V_{0,s}$ will always be identified with the vector space $L(V, \ldots, V; \mathbb{R})$ of s-linear maps $A \colon V \times \ldots \times V \longrightarrow \mathbb{R}$. For $A \in V_{0,s}$ and $B \in V_{0,t}$, the tensor product $A \otimes B \in V_{0,s+t}$ is then given by the simple formula

$$A \otimes B\,(v_1, \ldots, v_{s+t}) = A(v_1, \ldots, v_s)\, B(v_{s+1}, \ldots, v_{s+t})$$

for $v_1, \ldots, v_{s+t} \in V$.

C.2 Theorem (alternating multilinear maps) *For $A \in V_{0,s}$, the following properties are equivalent*:

(1) *A is alternating, that is,* $A(v_1, \dots, v_s) = 0$ *whenever* $v_i = v_j$ *for two indices* $i \neq j$;

(2) *A ist skew-symmetric, that is,* $A(v_{\tau(1)}, \dots, v_{\tau(s)}) = -A(v_1, \dots, v_s)$ *for every transposition* τ *of* $\{1, \dots, s\}$;

(3) $A(v_1, \dots, v_s) = 0$ *whenever* $v_1, \dots, v_s$ *are linearly dependent*;

(4) $A(v_1, \dots, v_s) = \det(a^i_j)\, A(w_1, \dots, w_s)$ *if* $v_j = \sum_{i=1}^s a^i_j w_i$ *for* $j = 1, \dots, s$.

We write $\Lambda_s(V^*)$ for the vector space of alternating $(0, s)$-tensors over V, and we put $\Lambda_0(V^*) := \mathbb{R}$. Note that $\Lambda_s(V^*) = \{0\}$ for $s > m = \dim(V)$.

C.3 Definition (exterior product) For $A \in \Lambda_s(V^*)$ and $B \in \Lambda_t(V^*)$, the *exterior product* (or *wedge product*) $A \wedge B \in \Lambda_{s+t}(V^*)$ is defined by

$$A \wedge B(v_1, \dots, v_{s+t}) := \sum_{\sigma \in S_{s,t}} \operatorname{sgn}(\sigma)\, A(v_{\sigma(1)}, \dots, v_{\sigma(s)})\, B(v_{\sigma(s+1)}, \dots, v_{\sigma(s+t)})$$

for $v_1, \dots, v_{s+t} \in V$, where $S_{s,t}$ denotes the set of all permutations $\sigma \in S_{s+t}$ such that $\sigma(1) < \dots < \sigma(s)$ and $\sigma(s+1) < \dots < \sigma(s+t)$.

The map $\wedge \colon \Lambda_s(V^*) \times \Lambda_t(V^*) \to \Lambda_{s+t}(V^*)$ is bilinear, and

$$B \wedge A = (-1)^{st} A \wedge B,$$

in particular $A \wedge A = 0$ if $A \in \Lambda_s(V^*)$ and s is odd. For $A \in \Lambda_s(V^*)$, $B \in \Lambda_t(V^*)$, and $C \in \Lambda_u(V^*)$,

$$(A \wedge B) \wedge C = A \wedge (B \wedge C).$$

If $\lambda_1, \dots, \lambda_s \in \Lambda_1(V^*) = V^*$, then $\lambda_1 \wedge \dots \wedge \lambda_s \in \Lambda_s(V^*)$ is given by

$$\begin{aligned}(\lambda_1 \wedge \dots \wedge \lambda_s)(v_1, \dots, v_s) &= \sum_{\sigma \in S_s} \operatorname{sgn}(\sigma)\, \lambda_1(v_{\sigma(1)}) \cdots \lambda_s(v_{\sigma(s)}) \\ &= \det(\lambda_i(v_j))\end{aligned}$$

for $v_1, \dots, v_s \in V$.

Now let $\{e_1, \dots, e_m\}$ be a basis of V, and let $\{\epsilon^1, \dots, \epsilon^m\}$ be the dual basis of V^*. For $1 \le i_1 < \dots < i_s \le m$ and $1 \le j_1, \dots, j_s \le m$,

$$\begin{aligned}&(\epsilon^{i_1} \wedge \dots \wedge \epsilon^{i_s})(e_{j_1}, \dots, e_{j_s}) \\ &\quad = \sum_{\sigma \in S_s} \operatorname{sgn}(\sigma)\, \delta^{i_1}_{j_{\sigma(1)}} \cdots \delta^{i_s}_{j_{\sigma(s)}} \\ &\quad = \begin{cases} \operatorname{sgn}(\sigma) & \text{if } (j_{\sigma(1)}, \dots, j_{\sigma(s)}) = (i_1, \dots, i_s), \\ 0 & \text{if } \{j_1, \dots, j_s\} \neq \{i_1, \dots, i_s\}. \end{cases}\end{aligned}$$

The set

$$\{\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_s} : 1 \le i_1 < \ldots < i_s \le m\}$$

forms a basis of $\Lambda_s(V^*)$, in particular $\dim(\Lambda_s(V^*)) = \binom{m}{s}$.

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