Existence and uniqueness of Haar integrals

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Abstract

This is an expository note providing an elementary constructive proof of the existence and uniqueness of a Haar integral on every locally compact and Hausdorff topological group.

In the following, G denotes a locally compact and Hausdorff topological group, and $C_c(G)$ denotes the vector space of continuous functions $f\colon G \to \mathbb{R}$ with compact support $\mathrm{spt}(f) := \overline{\{x \in G : f(x) \neq 0\}}$. For $a \in G$, we denote by $L_a\colon G \to G$ the homeomorphism of G that maps x to ax, and whose inverse L_a^{-1} is $L_{a^{-1}}$. A linear functional $\Lambda\colon C_c(G) \to \mathbb{R}$ is called

- *positive* if $\Lambda(f) \geq 0$ for all $f \in C_c^+(G) := \{f \in C_c(G) : f \geq 0\}$; and
- *left-invariant* if $\Lambda(f \circ L_a) = \Lambda(f)$ for all $f \in C_c(G)$ and $a \in G$.

A *left Haar integral* Λ on G is a left-invariant, positive and non-trivial (that is, not identically vanishing) linear functional $\Lambda\colon C_c(G) \to \mathbb{R}$.

Theorem 1. *There exists a left Haar integral Λ on G, unique up to multiplication by a positive constant. It satisfies $\Lambda(f) > 0$ for every $f \in C_c^+(G) \setminus \{0\}$.*

The existence of an essentially unique left Haar *measure* on G then follows immediately from the Riesz Representation Theorem (cf. [5, 2.14] and [6, Theorem 3.15]). Theorem 1 has its origins in [3] and is due to Weil [7], whose elegant proof depended, for the existence part, on the axiom of choice. Cartan [2] then gave a constructive existence proof inspired by Weil's argument for the uniqueness. The proof given below is another modification of Weil's approach (compare also [1], [4], [6]).

We start with a basic uniform continuity result. In the sequel, $\mathscr{U}$ denotes the collection of all open neighborhoods of the neutral element e in G.

Lemma 2. *Let $f \in C_c^+(G)$. Then there exists a function $\varrho \in C_c^+(G)$ with values in $[0,1]$ such that $\mathrm{spt}(f)$ is contained in the interior of the set $\varrho^{-1}\{1\}$, and for every such ϱ and every $\varepsilon > 0$ there exists a neighborhood $U \in \mathscr{U}$ such that $f(x) \leq f(y) + \varepsilon\varrho(y) \leq f(y) + \varepsilon$ whenever $x, y \in G$ and $x^{-1}y \in U$ or $y^{-1}x \in U$.*

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Proof. The existence of ϱ follows from Urysohn's Lemma (s. [5, 2.12]). Now let such a function ϱ and a constant $\varepsilon > 0$ be given. Let W denote the interior of $\varrho^{-1}\{1\}$, and put $K := \mathrm{spt}(f)$. For every $a \in K$, since f is continuous at a, there is an open neighborhood $V_a \subset W$ of a such that $|f(a)-f(b)| < \varepsilon/2$ for all $b \in V_a$. Furthermore, since the map $(x, y) \mapsto axy$ is continuous at (e, e), there exists $U_a \in \mathscr{U}$ such that $aU_aU_a \subset V_a$. The collection of open sets aU_a with $a \in K$ covers the compact set K, thus there is a finite set $A \subset K$ such that $K \subset \bigcup_{a\in A} aU_a$. Put $U' := \bigcap_{a\in A} U_a$ and $U := U' \cap (U')^{-1}$, and note that $U = U^{-1} \in \mathscr{U}$. Now let $x, y \in G$ be such that $x^{-1}y \in U$ (or $y^{-1}x = (x^{-1}y)^{-1} \in U$). Suppose $f(x) > 0$. Then $x \in K$, thus there is an $a \in A$ such that $x \in aU_a$ and hence $y = x(x^{-1}y) \in aU_aU$. It follows that $x, y \in V_a \subset W$, and $f(x) \le f(a) + \varepsilon/2 \le f(y) + \varepsilon = f(y) + \varepsilon\varrho(y)$. □

The proof of Theorem 1 examines certain functions $I\colon C_c^+(G) \to [0, \infty)$. Such a functional will be called

- *left-invariant* if $I(f \circ L_a) = I(f)$ for all $f \in C_c^+(G)$ and $a \in G$;
- *homogeneous* if $I(\lambda f) = \lambda I(f)$ for all $f \in C_c^+(G)$ and $\lambda \ge 0$;
- *subadditive* if $I(f + f') \le I(f) + I(f')$ for all $f, f' \in C_c^+(G)$; and
- *monotonic* if $I(f) \le I(g)$ whenever $f, g \in C_c^+(G)$ and $f \le g$.

We denote by $\mathscr{F}$ the set of all functions $I\colon C_c^+(G) \to [0, \infty)$ satisfying these four properties, and by $\mathscr{L}$ the set of all functions $I\colon C_c^+(G) \to [0, \infty)$ that are left-invariant, homogeneous, and additive. Note that $\mathscr{L} \subset \mathscr{F}$, for if $I \in \mathscr{L}$ and $f \le g$, then $I(f) \le I(f) + I(g - f) = I(g)$. The restriction of any left-invariant, positive linear functional on $C_c(G)$ to $C_c^+(G)$ belongs to $\mathscr{L}$. Conversely, given $I \in \mathscr{L}$, the extension to $C_c(G)$ obtained by putting $I(f) := I(f^+)-I(f^-)$ for $f \in C_c(G) \setminus C_c^+(G)$ is a left-invariant, positive linear functional.

For a neighborhood $U \in \mathscr{U}$, we put

$$\mathscr{P}(U) := \{g \in C_c^+(G) : g \not\equiv 0,\ \mathrm{spt}(g) \subset U\},$$

and we denote by $\mathscr{P}_*(U)$ the subset of all $g \in \mathscr{P}(U)$ such that $g(x^{-1}) = g(x)$ for all $x \in G$. Note that $\mathscr{P}(U)$ is non-empty by Urysohn's Lemma, and if $U = U^{-1} \in \mathscr{U}$ and $g \in \mathscr{P}(U)$, then $x \mapsto \max\{g(x), g(x^{-1})\}$ is an element of $\mathscr{P}_*(U)$.

Now we give the proof of Theorem 1, which is divided into five parts.

Part I. Let $g \in \mathscr{P}(G)$. For every $f \in C_c^+(G)$, we denote by $I_g(f)$ the infimum of all $s \ge 0$ for which there exist a finite set $A \subset G$ and constants $c_a \ge 0$, for $a \in A$, such that

$$f \le \sum_{a\in A} c_a g \circ L_a^{-1} \quad \text{and} \quad \sum_{a\in A} c_a \le s. \tag{1}$$

Due to the compactness of $\mathrm{spt}(f)$, the set of all such s is non-empty, so that $I_g(f)$ is finite. For instance, if V is the non-empty open set where $g > \|g\|_\infty/2$, then there is a finite set $A \subset G$ such that $\mathrm{spt}(f) \subset \bigcup_{a \in A} aV$ and hence $f \le \sum_{a \in A} cg \circ L_a^{-1}$ for $c := 2\|f\|_\infty/\|g\|_\infty$. Thus we have a function $I_g \colon C_c^+(G) \to [0,\infty)$, and it follows directly from the definition that $I_g \in \mathscr{F}$. Furthermore, for any $J \in \mathscr{F}$,

$$J(f) \le I_g(f)J(g) \quad \text{for all } f \in C_c^+(G), \tag{2}$$

as is easily seen by applying J to the first inequality in (1). For $J = \|\cdot\|_\infty$, (2) shows that $I_g(f) > 0$ whenever $f \in \mathscr{P}(G)$. It also follows that if $J(f) > 0$ for some $f \in C_c^+(G)$, then $J(g) > 0$ for all $g \in \mathscr{P}(G)$. This proves in particular the last assertion of the theorem.

We note further that *for every $f \in C_c^+(G)$ and every constant $r > 1$ there exist $\bar f \in C_c^+(G)$ and $U \in \mathscr{U}$ such that $\bar f > f$ on* $\mathrm{spt}(f)$*, $f(x) \le \bar f(y)$ whenever $x, y \in G$ and $x^{-1}y \in U$ or $y^{-1}x \in U$, and*

$$J(\bar f) \le rJ(f) \quad \textit{for all } J \in \mathscr{F}. \tag{3}$$

Suppose that $f \not\equiv 0$, and choose ϱ as in Lemma 2. Then let $\varepsilon > 0$ be such that $1 + \varepsilon I_f(\varrho) \le r$, and put $\bar f := f + \varepsilon\varrho$. Now the result follows from the lemma and the fact that $J(\bar f) \le J(f) + \varepsilon J(\varrho) \le (1 + \varepsilon I_f(\varrho))J(f)$ by (2). Note also that if $f \in \mathscr{P}_*(G)$, then by choosing $\varrho \in \mathscr{P}_*(G)$ one can arrange that $\bar f \in \mathscr{P}_*(G)$.

Part II. Next we show that *for every finite collection of functions $f_1, \dots, f_n \in C_c^+(G)$ and every $r > 1$ there is a neighborhood $V \in \mathscr{U}$ such that*

$$I_g(f_1) + \cdots + I_g(f_n) \le rI_g(f_1 + \cdots + f_n) \quad \textit{for all } g \in \mathscr{P}(V). \tag{4}$$

Put $f := f_1 + \cdots + f_n$. By (3) there is an $\bar f \in C_c^+(G)$ such that $\bar f > f$ on $\mathrm{spt}(f)$ and $J(\bar f) \le r^{1/2}J(f)$ for all $J \in \mathscr{F}$. Then there exist functions $\varrho_1, \dots, \varrho_n \in C_c^+(G)$ so that $f_i = \varrho_i \bar f$ and $\mathrm{spt}(\varrho_i) = \mathrm{spt}(f_i)$. Let $\varepsilon > 0$ be such that $1 + n\varepsilon \le r^{1/2}$. By Lemma 2 there exists $V \in \mathscr{U}$ such that $\varrho_i(y) \le \varrho_i(a) + \varepsilon$ whenever $a^{-1}y \in V$, for $i = 1, \dots, n$. Now let $g \in \mathscr{P}(V)$, and let s, A, c_a be such that the inequalities (1) hold with $\bar f$ in place of f. Then $(g \circ L_a^{-1})(y) > 0$ implies that $\varrho_i(y) \le \varrho_i(a) + \varepsilon$, so

$$f_i = \varrho_i \bar f \le \sum_{a \in A} c_a \varrho_i g \circ L_a^{-1} \le \sum_{a \in A} c_a(\varrho_i(a) + \varepsilon) g \circ L_a^{-1}$$

and therefore $I_g(f_i) \le \sum_a c_a(\varrho_i(a) + \varepsilon)$. Since $\sum_i \varrho_i \bar f = f \le \bar f$, we have $\sum_i \varrho_i \le 1$, thus it follows that

$$\sum_{i=1}^n I_g(f_i) \le (1 + n\varepsilon) \sum_{a \in A} c_a \le r^{1/2}s.$$

Taking the infimum over all such s we conclude that $\sum_i I_g(f_i) \le r^{1/2} I_g(\bar f)$. Since $I_g(\bar f) \le r^{1/2} I_g(f)$, this yields (4).

Part III. Next we establish a counterpart to (2): *Given $f \in C_c^+(G)$ and $r > 1$, there exists $U \in \mathscr{U}$ such that for every $g \in \mathscr{P}_*(U)$ there exists $W \in \mathscr{U}$ such that*

$$I_g(f)J(g) \le rJ(f) \quad \textit{for all } J \in \mathscr{L} \cup \{I_h : h \in \mathscr{P}(W)\}. \tag{5}$$

By (3) there exist $\bar{f} \in C_c^+(G)$ and $U \in \mathscr{U}$ such that $f(x) \leq \bar{f}(y)$ whenever $x^{-1}y \in U$, and $J(\bar{f}) \leq r^{1/3}J(f)$ whenever $J \in \mathscr{F}$. Let now $g \in \mathscr{P}_*(U)$, as in the assertion. Then, again by (3), there exist $\bar{g} \in \mathscr{P}_*(G)$ and $V \in \mathscr{U}$ such that $g(x^{-1}y) \leq \bar{g}(x^{-1}a)$ for $(x^{-1}a)^{-1}(x^{-1}y) = a^{-1}y \in V$, and $J(\bar{g}) \leq r^{1/3}J(g)$ for $J \in \mathscr{F}$. Since $\mathrm{spt}(\bar{f})$ is compact, there exists a finite set $A \subset G$ such that the collection of open sets aV with $a \in A$ covers $\mathrm{spt}(\bar{f})$. Then $\bar{f}$ can be decomposed by means of a partition of unity subordinate to this covering (cf. [5, 2.13]), thus $\bar{f} = \sum_{a\in A} \bar{f}_a$ for some functions $\bar{f}_a \in C_c^+(G)$ with $\mathrm{spt}(\bar{f}_a) \subset aV$. Now, for all $x, y \in G$,

$$f(x)g(x^{-1}y) \leq \sum_{a\in A} \bar{f}_a(y)g(x^{-1}y) \leq \sum_{a\in A} \bar{f}_a(y)\bar{g}(a^{-1}x); \tag{6}$$

the first inequality holds since $g(x^{-1}y) > 0$ implies that $f(x) \leq \bar{f}(y) = \sum_a \bar{f}_a(y)$, the second since $\bar{f}_a(y) > 0$ implies that $g(x^{-1}y) \leq \bar{g}(x^{-1}a) = \bar{g}(a^{-1}x)$. By Part II there exists $W \in \mathscr{U}$ such that $\sum_a I_h(\bar{f}_a) \leq r^{1/3}I_h(\bar{f})$ for all $h \in \mathscr{P}(W)$. Let now $J \in \mathscr{L} \cup \{I_h : h \in \mathscr{P}(W)\}$. Fix x for the moment and apply J to the functions of y on the left and right of (6). This yields

$$f(x)J(g) \leq \sum_{a\in A} J(\bar{f}_a)\bar{g}(a^{-1}x). \tag{7}$$

Then, applying I_g to the functions of x on either side of (7), and noting that $I_g(\bar{g}) \leq r^{1/3}I_g(g) \leq r^{1/3}$, we obtain $I_g(f)J(g) \leq r^{1/3} \sum_a J(\bar{f}_a)$. This sum is equal to $J(\bar{f})$ if J is additive and less than or equal to $r^{1/3}J(\bar{f})$ if $J = I_h$ with $h \in \mathscr{P}(W)$. Since $J(\bar{f}) \leq r^{1/3}J(f)$, this gives the result.

Part IV. Now we fix once and for all a reference function $\phi \in \mathscr{P}(G)$. Normalizing the functionals I_g, we note that, by (2),

$$\Lambda_g := \frac{1}{I_g(\phi)} I_g \leq I_\phi \quad \text{for all } g \in \mathscr{P}(G). \tag{8}$$

Let $f \in C_c^+(G)$ and $r > 1$. By (2) and Part III, there exists a neighborhood $U_r(f) \in \mathscr{U}$ such that for each $g \in \mathscr{P}_*(U_r(f))$ there exists $W \in \mathscr{U}$ such that the inequalities $J(f) \leq I_g(f)J(g) \leq rJ(f)$ and $J(\phi) \leq I_g(\phi)J(g) \leq rJ(\phi)$ hold simultaneously for all $J \in \mathscr{L} \cup \{I_h : h \in \mathscr{P}(W)\}$. Then it follows that

$$r^{-1}\Lambda_g(f) \leq \frac{J(f)}{J(\phi)} \leq r\Lambda_g(f) \quad \text{for all } J \in \mathscr{L} \cup \{I_h : h \in \mathscr{P}(W)\}. \tag{9}$$

From (8) and (9) we conclude that for any two left Haar integrals Λ, Λ' on G the quotients $\Lambda(f)/\Lambda(\phi)$ and $\Lambda'(f)/\Lambda'(\phi)$ agree. Thus Λ and Λ' are constant multiples of each other on $C_c^+(G)$ and hence also on $C_c(G)$. This proves the uniqueness assertion of Theorem 1.

Part V. Finally, we construct a left Haar integral on G. Let again $f \in C_c^+(G)$ and $r > 1$. Denote by $\mathscr{G}_r(f)$ the set of all $g \in \mathscr{P}(G)$ for which there exists a neighborhood $W \in \mathscr{U}$ such that

$$\Lambda_g(f) \leq r\Lambda_h(f) \quad \text{for all } h \in \mathscr{P}(W). \tag{10}$$

From the first inequality in (9) we know that $\mathcal{G}_r(f)$ contains $\mathcal{P}_*(U_r(f))$. Clearly $\mathcal{G}_r(f \circ L_a) = \mathcal{G}_r(f) = \mathcal{G}_r(\lambda f)$ for all $a \in G$ and $\lambda > 0$. Now put

$$\bar{\Lambda}_r(f) := \sup\{\Lambda_g(f) : g \in \mathcal{G}_r(f)\},$$

and note that this is finite by (8). The functional $\bar{\Lambda}_r \colon C_c^+(G) \to [0, \infty)$ is left-invariant and homogeneous. We claim that if $f, f' \in C_c^+(G)$, then

$$r^{-1}\bar{\Lambda}_r(f + f') \le \bar{\Lambda}_r(f) + \bar{\Lambda}_r(f') \le r^2\bar{\Lambda}_r(f + f'). \tag{11}$$

For every $g \in \mathcal{G}_r(f+f')$ there exists $W \in \mathcal{U}$ such that if $h \in \mathcal{P}(W)\cap\mathcal{G}_r(f)\cap\mathcal{G}_r(f')$, then $r^{-1}\Lambda_g(f + f') \le \Lambda_h(f + f') \le \Lambda_h(f) + \Lambda_h(f') \le \bar{\Lambda}_r(f) + \bar{\Lambda}_r(f')$. Hence the first inequality holds. Conversely, given $g \in \mathcal{G}_r(f)$ and $g' \in \mathcal{G}_r(f')$, by the definition of $\mathcal{G}_r(f)$ and $\mathcal{G}_r(f')$ and by Part II there exists $W \in \mathcal{U}$ such that if $h \in \mathcal{P}(W)\cap\mathcal{G}_r(f+f')$, then $\Lambda_g(f)+\Lambda_{g'}(f') \le r(\Lambda_h(f)+\Lambda_h(f')) \le r^2\Lambda_h(f+f') \le r^2\bar{\Lambda}_r(f + f')$. This yields the second inequality in (11).

Finally, if $1 < r' < r$, then clearly $\mathcal{G}_{r'}(f) \subset \mathcal{G}_r(f)$ and hence $\bar{\Lambda}_{r'}(f) \le \bar{\Lambda}_r(f)$. Thus the limit

$$\Lambda(f) := \lim_{r\to 1+} \bar{\Lambda}_r(f)$$

exists. The resulting functional $\Lambda \colon C_c^+(G) \to [0, \infty)$ is left-invariant and homogeneous. By virtue of (11), it is also additive, thus Λ belongs to $\mathcal{L}$ and extends to a left-invariant, positive linear functional on $C_c(G)$. This functional is non-trivial, in fact $\Lambda(\phi) = 1$, as $\Lambda_g(\phi) = 1$ for all $g \in \mathcal{P}(G)$.

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