Local currents in metric spaces

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In memoriam Juha Heinonen, 1960–2007

Abstract

Ambrosio and Kirchheim presented a theory of currents with finite mass in complete metric spaces. We develop a variant of the theory that does not rely on a finite mass condition, closely paralleling the classical Federer–Fleming theory. If the underlying metric space is an open subset of a Euclidean space, we obtain a natural chain monomorphism from general metric currents to general classical currents whose image contains the locally flat chains and which restricts to an isomorphism for locally normal currents. We give a detailed exposition of the slicing theory for locally normal currents with respect to locally Lipschitz maps, including the rectifiable slices theorem, and of the compactness theorem for locally integral currents in locally compact metric spaces, assuming only standard results from analysis and measure theory.

Contents

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Introduction

Currents in the sense of geometric measure theory, linear functionals on spaces of differential forms, were introduced by G. de Rham in 1955 for use in the theory of harmonic forms [12]. A few years later, H. Federer and W. H. Fleming devised the class of rectifiable currents, generalized oriented surfaces with integer multiplicities, and the subclass of integral currents, whose boundary is of the same type. Their fundamental paper [15] from 1960 also furnished the compactness theorem for integral currents and thereby a solution to the Plateau problem for surfaces of arbitrary dimension and codimension in Euclidean spaces. The theory of currents then rapidly developed into a powerful apparatus in the calculus of variations. Federer's monograph [14] gives a comprehensive account of the state of the subject prior to 1970. Since then, the theory has been extended in various directions and has found numerous applications in geometric analysis and Riemannian geometry, far beyond pure area minimization problems.

A breakthrough was achieved in 2000, when L. Ambrosio and B. Kirchheim [3], following ideas of E. De Giorgi [7], presented a theory of currents in complete metric spaces. This elegant approach employs $(m+1)$-tuples of real-valued Lipschitz functions in place of differential m-forms and provides some new insights even if the ambient space is Euclidean. Ambrosio and Kirchheim discovered new proofs of the boundary rectifiability theorem and the closure theorem for rectifiable currents, valid in any complete metric space. As an application, they obtained existence results for generalized Plateau problems in compact metric spaces and certain Banach spaces. The theory of metric currents has been further developed in [8], [32], [34], and some geometric applications have been found [33], [35]. Various interactions with other areas have emerged and should be further explored. In this context we just mention the functions of bounded higher variation [21], [8], [9], [28], [25], the recent progress on (generalized) flat chains [37], [38], [18], [1], the scans from [16], [10], [11], the differentiation theory on metric measure spaces [4], [22], [5], and the derivations from [30], [31]. We refer to [19] for an excellent survey of some of these and further related topics.

The metric currents considered by Ambrosio and Kirchheim have finite mass by definition. This a priori assumption plays a crucial role in their development of the theory, in particular it is used to derive the properties that qualify the functionals under consideration as analogues of classical currents. Here we present a somewhat different approach, strongly inspired by the work of Ambrosio and Kirchheim, but not relying on a finite mass condition. In addition, we switch to local formulations. At a later stage, this enables us to discuss metric currents with locally finite mass and locally rectifiable currents. For this purpose it is appropriate to assume the underlying metric space to be locally compact. However, once some basic properties are established, the theory readily extends to a theory of currents with locally compact support in arbitrary metric spaces. Furthermore, in the case of finite mass, it is also possible to dispense with the restriction on the support and to incorporate the class of Ambrosio–Kirchheim currents. In the second half of the paper, which discusses currents with locally finite mass, a number of results are local versions of those in [3]. However, in some instances, we give different arguments, using less prerequisites. A central aim was to provide a detailed account of the fundamentals of the theory, including a complete proof of the compactness theorem for locally integral currents in locally compact metric spaces, in a style readily accessible to non-specialists.

We now describe the contents of the paper in a more systematic way. Detailed references will be given later in the individual sections.

In Sect. 1 we fix the notation and gather a few basic facts from analysis and measure theory.

In Sect. 2 we turn to currents. A classical m-dimensional current in an open set $U \subset \mathbb{R}^n$ is a real-valued linear function on the space of compactly supported differential m-forms on U, continuous with respect to convergence of forms in a suitable C^∞-topology. Given a locally compact metric space X, we substitute differential m-forms by $(m+1)$-tuples $(f, \pi_1, \dots, \pi_m)$ of real-valued functions on X, where f is Lipschitz with compact support $\mathrm{spt}(f)$ and $\pi_1, \dots, \pi_m$ are locally Lipschitz. We denote by $\mathscr{D}^m(X)$ the set of all such tuples. The guiding principle is that if $X = U$ is an open subset of $\mathbb{R}^n$ and if $(f, \pi_1, \dots, \pi_m) \in C^\infty_{\mathrm{c}}(U) \times [C^\infty(U)]^m$, then this tuple represents the form $f\, d\pi_1 \wedge \ldots \wedge d\pi_m$. An m-dimensional metric current T in X is defined as an $(m+1)$-linear real-valued function on $\mathscr{D}^m(X)$, continuous with respect to convergence of tuples in a suitable topology involving locally uniform bounds on Lipschitz constants, and satisfying $T(f, \pi_1, \dots, \pi_m) = 0$ whenever some π_i is constant on a neighborhood of $\mathrm{spt}(f)$. The vector space of m-dimensional metric currents in X is denoted by $\mathscr{D}_m(X)$. The terminology is justified by the fact that the defining conditions of a metric current give rise to a set of further properties, corresponding to the usual rules of calculus for differential forms. Namely, every $T \in \mathscr{D}_m(X)$ is alternating in the m last arguments, and the following product rule holds: *If $(f, \pi_1, \dots, \pi_m) \in \mathscr{D}^m(X)$, and if $g\colon X \to \mathbb{R}$ is locally Lipschitz, then*

$$T(f, g\pi_1, \pi_2, \dots, \pi_m) = T(fg, \pi_1, \dots, \pi_m) + T(f\pi_1, g, \pi_2, \dots, \pi_m).$$

We also obtain a chain rule, a special case of which states that *if $(f, \pi) = (f, \pi_1, \dots, \pi_m) \in \mathscr{D}^m(X)$ and $g = (g_1, \dots, g_m) \in [C^{1,1}(\mathbb{R}^m)]^m$, i.e., the partial derivatives of g_i are locally Lipschitz, then*

$$T(f, g_1 \circ \pi, \dots, g_m \circ \pi) = T(f \det((Dg) \circ \pi), \pi_1, \dots, \pi_m).$$

Every function $u \in L^1_{\mathrm{loc}}(U)$ on an open set $U \subset \mathbb{R}^m$ induces a metric current $[u] \in \mathscr{D}_m(U)$ satisfying

$$[u](f, g) = \int_U uf \det(Dg)\, dx$$

for all $(f, g) = (f, g_1, \ldots, g_m) \in \mathscr{D}^m(U)$. This corresponds to the integration of a simple m-form over U. The chain rule plays a crucial role in the development of the theory. In particular, it is used to show (in Sect. 5) that *for every open set $U \subset \mathbb{R}^n$ and every m, there is an injective linear map C_m from $\mathscr{D}_m(U)$ into the space of general classical m-currents in U such that*

$$C_m(T)(f dg_1 \wedge \ldots \wedge dg_m) = T(f, g_1, \ldots, g_m)$$

for all $(f, g_1, \ldots, g_m) \in C_c^\infty(U) \times [C^\infty(U)]^m$. This makes the aforesaid guiding principle rigorous. Some more properties of these comparison maps C_m are mentioned further below.

In Sect. 3 we define the support $\mathrm{spt}(T) \subset X$ of a metric current T and discuss the boundary and push-forward operators. For a classical m-current $\bar{T}$, the boundary $\partial\bar{T}$ is the $(m-1)$-current satisfying $\partial\bar{T}(\phi) = \bar{T}(d\phi)$ for every $(m-1)$-form ϕ. Correspondingly, the boundary $\partial T \in \mathscr{D}_{m-1}(X)$ of a metric current $T \in \mathscr{D}_m(X)$ verifies

$$\partial T(f, \pi_1, \ldots, \pi_{m-1}) = T(\sigma, f, \pi_1, \ldots, \pi_{m-1})$$

for all $(f, \pi_1, \ldots, \pi_{m-1}) \in \mathscr{D}^{m-1}(X)$ and for all σ such that $\sigma = 1$ on some neighborhood of $\mathrm{spt}(f)$. We have $\partial \circ \partial = 0$, and, in case $X = U$ is an open set in $\mathbb{R}^n$, $\partial \circ C_m = C_{m-1} \circ \partial$, so that the C_m form a chain map. The push-forward $F_\# \bar{T}$ of a general classical current $\bar{T}$ is defined for every smooth map F whose restriction to the support of $\bar{T}$ is proper, and considerable efforts are required to extend the definition, for particular classes of currents, to locally Lipschitz maps. Given a metric current $T \in \mathscr{D}_m(X)$ and another locally compact metric space Y, the push-forward $F_\# T \in \mathscr{D}_m(Y)$ is defined for every locally Lipschitz map $F \colon D \to Y$ such that $\mathrm{spt}(T) \subset D \subset X$ and $F|_{\mathrm{spt}(T)}$ is proper. In case $D = X$, we have

$$F_\# T(f, \pi_1, \ldots, \pi_m) = T(\tilde{f}, \pi_1 \circ F, \ldots, \pi_m \circ F)$$

whenever $(f, \pi_1, \ldots, \pi_m) \in \mathscr{D}^m(Y)$ and $\tilde{f} \colon X \to \mathbb{R}$ is a compactly supported Lipschitz function that agrees with $f \circ F$ on $\mathrm{spt}(T)$.

In Sect. 4 we discuss the notion of mass. Given a metric current $T \in \mathscr{D}_m(X)$, we define its mass $\mathbf{M}_V(T)$ in an open set $V \subset X$ as the least number $M \in [0, \infty]$ such that

$$\sum_{\lambda \in \Lambda} T(f_\lambda, \pi^\lambda) \le M$$

whenever Λ is a finite set, $(f_\lambda, \pi^\lambda) \in \mathscr{D}^m(X)$, $\pi_1^\lambda, \ldots, \pi_m^\lambda$ are 1-Lipschitz, $\mathrm{spt}(f_\lambda) \subset V$, and $\sum_{\lambda \in \Lambda} |f_\lambda| \le 1$. If a metric current $T \in \mathscr{D}_m(X)$ has locally finite mass, then there is an associated Radon measure $\|T\|$ on X, characterized by $\|T\|(V) = \mathbf{M}_V(T)$ for all open sets $V \subset X$, and

$$T(f, \pi) \le \int_X |f| \, d\|T\|$$

whenever $(f, \pi) \in \mathscr{D}^m(X)$ *and the restrictions of* $\pi_1, \dots, \pi_m$ *to* $\operatorname{spt}(f)$ *are* 1*-Lipschitz.* This last inequality allows to extend T to all tuples (f, π) such that f is a bounded Borel function with compact support and $\pi_1, \dots, \pi_m$ are still locally Lipschitz. For a Borel set $B \subset X$, the restriction $T \lfloor B$ is then defined as the m-current satisfying

$$(T \lfloor B)(f, \pi) = T(\chi_B f, \pi)$$

for all $(f, \pi) \in \mathscr{D}^m(X)$, where χ_B is the characteristic function of B. For a locally bounded Borel function $g \colon X \to \mathbb{R}$, $T \lfloor g$ is defined similarly.

A metric m-current T is called (locally) normal if both the mass of T and the mass of ∂T are (locally) finite. A fundamental result about locally normal metric currents, proved in Sect. 5, is the compactness theorem: *If* $T_1, T_2, \ldots \in \mathscr{D}_m(X)$ *are currents with separable support, and if*

$$\sup_n (\mathbf{M}_V(T_n) + \mathbf{M}_V(\partial T_n)) < \infty$$

for every open set $V \subset X$ *with compact closure, then there is a subsequence* $T_{n(1)}, T_{n(2)}, \ldots$ *that converges weakly to some* $T \in \mathscr{D}_m(X)$*, i.e.,* $\lim_{i \to \infty} T_{n(i)}(f, \pi) = T(f, \pi)$ *for every* $(f, \pi) \in \mathscr{D}^m(X)$*.* Since $\mathbf{M}_V$ is lower semicontinuous with respect to weak convergence, and also $\partial T_{n(i)} \to \partial T$ weakly, the limit T is locally normal. In case $X = U$ is an open set in $\mathbb{R}^n$, the restriction of the comparison map C_m to the vector space of locally normal currents is an isomorphism onto the space of classical locally normal currents.

An important technique in the theory of currents consists in relating information on the structure of a current T to properties of lower dimensional slices of T in the level sets of a function. In Sect. 6 we discuss slicing of a locally normal current $T \in \mathscr{D}_m(X)$ with respect to a locally Lipschitz map $\pi \colon X \to \mathbb{R}^k$, where $1 \le k \le m$. Let $T_\pi \in \mathscr{D}_{m-k}(X)$ be the locally normal current satisfying

$$T_\pi(f, g_1, \ldots, g_{m-k}) = T(f, \pi_1, \ldots, \pi_k, g_1, \ldots, g_{m-k})$$

for all $(f, g) \in \mathscr{D}^{m-k}(X)$. *If* $\operatorname{spt}(T)$ *is separable, then for almost every* $y \in \mathbb{R}^k$ *there is a locally normal current in* $\mathscr{D}_{m-k}(X)$ *with support in* $\pi^{-1}\{y\} \cap \operatorname{spt}(T)$*, denoted by* $\langle T, \pi, y \rangle$*, such that*

$$\int_{\mathbb{R}^k} \langle T, \pi, y \rangle (f, g) \, dy = T_\pi(f, g)$$

for all $(f, g) \in \mathscr{D}^{m-k}(X)$*. Moreover, for every Borel set* $B \subset X$*,*

$$\int_{\mathbb{R}^k} \| \langle T, \pi, y \rangle \| (B) \, dy = \| T_\pi \| (B).$$

The first identity also holds more generally if f is a bounded Borel function with compact support.

Slicing is particulary important and useful when $k = 1$, for geometric applications, or when $k = m$. In the latter case, the slices are 0-dimensional, and π maps any compactly supported portion of T to a current of maximal dimension in $\mathbb{R}^m$. This situation is closely inspected in Sect. 7. The slicing theorem leads to a fundamental identity: *If $f \colon X \to \mathbb{R}$ is any bounded Borel function with compact support, then $\pi_\#(T \lfloor f) \in \mathscr{D}_m(\mathbb{R}^m)$ is a standard current $[u_f]$, for some $u_f \in L^1(\mathbb{R}^m)$, and*

$$\langle T, \pi, y\rangle(f) = u_f(y)$$

for almost every $y \in \mathbb{R}^m$. Moreover, if f is Lipschitz, then $[u_f]$ is normal, and u_f is a function of bounded variation. Such functions satisfy a Lipschitz condition outside a set of small Lebesgue measure. Exploiting the resulting Lipschitz property of the function $y \mapsto \langle T, \pi, y\rangle(f)$, we obtain a partial rectifiability result for every locally normal current T with separable support: *Let $A \subset \operatorname{spt}(T)$ be the set of all x such that $\{x\}$ is an atom of the corresponding slice $\langle T, \pi, \pi(x)\rangle$. Up to a set of $\|T_\pi\|$ measure zero, A can be represented as the union of countable many pairwise disjoint compact sets $B_i \subset A$ such that $\pi|_{B_i}$ is a bi-Lipschitz map into $\mathbb{R}^m$.*

In the final Sect. 8 we turn to rectifiable currents. For a general current $T \in \mathscr{D}_m(X)$ with locally finite mass, the measure $\|T\|$ may be diffused, so that T does not correspond to a generalized m-dimensional surface in any sense. We call T a locally integer rectifiable current if $\|T\|$ is concentrated on some countably m-rectifiable set E, i.e., the union of countably many Lipschitz images of subsets of $\mathbb{R}^m$, and if T satisfies the following integrality condition: Whenever $B \subset X$ is a Borel set with compact closure and $\pi \colon X \to \mathbb{R}^m$ is Lipschitz, then $\pi_\#(T \lfloor B) = [u_{B,\pi}]$ for some $u_{B,\pi} \in L^1(\mathbb{R}^m, \mathbb{Z})$. From these conditions it follows that the support of T is separable and that $\|T\|$ is absolutely continuous with respect to m-dimensional Hausdorff measure $\mathscr{H}^m$. The slicing theory is then supplemented with the rectifiable slices theorem, relying on the above partial rectifiability result and the identity $\langle T, \pi, y\rangle(\chi_B) = u_{B,\pi}(y)$: *Given a locally normal current $T \in \mathscr{D}_m(X)$ with separable support and $k \in \{1, \dots, m\}$, T is locally integer rectifiable if and only if for each Lipschitz map $\pi \colon X \to \mathbb{R}^k$, the $(m-k)$-dimensional slice $\langle T, \pi, y\rangle$ is locally integer rectifiable for almost every $y \in \mathbb{R}^k$.* We call $T \in \mathscr{D}_m(X)$ a locally integral current if both T and ∂T are locally integer rectifiable. In particular, every such T is locally normal. By the boundary rectifiability theorem, *every locally integer rectifiable and locally normal current is locally integral.* This follows easily from the rectifiable slices theorem, by induction on m. The closure theorem for locally integral currents states that *if $T_1, T_2, \ldots \in \mathscr{D}_m(X)$ is a sequence of locally integral currents converging weakly to $T \in \mathscr{D}_m(X)$, and if*

$$\sup_n (\mathbf{M}_V(T_n) + \mathbf{M}_V(\partial T_n)) < \infty$$

for every open set $V \subset X$ with compact closure, then T is locally integral. The proof is another simple inductive application of the rectifiable slices theorem.

By combining this result with the compactness theorem for locally normal currents mentioned earlier, we obtain the compactness theorem for locally integral currents.

Further results and applications will be discussed elsewhere.

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1 Preliminaries

We now fix the notation and collect some basic facts from analysis and measure theory. A few more prerequisites will be discussed in individual sections.

Given a point x in a metric space $X = (X, d)$, $\mathrm{B}(x, r) := \{y : d(x, y) \le r\}$ and $\mathrm{U}(x, r) := \{y : d(x, y) < r\}$ denote the closed and open ball, respectively, with center x and radius r.

1.1 Lipschitz maps

Let $X = (X, d)$ and $Y = (Y, d)$ be two metric spaces. For $l \in [0, \infty)$, a map $f \colon X \to Y$ is *l-Lipschitz* if

$$d(f(x), f(x')) \le l\, d(x, x')$$

for all $x, x' \in X$. The map $f \colon X \to Y$ is *Lipschitz* if the *Lipschitz constant*

$$\mathrm{Lip}(f) := \inf\{l \in [0, \infty) : f \text{ is } l\text{-Lipschitz}\}$$

is finite, and f is *locally Lipschitz* if every point in X has a neighborhood such that the restriction of f to this neighborhood is Lipschitz. Note that then $f|_K$ is Lipschitz for every compact set $K \subset X$. We let $\mathrm{Lip}_{\mathrm{loc}}(X, Y)$ be the set of locally Lipschitz maps from X into Y, $\mathrm{Lip}(X, Y)$ and $\mathrm{Lip}_l(X, Y)$ the subsets of Lipschitz maps and l-Lipschitz maps, respectively. In this notation, we abbreviate $(X, \mathbb{R})$ by (X). Note that $\mathrm{Lip}_{\mathrm{loc}}(X)$ forms an algebra, while $\mathrm{Lip}(X)$ is an algebra if and only if X is bounded. In fact, if $f, g \in \mathrm{Lip}(X)$ are two bounded Lipschitz functions on a metric space X, then the product fg is Lipschitz with Lipschitz constant

$$\mathrm{Lip}(fg) \le \|f\|_\infty \mathrm{Lip}(g) + \|g\|_\infty \mathrm{Lip}(f), \tag{1.1}$$

where $\|\cdot\|_\infty$ is the supremum norm.

A map $f\colon X \to Y$ is *bi-Lipschitz* if there is a constant $b \in [1, \infty)$ such that

$$b^{-1}\, d(x, x') \le d(f(x), f(x')) \le b\, d(x, x')$$

for all $x, x' \in X$.

If $A \subset X$ and $f \in \mathrm{Lip}_l(A)$, then there exists an extension $\bar{f} \in \mathrm{Lip}_l(X)$, i.e., $\bar{f}|_A = f$. In fact, one may simply define

$$\bar{f}(x) := \inf_{a \in A} (f(a) + l\, d(a, x)) \tag{1.2}$$

for $x \in X$. By applying this result to each component of a map $f = (f_1, \dots, f_m) \in \mathrm{Lip}_l(A, \mathbb{R}^m)$, one obtains an extension $\bar{f} \in \mathrm{Lip}_{\sqrt{m}l}(X, \mathbb{R}^m)$ of f.

Every uniformly continuous and bounded function $f\colon X \to \mathbb{R}$ is a uniform limit of a sequence of Lipschitz functions (see e.g. [17, Theorem 6.8]).

By Rademacher's theorem, every $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^m, \mathbb{R}^n)$ is differentiable at $\mathscr{L}^m$-almost all points of $\mathbb{R}^m$, where $\mathscr{L}^m$ denotes (outer) Lebesgue measure.

1.2 Borel functions and Baire functions

Given a topological space X, $\mathscr{B}^\infty_{\mathrm{loc}}(X)$ denotes the algebra of real-valued, locally bounded Borel functions on X, $\mathscr{B}^\infty(X)$ the subalgebra of bounded functions, and $\mathscr{B}^\infty_{\mathrm{c}}(X)$ the subalgebra of bounded and compactly supported functions.

A class Φ of real-valued functions on a set X is called a *Baire class* if the following holds: Whenever $f_1, f_2, \ldots \in \Phi$ and $f_i(x) \to g(x) \in \mathbb{R}$ for each $x \in X$, then $g \in \Phi$. In case X is a topological space, $f\colon X \to \mathbb{R}$ is a *Baire function* if it belongs to the smallest Baire class containing all continuous functions. Since the Borel functions form a Baire class, every Baire function is a Borel function. Conversely, if X is a metric space, then it is not difficult to see that characteristic functions of Borel sets are Baire functions and, hence, every Borel function is also a Baire function (cf. [14, 2.2.15]).

1.3 Outer measures and Radon measures

An *outer measure* ν on a set X is a set function $\nu\colon 2^X \to [0, \infty]$ such that $\nu(\emptyset) = 0$ and $\nu(A) \le \sum_{k=1}^\infty \nu(A_k)$ whenever $A \subset \bigcup_{k=1}^\infty A_k$. A set $A \subset X$ is called *ν-measurable* if $\nu(B) = \nu(B \cap A) + \nu(B \setminus A)$ for all $B \subset X$. The family of all ν-measurable sets forms a σ-algebra, and the restriction of ν to this σ-algebra is a measure. If $E \subset X$ is any set, then $\nu \llcorner E$ is the outer measure satisfying

$$(\nu \llcorner E)(A) = \nu(E \cap A)$$

for all $A \subset X$. Every ν-measurable set is also $(\nu \llcorner E)$-measurable. We say that ν is *concentrated* on E if $\nu(X \setminus E) = 0$ or, equivalently, $\nu = \nu \llcorner E$.

Let ν be an outer measure on a topological space X. The *support* $\mathrm{spt}(\nu)$ of ν in X is the closed set of all $x \in X$ such that $\nu(U) > 0$ for every neighborhood U of x. We call ν *Borel regular* if all Borel sets are ν-measurable and if every set $A \subset X$ is contained in a Borel set B with $\nu(B) = \nu(A)$. If ν is Borel regular and E is a Borel set, then $\nu \lfloor E$ is Borel regular.

Now let ν be an outer measure on a metric space X. Carathéodory's criterion says that if $\nu(A)+\nu(B) = \nu(A \cup B)$ for all $A, B \subset X$ with $\inf\{d(x,y) : x \in A,\ y \in B\} > 0$, then all Borel sets are ν-measurable. If ν is Borel regular and B is a ν-measurable set contained in the union of countably many open sets U_i with $\nu(U_i) < \infty$, and if $\epsilon > 0$, then there is an open set V such that $B \subset V$ and $\nu(V \setminus B) < \epsilon$.

An outer measure ν on a locally compact Hausdorff space X is called a *Radon measure* if Borel sets are ν-measurable, ν is finite on compact sets,

$$\nu(V) = \sup\{\nu(K) : K \subset X \text{ is compact, } K \subset V\}$$

for every open set $V \subset X$, and

$$\nu(A) = \inf\{\nu(V) : V \subset X \text{ is open, } A \subset V\}$$

for every set $A \subset X$. Then it is also true that if B is a ν-measurable set with $\nu(B) < \infty$, and if $\epsilon > 0$, then $\nu(B \setminus K) < \epsilon$ for some compact set $K \subset B$ (cf. [14, 2.2.5]).

For $m \in \mathbb{N}$, we denote by

$$\alpha_m := \mathscr{L}^m(\mathrm{B}(0,1))$$

the Lebesgue measure of the unit ball in $\mathbb{R}^m$, and we put $\alpha_0 := 1$. Given a metric space X, the *m-dimensional Hausdorff measure* of a set $A \subset X$ is defined by $\mathscr{H}^m(A) := \lim_{\delta \to 0+} \mathscr{H}^m_\delta(A)$, where $\mathscr{H}^m_\delta(A)$ is the infimum of $\sum_{C \in \mathscr{C}} \alpha_m(\mathrm{diam}(C)/2)^m$ over all countable coverings $\mathscr{C}$ of A with $\mathrm{diam}(C) := \sup\{d(x,y) : x, y \in C\} \le \delta$ for all $C \in \mathscr{C}$. For every m, $\mathscr{H}^m$ is a Borel regular outer measure on X. With the chosen normalization, $\mathscr{H}^m = \mathscr{L}^m$ on $\mathbb{R}^m$.

1.4 Maximal functions and Lebesgue points

Suppose that μ is a finite Borel measure on $\mathbb{R}^m$, i.e., a σ-additive function $\mu \colon \mathscr{B}_X \to [0,\infty)$, where $\mathscr{B}_X$ is the σ-algebra of Borel sets in X. We denote by $M_\mu \colon \mathbb{R}^m \to [0,\infty]$ the *maximal function* of μ, i.e.,

$$M_\mu(x) := \sup_{r>0} \frac{\mu(\mathrm{B}(x,r))}{\alpha_m r^m}.$$

A simple covering argument shows that

$$\mathscr{L}^m(\{x \in \mathbb{R}^m : M_\mu(x) > s\}) \le 3^m s^{-1} \mu(\mathbb{R}^m) \tag{1.3}$$

for all $s > 0$. In particular, $M_\mu(x) < \infty$ for $\mathscr{L}^m$-almost every $x \in \mathbb{R}^m$. Note also that M_μ is lower semicontinuous on $\mathbb{R}^m$.

We further recall that, given a function $u \in L^1(\mathbb{R}^m)$, $\mathscr{L}^m$-almost every $x \in \mathbb{R}^m$ is a *Lebesgue point* of u, i.e.,

$$\lim_{r \to 0} \frac{1}{\alpha_m r^m} \int_{\mathrm{B}(x,r)} |u(y) - u(x)| \, dy = 0.$$

Moreover, if we associate to each $x \in \mathbb{R}^m$ a sequence of Borel sets $E_i(x)$ with the property that $E_i(x) \subset \mathrm{B}(x, r_i(x))$ and $\mathscr{L}^m(E_i(X)) \ge \beta(x)\alpha_m r_i(x)^m$ for some $r_i(x) \to 0$ and $\beta(x) > 0$, then

$$u(x) = \lim_{i \to \infty} \frac{1}{\mathscr{L}^m(E_i(x))} \int_{E_i(x)} u(y) \, dy \tag{1.4}$$

at every Lebesgue point x of u, hence for $\mathscr{L}^m$-almost every $x \in \mathbb{R}^m$. (See e.g. the first section of [26, Ch. 7].)

1.5 Smoothing

We shall use the following basic facts regarding smoothing. Let $\eta \in C_{\mathrm{c}}^\infty(\mathbb{R}^m)$ be a mollifier, so that $\mathrm{spt}(\eta) \subset \mathrm{U}(0,1)$, $\eta(-z) = \eta(z) \ge 0$ for all $z \in \mathbb{R}^m$, and $\int_{\mathbb{R}^m} \eta(z) \, dz = 1$. Recall that for $g \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$, the convolution defined by

$$(\eta * g)(x) := \int_{\mathbb{R}^m} \eta(z) g(x - z) \, dz = \int_{\mathbb{R}^m} \eta(x - z) g(z) \, dz$$

for $x \in \mathbb{R}^m$ satisfies $\eta * g \in C^\infty(\mathbb{R}^m)$, and $\mathrm{spt}(\eta * g) \subset \mathrm{spt}(\eta) + \mathrm{spt}(g)$. If $g \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^m)$, then the partial derivatives of $\eta * g$ are given by

$$D_k(\eta * g) = \eta * D_k g, \tag{1.5}$$

$k = 1, \dots, m$. If g is bounded, then $\|\eta * g\|_\infty \le \|g\|_\infty$. If g is Lipschitz, then $\eta * g$ is Lipschitz, with constant

$$\mathrm{Lip}(\eta * g) \le \mathrm{Lip}(g). \tag{1.6}$$

Now put $\eta_j(z) := j^m \eta(jz)$ for $z \in \mathbb{R}^m$ and $j \in \mathbb{N}$, so that $\mathrm{spt}(\eta_j) \subset \mathrm{U}(0, 1/j)$ and $\int_{\mathbb{R}^m} \eta_j(z) \, dz = 1$. As $j \to \infty$, $(\eta_j * g)(x) \to g(x)$ whenever x is a Lebesgue point of g, moreover, the convergence is locally uniform if g is continuous. (See e.g. [14, 4.1.2].)

2 Metric currents

From now on, unless otherwise stated, X will always denote a *locally compact* metric space. We write $A \Subset X$ if $A \subset X$ and the closure of A is compact.

We let $\mathscr{D}(X)$ be the algebra of all $f \in \mathrm{Lip}(X)$ whose support $\mathrm{spt}(f)$ is compact; these will serve as test functions. For every compact set $K \subset X$

and every constant $l \geq 0$ we put $\mathrm{Lip}_{K,l}(X) := \{f \in \mathrm{Lip}_l(X) : \mathrm{spt}(f) \subset K\}$, so that $\mathscr{D}(X)$ is the union of all $\mathrm{Lip}_{K,l}(X)$. Then we equip $\mathscr{D}(X)$ with a locally convex vector space topology τ with respect to which

$$f_j \to f \quad \text{in } \mathscr{D}(X) \tag{2.1}$$

if and only if all f_j belong to some fixed $\mathrm{Lip}_{K,l}(X)$ and $f_j \to f$ pointwise on X for $j \to \infty$, which implies that $f_j \to f$ uniformly on X. Explicitly, this topology τ is given as follows. Let β be the collection of all absolutely convex sets $W \subset \mathscr{D}(X)$ with the following property: For every pair (K, l) and every $f \in W \cap \mathrm{Lip}_{K,l}(X)$, there is an $\epsilon > 0$ such that $g \in W$ whenever $g \in \mathrm{Lip}_{K,l}(X)$ and $\|f - g\|_\infty < \epsilon$. (Recall that W is absolutely convex if and only if $sf + tg \in W$ for all $f, g \in W$ and $s, t \in \mathbb{R}$ with $|s| + |t| \leq 1$.) Then β is a local base of τ at 0, thus τ is the collection of all unions of sets of the form $f + W$, where $f \in \mathscr{D}(X)$ and $W \in \beta$. (See e.g. [27, p. 152] for the corresponding construction in classical distribution theory.)

Similarly, we equip $\mathrm{Lip}_{\mathrm{loc}}(X)$ with a locally convex vector space topology with respect to which

$$\pi_j \to \pi \quad \text{in } \mathrm{Lip}_{\mathrm{loc}}(X) \tag{2.2}$$

if and only if for every compact set $K \subset X$ there is a constant l_K such that $\mathrm{Lip}(\pi_j|_K) \leq l_K$ for all j and $\pi_j \to \pi$ pointwise, hence uniformly, on K for $j \to \infty$. For a fixed compact set $K \subset X$, let β_K be the collection of all absolutely convex sets $W \subset \mathrm{Lip}_{\mathrm{loc}}(X)$ with the following property: For every $l \geq 0$ and every $\pi \in W$ with $\mathrm{Lip}(\pi|_K) \leq l$, there is an $\epsilon > 0$ such that $\rho \in W$ whenever $\rho \in \mathrm{Lip}_{\mathrm{loc}}(X)$, $\mathrm{Lip}(\rho|_K) \leq l$, and $\|(\pi - \rho)|_K\|_\infty < \epsilon$. The union of all β_K forms a local base of the topology of $\mathrm{Lip}_{\mathrm{loc}}(X)$. The topologies of $\mathscr{D}(X)$ and $\mathrm{Lip}_{\mathrm{loc}}(X)$ will only be used through (2.1) and (2.2).

We define the spaces

$$\mathscr{D}^0(X) := \mathscr{D}(X), \quad \mathscr{D}^m(X) := \mathscr{D}(X) \times [\mathrm{Lip}_{\mathrm{loc}}(X)]^m \quad (m \in \mathbb{N}),$$

which will serve as substitutes for the spaces of compactly supported m-forms. The guiding principle is that

$$(f, \pi_1, \dots, \pi_m) \in \mathscr{D}^m(X) \textit{ represents } f\, d\pi_1 \wedge \ldots \wedge d\pi_m \tag{2.3}$$

if X is an open subset of $\mathbb{R}^n$ and the $f, \pi_1, \dots, \pi_m$ are smooth. This correspondence will be made precise in Theorem 5.5. The space $\mathscr{D}^m(X)$ is equipped with the product topology. Thus,

$$(f^j, \pi_1^j, \dots, \pi_m^j) \to (f, \pi_1, \dots, \pi_m) \quad \text{in } \mathscr{D}^m(X) \tag{2.4}$$

if and only if $f^j \to f$ in $\mathscr{D}(X)$ and $\pi_i^j \to \pi_i$ in $\mathrm{Lip}_{\mathrm{loc}}(X)$ for $i = 1, \dots, m$.

Definition 2.1 (metric current). *For $m \in \{0\} \cup \mathbb{N}$, an m-dimensional* metric current *T in X is a function $T \colon \mathscr{D}^m(X) \to \mathbb{R}$ satisfying the following three conditions:*

(1) (multilinearity) T *is* $(m+1)$*-linear;*

(2) (continuity) $T(f^j, \pi_1^j, \dots, \pi_m^j) \to T(f, \pi_1, \dots, \pi_m)$ *if* $(f^j, \pi_1^j, \dots, \pi_m^j) \to (f, \pi_1, \dots, \pi_m)$ *in* $\mathscr{D}^m(X)$*;*

(3) (locality) *in case* $m \geq 1$, $T(f, \pi_1, \dots, \pi_m) = 0$ *whenever some* π_i *is constant on a neighborhood of* $\operatorname{spt}(f)$*.*

The vector space of m*-dimensional metric currents in* X *is denoted by*

$$\mathscr{D}_m(X).$$

We endow $\mathscr{D}_m(X)$ *with the locally convex* weak topology *with respect to which* $T_n \to T$ *if and only if*

$$T_n(f, \pi_1, \dots, \pi_m) \to T(f, \pi_1, \dots, \pi_m)$$

for every $(f, \pi_1, \dots, \pi_m) \in \mathscr{D}^m(X)$*.*

As a first consequence of the defining conditions, we note the following strict form of the locality property:

$$T(f, \pi_1, \dots, \pi_m) = 0 \textit{ whenever some } \pi_i \textit{ is constant on } \operatorname{spt}(f). \tag{2.5}$$

To see this, let $\beta_j \colon \mathbb{R} \to \mathbb{R}$ ($j \in \mathbb{N}$) be the 1-Lipschitz function satisfying $-\beta_j(-s) = \beta_j(s) = \max\{s - (1/j), 0\}$ for $s \geq 0$. Then $\beta_j \circ f \to f$ in $\mathscr{D}(X)$, and $\operatorname{spt}(f)$ contains a neighborhood of $\operatorname{spt}(\beta_j \circ f)$ for every j, so that (2.5) follows. An alternative proof uses the continuity in the respective argument. Suppose that $(\pi_i - c)|_{\operatorname{spt}(f)} = 0$ for some $c \in \mathbb{R}$. Then $\beta_j \circ (\pi_i - c)$ vanishes on some neighborhood of $\operatorname{spt}(f)$, and $\beta_j \circ (\pi_i - c) \to \pi_i - c$ in $\operatorname{Lip}_{\text{loc}}(X)$. This implies that $T(f, \pi_1, \dots, \pi_m) = 0$.

The following simple lemma shows that the space $[\mathscr{D}(X)]^{m+1}$ would serve the same purpose as $\mathscr{D}^m(X)$. However, the guiding principle (2.3) suggests to think of $\pi_1, \dots, \pi_m$ as coordinate functions, so that the choice of $\operatorname{Lip}_{\text{loc}}(X)$ instead of $\mathscr{D}(X)$ seems more appropriate. In the proof of Theorem 2.5 (chain rule) we shall also benefit from the fact that $\operatorname{Lip}_{\text{loc}}(\mathbb{R}^n)$ comprises the real polynomials in n variables.

Lemma 2.2. *Suppose* $T \colon [\mathscr{D}(X)]^{m+1} \to \mathbb{R}$ *is a function satisfying the three conditions of Definition 2.1 with* $[\mathscr{D}(X)]^{m+1}$ *in place of* $\mathscr{D}^m(X)$*. Then* T *extends uniquely to a current* $T \in \mathscr{D}_m(X)$*.*

In particular, every metric current $T \in \mathscr{D}_m(X)$ is determined by its values on $[\mathscr{D}(X)]^{m+1}$.

Proof. In view of conditions (1) and (3), the function T can be extended to $\mathscr{D}^m(X)$ so that

$$T(f, \pi_1, \dots, \pi_m) = T(f, \sigma\pi_1, \dots, \sigma\pi_m)$$

whenever $(f, \pi_1, \dots, \pi_m) \in \mathscr{D}^m(X)$, $\sigma \in \mathscr{D}(X)$, and $\sigma = 1$ on some neighborhood of $\mathrm{spt}(f)$. The right side is clearly independent of the choice of σ, for if $\tau \in \mathscr{D}(X)$ is another such function, then $(\sigma - \tau)\pi_i$ vanishes on a neighborhood of $\mathrm{spt}(f)$ for $i = 1, \dots, m$. Note also that whenever $K \subset X$ is compact, there is a $\sigma \in \mathscr{D}(X)$ such that $\sigma = 1$ on some neighborhood of K, since X is locally compact. Now each of the three properties of the given function T implies the respective property of the extended function via an appropriate choice of σ. □

By inserting a number of locally Lipschitz functions into T and keeping them fixed, we obtain a current of smaller dimension. This will often be used to simplify notation.

Definition 2.3. *For $T \in \mathscr{D}_m(X)$ and $(u, v) \in \mathrm{Lip}_{\mathrm{loc}}(X) \times [\mathrm{Lip}_{\mathrm{loc}}(X)]^k$, where $m \geq k \geq 0$, we define the current $T \lfloor (u, v) \in \mathscr{D}_{m-k}(X)$ by*

$$\begin{aligned}(T \lfloor (u, v))(f, g) &:= T(uf, v, g) \\ &= T(uf, v_1, \dots, v_k, g_1, \dots, g_{m-k})\end{aligned}$$

for $(f, g) \in \mathscr{D}^{m-k}(X)$.

In case $k = 0$, the definition simply reads

$$(T \lfloor u)(f, g) := T(uf, g).$$

In case $k \geq 1$, the placement of the functions $v_1, \dots, v_k$ on the right side is in accordance with the corresponding definition in the smooth case; it has the property that

$$(T \lfloor (1, v)) \lfloor (1, w) = T \lfloor (1, v, w). \tag{2.6}$$

The tuple $(u, v) \in \mathrm{Lip}_{\mathrm{loc}}(X) \times [\mathrm{Lip}_{\mathrm{loc}}(X)]^k$ corresponds to the k-form $u\, dv_1 \wedge \ldots \wedge dv_k$. It is clear that $T \lfloor (u, v)$ is indeed an element of $\mathscr{D}_{m-k}(X)$, as it would be with any other placement of $v_1, \dots, v_k$.

We now show that the defining properties of a general metric current give rise to a set of further properties, corresponding to the usual rules of calculus for differential forms.

Proposition 2.4 (alternating property and product rule)**.** *Suppose $T \in \mathscr{D}_m(X)$, $m \geq 1$, and $(f, \pi_1, \dots, \pi_m) \in \mathscr{D}^m(X)$.*

(1) *In case $m \geq 2$, if $\pi_i = \pi_j$ for some pair of distinct indices i, j, then*

$$T(f, \pi_1, \dots, \pi_m) = 0.$$

(2) *For all $g, h \in \mathrm{Lip}_{\mathrm{loc}}(X)$,*

$$T(f, gh, \pi_2, \dots, \pi_m) = T(fg, h, \pi_2, \dots, \pi_m) + T(fh, g, \pi_2, \dots, \pi_m).$$

Proof. To prove (1), it suffices to show that if $T \in \mathscr{D}_2(X)$ and $(f, \pi) \in \mathscr{D}^1(X)$, then $T(f, \pi, \pi) = 0$. For $k \in \mathbb{Z}$, let $\rho_k \colon \mathbb{R} \to \mathbb{R}$ be the piecewise affine 1-Lipschitz function with $\rho_k|_{[2k,2k+1]} = 1$ and $\operatorname{spt}(\rho_k) = [2k-1, 2k+2]$. Note that $\sum_{k\in\mathbb{Z}} \rho_k = 1$. Let $\sigma, \bar\sigma \colon \mathbb{R} \to \mathbb{R}$ denote the piecewise affine 4-Lipschitz functions such that $\sigma|_{\operatorname{spt}(\rho_k)} = 2k$ for k even and $\bar\sigma|_{\operatorname{spt}(\rho_k)} = 2k$ for k odd. Then

$$T(f, \sigma\circ\pi, \bar\sigma\circ\pi) = \sum_{k\in\mathbb{Z}} T((\rho_k\circ\pi)f, \sigma\circ\pi, \bar\sigma\circ\pi);$$

note that $(\rho_k \circ \pi) f = 0$ for almost all k since $\pi|_{\operatorname{spt}(f)}$ is bounded. By the strict locality property (2.5), each summand is zero because $\sigma\circ\pi$ or $\bar\sigma\circ\pi$ is constant on $\operatorname{spt}(\rho_k\circ\pi)$ for k even or odd, respectively. Hence $T(f, \sigma\circ\pi, \bar\sigma\circ\pi) = 0$. In the above definitions of the functions $\rho_k, \sigma, \bar\sigma$ we may equally well replace the unit by $1/j$, for $j \in \mathbb{N}$. The argument then shows that

$$T(f, \sigma_j\circ\pi, \bar\sigma_j\circ\pi) = 0,$$

where $\sigma_j(s) = \sigma(js)/j$ and $\bar\sigma_j(s) = \bar\sigma(js)/j$ for $s \in \mathbb{R}$. Letting j tend to ∞, we obtain $T(f, \pi, \pi) = 0$ by the continuity of T, since $\sigma_j \circ \pi \to \pi$ and $\bar\sigma_j \circ \pi \to \pi$ in $\operatorname{Lip}_{\mathrm{loc}}(X)$.

For the proof of (2), it suffices to show that if $T \in \mathscr{D}_1(X)$ and $(f, g) \in \mathscr{D}^1(X)$, then $T(f, g^2) = 2T(fg, g)$. Let $\rho_k, \sigma, \bar\sigma$ be defined as above. Since $\sigma\circ g|_{\operatorname{spt}(\rho_k\circ g)} = 2k$ for k even and $\bar\sigma\circ g|_{\operatorname{spt}(\rho_k\circ g)} = 2k$ for k odd, and since $(\rho_k\circ g)f = 0$ for almost all k, the multilinearity of T and (2.5) give

$$\begin{aligned}
&T(f, (\sigma\circ g)(\bar\sigma\circ g)) \\
&\quad = \sum_{k\in\mathbb{Z}} T((\rho_k\circ g)f, (\sigma\circ g)(\bar\sigma\circ g)) \\
&\quad = \sum_{k \text{ even}} 2kT((\rho_k\circ g)f, \bar\sigma\circ g) + \sum_{k \text{ odd}} 2kT((\rho_k\circ g)f, \sigma\circ g) \\
&\quad = \sum_{k\in\mathbb{Z}} 2kT((\rho_k\circ g)f, \sigma\circ g + \bar\sigma\circ g) \\
&\quad = T((\tau\circ g)f, (\sigma+\bar\sigma)\circ g)
\end{aligned}$$

for the piecewise affine 2-Lipschitz function $\tau := \sum_{k\in\mathbb{Z}} 2k\rho_k$, which satisfies $\tau|_{[2k,2k+1]} = 2k$ for $k \in \mathbb{Z}$. Rescaling by the factor $1/j$, as in the proof of (1), we obtain the identity

$$T(f, (\sigma_j\circ g)(\bar\sigma_j\circ g)) = T((\tau_j\circ g)f, (\sigma_j+\bar\sigma_j)\circ g),$$

where $\tau_j(s) = \tau(js)/j$ for $s \in \mathbb{R}$. Taking the limit for $j \to \infty$ we conclude that $T(f, g^2) = T(gf, 2g)$. $\square$

We now deduce a chain rule, which subsumes both the alternating property and the case $g = h$ of Proposition 2.4(2). For an open set $U \subset \mathbb{R}^n$, $C^{1,1}(U)$ denotes the space of all $g \in C^1(U)$ with partial derivatives $D_1 g, \dots, D_n g \in \operatorname{Lip}_{\mathrm{loc}}(U)$. For $n \ge m \ge 1$, we let $\Lambda(n, m)$ be the set of all strictly increasing maps $\lambda \colon \{1, \dots, m\} \to \{1, \dots, n\}$.

Theorem 2.5 (chain rule). *Suppose $m,n \geq 1$, $T \in \mathscr{D}_m(X)$, $U \subset \mathbb{R}^n$ is an open set, $f \in \mathscr{D}(X)$, $\pi = (\pi_1, \dots, \pi_n) \in \mathrm{Lip}_{\mathrm{loc}}(X,U)$, and $g = (g_1, \dots, g_m) \in [C^{1,1}(U)]^m$. If $n \geq m$, then*

$$T(f, g \circ \pi) = \sum_{\lambda \in \Lambda(n,m)} T\big(f \det\big[(D_{\lambda(k)} g_i) \circ \pi\big]_{i,k=1}^{m}, \pi_{\lambda(1)}, \dots, \pi_{\lambda(m)}\big).$$

If $n < m$, then $T(f, g \circ \pi) = 0$.

Proof. For illustration, suppose first that $n = m = 1$. In this case, the result says that

$$T(f, g \circ \pi) = T(f(g' \circ \pi), \pi) \tag{2.7}$$

whenever $T \in \mathscr{D}_1(X)$, $U \subset \mathbb{R}$ is an open set, $f \in \mathscr{D}(X)$, $\pi \in \mathrm{Lip}_{\mathrm{loc}}(X,U)$, and $g \in C^{1,1}(U)$. From the product rule, Proposition 2.4(2), we obtain the power rule

$$T(f, \pi^r) = T(f r \pi^{r-1}, \pi) \quad ((f,\pi) \in \mathscr{D}^1(X),\ r \in \mathbb{N}) \tag{2.8}$$

by induction on r. Hence, (2.7) holds if g is (the restriction of) a polynomial. Next, if $g \in C^2(U)$, there is a sequence of polynomials p_j such that $p_j \to g$, $p_j' \to g'$, and $p_j'' \to g''$ locally uniformly on U as $j \to \infty$. Then $p_j \circ \pi \to g \circ \pi$ and $p_j' \circ \pi \to g' \circ \pi$ in $\mathrm{Lip}_{\mathrm{loc}}(X)$, thus (2.7) follows by continuity. Finally, a smoothing argument extends the result to all $g \in C^{1,1}(U)$. For this, note that there is no loss of generality in assuming that $\mathrm{spt}(g)$ is compact; by (2.5), we may replace g by σg for any $\sigma \in C^{1,1}(U)$ such that $\sigma = 1$ on $\pi(\mathrm{spt}(f))$ and $\mathrm{spt}(\sigma)$ is compact.

Now let $n \geq m = 1$. We must show that

$$T(f, g \circ \pi) = \sum_{k=1}^{n} T\big(f((D_k g) \circ \pi), \pi_k\big) \tag{2.9}$$

whenever $T \in \mathscr{D}_1(X)$, $U \subset \mathbb{R}^n$ is an open set, $f \in \mathscr{D}(X)$, $\pi = (\pi_1, \dots, \pi_n) \in \mathrm{Lip}_{\mathrm{loc}}(X,U)$, and $g \in C^{1,1}(U)$. As above, the product rule implies that this identity holds if g is a polynomial in the variables $x_1, \dots, x_n$. Furthermore, if $g \in C^2(U)$, there is a sequence of polynomials $p_j = p_j(x_1, \dots, x_n)$ such that $p_j \to g$, $D_k p_j \to D_k g$, and $D_l D_k p_j \to D_l D_k g$ locally uniformly on U for all $k, l \in \{1, \dots, n\}$ (see e.g. [6, p. 57] for a direct proof). The continuity of T then yields (2.9), and a smoothing argument shows the same identity for all $g \in C^{1,1}(U)$.

Finally, the general result for $m, n \geq 1$ follows from (2.9), applied to each g_i, and the alternating property. □

Suppose U is an open subset of $\mathbb{R}^n$, $T \in \mathscr{D}_m(U)$, and π is the identity map on U, thus $\pi_i(x) = x_i$. In case $m > n$, the chain rule says that $T(f,g) = 0$ for all $(f,g) \in \mathscr{D}(U) \times [C^{1,1}(U)]^m$, and a smoothing argument then shows that $T = 0$. Hence

$$\mathscr{D}_m(U) = \{0\} \quad \text{for } m > n. \tag{2.10}$$

In case $m = n$, we obtain

$$T(f, g) = T(f \det(Dg), \pi) \tag{2.11}$$

for all $(f, g) \in \mathscr{D}(U) \times [C^{1,1}(U)]^m$. We now arrive at the first family of examples of metric currents, corresponding to the integration of a simple m-form over $U \subset \mathbb{R}^m$.

Proposition 2.6 (standard example)**.** *Suppose $U \subset \mathbb{R}^m$ is open, $m \geq 1$. Every function $u \in L^1_{\mathrm{loc}}(U)$ induces a current $[u] \in \mathscr{D}_m(U)$ satisfying*

$$[u](f, g) = \int_U uf \det(Dg)\, dx$$

for all $(f, g) \in \mathscr{D}^m(U)$.

Clearly $[u]$ is $(m+1)$-linear and satisfies the locality condition. The continuity follows from a well-known property of mappings in $[W^{1,\infty}_{\mathrm{loc}}(U)]^m$ (see e.g. [2, Theorem 2.16]). Since these examples will play a crucial role, we include a proof below. For reasons of consistency (cf. Definition 8.1), we also extend the notation to the case $m = 0$: Then $\mathbb{R}^0 = \{0\}$, $u \in L^1_{\mathrm{loc}}(\mathbb{R}^0)$ assigns the constant $u(0) \in \mathbb{R}$, and $[u] \in \mathscr{D}_0(\mathbb{R}^0)$ is the current satisfying $[u](f) = u(0)f(0)$ for all $f \in \mathscr{D}(\mathbb{R}^0)$.

Proof. We verify the continuity of $[u]$. Suppose $(f^j, g^j) \to (f, g)$ in $\mathscr{D}^m(U)$. Then there exist an open set $V \Subset U$ and a constant l such that $\mathrm{spt}(f^j) \subset V$ and $\mathrm{Lip}(f^j) \leq l$ for all j, and $f^j \to f$ uniformly; moreover, for $i = 1, \dots, m$, $\mathrm{Lip}(g^j_i|_V) \leq l$ for all j, and $g^j_i|_V \to g_i|_V$ uniformly. Put $h^j_i := g^j_i - g_i$. Then we have

$$\begin{aligned}
&[u](f^j, g^j) - [u](f, g) \\
&\quad = [u](f^j - f, g^j) + \sum_{i=1}^m [u](f, g_1, \dots, g_{i-1}, h^j_i, g^j_{i+1}, \dots, g^j_m).
\end{aligned}$$

Since the sequence $(\det(Dg^j))_{j \in \mathbb{N}}$ is bounded in $L^\infty(V)$, the first term on the right side clearly tends to 0 for $j \to \infty$. Now consider the summand for $i = 1$; the other summands are treated similarly. Since $uf \in L^1(V)$, we want to show that

$$\lim_{j \to \infty} \int_V v \det(D(h^j_1, g^j_2, \dots, g^j_m))\, dx = 0 \tag{2.12}$$

for all $v \in L^1(V)$. As $C^1_c(V)$ is dense in $L^1(V)$ and the sequence of determinants is bounded in $L^\infty(V)$, it suffices to prove (2.12) with v replaced by $w \in C^1_c(V)$. We claim that

$$\int_V w \det(D(h^j_1, g^j_2, \dots, g^j_m))\, dx = -\int_V h^j_1 \det(D(w, g^j_2, \dots, g^j_m))\, dx. \tag{2.13}$$

If $h_1^j, g_2^j, \dots, g_m^j \in C^2(V)$, then $\int_V d(wh_1^j\, dg_2^j \wedge \ldots \wedge dg_m^j) = 0$ and hence

$$\int_V w\, dh_1^j \wedge dg_2^j \wedge \ldots \wedge dg_m^j = -\int_V h_1^j\, dw \wedge dg_2^j \wedge \ldots \wedge dg_m^j,$$

which is just a restatement of (2.13). Now a smoothing argument relying on the bounded convergence theorem shows (2.13) in the general case. Since $h_1^j|_V \to 0$ uniformly, the right side of (2.13) tends to zero for $j \to \infty$. Thus (2.12) holds with v replaced by $w \in C_c^1(V)$. □

We conclude this section with some comments regarding Proposition 2.4 and Theorem 2.5. The proof of the alternating property does not require the continuity of T in the first argument and corresponds essentially to the last paragraph on p. 17 in [3]. In contrast, both the product and the chain rule depend on the joint continuity condition, Definition 2.1(2). To exemplify this, consider the functional $T\colon \mathscr{D}^1(\mathbb{R}) \to \mathbb{R}$ defined by

$$T(f,\pi) := \int_{\mathbb{R}} f'\pi'\, dx.$$

Clearly T is 2-linear and satisfies the locality condition. Moreover, an approximation and integration by parts argument as in the proof of Proposition 2.6 shows that if $(f^j,\pi^j) \to (f,\pi)$ in $\mathscr{D}^1(\mathbb{R})$, then $T(f^j,\pi) \to T(f,\pi)$ and $T(f,\pi^j) \to T(f,\pi)$. Yet, T fails to be a 1-dimensional metric current. For instance, if $f^j(x) = \chi_{[0,2\pi]}(x)\sin(jx)/j$, then $f^j \to 0$ in $\mathscr{D}(\mathbb{R})$, while $T(f^j,f^j) = \pi$ for all $j \in \mathbb{N}$. Neither the product rule nor the chain rule holds for T. In [3, Theorem 3.5], the proofs of the corresponding identities rely on the finite mass axiom and use piecewise affine approximation rather than polynomial approximation.

3 Support, boundary, and push-forward

For classical currents, these notions are defined in duality with support, exterior derivative, and pull-back of forms. Similar constructions apply in the metric context.

Definition 3.1 (support). *Given a current $T \in \mathscr{D}_m(X)$, $m \geq 0$, its* support $\mathrm{spt}(T)$ *in X is the intersection of all closed sets $C \subset X$ with the property that $T(f,\pi) = 0$ whenever $(f,\pi) \in \mathscr{D}^m(X)$ with* $\mathrm{spt}(f) \cap C = \emptyset$.

The definition is justified by the next lemma, whose proof employs Lipschitz partitions of unity. This is made possible by the fact that the functions in $\mathscr{D}(X)$ have compact support. In [3], the support of a current T is defined as the support of the associated finite Borel measure $\|T\|$.

Lemma 3.2 (support). *Suppose that $T \in \mathscr{D}_m(X)$, $m \geq 0$.*

(1) *The support* $\operatorname{spt}(T)$ *equals the set of all* $x \in X$ *such that for every* $\epsilon > 0$ *there exists an* $(f, \pi) \in \mathscr{D}^m(X)$ *with* $\operatorname{spt}(f) \subset \mathrm{B}(x, \epsilon)$ *and* $T(f, \pi) \neq 0$.

(2) *If* $f|_{\operatorname{spt}(T)} = 0$, *then* $T(f, \pi_1, \ldots, \pi_m) = 0$.

(3) *In case* $m \geq 1$, $T(f, \pi_1, \ldots, \pi_m) = 0$ *whenever some* π_i *is constant on* $\{f \neq 0\} \cap \operatorname{spt}(T)$.

This shows in particular that $T(f, \pi_1, \ldots, \pi_m)$ depends only on the restrictions of $f, \pi_1, \ldots, \pi_m$ to $\operatorname{spt}(T)$.

Proof. Let Σ be the set described in (1). Suppose that $x \notin \operatorname{spt}(T)$. There is a closed set C with the property stated in Definition 3.1 such that $x \notin C$. Then there is an $\epsilon > 0$ such that $T(f, \pi) = 0$ whenever $\operatorname{spt}(f) \subset \mathrm{B}(x, \epsilon)$. This shows that $x \notin \Sigma$, so $\Sigma \subset \operatorname{spt}(T)$.

Next we prove that $T(f, \pi) = 0$ whenever $\operatorname{spt}(f) \cap \Sigma = \emptyset$. Since $\operatorname{spt}(f)$ is a compact subset of $X \setminus \Sigma$, there exist finitely many open balls $U_1, \ldots, U_N$ such that $\operatorname{spt}(f) \subset \bigcup_{k=1}^N U_k$ and $T(g, \pi) = 0$ whenever $\{g \neq 0\} \subset U_k$ for some k. Decomposing f by means of a Lipschitz partition of unity $(\rho_k)_{k=1}^N$ on $\operatorname{spt}(f)$ with $\{\rho_k \neq 0\} \subset U_k$ we see that $T(f, \pi) = 0$. As Σ is closed, this shows in particular that $\operatorname{spt}(T) \subset \Sigma$.

For (2), let β_j be the function defined after (2.5), $j \in \mathbb{N}$. If $f|_{\operatorname{spt}(T)} = 0$, then $\operatorname{spt}(\beta_j \circ f) \cap \operatorname{spt}(T) = \emptyset$. The argument of the previous paragraph then shows that $T(\beta_j \circ f, \pi) = 0$ for all j, thus $T(f, \pi) = 0$ by the continuity of T.

To prove (3), by the linearity, locality, and the alternating property of T it suffices to show that $T(f, \pi_1, \ldots, \pi_m) = 0$ if $\pi_1 = 0$ on $K := \operatorname{spt}\big(f|_{\operatorname{spt}(T)}\big)$. Then $\operatorname{spt}(\beta_j \circ \pi_1) \cap K = \emptyset$ for β_j as above, $j \in \mathbb{N}$. For fixed j, since K is compact, there is a function $\sigma \in \mathscr{D}(X)$ such that $\sigma|_K = 1$ and $\beta_j \circ \pi_1 = 0$ on some neighborhood of $\operatorname{spt}(\sigma)$. Then $(1 - \sigma) f|_{\operatorname{spt}(T)} = 0$, hence

$$T(f, \beta_j \circ \pi_1, \pi_2, \ldots, \pi_m) = T(\sigma f, \beta_j \circ \pi_1, \pi_2, \ldots, \pi_m) = 0$$

by (2) and the locality of T. Since $\beta_j \circ \pi_1 \to \pi_1$ in $\operatorname{Lip}_{\mathrm{loc}}(X)$ for $j \to \infty$, we have $T(f, \pi_1, \ldots, \pi_m) = 0$. □

Suppose A is a closed subset of X, and $T_A \in \mathscr{D}_m(A)$. For each $f \in \mathscr{D}(X)$, the support of $f|_A$ in A is compact. Hence one obtains a current $T \in \mathscr{D}_m(X)$ by defining

$$T(f, \pi_1, \ldots, \pi_m) := T_A(f|_A, \pi_1|_A, \ldots, \pi_m|_A) \tag{3.1}$$

for all $(f, \pi_1, \ldots, \pi_m) \in \mathscr{D}^m(X)$. Clearly $\operatorname{spt}(T) = \operatorname{spt}(T_A)$. Conversely, the following holds.

Proposition 3.3. *Let* $T \in \mathscr{D}_m(X)$, $m \geq 0$, *and let* $A \subset X$ *be a locally compact set containing* $\operatorname{spt}(T)$. *Then there is a unique current* $T_A \in \mathscr{D}_m(A)$ *with the property that*

$$T_A(f, \pi_1, \ldots, \pi_m) = T(\bar{f}, \bar{\pi}_1, \ldots, \bar{\pi}_m)$$

whenever $(f, \pi_1, \ldots, \pi_m) \in \mathscr{D}^m(A)$, $(\bar{f}, \bar{\pi}_1, \ldots, \bar{\pi}_m) \in \mathscr{D}^m(X)$, $\bar{f}|_A = f$, *and* $\bar{\pi}_i|_A = \pi_i$ *for* $i = 1, \ldots, m$. *Moreover,* $\operatorname{spt}(T_A) = \operatorname{spt}(T)$.

In particular, every current $T \in \mathscr{D}_m(X)$ may be viewed as a current in its own support, i.e., as an element of $\mathscr{D}_m(\operatorname{spt}(T))$.

Proof. For every compact set $K \subset A$, every $l \geq 0$, and every $c > 0$ there exist a compact set $K' \subset X$ containing K, an $l' \geq l$, and an operator

$$E \colon \bigl\{ f \in \operatorname{Lip}_{K,l}(A) : \|f\|_\infty \leq c \bigr\} \to \operatorname{Lip}_{K',l'}(X)$$

such that $(Ef)|_A = f$, $\|Ef\|_\infty = \|f\|_\infty$, and $\|Ef - Eg\|_\infty = \|f - g\|_\infty$. In fact, one may choose $\sigma \in \mathscr{D}(X)$ such that $0 \leq \sigma \leq 1$ and $\sigma|_K = 1$ and define

$$(Ef)(x) := \sigma(x) \min\bigl\{ \|f\|_\infty, \inf_{a \in A}(f(a) + l\, d(a, x)) \bigr\},$$

cf. Sect. 1.1. This has the required properties, with $K' = \operatorname{spt}(\sigma)$ and $l' = l + c \operatorname{Lip}(\sigma)$. Now the result follows easily from Lemma 2.2 and Lemma 3.2. □

Suppose for the moment that X is an arbitrary metric space. In view of (3.1) and Proposition 3.3, it is possible to define the vector space $\mathscr{D}_m(X)$ of general metric m-currents in X as follows: An element

$$\mathbf{T} \in \mathscr{D}_m(X) \tag{3.2}$$

is a pair $\mathbf{T} = (X_\mathbf{T}, T)$ consisting of a closed and locally compact set $X_\mathbf{T} \subset X$ and a current $T \in \mathscr{D}_m(X_\mathbf{T})$ (Definition 2.1) with $\operatorname{spt}(T) = X_\mathbf{T}$. For $\alpha \in \mathbb{R} \setminus \{0\}$, $\alpha\mathbf{T} := (X_\mathbf{T}, \alpha T)$, and $0\mathbf{T} := (\emptyset, 0)$. To form the sum of two elements $\mathbf{T}, \mathbf{T}' \in \mathscr{D}_m(X)$, regard both T and T' as currents in $X_\mathbf{T} \cup X_{\mathbf{T}'}$, put $X_{\mathbf{T}+\mathbf{T}'} := \operatorname{spt}(T + T') \subset X_\mathbf{T} \cup X_{\mathbf{T}'}$, and then interpret $T + T'$ as a current in $X_{\mathbf{T}+\mathbf{T}'}$. With this understood, we may write again T instead of $\mathbf{T}$ and X_T or $\operatorname{spt}(T)$ instead of $X_\mathbf{T}$. We shall briefly return to this discussion at the end of Sect. 4, but otherwise we shall not pursue it in the present paper.

We now proceed to the definition of the boundary of a metric m-current in the locally compact space X, which is easily seen to be an $(m-1)$-current.

Definition 3.4 (boundary)**.** *The* boundary *of a current* $T \in \mathscr{D}_m(X)$, $m \geq 1$, *is the current* $\partial T \in \mathscr{D}_{m-1}(X)$ *defined by*

$$\partial T(f, \pi_1, \ldots, \pi_{m-1}) := T(\sigma, f, \pi_1, \ldots, \pi_{m-1})$$

for $(f, \pi_1, \ldots, \pi_{m-1}) \in \mathscr{D}^{m-1}(X)$, *where* $\sigma \in \mathscr{D}(X)$ *is any function such that* $\sigma = 1$ *on* $\{f \neq 0\} \cap \operatorname{spt}(T)$.

If $\tau \in \mathscr{D}(X)$ is another such function, then f vanishes on $\{\sigma - \tau \neq 0\} \cap \operatorname{spt}(T)$, thus $T(\sigma - \tau, f, \pi_1, \ldots, \pi_{m-1}) = 0$ by Lemma 3.2(3). This shows that ∂T is well-defined. Clearly ∂T is multilinear and continuous. To verify the locality of ∂T, suppose that some π_i is constant on a neighborhood of spt(f), choose σ such that π_i is constant on a neighborhood of spt(σ), and use the locality of T. We have

$$\operatorname{spt}(\partial T) \subset \operatorname{spt}(T), \tag{3.3}$$

and if $A \subset X$ and T_A are as in Proposition 3.3, then

$$\partial(T_A) = (\partial T)_A. \tag{3.4}$$

If $m \geq 2$, then

$$\partial(\partial T) = 0. \tag{3.5}$$

To see this, let $(f, \pi) \in \mathscr{D}^{m-2}(X)$, and choose $\rho, \sigma, \tau \in \mathscr{D}(X)$ such that $\rho|_{\operatorname{spt}(f)} = 1$, $\sigma|_{\operatorname{spt}(\rho)} = 1$, and $\tau|_{\operatorname{spt}(\sigma)} = 1$. Then

$$\partial(\partial T)(f, \pi) = \partial T(\sigma, f, \pi) = T(\tau, \sigma, f, \pi) = T(\rho, \sigma, f, \pi) = 0;$$

the third equality holds since $f|_{\operatorname{spt}(\tau-\rho)} = 0$, the last since $\sigma|_{\operatorname{spt}(\rho)} = 1$. The operator $\partial \colon \mathscr{D}_m(X) \to \mathscr{D}_{m-1}(X)$ is linear, and if $T_n \to T$ weakly in $\mathscr{D}_m(X)$, then $\partial T_n \to \partial T$ weakly in $\mathscr{D}_{m-1}(X)$.

The following lemma corresponds to the identity

$$(\partial T) \lfloor \phi = T \lfloor d\phi + (-1)^k \partial(T \lfloor \phi)$$

for a classical m-current T and k-form ϕ, cf. [14, p. 356].

Lemma 3.5. *For $T \in \mathscr{D}_m(X)$ and $(u, v) \in \operatorname{Lip}_{\text{loc}}(X) \times [\operatorname{Lip}_{\text{loc}}(X)]^k$, where $m > k \geq 0$, the identity*

$$(\partial T) \lfloor (u, v) = T \lfloor (1, u, v) + (-1)^k \partial(T \lfloor (u, v))$$

holds.

In case $k = 0$, the identity simply reads

$$(\partial T) \lfloor u = T \lfloor (1, u) + \partial(T \lfloor u). \tag{3.6}$$

Note also that

$$(\partial T) \lfloor (1, v) = (-1)^k \partial(T \lfloor (1, v)) \tag{3.7}$$

since $T \lfloor (1, 1, v) = 0$ by the locality of T.

Proof. Let $(f, g) \in \mathscr{D}^{m-k-1}(X)$, and choose $\sigma \in \mathscr{D}(X)$ with $\sigma|_{\operatorname{spt}(f)} = 1$. Then

$$\begin{aligned}((\partial T) \lfloor (u, v))(f, g) &= \partial T(uf, v, g) \\ &= T(\sigma, uf, v, g) \\ &= T(\sigma f, u, v, g) + T(\sigma u, f, v, g) \\ &= T(f, u, v, g) + (-1)^k T(\sigma u, v, f, g) \\ &= (T \lfloor (1, u, v))(f, g) + (-1)^k (T \lfloor (u, v))(\sigma, f, g) \\ &= (T \lfloor (1, u, v))(f, g) + (-1)^k \partial(T \lfloor (u, v))(f, g);\end{aligned}$$

the third step uses Proposition 2.4(2), the fourth 2.4(1). □

Next we define push-forwards of metric currents under locally Lipschitz maps. Since the test functions have compact support, we need to restrict to proper maps, as in the classical case (cf. [14, p. 359 and 4.1.14]).

Definition 3.6 (push-forward)**.** *Suppose that* $T \in \mathscr{D}_m(X)$*,* $m \geq 0$*,* $A \subset X$ *is a locally compact set containing* $\mathrm{spt}(T)$*,* Y *is another locally compact metric space, and* $F \in \mathrm{Lip}_{\mathrm{loc}}(A, Y)$ *is proper, i.e.,* $F^{-1}(K)$ *is compact whenever* $K \subset Y$ *is compact. The* push-forward of T via F *is the current* $F_\# T \in \mathscr{D}_m(Y)$ *defined by*

$$F_\# T(f, \pi_1, \dots, \pi_m) := T_A(f \circ F, \pi_1 \circ F, \dots, \pi_m \circ F)$$

for $(f, \pi_1, \dots, \pi_m) \in \mathscr{D}^m(Y)$*, where* $T_A \in \mathscr{D}_m(A)$ *is as in Proposition 3.3.*

Note that $f \circ F \in \mathscr{D}(A)$ since F is locally Lipschitz and proper. One readily verifies that $F_\# T$ is a metric current. Since F is a proper continuous map, $F(\mathrm{spt}(T))$ is closed in Y, and

$$\mathrm{spt}(F_\# T) \subset F(\mathrm{spt}(T)). \tag{3.8}$$

If $m \geq 1$, then

$$\partial(F_\# T) = F_\#(\partial T). \tag{3.9}$$

To see this, let $(f, \pi) \in \mathscr{D}^{m-1}(Y)$, and choose $\sigma \in \mathscr{D}(Y)$ such that $\sigma|_{\mathrm{spt}(f)} = 1$; then

$$\begin{aligned}\partial(F_\# T)(f, \pi) &= (F_\# T)(\sigma, f, \pi) = T_A(\sigma \circ F, f \circ F, \pi \circ F) \\ &= \partial(T_A)(f \circ F, \pi \circ F) = (\partial T)_A(f \circ F, \pi \circ F) = F_\#(\partial T)(f, \pi).\end{aligned}$$

If Z is another locally compact metric space and $G \in \mathrm{Lip}_{\mathrm{loc}}(Y, Z)$ is proper, then

$$G_\#(F_\# T) = (G \circ F)_\# T. \tag{3.10}$$

If $F \in \mathrm{Lip}_{\mathrm{loc}}(X, Y)$ is proper, then the operator $F_\# \colon \mathscr{D}_m(X) \to \mathscr{D}_m(Y)$ is linear, and $F_\# T_n \to F_\# T$ weakly in $\mathscr{D}_m(Y)$ whenever $T_n \to T$ weakly in $\mathscr{D}_m(X)$. Finally, suppose $F \in \mathrm{Lip}_{\mathrm{loc}}(D, Y)$ for some set $D \subset X$ containing $\mathrm{spt}(T)$, and $F|_{\mathrm{spt}(T)}$ is proper. In this situation, we put

$$F_\# T := \big(F|_{\mathrm{spt}(T)}\big)_\# T; \tag{3.11}$$

this is consistent with the above definition in case D is locally compact and F is proper. When $F \in \mathrm{Lip}_{\mathrm{loc}}(X, Y)$ and $F|_{\mathrm{spt}(T)}$ is proper, it follows that

$$F_\# T(f, \pi_1, \dots, \pi_m) = T(\sigma(f \circ F), \pi_1 \circ F, \dots, \pi_m \circ F) \tag{3.12}$$

for $(f, \pi_1, \dots, \pi_m) \in \mathscr{D}^m(Y)$ and any $\sigma \in \mathscr{D}(X)$ such that $\sigma = 1$ on $\{f \circ F \neq 0\} \cap \mathrm{spt}(T)$.

We compute the push-forward of a current $[u]$ as in Proposition 2.6 (standard example).

Lemma 3.7. *Suppose* $u \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$, $m \geq 1$, $F \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^m, \mathbb{R}^m)$, *and* $F|_{\mathrm{spt}(u)}$ *is proper. Then* $F_\#[u] = [v]$, *where* $v \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ *satisfies*

$$v(y) = \sum_{x \in F^{-1}\{y\}} u(x) \,\mathrm{sign}(\det(DF(x)))$$

for $\mathscr{L}^m$*-almost every* $y \in \mathbb{R}^m$.

Proof. Let $(f, \pi) \in \mathscr{D}^m(\mathbb{R}^m)$. Then

$$F_\#[u](f, \pi) = \int_{\mathbb{R}^m} u(f \circ F) \det(D(\pi \circ F)) \, dx = \int_{\mathbb{R}^m} h(x) \, |\det(DF(x))| \, dx$$

for $h(x) := u(x) f(F(x)) \det(D\pi(F(x))) \,\mathrm{sign}(\det(DF(x)))$. By the change of variables formula (cf. [14, Theorem 3.2.3(2)] or [13, 3.3.3], the case $n = m$),

$$\begin{aligned} F_\#[u](f, \pi) &= \int_{\mathbb{R}^m} \sum_{x \in F^{-1}\{y\}} h(x) \, dy = \int_{\mathbb{R}^m} v(y) f(y) \det(D\pi(y)) \, dy \\ &= [v](f, \pi). \end{aligned}$$

This proves the lemma. □

4 Mass

We now define the mass of a metric current. Our approach is inspired by both the classical definition (recalled in (5.2)) and [3, Proposition 2.7]. Currents with locally finite mass will be of particular interest.

Definition 4.1 (mass). *For* $T \in \mathscr{D}_m(X)$, $m \geq 0$, *and every open set* $V \subset X$, *we define the* mass $\mathbf{M}_V(T)$ *of* T *in* V *as the least number* $M \in [0, \infty]$ *such that*

$$\sum_{\lambda \in \Lambda} T(f_\lambda, \pi^\lambda) \leq M$$

whenever Λ *is a finite set,* $(f_\lambda, \pi^\lambda) = (f_\lambda, \pi_1^\lambda, \ldots, \pi_m^\lambda) \in \mathscr{D}(X) \times [\mathrm{Lip}_1(X)]^m$, $\mathrm{spt}(f_\lambda) \subset V$, *and* $\sum_{\lambda \in \Lambda} |f_\lambda| \leq 1$. *The number* $\mathbf{M}(T) := \mathbf{M}_X(T)$ *is the* total mass *of* T. *We denote by*

$$\mathbf{M}_{m,\mathrm{loc}}(X)$$

the vector space of all $T \in \mathscr{D}_m(X)$ *such that* $\mathbf{M}_V(T) < \infty$ *for every open set* $V \Subset X$, *and we put*

$$\mathbf{M}_m(X) := \{T \in \mathscr{D}_m(X) : \mathbf{M}(T) < \infty\}.$$

Furthermore, we define

$$\|T\|(A) := \inf\{\mathbf{M}_V(T) : V \subset X \text{ is open, } A \subset V\}$$

for $T \in \mathscr{D}_m(X)$ *and every set* $A \subset X$.

Note that for $T \in \mathscr{D}_0(X)$,

$$\mathbf{M}_V(T) = \sup\{T(f) : f \in \mathscr{D}(X),\, \mathrm{spt}(f) \subset V,\, |f| \le 1\},$$

and by the continuity of T it follows that

$$\mathbf{M}_V(T) = \sup\{T(f) : f \in \mathscr{D}(X),\, |f| \le \chi_V\}. \tag{4.1}$$

If $T \in \mathscr{D}_m(X)$ and $A \subset X$ is open, then clearly

$$\|T\|(A) = \mathbf{M}_A(T). \tag{4.2}$$

The mass is lower semicontinuous with respect to weak convergence: If $T_n \to T$ weakly in $\mathscr{D}_m(X)$, then

$$\mathbf{M}_V(T) \le \liminf_{n\to\infty} \mathbf{M}_V(T_n) \tag{4.3}$$

for every open set $V \subset X$. For $T, T' \in \mathscr{D}_m(X)$ and $\alpha \in \mathbb{R}$, we have

$$\|\alpha T\| = |\alpha|\|T\|, \quad \|T + T'\| \le \|T\| + \|T'\|, \tag{4.4}$$

and $\|T\| = 0$ if and only if $T = 0$. In particular, $\mathbf{M}$ is a norm on $\mathbf{M}_m(X)$.

Proposition 4.2. *For $m \ge 0$, $(\mathbf{M}_m(X), \mathbf{M})$ is a Banach space.*

Proof. Let $(T_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $(\mathbf{M}_m(X), \mathbf{M})$. For every $\epsilon > 0$ there exists an index k_ϵ such that

$$|T_k(f,\pi) - T_l(f,\pi)| = |(T_k - T_l)(f,\pi)| \le \mathbf{M}(T_k - T_l) \le \epsilon$$

whenever $k, l \ge k_\epsilon$ and $(f,\pi) \in \mathscr{D}(X) \times [\mathrm{Lip}_1(X)]^m$ with $|f| \le 1$. In particular, $(T_k(f,\pi))_{k\in\mathbb{N}}$ is a Cauchy sequence for every such (f,π). It follows that there is an $(m+1)$-linear function $T\colon \mathscr{D}^m(X) \to \mathbb{R}$ such that $\lim_{k\to\infty} T_k(f,\pi) = T(f,\pi)$ for all $(f,\pi) \in \mathscr{D}^m(X)$ and T satisfies the locality condition (Definition 2.1(3)). Moreover, for $\epsilon > 0$ and k_ϵ as above, we have

$$|T_k(f,\pi) - T(f,\pi)| \le \epsilon$$

whenever $k \ge k_\epsilon$ and $(f,\pi) \in \mathscr{D}(X) \times [\mathrm{Lip}_1(X)]^m$ with $|f| \le 1$. It suffices to verify the continuity of T on the set of all such (f,π). Suppose that $(f,\pi), (f^1,\pi^1), (f^2,\pi^2), \dots$ belong to this set and $(f^j,\pi^j) \to (f,\pi)$ in $\mathscr{D}^m(X)$. Given $\epsilon > 0$, there is an index j_ϵ such that $|T_{k_\epsilon}(f^j,\pi^j) - T_{k_\epsilon}(f,\pi)| \le \epsilon$ for all $j \ge j_\epsilon$; then

$$\begin{aligned} &|T(f^j,\pi^j) - T(f,\pi)| \\ &\quad \le |T(f^j,\pi^j) - T_{k_\epsilon}(f^j,\pi^j)| + \epsilon + |T_{k_\epsilon}(f,\pi) - T(f,\pi)| \le 3\epsilon. \end{aligned}$$

Hence T is a current. Whenever Λ is a finite set, $(f_\lambda, \pi^\lambda) \in \mathscr{D}(X) \times [\mathrm{Lip}_1(X)]^m$ for $\lambda \in \Lambda$, and $\sum_{\lambda\in\Lambda} |f_\lambda| \le 1$, we have

$$\sum_{\lambda\in\Lambda} (T_k - T_l)(f_\lambda, \pi^\lambda) \le \mathbf{M}(T_k - T_l) \le \epsilon$$

for $k, l \ge k_\epsilon$, hence $\sum_{\lambda\in\Lambda}(T_k - T)(f_\lambda, \pi^\lambda) \le \epsilon$ for $k \ge k_\epsilon$. We conclude that $\mathbf{M}(T_k - T) \le \epsilon$ for $k \ge k_\epsilon$. Thus $T \in \mathbf{M}_m(X)$ and $T_k \to T$ in $(\mathbf{M}_m(X), \mathbf{M})$. □

Let $U \subset \mathbb{R}^m$ be an open set. For $T \in \mathscr{D}_m(U)$, it follows from Theorem 2.5 (chain rule) that

$$\mathbf{M}_V(T) = \sup\{T(f, \mathrm{id}) : f \in \mathscr{D}(X),\, \mathrm{spt}(f) \subset V,\, |f| \le 1\}$$

for every open set $V \subset U$. Given a current $[u]$ as in Proposition 2.6 (standard example), where $u \in L^1_{\mathrm{loc}}(U)$, we have

$$\mathbf{M}_V([u]) = \sup\left\{ \int_V uf\,dx : f \in \mathscr{D}(V),\, |f| \le 1 \right\} = \int_V |u|\,dx \tag{4.5}$$

for every open set $V \subset U$. In particular, $[u] \in \mathbf{M}_{m,\mathrm{loc}}(U)$.

Theorem 4.3 (mass). *Let $T \in \mathscr{D}_m(X)$, $m \ge 0$.*

(1) *The function $\|T\| \colon 2^X \to [0, \infty]$ (cf. Definition 4.1) is a Borel regular outer measure.*

(2) *We have $\mathrm{spt}(\|T\|) = \mathrm{spt}(T)$ and $\|T\|(X \setminus \mathrm{spt}(T)) = 0$.*

(3) *For every open set $V \subset X$,*

$$\|T\|(V) = \sup\{\|T\|(K) : K \subset X \text{ is compact, } K \subset V\}.$$

(4) *If $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$, then $\|T\|$ is a Radon measure, and*

$$|T(f, \pi)| \le \prod_{i=1}^m \mathrm{Lip}\big(\pi_i|_{\mathrm{spt}(f)}\big) \int_X |f|\,d\|T\|$$

for all $(f, \pi) = (f, \pi_1, \dots, \pi_m) \in \mathscr{D}^m(X)$.

In case $m = 0$, the inequality in (4) reads

$$|T(f)| \le \int_X |f|\,d\|T\|.$$

In case $m \ge 1$, it holds as well with $\{f \ne 0\} \cap \mathrm{spt}(T)$ in place of $\mathrm{spt}(f)$, by Lemma 3.2(3) and the extendability of Lipschitz functions. Properties (1), (2), and (3) hold in general for every $(m+1)$-linear functional $T \colon \mathscr{D}^m(X) \to \mathbb{R}$; for $m = 0$ the same argument occurs in the proof of the Riesz representation theorem (see e.g. [29, Theorem 4.1]). The proof of (4) corresponds essentially to that of [3, (2.8)].

Proof. Clearly $\|T\|(\emptyset) = 0$. We claim that

$$\|T\|(V) \le \sum_{k=1}^{\infty} \|T\|(V_k)$$

whenever $V, V_1, V_2, \ldots \subset X$ are open and $V \subset \bigcup_{k=1}^{\infty} V_k$. Let Λ and (f_λ, π^λ) be given as in the definition of $\mathbf{M}_V(T)$. There is an index N such that $\bigcup_{k=1}^{N} V_k$ contains the compact set $K := \bigcup_{\lambda \in \Lambda} \operatorname{spt}(f_\lambda)$. Then there exist $\rho_1, \ldots, \rho_N \in \mathscr{D}(X)$ such that $\sum_{k=1}^{N} \rho_k = 1$ on K and $0 \le \rho_k \le 1$, $\operatorname{spt}(\rho_k) \subset V_k$ for $k = 1, \ldots, N$. Then

$$\sum_{\lambda \in \Lambda} T(f_\lambda, \pi^\lambda) = \sum_{\lambda \in \Lambda} \sum_{k=1}^{N} T(\rho_k f_\lambda, \pi^\lambda) = \sum_{k=1}^{N} \sum_{\lambda \in \Lambda} T(\rho_k f_\lambda, \pi^\lambda) \le \sum_{k=1}^{N} \|T\|(V_k)$$

since $\operatorname{spt}(\rho_k f_\lambda) \subset V_k$ and $\sum_{\lambda \in \Lambda} |\rho_k f_\lambda| \le 1$ for $k = 1, \ldots, N$, proving the claim. It follows that $\|T\|$ is an outer measure, and $\|T\|(A \cup B) = \|T\|(A) + \|T\|(B)$ whenever $\inf\{d(x,y) : x \in A, y \in B\} > 0$. By Carathéodory's criterion, every Borel set is $\|T\|$-measurable. If A is an arbitrary set, then A is contained in a G_δ set B with $\|T\|(B) = \|T\|(A)$, by the definition of $\|T\|(A)$. Thus $\|T\|$ is Borel regular. This proves (1).

Assertion (2) follows from Lemma 3.2.

For (3), given an open set V, let $\alpha < \|T\|(V)$. Then there exist a finite set Λ and $(f_\lambda, \pi^\lambda) \in \mathscr{D}(X) \times [\operatorname{Lip}_1(X)]^m$ such that $K := \bigcup_{\lambda \in \Lambda} \operatorname{spt}(f_\lambda) \subset V$, $\sum_{\lambda \in \Lambda} |f_\lambda| \le 1$, and $s := \sum_{\lambda \in \Lambda} T(f_\lambda, \pi^\lambda) \ge \alpha$. For every open set U containing K we have $\|T\|(U) \ge s \ge \alpha$, hence $\|T\|(K) \ge \alpha$.

It remains to prove (4). If $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$, then $\|T\|$ is finite on compact sets, thus T is a Radon measure. For the integral estimate, we first consider the case $m = 0$, so that $T \in \mathbf{M}_{0,\mathrm{loc}}(X)$. Assuming without loss of generality that $f \ge 0$, we put $f_s := \min\{f, s\}$ and observe that, by (4.1),

$$|T(f_t) - T(f_s)| = |T(f_t - f_s)| \le \|T\|(\{f > s\})(t - s)$$

whenever $0 \le s < t$. Hence $s \mapsto T(f_s)$ is a Lipschitz function with $|(d/ds)T(f_s)| \le \|T\|(\{f > s\})$ for almost every $s \ge 0$. Since $T(f) = T(f) - T(f_0) = \int_0^\infty (d/ds)T(f_s)\, ds$, we conclude that

$$|T(f)| \le \int_0^\infty \left| \frac{d}{ds} T(f_s) \right| ds \le \int_0^\infty \|T\|(\{f > s\})\, ds = \int_X f \, d\|T\|$$

(for the last step, see e.g. [26, Theorem 8.16]). Now let $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$, $m \ge 1$. Let first $(f, \pi) \in \mathscr{D}(X) \times [\operatorname{Lip}_1(X)]^m$, and consider $T_\pi := T \lfloor (1, \pi) \in \mathscr{D}_0(X)$. Then $\|T_\pi\| \le \|T\|$, thus $T_\pi \in \mathbf{M}_{0,\mathrm{loc}}(X)$, and

$$|T(f, \pi)| = |T_\pi(f)| \le \int_X |f| \, d\|T_\pi\| \le \int_X |f| \, d\|T\|.$$

Finally, given $(f, \pi) \in \mathscr{D}^m(X)$, there exists $\tilde{\pi} \in [\operatorname{Lip}(X)]^m$ such that $\tilde{\pi} = \pi$ on $\operatorname{spt}(f)$ and $\operatorname{Lip}(\tilde{\pi}_i) = \operatorname{Lip}\big(\pi_i|_{\operatorname{spt}(f)}\big)$ for $i = 1, \ldots, m$. Then $T(f, \pi) = T(f, \tilde{\pi})$ by the strict locality property (2.5), and the result follows. $\square$

Recall from Sect. 1.2 that we denote by $\mathscr{B}^\infty_{\rm c}(X)$ the algebra of all bounded Borel functions $f\colon X \to \mathbb{R}$ such that $\operatorname{spt}(f)$ is compact. From Theorem 4.3(4) it follows that every $T \in \mathbf{M}_{m,\rm loc}(X)$, $m \ge 0$, naturally extends to a function

$$T\colon \mathscr{B}^\infty_{\rm c}(X) \times [\operatorname{Lip}_{\rm loc}(X)]^m \to \mathbb{R}. \tag{4.6}$$

To see this, note that $\mathscr{D}(X)$ is dense in $L^1(\|T\|)$ (since $C_{\rm c}(X)$ is, see e.g. [26, 3.14], and since every element of $C_{\rm c}(X)$ is a uniform limit of a sequence of Lipschitz functions, cf. Sect. 1.1). Hence, whenever $f \in \mathscr{B}^\infty_{\rm c}(X) \subset L^1(\|T\|)$ and $U \Subset X$ is a neighborhood of $\operatorname{spt}(f)$, there is a sequence $(g_k)_{k\in\mathbb{N}}$ in $\mathscr{D}(X)$ such that $g_k \to f$ in $L^1(\|T\|)$ and $\operatorname{spt}(g_k) \subset U$ for all k. By Theorem 4.3(4), for every $\pi \in [\operatorname{Lip}_{\rm loc}(X)]^m$, $(T(g_k,\pi))_{k\in\mathbb{N}}$ is a Cauchy sequence whose limit is independent of the choice of $(g_k)_{k\in\mathbb{N}}$. Then $T(f,\pi)$ is defined to be this limit.

Theorem 4.4 (extended functional)**.** *Let $T \in \mathbf{M}_{m,\rm loc}(X)$, $m \ge 0$. The extension $T\colon \mathscr{B}^\infty_{\rm c}(X) \times [\operatorname{Lip}_{\rm loc}(X)]^m \to \mathbb{R}$ possesses the following properties:*

(1) (multilinearity) *T is $(m+1)$-linear on $\mathscr{B}^\infty_{\rm c}(X) \times [\operatorname{Lip}_{\rm loc}(X)]^m$.*

(2) (continuity) *$T(f^j,\pi^j) \to T(f,\pi)$ whenever $f, f^1, f^2, \ldots \in \mathscr{B}^\infty_{\rm c}(X)$, $\sup_j \|f^j\|_\infty < \infty$, $\bigcup_j \operatorname{spt}(f^j) \subset K$ for some compact set $K \subset X$, $f^j \to f$ pointwise on X, and $\pi^j \to \pi$ in $[\operatorname{Lip}_{\rm loc}(X)]^m$.*

(3) (locality) *In case $m \ge 1$, $T(f,\pi) = 0$ whenever some π_i is constant on the support of $f \in \mathscr{B}^\infty_{\rm c}(X)$.*

(4) *For all $(f,\pi) \in \mathscr{B}^\infty_{\rm c}(X) \times [\operatorname{Lip}_{\rm loc}(X)]^m$,*

$$|T(f,\pi)| \le \prod_{i=1}^m \operatorname{Lip}\bigl(\pi_i|_{\operatorname{spt}(f)}\bigr) \int_X |f|\, d\|T\|.$$

Obviously (4) subsumes (3). Moreover, whenever a functional T satisfying (4) is linear in the first argument and continuous with respect to the convergence in $[\operatorname{Lip}_{\rm loc}(X)]^m$, then T fulfils (2) (cf. (4.9) below). As a consequence of (4) and Theorem 4.3(2), we also have $T(f - \tilde f, \pi) = 0$ and hence

$$T(f,\pi) = T(\tilde f, \pi) \tag{4.7}$$

whenever $f, \tilde f \in \mathscr{B}^\infty_{\rm c}(X)$ agree on $\operatorname{spt}(T)$.

Proof. Clearly the extended functional T is $(m+1)$-linear, and

$$|T(f,\pi)| \le \prod_{i=1}^m \operatorname{Lip}(\pi_i|_U) \int_X |f|\, d\|T\| \tag{4.8}$$

whenever $(f,\pi) \in \mathscr{B}^\infty_{\rm c}(X) \times [\operatorname{Lip}_{\rm loc}(X)]^m$ and $U \Subset X$ is a neighborhood of $\operatorname{spt}(f)$.

Given $(f, \pi), (f^1, \pi^1), (f^2, \pi^2), \ldots$ and K as in (2), choose a neighborhood $U \Subset X$ of K. Since $\pi^j \to \pi$ in $[\mathrm{Lip}_{\mathrm{loc}}(X)]^m$, there is a constant l such that $\sup_j \mathrm{Lip}(\pi_i^j|_U) \le l$ and $\mathrm{Lip}(\pi_i|_U) \le l$ for $i = 1, \ldots, m$. We have

$$|T(f^j, \pi^j) - T(f, \pi)| \le |T(f^j - f, \pi^j)| + |T(f, \pi^j) - T(f, \pi)|, \tag{4.9}$$

and $|T(f^j - f, \pi^j)| \le l^m \int_X |f^j - f| \, d\|T\| \to 0$, by the bounded convergence theorem. For the second term on the right side, given $\epsilon > 0$, choose $g \in \mathscr{D}(X)$ such that $\mathrm{spt}(g) \subset U$ and $l^m \int_X |f - g| \, d\|T\| < \epsilon/3$. For j sufficiently large, $|T(g, \pi^j) - T(g, \pi)| < \epsilon/3$ by the continuity of T, hence

$$\begin{aligned} &|T(f, \pi^j) - T(f, \pi)| \\ &\quad \le |T(f - g, \pi^j)| + |T(f - g, \pi)| + |T(g, \pi^j) - T(g, \pi)| < \epsilon. \end{aligned}$$

This shows (2).

By the locality of T, clearly $T(f, \pi) = 0$ if some π_i is constant on a neighborhood of the support of $f \in \mathscr{B}_{\mathrm{c}}^\infty(X)$. Hence, for the proof of (3), there is no loss of generality in assuming that $\pi_i = 0$ on $\mathrm{spt}(f)$. Let β_j be the function defined after (2.5). Then $\beta_j \circ \pi_i = 0$ on some neighborhood of $\mathrm{spt}(f)$, and letting j tend to infinity we obtain $T(f, \pi) = 0$ by means of (2).

Finally, (4) follows from (4.8) by (3) and the extendability of Lipschitz functions. □

The extension of T allows to define $T \lfloor u$ more generally for locally bounded Borel functions u, in particular for characteristic functions of Borel sets. This complements Definition 2.3.

Definition 4.5. *For $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$ and $(u, v) \in \mathscr{B}_{\mathrm{loc}}^\infty(X) \times [\mathrm{Lip}_{\mathrm{loc}}(X)]^k$, where $m \ge k \ge 0$, we define $T \lfloor (u, v) \in \mathbf{M}_{m-k,\mathrm{loc}}(X)$ by the same equation as in Definition 2.3. For a Borel set $B \subset X$,*

$$T \lfloor B := T \lfloor \chi_B.$$

Clearly $T \lfloor (u, v)$ is a current, and it follows from Theorem 4.4(4) that

$$\mathbf{M}_V(T \lfloor (u, v)) \le \prod_{i=1}^k \mathrm{Lip}(v_i|_V) \int_V |u| \, d\|T\| \tag{4.10}$$

for all open sets $V \subset X$ (meaning $\mathbf{M}_V(T \lfloor u) \le \int_V |u| \, d\|T\|$ in case $k = 0$). Since the right side is finite if the closure of V is compact, we have $T \lfloor (u, v) \in \mathbf{M}_{m-k,\mathrm{loc}}(X)$.

The next lemma gives information on push-forwards of currents with locally finite mass.

Lemma 4.6. *Suppose $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$, $m \ge 0$, Y is another locally compact metric space, $F \in \mathrm{Lip}_{\mathrm{loc}}(X, Y)$, and $F|_{\mathrm{spt}(T)}$ is proper. Then $F_\# T \in \mathbf{M}_{m,\mathrm{loc}}(Y)$, and the following properties hold:*

(1) *For all* $(f,\pi) \in \mathscr{B}^\infty_{\rm c}(Y) \times [\mathrm{Lip}_{\rm loc}(Y)]^m$ *and any* $\sigma \in \mathscr{B}^\infty_{\rm c}(X)$ *such that* $\sigma = 1$ *on* $\{f \circ F \neq 0\} \cap \mathrm{spt}(T)$,

$$F_\# T(f,\pi) = T(\sigma(f \circ F), \pi \circ F).$$

(2) *For every Borel set* $B \subset Y$,

$$\mathbf{M}((F_\# T)\lfloor B) \leq \mathrm{Lip}\big(F|_{F^{-1}(B)\cap \mathrm{spt}(T)}\big)^m \|T\|(F^{-1}(B)).$$

Proof. Let $V \Subset Y$ be an open set, and choose $\sigma \in \mathscr{D}(X)$ such that $\sigma = 1$ on $F^{-1}(V) \cap \mathrm{spt}(T)$. It follows from (3.12) and Theorem 4.3(4) that

$$\mathbf{M}_V(F_\# T) \leq \mathrm{Lip}\big(F|_{\mathrm{spt}(\sigma)}\big)^m \|T\|(F^{-1}(V)).$$

Hence $F_\# T \in \mathbf{M}_{m,\rm loc}(Y)$.

To prove (1), fix $\pi \in [\mathrm{Lip}_{\rm loc}(Y)]^m$ and $\rho \in \mathscr{D}(Y)$, $\rho \geq 0$. Choose $\tau \in \mathscr{D}(X)$ such that $\tau = 1$ on $\{\rho \circ F \neq 0\} \cap \mathrm{spt}(T)$, and denote by Φ the set of all $f \in \mathscr{B}^\infty_{\rm c}(Y)$ such that $|f| \leq \rho$ and $F_\# T(f,\pi) = T(\tau(f \circ F), \pi \circ F)$. It follows from Theorem 4.4(2) that Φ is a Baire class. By (3.12), Φ contains all $f \in \mathscr{D}(Y)$ with $|f| \leq \rho$ and therefore consists of all $f \in \mathscr{B}^\infty_{\rm c}(Y)$ with $|f| \leq \rho$, cf. Sect. 1.2. In view of (4.7), this gives (1).

For (2), suppose $(f,\pi) \in \mathscr{D}(Y) \times [\mathrm{Lip}_1(Y)]^m$, and let σ be the characteristic function of $F^{-1}(B) \cap \{f \circ F \neq 0\} \cap \mathrm{spt}(T)$. By (1) and Theorem 4.4(4),

$$\begin{aligned}((F_\# T)\lfloor B)(f,\pi) &= F_\# T(\chi_B f, \pi) = T(\sigma(f \circ F), \pi \circ F) \\ &\leq \mathrm{Lip}\big(F|_{\mathrm{spt}(\sigma)}\big)^m \int_{F^{-1}(B)} |f \circ F| \, d\|T\|.\end{aligned}$$

This yields the result. □

For a current $T \in \mathbf{M}_{m,\rm loc}(X)$ and a Borel set $B \subset X$, we always have $\mathbf{M}(T\lfloor B) \leq \|T\|(B)$. Equality holds, for instance, if $\mathrm{spt}(T)$ is separable; this is shown by the following lemma.

Lemma 4.7 (characterizing $\|T\|$)**.** *Suppose* $T \in \mathbf{M}_{m,\rm loc}(X)$, $m \geq 0$, *and* $B \subset X$ *is either a Borel set that is* σ*-finite with respect to* $\|T\|$ *or an open set. Then* $\|T\|(B)$ *is the least number such that*

$$\sum_{\lambda \in \Lambda} T(f_\lambda, \pi^\lambda) \leq \|T\|(B)$$

whenever Λ *is a finite set,* $(f_\lambda, \pi^\lambda) \in \mathscr{B}^\infty_{\rm c}(X) \times [\mathrm{Lip}_1(X)]^m$, *and* $\sum_{\lambda\in\Lambda} |f_\lambda| \leq \chi_B$. *Moreover*

$$\|T\|\lfloor B = \|T\lfloor B\|,$$

in particular $\|T\|(B) = \mathbf{M}(T\lfloor B)$.

In case $m = 0$, this says that

$$\|T\|(B) = \sup\{T(f) : f \in \mathscr{B}^\infty_{\mathrm{c}}(X),\ |f| \le \chi_B\}. \tag{4.11}$$

In particular,

$$\|T\|(\{x\}) = |T(\chi_{\{x\}})| \tag{4.12}$$

for every $x \in X$.

The following simple example illustrates the σ-finiteness assumption in the lemma. Let R be any uncountable discrete space, and equip $X = R \times (-1, 1)$ with the metric d defined by

$$d((r,s),(r',s')) := \begin{cases} |s - s'| & \text{if } r = r', \\ 1 & \text{if } r \ne r'. \end{cases}$$

Note that X is locally compact. Let $T \in \mathbf{M}_{1,\mathrm{loc}}(X)$ be the current satisfying

$$T(f,\pi) = \int_X f(r,s) \frac{d\pi}{ds}(r,s)\, d\mathscr{H}^1(r,s)$$

for all $(f,\pi) \in \mathscr{D}^1(X)$; then $\|T\| = \mathscr{H}^1$ (compare (4.5)). Since R is uncountable, the closed set $B := R \times \{0\}$ is not σ-finite with respect to $\|T\|$, hence $\|T\| \lfloor B$ is not σ-finite. On the other hand, $(\|T\| \lfloor B)(K) = 0$ for all compact sets $K \subset X$, and $T \lfloor B = 0$.

Proof of Lemma 4.7. We show the first part. By Theorem 4.4(4), the inequality always holds. To see that $\|T\|(B)$ is the least number with this property, let $\epsilon > 0$, and choose an open set V such that $B \subset V$ and $\|T\|(V \setminus B) \le \epsilon$. Note that this is possible by the assumption on B. Let $\alpha < \|T\|(V)$. Then there exist Λ and (f_λ, π^λ) as in the definition of $\mathbf{M}_V(T)$ such that

$$\begin{aligned} \alpha \le \sum_{\lambda\in\Lambda} T(f_\lambda, \pi^\lambda) &= \sum_{\lambda\in\Lambda} T(\chi_B f_\lambda, \pi^\lambda) + \sum_{\lambda\in\Lambda} T(\chi_{V\setminus B} f_\lambda, \pi^\lambda) \\ &\le \|T\|(B) + \epsilon. \end{aligned}$$

This gives the result.

For the second part, it suffices to prove that $\|T\|(B \cap A) = \|T \lfloor B\|(A)$ for every Borel set $A \subset X$. To verify this equality, apply the result of the first part to either side. $\square$

In (4.6) we extended the elements of $\mathbf{M}_{m,\mathrm{loc}}(X)$ to $\mathscr{B}^\infty_{\mathrm{c}}(X) \times [\mathrm{Lip}_{\mathrm{loc}}(X)]^m$. We now extend currents of finite total mass in another direction. This will also establish the connection to the metric currents of [3].

Let $T \in \mathbf{M}_m(X)$. We consider the restriction of T to $\mathscr{D}(X) \times [\mathrm{Lip}(X)]^m$, which determines T uniquely (compare Lemma 2.2). Since $\|T\|$ is finite, $L^1(\|T\|)$ contains the algebra $\mathscr{B}^\infty(X)$ of bounded Borel functions on X. As in Theorem 4.4, it follows from Theorem 4.3(4) that the restriction of T extends to a function

$$T\colon \mathscr{B}^\infty(X) \times [\mathrm{Lip}(X)]^m \to \mathbb{R} \tag{4.13}$$

with the following properties:

(1) (multilinearity) T is $(m+1)$-linear.

(2) (continuity) $T(f^j, \pi^j) \to T(f, \pi)$ whenever $(f, \pi), (f^1, \pi^1), (f^2, \pi^2), \ldots \in \mathscr{B}^\infty(X) \times [\mathrm{Lip}_l(X)]^m$ for some $l \ge 0$, $\sup_j \|f^j\|_\infty < \infty$, and $(f^j, \pi^j) \to (f, \pi)$ pointwise on X.

(3) (locality) In case $m \ge 1$, $T(f, \pi) = 0$ whenever some π_i is constant on $\mathrm{spt}(f)$.

(4) For all $(f, \pi) \in \mathscr{B}^\infty(X) \times [\mathrm{Lip}(X)]^m$,

$$|T(f,\pi)| \le \prod_{i=1}^m \mathrm{Lip}\big(\pi_i|_{\mathrm{spt}(f)}\big) \int_X |f| \, d\|T\|.$$

Furthermore, by (4) and Lemma 4.7, for every Borel set $B \subset X$, $\|T\|(B)$ is the least number such that

$$\sum_{\lambda \in \Lambda} T(f_\lambda, \pi^\lambda) \le \|T\|(B) \tag{4.14}$$

whenever Λ is a finite set, $(f_\lambda, \pi^\lambda) \in \mathscr{B}^\infty(X) \times [\mathrm{Lip}_1(X)]^m$, and $\sum_{\lambda\in\Lambda} |f_\lambda| \le \chi_B$.

Suppose now, for the remaining part of this section, that X is an arbitrary metric space. Combining the extension just described with the discussion of (3.2), we obtain a corresponding normed space $(\mathbf{M}_m(X), \mathbf{M})$ of currents with finite mass and locally compact support in X. By definition, an element T of $\mathbf{M}_m(X)$ is a functional on $\mathscr{B}^\infty(X_T) \times [\mathrm{Lip}(X_T)]^m$ for some closed and locally compact set $X_T \subset X$. However, T may now equally well be viewed as a functional on $\mathscr{B}^\infty(X) \times [\mathrm{Lip}(X)]^m$ (compare (3.1) and Proposition 3.3). A current

$$T \in \mathbf{M}_m(X)$$

is then a function as in (4.13) satisfying (1)–(4) (now for arbitrary X), where $\|T\|$ is a finite Borel regular outer measure that is concentrated on its locally compact support $\mathrm{spt}(\|T\|)$ and characterized by (4.14). Since $\|T\|$ is finite, $\mathrm{spt}(\|T\|)$ is separable (cf. [14, Theorem 2.2.16]) and hence also σ-compact. In contrast to Proposition 4.2, the space $(\mathbf{M}_m(X), \mathbf{M})$ of all such T, where $\mathbf{M}(T) = \|T\|(X)$, is no longer complete in general.

We now denote by $(\mathbf{M}_m^{\mathrm{AK}}(X), \mathbf{M})$ the Banach space of all m-currents in the sense of Ambrosio and Kirchheim, viewed as functionals on $\mathscr{B}^\infty(X) \times [\mathrm{Lip}(X)]^m$, cf. [3, Theorem 3.5]. A current

$$T \in \mathbf{M}_m^{\mathrm{AK}}(X)$$

is a function as in (4.13) satisfying (1)–(3), moreover there exist a σ-compact set $\Sigma \subset X$ and a finite Borel measure μ on X such that $\mu(X \setminus \Sigma) = 0$ and (4) holds with μ in place of $\|T\|$. There is a minimal Borel measure μ_T with this property (denoted $\|T\|$ in [3]), and $\mathbf{M}(T) := \mu_T(X)$. Regarding the existence of Σ, see [3, Lemma 2.9] and the remark thereafter; note that this lemma requires completeness of the underlying metric space.

We now verify that $(\mathbf{M}_m(X), \mathbf{M})$ is a dense subspace of $(\mathbf{M}_m^{\mathrm{AK}}(X), \mathbf{M})$, so that

$$(\mathbf{M}_m^{\mathrm{AK}}(X), \mathbf{M}) \textit{ is the completion of } (\mathbf{M}_m(X), \mathbf{M}). \tag{4.15}$$

Clearly $\mathbf{M}_m(X) \subset \mathbf{M}_m^{\mathrm{AK}}(X)$, and for $T \in \mathbf{M}_m(X)$, (4.14) readily implies that $\|T\|(B) = \mu_T(B)$ for every Borel set $B \subset X$, thus the two definitions of $\mathbf{M}(T)$ agree. Given $T \in \mathbf{M}_m^{\mathrm{AK}}(X)$, there exist compact sets $K_1 \subset K_2 \subset \dots$ in X such that $\mu_T(X \setminus \Sigma) = 0$ for $\Sigma := \bigcup_{k=1}^{\infty} K_k$. The restrictions $T \lfloor K_k$ form a sequence in $\mathbf{M}_m(X)$ converging in $(\mathbf{M}_m^{\mathrm{AK}}(X), \mathbf{M})$ to T.

In the sequel, X will again denote a locally compact metric space.

5 Normal currents

We turn to the chain complex of normal currents. In the first part of this section we prove the compactness theorem for locally normal metric currents. Then we compare metric currents in an open set $U \subset \mathbb{R}^n$ with classical currents, in particular we establish an isomorphism for locally normal currents.

Definition 5.1 (normal current)**.** *For $T \in \mathscr{D}_m(X)$ and every open set $V \subset X$, define*

$$\mathbf{N}_V(T) := \mathbf{M}_V(T) + \mathbf{M}_V(\partial T)$$

if $m \geq 1$ and $\mathbf{N}_V(T) := \mathbf{M}_V(T)$ if $m = 0$, and let $\mathbf{N}(T) := \mathbf{N}_X(T)$. The vector space

$$\mathbf{N}_{m,\mathrm{loc}}(X)$$

of m-dimensional locally normal currents *in X consists of all $T \in \mathscr{D}_m(X)$ such that $\mathbf{N}_V(T) < \infty$ for all open sets $V \Subset X$. An m-dimensional* normal current *in X is an element of*

$$\mathbf{N}_m(X) := \{T \in \mathscr{D}_m(X) : \mathbf{N}(T) < \infty\}.$$

Note that $(\mathbf{N}_m(X), \mathbf{N})$ is a Banach space, cf. Proposition 4.2. If $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$ and $(u, v) \in \mathrm{Lip}_{\mathrm{loc}}(X) \times [\mathrm{Lip}_{\mathrm{loc}}(X)]^k$, where $m > k \geq 0$, then

$$\partial(T \lfloor (u, v)) = (-1)^k \big((\partial T) \lfloor (u, v) - T \lfloor (1, u, v)\big)$$

by Lemma 3.5. Applying (4.10) twice, we get

$$\mathbf{M}_V(\partial(T \lfloor (u, v))) \leq \prod_{i=1}^{k} \mathrm{Lip}(v_i|_V) \left(\int_V |u| \, d\|\partial T\| + \mathrm{Lip}(u|_V) \|T\|(V) \right) \tag{5.1}$$

for every open set $V \subset X$. Together with (4.10), this shows that $T\lfloor(u,v) \in \mathbf{N}_{m-k,\mathrm{loc}}(X)$. By Lemma 4.6 and (3.9), push-forwards of locally normal currents are locally normal.

The following result corresponds to [3, Proposition 5.1].

Lemma 5.2 (uniform continuity of normal currents)**.** *Let $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$, $m \geq 1$.*

(1) *For every $(f,g) \in \mathscr{D}^m(X)$ with $g_2, \dots, g_m \in \mathrm{Lip}_1(X)$,*

$$|T(f,g)| \leq \mathrm{Lip}(f) \int_{\mathrm{spt}(f)} |g_1| \, d\|T\| + \int_X |fg_1| \, d\|\partial T\|.$$

(2) *For all $(f,g), (\tilde{f},\tilde{g}) \in \mathscr{D}(X) \times [\mathrm{Lip}_1(X)]^m$,*

$$\begin{aligned} \big|T(f,g) - T(\tilde{f},\tilde{g})\big| &\leq \int_X |f - \tilde{f}| \, d\|T\| \\ &+ \sum_{i=1}^m \bigg(\mathrm{Lip}(f) \int_{\mathrm{spt}(f)} |g_i - \tilde{g}_i| \, d\|T\| + \int_X |f||g_i - \tilde{g}_i| \, d\|\partial T\| \bigg). \end{aligned}$$

Proof. For the proof of (1) we assume $m = 1$. Let $(f,g) \in \mathscr{D}^1(X)$, and choose $\sigma \in \mathscr{D}(X)$ with $\sigma|_{\mathrm{spt}(f)} = 1$. By the product rule,

$$|T(f,g)| \leq |T(\sigma g, f)| + |T(\sigma, fg)| = |T(\sigma g, f)| + |\partial T(fg)|,$$

hence

$$|T(f,g)| \leq \mathrm{Lip}(f) \int_{\mathrm{spt}(f)} |g| \, d\|T\| + \int_X |fg| \, d\|\partial T\|.$$

For the proof of (2) we observe that

$$\begin{aligned} &T(f,g) - T(\tilde{f},\tilde{g}) \\ &\quad = T(f - \tilde{f}, \tilde{g}) + \sum_{i=1}^m T(f, \tilde{g}_1, \dots, \tilde{g}_{i-1}, g_i - \tilde{g}_i, g_{i+1}, \dots, g_m), \end{aligned}$$

then we apply the alternating property and (1) to each summand. □

Lemma 5.2 yields the following proposition, which will be used repeatedly in the sequel.

Proposition 5.3 (convergence criterion)**.** *Suppose X is compact, $\mathscr{F} \subset \mathrm{Lip}_1(X)$ is dense in $\mathrm{Lip}_1(X)$ with respect to the metric induced by $\|\cdot\|_\infty$, and $(T_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbf{N}_m(X)$, $m \geq 0$, with $M := \sup_n \mathbf{N}(T_n) < \infty$ and the property that the limit $\lim_{n\to\infty} T_n(f,g)$ exists for all $(f,g) \in \mathscr{F} \times \mathscr{F}^m$. Then $(T_n)_{n\in\mathbb{N}}$ converges weakly to some $T \in \mathbf{N}_m(X)$.*

Proof. We assume $m \geq 1$; the case $m = 0$ is similar but easier. Let $(f, g), (\tilde{f}, \tilde{g}) \in \mathscr{D}(X) \times [\mathrm{Lip}_1(X)]^m$, and define

$$R(f, g, \tilde{f}, \tilde{g}) := \|f - \tilde{f}\|_\infty + \max\{\|f\|_\infty, \mathrm{Lip}(f)\} \sum_{i=1}^{m} \|g_i - \tilde{g}_i\|_\infty.$$

For every $n \in \mathbb{N}$, Lemma 5.2(2) gives

$$|T_n(f, g) - T_n(\tilde{f}, \tilde{g})| \leq R(f, g, \tilde{f}, \tilde{g})\mathbf{N}(T_n) \leq R(f, g, \tilde{f}, \tilde{g})M.$$

Fix $(f, g) \in \mathrm{Lip}_1(X) \times [\mathrm{Lip}_1(X)]^m$ for the moment. Given $\epsilon > 0$, there is $(\tilde{f}, \tilde{g}) \in \mathscr{F} \times \mathscr{F}^m$ such that $R(f, g, \tilde{f}, \tilde{g})M \leq \epsilon$, and there is an index n_0 such that $|T_n(\tilde{f}, \tilde{g}) - T_{n'}(\tilde{f}, \tilde{g})| \leq \epsilon$ for all $n, n' \geq n_0$. Then $|T_n(f, g) - T_{n'}(f, g)| \leq 3\epsilon$ for all $n, n' \geq n_0$, so $(T_n(f, g))_{n \in \mathbb{N}}$ is a Cauchy sequence. Define $T \colon \mathrm{Lip}_1(X) \times [\mathrm{Lip}_1(X)]^m \to \mathbb{R}$ such that

$$T(f, g) = \lim_{n \to \infty} T_n(f, g)$$

for all $(f, g) \in \mathrm{Lip}_1(X) \times [\mathrm{Lip}_1(X)]^m$. It follows that $|T(f, g) - T(\tilde{f}, \tilde{g})| \leq R(f, g, \tilde{f}, \tilde{g})M$ for all $(f, g), (\tilde{f}, \tilde{g}) \in \mathrm{Lip}_1(X) \times [\mathrm{Lip}_1(X)]^m$. Now it is clear that T extends uniquely to a current $T \in \mathscr{D}_m(X)$, and $T_n \to T$ weakly. By the lower semicontinuity of mass, $T \in \mathbf{N}_m(X)$. □

We now arrive at a fundamental result for locally normal currents in a locally compact metric space X.

Theorem 5.4 ($\mathbf{N}_{m,\mathrm{loc}}$ compactness)**.** *Suppose $(T_n)_{n \in \mathbb{N}}$ is a sequence in $\mathbf{N}_{m,\mathrm{loc}}(X)$, $m \geq 0$, such that each $\mathrm{spt}(T_n)$ is separable and $\sup_n \mathbf{N}_V(T_n) < \infty$ for every open set $V \Subset X$. Then there is a subsequence $(T_{n(i)})_{i \in \mathbb{N}}$ that converges weakly to some $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$.*

Compare [3, Theorem 5.2].

Proof. Assume first that X is compact, so that $M := \sup_n \mathbf{N}(T_n) < \infty$. Choose a countable set $\mathscr{F} \subset \mathrm{Lip}_1(X)$ as in Proposition 5.3. For every $(f, g) \in \mathscr{F} \times \mathscr{F}^m$,

$$|T_n(f, g)| \leq \|f\|_\infty \mathbf{M}(T_n) \leq \|f\|_\infty M$$

for all n. A diagonal process then yields a subsequence $(T_{n(i)})_{i \in \mathbb{N}}$ such that the limit $\lim_{i \to \infty} T_{n(i)}(f, g)$ exists for every $(f, g) \in \mathscr{F} \times \mathscr{F}^m$. By Proposition 5.3, this subsequence converges weakly to some $T \in \mathbf{N}_m(X)$.

In the general case, the closure of $\bigcup_{n \in \mathbb{N}} \mathrm{spt}(T_n)$ is separable, thus there is no loss of generality in assuming that X itself is separable. Then there exists a countable set $\Sigma \subset \mathscr{D}(X)$ such that for every compact set $K \subset X$ there is a $\sigma \in \Sigma$ with $\sigma|_K = 1$. Note that

$$\mathbf{N}(T_n \llcorner \sigma) \leq (\|\sigma\|_\infty + \mathrm{Lip}(\sigma))\mathbf{N}_V(T_n)$$

whenever $\sigma \in \mathscr{D}(X)$ and $V \subset X$ is an open set containing $\operatorname{spt}(\sigma)$. Hence, for each $\sigma \in \Sigma$, we may apply the first part to the restrictions $T_n \lfloor \sigma$, temporarily viewed as currents in the compact set $\operatorname{spt}(\sigma)$. In combination with a diagonal process, this allows to extract a subsequence $(T_{n(i)})_{i\in\mathbb{N}}$ such that for every $\sigma \in \Sigma$, the $T_{n(i)} \lfloor \sigma$ converge weakly to some $T_\sigma \in \mathbf{N}_m(X)$. Finally, for every $(f,g) \in \mathscr{D}^m(X)$, choose $\sigma \in \Sigma$ with $\sigma|_{\operatorname{spt}(f)} = 1$ and put $T(f,g) := T_\sigma(f,g)$. This defines a locally normal current T in X, and $T_{n(i)} \to T$ weakly. □

In the context of classical currents, there is a similar compactness theorem for currents with locally finite mass, cf. [29, Lemma 26.14]. Such a result is not available for metric currents. For instance, let $(\eta_j)_{j\in\mathbb{N}}$ be a sequence of mollifiers on $\mathbb{R}^m$, as in Sect. 1.5. The corresponding currents $[\eta_j] \in \mathscr{D}_m(\mathbb{R}^m)$ (cf. Proposition 2.6) satisfy $\mathbf{M}([\eta_j]) = 1$ and $\operatorname{spt}([\eta_j]) \subset \mathrm{U}(0, 1/j)$. However, no subsequence converges weakly to a current in $\mathscr{D}_m(\mathbb{R}^m)$, for there is no metric m-current for $m \geq 1$ whose support is a single point, cf. Lemma 3.2(3). Viewed as classical currents, the $[\eta_j]$ converge weakly to the classical m-current $\bar{T}$ satisfying $\bar{T}(f\,dx_1 \wedge \ldots \wedge dx_m) = f(0)$ for all $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^m)$.

We now examine the relation between metric and classical currents more closely. To this end we first recall a number of definitions and results from [14]. We adopt the notation from there, except that we put a bar on the symbols $\mathscr{D}, \mathbf{M}, \mathbf{N}, \mathbf{F}$ to distinguish those spaces of forms, currents, and seminorms from their metric analogues.

For an open subset U of $\mathbb{R}^n$, $\bar{\mathscr{D}}^m(U)$ denotes the vector space of compactly supported C^∞ m-forms on U, endowed with the usual locally convex C^∞ topology, and $\bar{\mathscr{D}}_m(U)$ denotes the dual space, consisting of all m-dimensional currents in U, cf. [29, p. 131f] and [14, 4.1.7]. The *comass* $\|\phi\|$ of an m-covector $\phi \in \Lambda^m\mathbb{R}^n$ is the supremum of $\langle \xi, \phi\rangle$ over all simple m-vectors $\xi \in \Lambda_m\mathbb{R}^n$ with Euclidean norm $|\xi| \leq 1$, where $\langle\cdot,\cdot\rangle$ denotes the dual pairing. The *mass* $\|\xi\|$ of an m-vector $\xi \in \Lambda_m\mathbb{R}^n$ is the supremum of $\langle \xi, \phi\rangle$ over all $\phi \in \Lambda^m\mathbb{R}^n$ with comass $\|\phi\| \leq 1$, cf. [14, 1.8.1]. The *comass norm* of an m-form $\phi \in \bar{\mathscr{D}}^m(U)$ is given by

$$\bar{\mathbf{M}}(\phi) = \sup\nolimits_{x\in U} \|\phi(x)\|.$$

Let $T \in \bar{\mathscr{D}}_m(U)$. For every open set $V \subset U$, put

$$\bar{\mathbf{M}}_V(T) := \sup\bigl\{T(\phi) : \phi \in \bar{\mathscr{D}}^m(U),\ \operatorname{spt}(\phi) \subset V,\ \bar{\mathbf{M}}(\phi) \leq 1\bigr\}. \tag{5.2}$$

The number $\bar{\mathbf{M}}(T) := \bar{\mathbf{M}}_U(T) \in [0,\infty]$ is the *mass* of T. As in Theorem 4.3, one obtains a Borel regular outer measure $\|T\|$ such that

$$\|T\|(A) = \inf\bigl\{\bar{\mathbf{M}}_V(T) : V \subset U \text{ is open, } A \subset V\bigr\}$$

for every set $A \subset U$. If $\|T\|$ is locally finite, then it is a Radon measure, and there exists a $\|T\|$-measurable m-vector field ξ on U so that $\|\xi(x)\| = 1$ for $\|T\|$-almost every $x \in U$ and

$$T(\phi) = \int_U \langle \xi(x), \phi(x)\rangle\, d\|T\|(x) \leq \int_U \|\phi(x)\|\, d\|T\|(x)$$

for all $\phi \in \bar{\mathscr{D}}^m(U)$, cf. [14, 4.1.5 and p. 349] and [29, 26.4–26.8]. The seminorm $\bar{\mathbf{N}}$ on $\bar{\mathscr{D}}_m(U)$ and the space $\bar{\mathbf{N}}_{m,\mathrm{loc}}(U)$ of *locally normal currents* are defined in analogy to Definition 5.1. For a compact set $K \subset U$, $\bar{\mathbf{N}}_{m,K}(U)$ denotes the set of all $T \in \bar{\mathscr{D}}_m(U)$ with $\mathrm{spt}(T) \subset K$ and $\bar{\mathbf{N}}(T) < \infty$, cf. [14, p. 358].

The *flat seminorm* of a form $\phi \in \bar{\mathscr{D}}^m(U)$ relative to a compact set $K \subset U$ is given by

$$\bar{\mathbf{F}}_K(\phi) = \sup\bigl\{\sup_{x\in K} \|\phi(x)\|, \sup_{x\in K} \|d\phi(x)\|\bigr\},$$

the respective *flat seminorm* of a current $T \in \bar{\mathscr{D}}_m(U)$ by

$$\bar{\mathbf{F}}_K(T) = \sup\bigl\{T(\phi) : \phi \in \bar{\mathscr{D}}^m(U),\, \bar{\mathbf{F}}_K(\phi) \le 1\bigr\}.$$

Note that $\bar{\mathbf{F}}_K(\partial T) \le \bar{\mathbf{F}}_K(T)$ for $m \ge 1$. If $\bar{\mathbf{F}}_K(T) < \infty$, then $\mathrm{spt}(T) \subset K$. If $T \in \bar{\mathscr{D}}_m(U)$ and $\mathrm{spt}(T) \subset K$, then

$$\bar{\mathbf{F}}_K(T) \le \bar{\mathbf{M}}(T - \partial S) + \bar{\mathbf{M}}(S)$$

for all $S \in \bar{\mathscr{D}}_{m+1}(U)$ with $\mathrm{spt}(S) \subset K$, and equality holds for at least one such S. The $\bar{\mathbf{F}}_K$-closure of $\bar{\mathbf{N}}_{m,K}(U)$ in $\bar{\mathscr{D}}_m(U)$ is denoted by $\bar{\mathbf{F}}_{m,K}(U)$. The space $\bar{\mathbf{F}}_m(U)$ of *flat chains* with compact support in U is the union of all $\bar{\mathbf{F}}_{m,K}(U)$. The space $\bar{\mathbf{F}}_{m,\mathrm{loc}}(U)$ of *locally flat chains* consists of all $T \in \bar{\mathscr{D}}_m(U)$ such that $T \lfloor \sigma \in \bar{\mathbf{F}}_m(U)$ for every $\sigma \in C_{\mathrm{c}}^\infty(U)$, cf. [14, 4.1.12].

Finally, we note that the set $\bigl\{T \in \bar{\mathbf{F}}_{m,K}(U) : \bar{\mathbf{M}}(T) < \infty\bigr\}$ equals the $\bar{\mathbf{M}}$-closure of $\bar{\mathbf{N}}_{m,K}(U)$ in $\bar{\mathscr{D}}_m(U)$, cf. [14, 4.1.17], and

$$\begin{aligned}\bar{\mathbf{F}}_{m,K}(U) = \bigl\{R + \partial S : & R \in \bar{\mathbf{F}}_{m,K}(U), \bar{\mathbf{M}}(R) < \infty, \\ & S \in \bar{\mathbf{F}}_{m+1,K}(U),\, \bar{\mathbf{M}}(S) < \infty\bigr\}, \end{aligned} \tag{5.3}$$

cf. [14, p. 382].

Theorem 5.5 (comparison map)**.** *Let $U \subset \mathbb{R}^n$ be an open set, $n \ge 1$. For every $m \ge 0$, there exists an injective linear map $C_m \colon \mathscr{D}_m(U) \to \bar{\mathscr{D}}_m(U)$ such that*

$$C_m(T)(f\, dg_1 \wedge \ldots \wedge dg_m) = T(f, g_1, \ldots, g_m)$$

for all $(f, g_1, \ldots, g_m) \in C_{\mathrm{c}}^\infty(U) \times [C^\infty(U)]^m$. The following properties hold:

(1) *For $m \ge 1$, $\partial \circ C_m = C_{m-1} \circ \partial$.*

(2) *For all $T \in \mathscr{D}_m(U)$, $\|T\| \le \|C_m(T)\| \le \binom{n}{m}\|T\|$.*

(3) *The restriction of C_m to $\mathbf{N}_{m,\mathrm{loc}}(U)$ is an isomorphism onto $\bar{\mathbf{N}}_{m,\mathrm{loc}}(U)$.*

(4) *The image of C_m contains the space $\bar{\mathbf{F}}_{m,\mathrm{loc}}(U)$ of m-dimensional locally flat chains in U.*

For currents with finite mass and compact support in $\mathbb{R}^n$ this result was proved by Ambrosio and Kirchheim in [3, Theorem 11.1]. They conjectured that the image under C_m of $\{T \in \mathbf{M}_m(\mathbb{R}^n) : \mathrm{spt}(T) \text{ is compact}\}$ coincides with the space of m-dimensional flat chains with finite mass and compact support in $\mathbb{R}^n$. In view of (4) and the many analogous properties of $\mathscr{D}_m(U)$ and $\bar{\mathbf{F}}_{m,\mathrm{loc}}(U)$, one may similarly ask whether $C_m(\mathscr{D}_m(U)) = \bar{\mathbf{F}}_{m,\mathrm{loc}}(U)$.

Proof. In case $m > n$, $\mathscr{D}_m(U) = \{0\}$ by (2.10), and also $\bar{\mathscr{D}}_m(U) = \{0\}$. Thus C_m is the trivial map in this case.

In case $m = 0$, given $T \in \mathscr{D}_0(U)$, $C_0(T)$ is the functional satisfying $C_0(T)(f) = T(f)$ for all $f \in \bar{\mathscr{D}}^0(U) = C^\infty_{\mathrm{c}}(U)$. The continuity property of T implies that $C_0(T)$ is sequentially continuous on $\bar{\mathscr{D}}^0(U)$. This yields $C_0(T) \in \bar{\mathscr{D}}_0(U)$ (cf. [27, Theorem 6.6]).

Now let $T \in \mathscr{D}_m(U)$, $1 \le m \le n$. To define $C_m(T)$, write $\phi \in \bar{\mathscr{D}}^m(U)$ as $\phi = \sum_{\lambda\in\Lambda(n,m)} \phi_\lambda \, dx_{\lambda(1)} \wedge \ldots \wedge dx_{\lambda(m)}$, $\phi_\lambda \in C^\infty_{\mathrm{c}}(U)$, and put

$$C_m(T)(\phi) := \sum_{\lambda\in\Lambda(n,m)} T(\phi_\lambda, \pi_{\lambda(1)}, \ldots, \pi_{\lambda(m)}),$$

where $\pi_i \colon U \to \mathbb{R}$ is the ith coordinate projection, $\pi_i(x) = x_i$. As above, due to the continuity of T in the first argument, this defines an element of $\bar{\mathscr{D}}_m(U)$. For $(f, g_1, \ldots, g_m) \in C^\infty_{\mathrm{c}}(U) \times [C^\infty(U)]^m$, the coefficients of the form $\phi = f\, dg_1 \wedge \ldots \wedge dg_m$ are given by $\phi_\lambda = f \det[D_{\lambda(k)} g_i]_{i,k=1}^m$. Thus

$$C_m(T)(\phi) = \sum_{\lambda\in\Lambda(n,m)} T(\phi_\lambda, \pi_{\lambda(1)}, \ldots, \pi_{\lambda(m)}) = T(f, g_1, \ldots, g_m)$$

by Theorem 2.5 (chain rule).

As for the injectivity of C_m in the case $0 \le m \le n$, it suffices to note that for every nonzero $T \in \mathscr{D}_m(U)$ one finds, by approximation, an $(f, g_1, \ldots, g_m) \in C^\infty_{\mathrm{c}}(U) \times [C^\infty(U)]^m$ with

$$C_m(T)(f\, dg_1 \wedge \ldots \wedge dg_m) = T(f, g_1, \ldots, g_m) \neq 0.$$

For (1), let $T \in \mathscr{D}_m(U)$ and $(f, g_1, \ldots, g_{m-1}) \in C^\infty_{\mathrm{c}}(U) \times [C^\infty(U)]^{m-1}$, and choose $\sigma \in C^\infty_{\mathrm{c}}(U)$ with $\sigma|_{\mathrm{spt}(f)} = 1$. Then

$$\begin{aligned}\partial(C_m(T))(f\, dg_1 \wedge \ldots \wedge dg_{m-1}) &= C_m(T)(\sigma\, df \wedge dg_1 \wedge \ldots \wedge dg_{m-1}) \\ &= T(\sigma, f, g_1, \ldots, g_{m-1}) \\ &= \partial T(f, g_1, \ldots, g_{m-1}) \\ &= C_{m-1}(\partial T)(f\, dg_1 \wedge \ldots \wedge dg_{m-1}).\end{aligned}$$

To prove (2), let $T \in \mathscr{D}_m(U)$, and let $V \subset U$ be an open set. If Λ is a finite set, $(f_\lambda, g^\lambda) \in C^\infty_{\mathrm{c}}(U) \times [C^\infty(U) \cap \mathrm{Lip}_1(U)]^m$, $\mathrm{spt}(f_\lambda) \subset V$, and

$\sum_{\lambda\in\Lambda}|f_\lambda|\le 1$, then the form $\phi := \sum_{\lambda\in\Lambda} f_\lambda\, dg_1^\lambda\wedge\ldots\wedge dg_m^\lambda \in \bar{\mathscr{D}}^m(U)$ has comass norm $\bar{\mathbf{M}}(\phi)\le 1$, hence

$$\sum_{\lambda\in\Lambda} T(f_\lambda, g^\lambda) = C_m(T)(\phi) \le \bar{\mathbf{M}}_V(C_m(T)).$$

This implies that $\mathbf{M}_V(T)\le \bar{\mathbf{M}}_V(C_m(T))$. Conversely, for every form $\phi = \sum_{\lambda\in\Lambda(n,m)}\phi_\lambda\, dx_{\lambda(1)}\wedge\ldots\wedge dx_{\lambda(m)}\in\bar{\mathscr{D}}^m(U)$ with $\operatorname{spt}(\phi)\subset V$ and $\bar{\mathbf{M}}(\phi)\le 1$, we have $|\phi_\lambda|\le 1$ for all $\lambda\in\Lambda(n,m)$, so that

$$C_m(T)(\phi) = \sum_{\lambda\in\Lambda(n,m)} T(\phi_\lambda, \pi_{\lambda(1)},\ldots,\pi_{\lambda(m)}) \le \binom{n}{m}\mathbf{M}_V(T).$$

Hence $\bar{\mathbf{M}}_V(C_m(T))\le\binom{n}{m}\mathbf{M}_V(T)$. This yields (2).

We prove (3). From (1) and (2) it follows that $C_m(T)\in\bar{\mathbf{N}}_{m,\mathrm{loc}}(U)$ if and only if $T\in\mathbf{N}_{m,\mathrm{loc}}(U)$. Hence, since C_m is injective, it suffices to construct a map $\bar{C}_m\colon\bar{\mathbf{N}}_{m,\mathrm{loc}}(U)\to\mathscr{D}_m(U)$ such that $C_m(\bar{C}_m(\bar{T}))=\bar{T}$ for all $\bar{T}\in\bar{\mathbf{N}}_{m,\mathrm{loc}}(U)$. Let $\bar{T}\in\bar{\mathbf{N}}_{m,\mathrm{loc}}(U)$. We first observe that whenever $(f,g),(\tilde{f},\tilde{g})\in C_c^\infty(U)\times[C^\infty(U)\cap\operatorname{Lip}_1(U)]^m$, then

$$\begin{aligned}\big|\bar{T}(f\,dg_1\wedge\ldots\wedge dg_m) - \bar{T}(\tilde{f}\,d\tilde{g}_1\wedge\ldots\wedge d\tilde{g}_m)\big| &\le \int_U |f-\tilde{f}|\,d\|\bar{T}\| \\ +\sum_{i=1}^m\bigg(\operatorname{Lip}(f)\int_{\operatorname{spt}(f)}|g_i-\tilde{g}_i|\,d\|\bar{T}\| &+ \int_U |f||g_i-\tilde{g}_i|\,d\|\partial\bar{T}\|\bigg); \qquad (5.4)\end{aligned}$$

this is just the "classical" analogue of Lemma 5.2(2). To define $\bar{C}_m(\bar{T})$, let $(f,g)\in\mathscr{D}(U)\times[\operatorname{Lip}_1(U)]^m$, and choose a sequence $((f^k,g^k))_{k\in\mathbb{N}}$ in $C_c^\infty(U)\times[C^\infty(U)\cap\operatorname{Lip}_1(U)]^m$ such that $f^k\to f$ in $\mathscr{D}(U)$ and $g_i^k\to g_i$ locally uniformly for $i=1,\ldots,m$. It follows from (5.4) that $(\bar{T}(f^k\,dg_1^k\wedge\ldots\wedge dg_m^k))_{k\in\mathbb{N}}$ is a Cauchy sequence whose limit is independent of the choice of the sequence $((f^k,g^k))_{k\in\mathbb{N}}$. Put

$$\bar{C}_m(\bar{T})(f,g) := \lim_{k\to\infty}\bar{T}(f^k\,dg_1^k\wedge\ldots\wedge dg_m^k).$$

Let $(\tilde{f},\tilde{g})\in\mathscr{D}(U)\times[\operatorname{Lip}_1(U)]^m$ be another such tuple. By choosing the approximating sequences appropriately, we see that (5.4) holds in the limit, i.e., $\big|\bar{C}_m(\bar{T})(f,g)-\bar{C}_m(\bar{T})(\tilde{f},\tilde{g})\big|$ is less than or equal to the expression on the right side of (5.4). Now it is clear that $\bar{C}_m(\bar{T})$ extends to a current $\bar{C}_m(\bar{T})\in\mathscr{D}_m(U)$ satisfying

$$\bar{C}_m(\bar{T})(f,g) = \bar{T}(f\,dg_1\wedge\ldots\wedge dg_m)$$

for all $(f,g)\in C_c^\infty(U)\times[C^\infty(U)]^m$. As the left side of this last equality equals $C_m(\bar{C}_m(\bar{T}))(f\,dg_1\wedge\ldots\wedge dg_m)$, we have $C_m(\bar{C}_m(\bar{T}))=\bar{T}$.

It remains to prove (4). First we observe that for every compact set $K\subset U$, the restriction of $\bar{C}_m$ to $\bar{\mathbf{N}}_{m,K}(U)$ naturally extends to a map from the set $\{\bar{T} \in \bar{\mathbf{F}}_{m,K}(U) : \bar{\mathbf{M}}(\bar{T}) < \infty\}$, which equals the $\bar{\mathbf{M}}$-closure of $\bar{\mathbf{N}}_{m,K}(U)$ in $\bar{\mathscr{D}}_m(U)$, into $\mathbf{M}_m(U)$. Since the restriction of $\bar{C}_m$ is 1-Lipschitz with respect to $\bar{\mathbf{M}}$ and $\mathbf{M}$, this follows by the completeness of $(\mathbf{M}_m(U), \mathbf{M})$ (cf. Proposition 4.2). For every $\bar{T} \in \bar{\mathbf{F}}_{m,K}(U)$ with $\bar{\mathbf{M}}(\bar{T}) < \infty$, we still have $C_m(\bar{C}_m(\bar{T})) = \bar{T}$. Next, let $\bar{T} \in \bar{\mathbf{F}}_{m,K}(U)$. Choose $\bar{R} \in \bar{\mathbf{F}}_{m,K}(U)$ with $\bar{\mathbf{M}}(\bar{R}) < \infty$ and $\bar{S} \in \bar{\mathbf{F}}_{m+1,K}(U)$ with $\bar{\mathbf{M}}(\bar{S}) < \infty$ such that $\bar{T} = \bar{R} + \partial\bar{S}$, cf. (5.3), and put $\bar{C}_m(\bar{T}) := \bar{C}_m(\bar{R}) + \partial(\bar{C}_{m+1}(\bar{S})) \in \mathscr{D}_m(U)$. Then

$$C_m(\bar{C}_m(\bar{T})) = C_m(\bar{C}_m(\bar{R})) + \partial\big(C_{m+1}(\bar{C}_{m+1}(\bar{S}))\big) = \bar{T}.$$

Since C_m is injective, this identity also shows that $\bar{C}_m(\bar{T})$ is well-defined. Finally, suppose that $\bar{T} \in \bar{\mathbf{F}}_{m,\mathrm{loc}}(U)$. Given $(f, g_1, \ldots, g_m) \in \mathscr{D}^m(U)$, choose $\sigma \in C_c^\infty(U)$ such that $\sigma = 1$ on some neighborhood of $\mathrm{spt}(f)$ and put $\bar{C}_m(\bar{T})(f, g_1, \ldots, g_m) := \bar{C}_m(\bar{T} \lfloor \sigma)(f, g_1, \ldots, g_m)$. It is readily checked that this yields a well-defined $\bar{C}_m(\bar{T}) \in \mathscr{D}_m(U)$ with $C_m(\bar{C}_m(\bar{T})) = \bar{T}$. □

For an example of a current $T \in \mathbf{N}_{m,\mathrm{loc}}(\mathbb{R}^n)$ with $\|T\| \neq \|C_m(T)\|$, let $T \in \mathbf{N}_{2,\mathrm{loc}}(\mathbb{R}^4)$ be defined by

$$T(f, g, h) = \int_{\mathbb{R}^4} f\big(D_1 g D_2 h - D_2 g D_1 h + D_3 g D_4 h - D_4 g D_3 h\big)\, dx$$

for $(f, g, h) \in \mathscr{D}^2(\mathbb{R}^4)$. The corresponding classical current $\bar{T} = C_2(T) \in \bar{\mathbf{N}}_{2,\mathrm{loc}}(\mathbb{R}^4)$ is given by

$$\bar{T}(\phi) = \int_{\mathbb{R}^4} \langle e_1 \wedge e_2 + e_3 \wedge e_4, \phi\rangle\, dx$$

for $\phi \in \bar{\mathscr{D}}^2(\mathbb{R}^4)$. Note that these currents have boundary zero. To compute the mass, use the following inequality: If v, w are vectors in $\mathbb{R}^4$ with Euclidean norm $|v|, |w| \leq 1$, then

$$|v_1 w_2 - v_2 w_1 + v_3 w_4 - v_4 w_3| = |\langle(-v_2, v_1, -v_4, v_3), w\rangle| \leq 1,$$

with equality if and only if $w = \pm(-v_2, v_1, -v_4, v_3)$. It follows that $\|T\| = \mathscr{L}^4$. The same inequality also shows that the form $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ has comass norm 1 (a special case of Wirtinger's inequality, cf. [14, p. 40]), which leads to $\|\bar{T}\| = 2\mathscr{L}^4$.

6 Slicing

We now develop the slicing theory for locally normal m-currents in X with respect to a locally Lipschitz map $\pi \colon X \to \mathbb{R}^k$, where $1 \leq k \leq m$. The slices are currents of dimension $m - k$ in the level sets of π. General references are [14, 4.2.1 and 4.3.1–5], [29, 28.6–28.10] for classical currents and [3, pp. 31–36] for metric currents of finite mass. We first treat the case $k = 1$, which suffices for most geometric applications. The general case will be relevant in Sect. 7 and in Theorems 8.4 and 8.5.

Definition 6.1 (codimension one slices of normal currents)**.** *Suppose* $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$, $m \geq 1$, $\pi \in \mathrm{Lip}_{\mathrm{loc}}(X)$, *and* $s \in \mathbb{R}$. *The* left-hand *and* right-hand slices *of* T *at* s *with respect to* π *are the currents in* $\mathscr{D}_{m-1}(X)$ *defined by*

$$\begin{aligned}\langle T, \pi, s-\rangle &:= \partial(T\lfloor\{\pi < s\}) - (\partial T)\lfloor\{\pi < s\} \\ &= (\partial T)\lfloor\{\pi \geq s\} - \partial(T\lfloor\{\pi \geq s\}), \\ \langle T, \pi, s+\rangle &:= \partial(T\lfloor\{\pi \leq s\}) - (\partial T)\lfloor\{\pi \leq s\} \\ &= (\partial T)\lfloor\{\pi > s\} - \partial(T\lfloor\{\pi > s\}),\end{aligned}$$

respectively.

Note that since $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$ and $\partial T \in \mathbf{M}_{m-1,\mathrm{loc}}(X)$, the restrictions $T\lfloor B$ and $(\partial T)\lfloor B$ are defined for every Borel set $B \subset X$. Moreover, by Theorem 4.4(1), $T\lfloor B + T\lfloor(X \setminus B) = T$. If $(\|T\| + \|\partial T\|)(\pi^{-1}\{s\}) = 0$, then $T\lfloor\pi^{-1}\{s\} = 0$ and $(\partial T)\lfloor\pi^{-1}\{s\} = 0$, thus $\langle T, \pi, s-\rangle = \langle T, \pi, s+\rangle$. It follows that if $\mathrm{spt}(T)$ is separable, then

$$\langle T, \pi, s-\rangle = \langle T, \pi, s+\rangle \tag{6.1}$$

for all but countably many $s \in \mathbb{R}$. We now focus on right-hand slices.

From the two representations of $\langle T, \pi, s+\rangle$ we see that

$$\mathrm{spt}(\langle T, \pi, s+\rangle) \subset \pi^{-1}\{s\} \cap \mathrm{spt}(T) \tag{6.2}$$

for every $s \in \mathbb{R}$. If $m \geq 2$, then

$$\langle \partial T, \pi, s+\rangle = -\partial\langle T, \pi, s+\rangle \tag{6.3}$$

for every $s \in \mathbb{R}$.

There is a useful characterization of $\langle T, \pi, s+\rangle$ as a weak limit. For this we approximate the characteristic function $\chi_{\{\pi>s\}}$ by a family $(u_{s,\delta})_{\delta>0}$ in $\mathrm{Lip}_{\mathrm{loc}}(X)$ such that

$$0 \leq \chi_{\{\pi>s\}} - u_{s,\delta} \leq \chi_{\{s<\pi<s+\delta\}}$$

on X. (A natural choice is $u_{s,\delta} := \gamma_{s,\delta} \circ \pi$ for the piecewise affine $(1/\delta)$-Lipschitz function $\gamma_{s,\delta} \colon \mathbb{R} \to \mathbb{R}$ with $\gamma_{s,\delta}|_{(-\infty,s]} = 0$ and $\gamma_{s,\delta}|_{[s+\delta,\infty)} = 1$.) By (3.6), $T\lfloor(1, u_{s,\delta}) = (\partial T)\lfloor u_{s,\delta} - \partial(T\lfloor u_{s,\delta})$, hence

$$\langle T, \pi, s+\rangle - T\lfloor(1, u_{s,\delta}) = (\partial T)\lfloor(\chi_{\{\pi>s\}} - u_{s,\delta}) - \partial(T\lfloor(\chi_{\{\pi>s\}} - u_{s,\delta})).$$

It follows that

$$\langle T, \pi, s+\rangle = \lim_{\delta\to 0+} T\lfloor(1, u_{s,\delta}) \tag{6.4}$$

for the weak limit.

Theorem 6.2 (codimension one slices of normal currents)**.** *Suppose* $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$, $m \geq 1$, *and* $\pi \in \mathrm{Lip}_{\mathrm{loc}}(X)$.

(1) *For every* $s \in \mathbb{R}$ *and every open set* $V \subset X$,

$$\|\langle T, \pi, s+\rangle\|(V) \le \liminf_{\delta \to 0+} \frac{1}{\delta} \|T \lfloor (1, \pi)\|(V \cap \{s < \pi < s + \delta\}).$$

(2) *For all* $(f, g) \in \mathscr{D}^{m-1}(X)$,

$$\int_{\mathbb{R}} \langle T, \pi, s+\rangle(f, g)\, ds = (T \lfloor (1, \pi))(f, g).$$

(3) *For every* $\|T \lfloor (1, \pi)\|$*-measurable set* $B \subset X$ *with* $\|T \lfloor (1, \pi)\|(B) < \infty$,

$$\int_{\mathbb{R}} \|\langle T, \pi, s+\rangle\|(B)\, ds = \|T \lfloor (1, \pi)\|(B).$$

(4) *If* $\operatorname{spt}(T)$ *is separable, then* $\langle T, \pi, s+\rangle \in \mathbf{N}_{m-1,\mathrm{loc}}(X)$ *for almost every* $s \in \mathbb{R}$. *If* $\pi|_{\operatorname{spt}(T)}$ *is proper, or if* $T \in \mathbf{N}_m(X)$ *and* $\pi \in \operatorname{Lip}(X)$, *then* $\langle T, \pi, s+\rangle \in \mathbf{N}_{m-1}(X)$ *for almost every* $s \in \mathbb{R}$.

Proof. For every $\delta > 0$, fix a non-decreasing function $\gamma_\delta \in C^{1,1}(\mathbb{R})$ such that $\gamma_\delta|_{(-\infty,0]} = 0$ and $\gamma_\delta|_{[\delta,\infty)} = 1$. Let $s \in \mathbb{R}$. Define $\gamma_{s,\delta}(t) := \gamma_\delta(t-s)$ for $t \in \mathbb{R}$, and put $u_{s,\delta} := \gamma_{s,\delta} \circ \pi$. By Theorem 2.5 (chain rule),

$$T \lfloor (1, u_{s,\delta}) = T \lfloor (\gamma'_{s,\delta} \circ \pi, \pi) = (T \lfloor (1, \pi)) \lfloor (\gamma'_{s,\delta} \circ \pi).$$

Hence, for every open set $V \subset X$, (4.10) gives

$$\begin{aligned}\|T \lfloor (1, u_{s,\delta})\|(V) &\le \int_V \gamma'_{s,\delta} \circ \pi \, d\|T \lfloor (1, \pi)\| \\ &\le \operatorname{Lip}(\gamma_{s,\delta}) \|T \lfloor (1, \pi)\|(V \cap \{s < \pi < s + \delta\}),\end{aligned}$$

since $\gamma'_{s,\delta} \circ \pi$ is zero outside $\{s < \pi < s + \delta\}$. Choosing the function γ_δ appropriately, with $\operatorname{Lip}(\gamma_\delta) \le 1/\delta + \delta$ say, we obtain (1) by (6.4) and the lower semicontinuity of mass.

For the proof of (2) we assume $(f, g) \in \mathscr{D}(X) \times [\operatorname{Lip}_1(X)]^{m-1}$. Choose $a \in \mathbb{R}$ such that $\operatorname{spt}(f) \subset \{\pi > a\}$. If $\delta > 0$ is fixed, then the function $s \mapsto T(f, u_{s,\delta}, g)$ is continuous, and an approximation argument using simple functions shows that

$$\int_a^\infty T(f, u_{s,\delta}, g)\, ds = T\biggl(f, \int_a^\infty u_{s,\delta}\, ds,\, g\biggr).$$

Now we let δ tend to 0. We know that then $T(f, u_{s,\delta}, g) \to \langle T, \pi, s+\rangle(f, g)$ for every $s \in \mathbb{R}$. Moreover, since $|u_{s,\delta}| \le 1$, Lemma 5.2(1) yields the uniform bound $|T(f, u_{s,\delta}, g)| \le \operatorname{Lip}(f)\|T\|(\operatorname{spt}(f)) + \int_X |f|\, d\|\partial T\|$. Hence, $s \mapsto \langle T, \pi, s+\rangle(f, g)$ is a bounded Borel function with compact support, and

$$\lim_{\delta \to 0+} \int_a^\infty T(f, u_{s,\delta}, g)\, ds = \int_a^\infty \langle T, \pi, s+\rangle(f, g)\, ds = \int_{\mathbb{R}} \langle T, \pi, s+\rangle(f, g)\, ds.$$

On the other hand, the functions $t \mapsto \int_a^\infty \gamma_{s,\delta}(t)\,ds = \int_{-\infty}^{t-a} \gamma_\delta(r)\,dr$ are 1-Lipschitz for all $\delta > 0$ and converge uniformly to $t \mapsto \max\{t-a, 0\}$, as $\delta \to 0$. It follows that

$$\lim_{\delta\to 0+} T\left(f, \int_a^\infty u_{s,\delta}\,ds, g\right) = T(f, \pi - a, g) = T(f, \pi, g) = (T\lfloor(1,\pi))(f,g).$$

This proves (2).

Suppose $V \subset X$ is an open set such that $\|T\lfloor(1,\pi)\|(V) < \infty$. Then the function $s \mapsto \|T\lfloor(1,\pi)\|(V \cap \{\pi < s\})$ is non-decreasing, hence almost everywhere differentiable. From (1) it follows that

$$\|\langle T, \pi, s+\rangle\|(V) \le \frac{d}{ds}\|T\lfloor(1,\pi)\|(V \cap \{\pi < s\}) \tag{6.5}$$

for almost every $s \in \mathbb{R}$. Hence

$$\int_{\mathbb{R}}^{*} \|\langle T, \pi, s+\rangle\|(V)\,ds \le \|T\lfloor(1,\pi)\|(V),$$

where $\int_{\mathbb{R}}^*$ denotes the upper integral. The reverse inequality

$$\int_{*\mathbb{R}} \|\langle T, \pi, s+\rangle\|(V)\,ds \ge \|T\lfloor(1,\pi)\|(V)$$

for the lower integral is a direct consequence of (2). This shows (3) for every open set V with $\|T\lfloor(1,\pi)\|(V) < \infty$. Since every compact set $K \subset X$ is a difference of two such open sets, the same identity holds for all compact sets, and we obtain (3) by approximation.

Finally, (4) follows easily from (3), or just (6.5), together with the corresponding result for $\partial\langle T, \pi, s+\rangle = -\langle \partial T, \pi, s+\rangle$ in case $m \ge 2$. □

Now we pass to slices of codimension k, for $1 \le k \le m$. Our approach is similar to that of [14, 4.3.1], where slices are defined for arbitrary locally flat chains (cf. the last remark on p. 437 in [14]). Definition 6.3 generalizes (6.4) and applies to all metric currents, however we shall prove the existence of the weak limits in question only for locally normal currents.

For $s \in \mathbb{R}$ and $\delta > 0$, let now $\gamma_{s,\delta}\colon \mathbb{R} \to \mathbb{R}$ be the piecewise affine $(1/\delta)$-Lipschitz function with $\gamma_{s,\delta}|_{(-\infty,s]} = 0$ and $\gamma_{s,\delta}|_{[s+\delta,\infty)} = 1$. Then, for $y = (y_1, \dots, y_k) \in \mathbb{R}^k$ and $\delta > 0$, define $\gamma_{y,\delta}\colon \mathbb{R}^k \to \mathbb{R}^k$ such that

$$\gamma_{y,\delta}(z) = (\gamma_{y_1,\delta}(z_1), \dots, \gamma_{y_k,\delta}(z_k))$$

for all $z = (z_1, \dots, z_k) \in \mathbb{R}^k$.

Definition 6.3 (slices)**.** *Suppose* $1 \le k \le m$*,* $\pi \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R}^k)$*, and* $T \in \mathscr{D}_m(X)$*. We define the* slice *of* T *at* $y \in \mathbb{R}^k$ *with respect to* π *as the weak limit*

$$\langle T, \pi, y\rangle := \lim_{\delta\to 0+} T\lfloor(1, \gamma_{y,\delta}\circ\pi)$$

whenever it exists and defines an element of $\mathscr{D}_{m-k}(X)$*.*

In view of (6.4), in case $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$ and $k = 1$ we have

$$\langle T, \pi, y \rangle = \langle T, \pi, y+ \rangle \tag{6.6}$$

for every $y \in \mathbb{R}$. (We could equally well arrange that $\langle T, \pi, y \rangle = (\langle T, \pi, y- \rangle + \langle T, \pi, y+ \rangle)/2$, as in [14, 4.3.4], by choosing a more symmetric definition of $\langle T, \pi, y \rangle$.) Before proving the existence of slices of locally normal currents in the case $k > 1$ we discuss a few properties that can be derived directly from the definition.

Let $y = (y_1, \dots, y_k) \in \mathbb{R}^k$, and suppose that $\langle T, \pi, y \rangle \in \mathscr{D}_{m-k}(X)$ exists. For $\delta > 0$, consider the cube

$$C(y, \delta) := [y_1, y_1 + \delta] \times \ldots \times [y_k, y_k + \delta] \subset \mathbb{R}^k.$$

A simple Lipschitz partition of unity argument on $\mathbb{R}^k \setminus C(y, \delta)$ shows that

$$\operatorname{spt}(T \lfloor (1, \gamma_{y,\delta} \circ \pi)) \subset \pi^{-1}(C(y, \delta)) \cap \operatorname{spt}(T), \tag{6.7}$$

so that

$$\operatorname{spt}(\langle T, \pi, y \rangle) \subset \pi^{-1}\{y\} \cap \operatorname{spt}(T). \tag{6.8}$$

If $m > k$, then $(\partial T) \lfloor (1, \gamma_{y,\delta} \circ \pi) = (-1)^k \partial (T \lfloor (1, \gamma_{y,\delta} \circ \pi))$ by (3.7), hence

$$\langle \partial T, \pi, y \rangle = (-1)^k \partial \langle T, \pi, y \rangle. \tag{6.9}$$

Whenever $0 \le l \le m - k$ and $(u, v) \in \mathrm{Lip}_{\mathrm{loc}}(X) \times [\mathrm{Lip}_{\mathrm{loc}}(X)]^l$, the alternating property gives $(T \lfloor (u, v)) \lfloor (1, \gamma_{y,\delta} \circ \pi) = (-1)^{kl} (T \lfloor (1, \gamma_{y,\delta} \circ \pi)) \lfloor (u, v)$, so

$$\langle T \lfloor (u, v), \pi, y \rangle = (-1)^{kl} \langle T, \pi, y \rangle \lfloor (u, v), \tag{6.10}$$

in particular $\langle T \lfloor u, \pi, y \rangle = \langle T, \pi, y \rangle \lfloor u$.

Theorem 6.4 (slicing)**.** *Suppose* $1 \le k \le m$, $\pi \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R}^k)$, $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$*, and* $\operatorname{spt}(T)$ *is separable.*

(1) *For* $\mathscr{L}^k$*-almost every* $y \in \mathbb{R}^k$*, the slice* $\langle T, \pi, y \rangle \in \mathscr{D}_{m-k}(X)$ *exists and is locally normal.*

(2) *For all* $(f, g) \in \mathscr{B}_c^\infty(X) \times [\mathrm{Lip}_{\mathrm{loc}}(X)]^{m-k}$,

$$\int_{\mathbb{R}^k} \langle T, \pi, y \rangle (f, g)\, dy = (T \lfloor (1, \pi))(f, g).$$

(3) *For every* $\|T \lfloor (1, \pi)\|$*-measurable set* $B \subset X$,

$$\int_{\mathbb{R}^k} \|\langle T, \pi, y \rangle\|(B)\, dy = \|T \lfloor (1, \pi)\|(B).$$

Proof. We proceed by induction on k.

Let $k = 1$. We know from (6.6) and Theorem 6.2 that (1) holds, (2) holds for all $(f, g) \in \mathscr{D}^{m-1}(X)$, and (3) holds for all $\|T \lfloor (1, \pi)\|$-measurable sets with finite measure. Since $\mathrm{spt}(T)$ is separable by assumption, $\mathrm{spt}(T)$ is σ-compact, hence $\|T \lfloor (1, \pi)\|$ is σ-finite, and (3) follows. To prove (2) in the general case, fix $g \in [\mathrm{Lip}_1(X)]^{m-1}$ and $\sigma \in \mathscr{D}(X)$, $0 \le \sigma \le 1$, and let $K := \mathrm{spt}(\sigma)$. Denote by Φ the set of all $f \in \mathscr{B}^\infty_{\mathrm{c}}(X)$ with $|f| \le \sigma$ such that (2) holds for (f, g). Suppose that $f_1, f_2, \ldots \in \Phi$, $f \in \mathscr{B}^\infty_{\mathrm{c}}(X)$, and $f_j \to f$ pointwise on X. For almost every $y \in \mathbb{R}$, $|\langle T, \pi, y\rangle(f_j, g)| \le \|\langle T, \pi, y\rangle\|(K)$ for all j. Moreover, by (3), $\int_{\mathbb{R}} \|\langle T, \pi, y\rangle\|(K)\, dy = \|T \lfloor (1, \pi)\|(K) < \infty$. Now the bounded convergence theorem and Theorem 4.4(2) imply that $f \in \Phi$, so that Φ is a Baire class. Since Φ contains all $f \in \mathscr{D}(X)$ with $|f| \le \sigma$, Φ is the class of all $f \in \mathscr{B}^\infty_{\mathrm{c}}(X)$ with $|f| \le \sigma$, cf. Sect. 1.2. This yields (2).

Now let $k \ge 2$, and suppose (1), (2), and (3) hold with $k' := k-1$ in place of k. We write π' for $(\pi_1, \ldots, \pi_{k'})$, so that $\pi = (\pi', \pi_k)$. There is a Borel set $D' \subset \mathbb{R}^{k'}$ with $\mathscr{L}^{k'}(\mathbb{R}^{k'} \setminus D') = 0$ such that $\langle T, \pi', y'\rangle$ exists as an element of $\mathbf{N}_{m-k',\mathrm{loc}}(X)$ for every $y' = (y_1, \ldots, y_{k'}) \in D'$. By (6.6),

$$T_y := \big\langle \langle T, \pi', y'\rangle, \pi_k, y_k \big\rangle \in \mathscr{D}_{m-k}(X)$$

then exists for every $y = (y', y_k) \in D' \times \mathbb{R}$. Moreover, for fixed $(f, g) \in \mathscr{D}^{m-k}(X)$, the function $y \mapsto T_y(f, g)$ is Borel measurable on $D' \times \mathbb{R}$, since

$$T_y(f, g) = \lim_{\delta \to 0+} \lim_{\delta' \to 0+} T(f, \gamma_{y',\delta'} \circ \pi', \gamma_{y_k,\delta} \circ \pi_k, g)$$

and $y \mapsto T(f, \gamma_{y',\delta'} \circ \pi', \gamma_{y_k,\delta} \circ \pi_k, g)$ is continuous on $\mathbb{R}^k$ for fixed $\delta', \delta > 0$. Since each $\|T_y\|$ is concentrated on the separable set $\mathrm{spt}(T)$, it also follows that for every open set $V \subset X$ the function $y \mapsto \|T_y\|(V) \in [0, \infty]$ is Borel measurable on $D' \times \mathbb{R}$. Hence, by Fubini's theorem, the result in the case $k = 1$, (6.10), and the induction assumption,

$$\begin{aligned}
\int_{\mathbb{R}^k} \|T_y\|(V)\, dy &= \int_{\mathbb{R}^{k'}} \int_{\mathbb{R}} \big\|\big\langle \langle T, \pi', y'\rangle, \pi_k, y_k \big\rangle\big\|(V)\, dy_k\, dy' \\
&= \int_{\mathbb{R}^{k'}} \|\langle T, \pi', y'\rangle \lfloor (1, \pi_k)\|(V)\, dy' \\
&= \int_{\mathbb{R}^{k'}} \|\langle T \lfloor (1, \pi_k), \pi', y'\rangle\|(V)\, dy' \\
&= \|(T \lfloor (1, \pi_k)) \lfloor (1, \pi')\|(V) \\
&= \|T \lfloor (1, \pi)\|(V).
\end{aligned}$$

Reasoning as in the proof of Theorem 6.2 and using again the σ-compactness of $\mathrm{spt}(T)$ we obtain

$$\int_{\mathbb{R}^k} \|T_y\|(B)\, dy = \|T \lfloor (1, \pi)\|(B) \tag{6.11}$$

for every $\|T\lfloor(1,\pi)\|$-measurable set $B \subset X$. Now it follows that $T_y \in \mathbf{M}_{m-k,\mathrm{loc}}(X)$ for almost every $y \in \mathbb{R}^k$, and for fixed $(f,g) \in \mathscr{D}^{m-k}(X)$, the function $y \mapsto T_y(f,g)$ is in $L^1(\mathbb{R}^k)$. By Fubini's theorem, the result in the case $k=1$, and the induction assumption,

$$\begin{aligned}\int_{\mathbb{R}^k} T_y(f,g)\,dy &= \int_{\mathbb{R}^{k'}}\int_{\mathbb{R}} \big\langle\langle T,\pi',y'\rangle,\pi_k,y_k\big\rangle(f,g)\,dy_k\,dy' \\ &= \int_{\mathbb{R}^{k'}} \langle T,\pi',y'\rangle(f,\pi_k,g)\,dy' \\ &= T(f,\pi',\pi_k,g) \\ &= (T\lfloor(1,\pi))(f,g).\end{aligned}$$

As in the case $k=1$, a Baire class argument then shows that

$$\int_{\mathbb{R}^k} T_y(f,g)\,dy = (T\lfloor(1,\pi))(f,g) \tag{6.12}$$

for all $(f,g) \in \mathscr{B}^\infty_{\mathrm{c}}(X)\times[\mathrm{Lip}_{\mathrm{loc}}(X)]^{m-k}$. Given $\phi \in \mathscr{B}^\infty_{\mathrm{loc}}(\mathbb{R}^k)$, we may replace f by $(\phi\circ\pi)f$; since $\phi\circ\pi = \phi(y)$ on $\mathrm{spt}(T_y) \subset \pi^{-1}\{y\}$, this yields the identity

$$\int_{\mathbb{R}^k} \phi(y)T_y(f,g)\,dy = (T\lfloor(\phi\circ\pi,\pi))(f,g). \tag{6.13}$$

Now we show that for almost every $y \in \mathbb{R}^k$, the slice $\langle T,\pi,y\rangle$ exists as an element of $\mathbf{N}_{m-k,\mathrm{loc}}(X)$ and coincides with T_y. By (6.12) and (6.11), this will complete the proof of the theorem.

We first assume that X is compact, so that T is normal and π is Lipschitz. Then we fix a countable set $\mathscr{F} \subset \mathrm{Lip}_1(X)$ that is dense in $\mathrm{Lip}_1(X)$ with respect to $\|\cdot\|_\infty$. Put $T_\pi := T\lfloor(1,\pi) \in \mathbf{N}_{m-k}(X)$ for the moment, and let $y \in \mathbb{R}^k$ and $\delta > 0$. Using (6.7), (4.7), and Theorem 4.4 we get

$$\begin{aligned}T\lfloor(1,\gamma_{y,\delta}\circ\pi) &= T\lfloor(\chi_{C(y,\delta)}\circ\pi,\gamma_{y,\delta}\circ\pi) \\ &= \frac{1}{\delta^k}T\lfloor(\chi_{C(y,\delta)}\circ\pi,\pi) = \frac{1}{\delta^k}T_\pi\lfloor\pi^{-1}(C(y,\delta)).\end{aligned} \tag{6.14}$$

Similarly, if $m > k$, applying (3.7) twice we obtain

$$\begin{aligned}\partial\big(T\lfloor(1,\gamma_{y,\delta}\circ\pi)\big) &= (-1)^k(\partial T)\lfloor(1,\gamma_{y,\delta}\circ\pi) \\ &= (-1)^k\frac{1}{\delta^k}\big((\partial T)\lfloor(1,\pi)\big)\lfloor\pi^{-1}(C(y,\delta)) = \frac{1}{\delta^k}(\partial T_\pi)\lfloor\pi^{-1}(C(y,\delta)).\end{aligned}$$

For every Borel set $C \subset \mathbb{R}^k$, put

$$\mu(C) := (\|T_\pi\| + \|\partial T_\pi\|)(\pi^{-1}(C))$$

if $m > k$, and $\mu(C) := \|T_\pi\|(\pi^{-1}(C))$ if $m = k$. This defines a finite Borel measure μ on $\mathbb{R}^k$. Let $M_\mu\colon \mathbb{R}^k \to [0,\infty]$ be the maximal function of μ. Then

$$\mathbf{N}\big(T\lfloor(1,\gamma_{y,\delta}\circ\pi)\big) = \frac{1}{\delta^k}\mu(C(y,\delta)) \le \alpha_k k^{k/2}M_\mu(y).$$

For all $(f, g) \in \mathscr{D}^{m-k}(X)$, (6.14) and (6.13) give

$$\big(T\lfloor (1, \gamma_{y,\delta} \circ \pi)\big)(f, g) = \frac{1}{\delta^k}\big(T\lfloor (\chi_{C(y,\delta)} \circ \pi, \pi)\big)(f, g)$$
$$= \frac{1}{\delta^k} \int_{C(y,\delta)} T_z(f, g)\, dz. \qquad (6.15)$$

Almost every $y \in \mathbb{R}^k$ satisfies $M_\mu(y) < \infty$ and is a Lebesgue point of $z \mapsto T_z(f, g)$ for all $(f, g) \in \mathscr{F} \times \mathscr{F}^{m-k}$. Applying Proposition 5.3 (convergence criterion) and (1.4) we conclude that for every such y, the slice $\langle T, \pi, y\rangle$ exists as an element of $\mathbf{N}_{m-k}(X)$ and coincides with T_y.

In the general case, when X is locally compact and $\operatorname{spt}(T)$ is separable, there is no loss of generality in assuming X itself to be separable. Then we choose a countable set $\Sigma \subset \mathscr{D}(X)$ as in the proof of Theorem 5.4. The above argument together with (6.10) then shows that for almost every $y \in \mathbb{R}^k$, the slice $\langle T\lfloor \sigma, \pi, y\rangle$ exists as an element of $\mathbf{N}_{m-k}(X)$ and coincides with $T_y\lfloor \sigma$ for every $\sigma \in \Sigma$. The general result follows. □

The following theorem will be used in the proof of Theorem 8.5 (rectifiable slices).

Theorem 6.5 (iterated slices)**.** *Suppose* $k, k' \geq 1$, $m \geq k + k'$, $\pi \in \operatorname{Lip}_{\mathrm{loc}}(X, \mathbb{R}^k)$, $\pi' \in \operatorname{Lip}_{\mathrm{loc}}(X, \mathbb{R}^{k'})$, $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$, *and* $\operatorname{spt}(T)$ *is separable. Then*

$$\langle T, (\pi, \pi'), (y, y')\rangle = \big\langle \langle T, \pi, y\rangle, \pi', y'\big\rangle$$

for $\mathscr{L}^{k+k'}$*-almost every* $(y, y') \in \mathbb{R}^{k+k'}$.

In case $k' = 1$ the result is clear from the preceding proof. Our argument for the general case follows [14, Theorem 4.3.5].

Proof. We assume that X is compact, so that $T \in \mathbf{N}_m(X)$ and $(\pi, \pi') \in \operatorname{Lip}(X, \mathbb{R}^{k+k'})$. The theorem easily follows from the result in this special case. We fix a countable set $\mathscr{F} \subset \operatorname{Lip}_1(X)$ that is dense in $\operatorname{Lip}_1(X)$ with respect to $\|\cdot\|_\infty$.

From Theorem 6.4 and Fubini's theorem we conclude that there is a Borel set $D_1 \subset \mathbb{R}^{k+k'}$ with $\mathscr{L}^{k+k'}(\mathbb{R}^{k+k'} \setminus D_1) = 0$ such that whenever $(y, y') \in D_1$, both $\langle T, \pi, y\rangle$ and $\langle T, (\pi, \pi'), (y, y')\rangle$ exist and are normal currents, $\mathscr{L}^{k'}((\{y\} \times \mathbb{R}^{k'}) \setminus D_1) = 0$, and

$$\int_{\mathbb{R}^{k'}} \mathbf{N}\big(\langle T, (\pi, \pi'), (y, z')\rangle\big)\, dz' < \infty. \qquad (6.16)$$

For $\delta' > 0$ and $(y, y') \in D_1$ we define a functional $A_{\delta'}(y, y')$ on $\mathscr{D}^{m-k-k'}(X)$ by

$$A_{\delta'}(y, y')(f, g) := \frac{1}{(\delta')^k} \int_{C(y',\delta')} \langle T, (\pi, \pi'), (y, z')\rangle(f, g)\, dz'.$$

Now (6.16) with $\mathbf{M}$ in place of $\mathbf{N}$ shows that $\langle T, (\pi, \pi'), (y, \cdot)\rangle(f, g) \in L^1(\mathbb{R}^{k'})$ and, in conjunction with the bounded convergence theorem, that $A_{\delta'}(y, y')$ satisfies the continuity condition for metric currents. Hence $A_{\delta'}(y, y')$ is a current; since

$$\mathbf{N}(A_{\delta'}(y, y')) \le \frac{1}{(\delta')^k} \int_{C(y',\delta')} \mathbf{N}\big(\langle T, (\pi, \pi'), (y, z')\rangle\big)\, dz', \tag{6.17}$$

it is normal by (6.16). Now let $\delta, \delta' > 0$ and $(y, y') \in D_1$. For all $(f, g) \in \mathscr{D}^{m-k-k'}(X)$, we obtain

$$\begin{aligned} &T(f, \gamma_{y,\delta} \circ \pi, \gamma_{y',\delta'} \circ \pi', g) \\ &\quad = \frac{1}{\delta^k (\delta')^k} \int_{C(y,\delta)\times C(y',\delta')} \langle T, (\pi, \pi'), (z, z')\rangle(f, g)\, d(z, z') \\ &\quad = \frac{1}{\delta^k} \int_{C(y,\delta)} A_{\delta'}(z, y')(f, g)\, dz \end{aligned} \tag{6.18}$$

by the same argument as for (6.15), and by Fubini's theorem. Note that the left side of this identity is continuous in (y, y'), moreover the limit for $\delta \to 0$ and for fixed $\delta' > 0$ is equal to $\langle T, \pi, y\rangle(f, \gamma_{y',\delta'} \circ \pi', g)$. Let D_2 be the Borel set of all $(y, y') \in D_1$ where the limit of (6.18) equals $A_{\delta'}(y, y')(f, g)$ for all rational numbers $\delta' > 0$ and $(f, g) \in \mathscr{F} \times \mathscr{F}^{m-k-k'}$. Then $\mathscr{L}^{k+k'}(D_1 \setminus D_2) = 0$, and for every $(y, y') \in D_2$, the equality

$$\langle T, \pi, y\rangle(f, \gamma_{y',\delta'} \circ \pi', g) = A_{\delta'}(y, y')(f, g) \tag{6.19}$$

holds for all rational numbers $\delta' > 0$ and $(f, g) \in \mathscr{F} \times \mathscr{F}^{m-k-k'}$, hence for all $\delta' > 0$ and $(f, g) \in \mathscr{D}^{m-k-k'}(X)$ by continuity and multilinearity. Now we pass to the limit for $\delta' \to 0$. Let D_3 be the Borel set of all $(y, y') \in D_2$ where the limit of the right side of (6.19) coincides with $\langle T, (\pi, \pi'), (y, y')\rangle(f, g)$ for all $(f, g) \in \mathscr{F} \times \mathscr{F}^{m-k-k'}$ and where

$$\sup_{\delta'>0} \frac{1}{(\delta')^k} \int_{C(y',\delta')} \mathbf{N}\big(\langle T, (\pi, \pi'), (y, z')\rangle\big)\, dz' < \infty. \tag{6.20}$$

Then $\mathscr{L}^{k+k'}(D_2 \setminus D_3) = 0$, and

$$\lim_{\delta'\to 0} \big(\langle T, \pi, y\rangle \lfloor (1, \gamma_{y',\delta'} \circ \pi')\big)(f, g) = \langle T, (\pi, \pi'), (y, y')\rangle(f, g)$$

for all $(y, y') \in D_3$ and $(f, g) \in \mathscr{F} \times \mathscr{F}^{m-k-k'}$. Combining (6.19), (6.17), and (6.20), we have $\sup_{\delta'>0} \mathbf{N}\big(\langle T, \pi, y\rangle \lfloor (1, \gamma_{y',\delta'} \circ \pi')\big) < \infty$, so it follows from Proposition 5.3 (convergence criterion) that

$$\big\langle \langle T, \pi, y\rangle, \pi', y' \big\rangle = \langle T, (\pi, \pi'), (y, y')\rangle$$

for all $(y, y') \in D_3$. $\square$

We conclude this section with a result that will be used in the proof of Theorem 8.9 (closure theorem).

Proposition 6.6 (slicing convergent sequences)**.** *Suppose* $m \ge 1$*,* $(T_n)_{n\in\mathbb{N}}$ *is a sequence in* $\mathbf{N}_{m,\mathrm{loc}}(X)$ *that converges weakly to some* $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$*, each* $\mathrm{spt}(T_n)$ *is separable, and* $\sup_n \mathbf{N}_V(T_n) < \infty$ *for every open set* $V \Subset X$*. Let* $\pi \in \mathrm{Lip}_{\mathrm{loc}}(X)$*. Then for* $\mathscr{L}^1$*-almost every* $s \in \mathbb{R}$ *there is a subsequence* $(T_{n(i)})_{i\in\mathbb{N}}$ *such that*

$$\langle T_{n(i)}, \pi, s\rangle \to \langle T, \pi, s\rangle$$

weakly and $\sup_i \mathbf{N}_V(\langle T_{n(i)}, \pi, s\rangle) < \infty$ *for every open set* $V \Subset X$*.*

Compare [36, p. 208] and [3, Proposition 8.3].

Proof. We assume without loss of generality that X is compact.

We first show the existence of a subsequence $(T_{n(i)})_{i\in\mathbb{N}}$ such that for all but countably many $s \in \mathbb{R}$, $\langle T_{n(i)}, \pi, s\rangle \to \langle T, \pi, s\rangle$ weakly for $i \to \infty$. Consider the measures μ_n on $\mathbb{R}$ defined by $\mu_n(B) := (\|T_n\| + \|\partial T_n\|)(\pi^{-1}(B))$ for all Borel sets $B \subset \mathbb{R}$. Note that $\sup_n \mu_n(\mathbb{R}) < \infty$. Choose a subsequence $(\mu_{n(i)})_{i\in\mathbb{N}}$ that converges weakly* to some finite Borel measure μ on $\mathbb{R}$ (see e.g. [13, Sect. 1.9]). Now let $s \in \mathbb{R}$ be such that $\mu(\{s\}) = 0$. Suppose $(f, g) \in \mathscr{D}(X) \times [\mathrm{Lip}_1(X)]^m$, and $|f| \le 1$. Given $\epsilon > 0$, choose $\delta > 0$ such that $\|T\|(\{s < \pi < s+\delta\}) < \epsilon/3$ and $\mu([s, s+\delta]) < \epsilon/3$, and let $u_{s,\delta} \in \mathrm{Lip}(X)$ be given as in (6.4). Then

$$\begin{aligned}\big|(T \lfloor \chi_{\{\pi>s\}})(f,g) - (T \lfloor u_{s,\delta})(f,g)\big| &= |T((\chi_{\{\pi>s\}} - u_{s,\delta})f, g)| \\ &\le \|T\|(\{s < \pi < s+\delta\}) < \epsilon/3.\end{aligned}$$

Since $\limsup_{i\to\infty} \mu_{n(i)}([s, s+\delta]) \le \mu([s, s+\delta])$, it follows similarly that

$$\big|(T_{n(i)} \lfloor \chi_{\{\pi>s\}})(f,g) - (T_{n(i)} \lfloor u_{s,\delta})(f,g)\big| \le \mu_{n(i)}([s, s+\delta]) < \epsilon/3$$

for all sufficiently large i. Moreover, since $T_{n(i)} \to T$ weakly,

$$\big|(T_{n(i)} \lfloor u_{s,\delta})(f,g) - (T \lfloor u_{s,\delta})(f,g)\big| < \epsilon/3$$

for i sufficiently large. Combining these estimates we conclude that

$$T_{n(i)} \lfloor \chi_{\{\pi>s\}} \to T \lfloor \chi_{\{\pi>s\}}$$

weakly for $i \to \infty$, and a completely analogous argument shows that

$$(\partial T_{n(i)}) \lfloor \chi_{\{\pi>s\}} \to (\partial T) \lfloor \chi_{\{\pi>s\}}$$

weakly. By (6.6), $\langle T_{n(i)}, \pi, s\rangle \to \langle T, \pi, s\rangle$ weakly for $i \to \infty$.

Applying Theorem 6.2(3) and (6.3) we obtain

$$\begin{aligned}\int_{\mathbb{R}} \liminf_{n\to\infty} \mathbf{N}(\langle T_n, \pi, s\rangle)\, ds &\le \liminf_{n\to\infty} \int_{\mathbb{R}} \mathbf{N}(\langle T_n, \pi, s\rangle)\, ds \\ &\le \mathrm{Lip}(\pi) \sup\nolimits_n \mathbf{N}(T_n) < \infty.\end{aligned}$$

Hence, for $\mathscr{L}^1$-almost every $s \in \mathbb{R}$, there is a subsequence $(T_{n(i)})_{i\in\mathbb{N}}$ such that $\sup_i \mathbf{N}(\langle T_{n(i)}, \pi, s\rangle) < \infty$. Together with the first part of the proof this yields the result. □

7 Projections to Euclidean spaces

The main purpose of this section is to establish Theorem 7.6, the central piece of the proof of the rectifiablity criterion used later on, Theorem 8.4. We start by recalling a few basic facts about functions of bounded variation (cf. [13, Ch. 5], [29, §6]).

Let $U \subset \mathbb{R}^m$ be an open set, $m \geq 1$, and let $u \in L^1_{\mathrm{loc}}(U)$. For an open set $A \subset U$,

$$\mathbf{V}_A(u) := \sup\Big\{ \int_A u \operatorname{div}(\varphi)\, dx : \varphi \in C^1_{\mathrm{c}}(A, \mathbb{R}^m),\ |\varphi| \leq 1 \Big\}$$

defines the *variation of* u *in* A. The number $\mathbf{V}_A(u)$ is unchanged if $C^1_{\mathrm{c}}(A,\mathbb{R}^m)$ is replaced by $C^\infty_{\mathrm{c}}(A,\mathbb{R}^m)$ or $\mathrm{Lip}_{\mathrm{c}}(A,\mathbb{R}^m) = [\mathscr{D}(A)]^m$. Then u is a *function of locally bounded variation in* U if $\mathbf{V}_A(u) < \infty$ for every open set $A \Subset U$, and $\mathrm{BV}_{\mathrm{loc}}(U)$ denotes the vector space of all such functions. Similarly, u is a *function of bounded variation in* U if $u \in L^1(U)$ and $\mathbf{V}_U(u) < \infty$, and $\mathrm{BV}(U)$ denotes the vector space of all such functions. Note that for $u \in \mathrm{Lip}_{\mathrm{loc}}(U)$, the integration by parts formula

$$\int_A u \operatorname{div}(\varphi)\, dx = -\int_A \langle \nabla u, \varphi \rangle\, dx$$

holds for all $\varphi \in C^1_{\mathrm{c}}(A,\mathbb{R}^m)$, which implies that $\mathbf{V}_A(u) = \int_A |\nabla u|\, dx$. In particular, $\mathrm{Lip}_{\mathrm{loc}}(U) \subset \mathrm{BV}_{\mathrm{loc}}(U)$. The representation theorem for $u \in \mathrm{BV}_{\mathrm{loc}}(U)$ says that there exist a Radon measure on U, denoted by $|Du|$, and a $|Du|$-measurable vector field τ on U such that $|\tau(x)| = 1$ for $|Du|$-almost every $x \in U$ and

$$\int_U u \operatorname{div}(\varphi)\, dx = -\int_U \langle \varphi, \tau \rangle\, d|Du| \tag{7.1}$$

for $\varphi \in C^1_{\mathrm{c}}(U,\mathbb{R}^m)$ or even $\varphi \in \mathrm{Lip}_{\mathrm{c}}(U,\mathbb{R}^m) = [\mathscr{D}(U)]^m$. The Radon measure $|Du|$ is characterized by $|Du|(A) = \mathbf{V}_A(u)$ for all open sets $A \subset U$.

Functions of bounded variation will be used through the following known fact (cf. the proof of [3, Lemma 7.3] and the references there), which expresses a Lipschitz property in terms of the maximal function of the variation measure.

Lemma 7.1. *Suppose that* $u \in \mathrm{BV}(\mathbb{R}^m)$, $m \geq 1$. *Whenever* x, x' *are two Lebesgue points of* u, *then*

$$|u(x) - u(x')| \leq c_m \big(M_{|Du|}(x) + M_{|Du|}(x') \big) |x - x'|$$

for some constant c_m *depending only on* m.

Thus, by (1.3), u is $2c_m s$-Lipschitz outside a set of Lebesgue measure at most $3^m s^{-1} |Du|(\mathbb{R}^m)$, for any $s > 0$.

Proof. Suppose first that $u \in C^1(\mathbb{R}^m)$ in addition, and let $r > 0$. Since $u(z) - u(0) = \int_0^1 \langle \nabla u(tz), z \rangle \, dt$ for $z \in \mathbb{R}^m$, it follows that

$$\begin{aligned}\int_{\mathrm{B}(0,r)} \frac{|u(z)-u(0)|}{|z|} \, dz &\le \int_{\mathrm{B}(0,r)} \int_0^1 |\nabla u(tz)| \, dt \, dz \\ &= \int_0^1 \int_{\mathrm{B}(0,r)} |\nabla u(tz)| \, dz \, dt \\ &= \int_0^1 \frac{1}{t^m} \int_{\mathrm{B}(0,tr)} |\nabla u(z)| \, dz \, dt \\ &\le \alpha_m r^m M_{|Du|}(0).\end{aligned}$$

For a general $u \in \mathrm{BV}(\mathbb{R}^m)$, a smoothing argument then shows that

$$\int_{\mathrm{B}(x,r)} \frac{|u(z)-u(x)|}{|z-x|} \, dz \le \alpha_m r^m M_{|Du|}(x)$$

for every Lebesgue point x of u (cf. [2, Remark 3.8]).

Now suppose that $x, x' \in \mathbb{R}^m$ are two Lebesgue points of u, and $r := |x - x'| > 0$. Put $c := 2\alpha_m r^m / \mathscr{L}^m(\mathrm{B}(x,r) \cap \mathrm{B}(x',r))$,

$$A := \bigl\{ z \in \mathrm{B}(x,r) : |u(x) - u(z)| > c M_{|Du|}(x) |x - z| \bigr\},$$

and define A' similarly with x' in place of x. The above estimate implies that $\mathscr{L}^m(A), \mathscr{L}^m(A') < \alpha_m r^m / c$ and thus

$$\mathscr{L}^m(A \cup A') < 2\alpha_m r^m / c = \mathscr{L}^m(\mathrm{B}(x,r) \cap \mathrm{B}(x',r)).$$

In particular, $\mathrm{B}(x,r) \cap \mathrm{B}(x',r) \setminus (A \cup A')$ is non-empty; for any z in this set we get

$$\begin{aligned}|u(x) - u(x')| &\le |u(x) - u(z)| + |u(x') - u(z)| \\ &\le c M_{|Du|}(x) \, |x - z| + c M_{|Du|}(x') \, |x' - z|.\end{aligned}$$

Since $|x - z|, |x' - z| \le r = |x - x'|$, the result follows. □

The next result relates normal currents of the type described in Proposition 2.6 (standard example) to functions of bounded variation, cf. [29, Remark 26.28] and [3, Theorem 3.7].

Theorem 7.2 (normal m-currents in $\mathbb{R}^m$)**.** *Let $U \subset \mathbb{R}^m$ be an open set, $m \ge 1$.*

(1) *If $u \in L^1_{\mathrm{loc}}(U)$, then $\mathbf{M}_A(\partial[u]) = \mathbf{V}_A(u)$ for every open set $A \subset U$.*

(2) *If $T \in \mathbf{N}_{m,\mathrm{loc}}(U)$, then $T = [u]$ for some $u \in \mathrm{BV}_{\mathrm{loc}}(U)$, and $\|\partial T\| = |Du|$.*

In particular, if $T \in \mathbf{N}_m(U)$, then $T = [u]$ for some $u \in \mathrm{BV}(U)$.

Proof. Let $u \in L^1_{\mathrm{loc}}(U)$. Denote by π the identity map on U, $\pi_k(x) = x_k$. Let $\varphi \in C^\infty_{\mathrm{c}}(U, \mathbb{R}^m)$, and choose $\sigma \in \mathscr{D}(U)$ such that $\sigma|_{\mathrm{spt}(\varphi)} = 1$. By Theorem 2.5 (chain rule),

$$\begin{aligned}\int_U u \operatorname{div}(\varphi)\, dx &= [u](\operatorname{div}(\varphi), \pi) \\ &= \sum_{k=1}^m (-1)^{k-1} [u](\sigma, \varphi_k, \pi_1, \dots, \pi_{k-1}, \pi_{k+1}, \dots, \pi_m) \\ &= \sum_{k=1}^m (-1)^{k-1} \partial[u](\varphi_k, \pi_1, \dots, \pi_{k-1}, \pi_{k+1}, \dots, \pi_m).\end{aligned}$$

Assuming that $A \subset U$ is an open set with $\mathbf{M}_A(\partial[u]) < \infty$, we conclude that $\mathbf{V}_A(u) \le m\mathbf{M}_A(\partial[u]) < \infty$. In particular $u \in \mathrm{BV}(A)$, and the representation formula (7.1) holds on A. Let τ be a corresponding vector field on A. To show that in fact $\mathbf{V}_A(u) \le \mathbf{M}_A(\partial[u])$, let $v \in \mathbb{R}^m$ be a unit vector, and complete it to an orthonormal basis $(v, e_1, \dots, e_{m-1})$ of $\mathbb{R}^m$. Define $g_i \in \mathrm{Lip}_1(U)$ by $g_i(x) := \langle x, e_i \rangle$. For $f, \sigma \in \mathscr{D}(U)$ with $\mathrm{spt}(f) \subset A$ and $\sigma|_{\mathrm{spt}(f)} = 1$, we have

$$\begin{aligned}\int_A \langle fv, -\tau \rangle\, d|Du| = \int_A u D_v f\, dx &= [u](\sigma, f, g_1, \dots, g_{m-1}) \\ &= \partial[u](f, g_1, \dots, g_{m-1}) \le \int_A |f|\, d\|\partial[u]\|.\end{aligned}$$

Now an approximation argument shows that $|Du|(A) \le \|\partial[u]\|(A)$, thus $\mathbf{V}_A(u) \le \mathbf{M}_A(\partial[u])$.

To prove the reverse inequality, suppose that $g_1, \dots, g_{m-1} \in C^2(U)$. Consider the matrix $D(g_1, \dots, g_{m-1})$, and let M_k be the $(m-1) \times (m-1)$ minor obtained by deleting the kth column. Let φ be the C^1 vector field on U with components $\varphi_k = (-1)^{k-1} \det(M_k)$. Note that $\operatorname{div}(\varphi) = 0$ since

$$0 = d(dg_1 \wedge \ldots \wedge dg_{m-1}) = \operatorname{div}(\varphi)\, dx_1 \wedge \ldots \wedge dx_m.$$

Now let $f \in C^1_{\mathrm{c}}(U)$. Then

$$\det(D(f, g_1, \dots, g_{m-1})) = \langle \nabla f, \varphi \rangle = \operatorname{div}(f\varphi) - f \operatorname{div}(\varphi) = \operatorname{div}(f\varphi)$$

on U. Hence, for any $\sigma \in \mathscr{D}(U)$ with $\sigma|_{\mathrm{spt}(f)} = 1$,

$$\partial[u](f, g_1, \dots, g_{m-1}) = [u](\sigma, f, g_1, \dots, g_{m-1}) = \int_U u \operatorname{div}(f\varphi)\, dx.$$

If, in addition, $g_1, \dots, g_{m-1} \in \mathrm{Lip}_1(U)$, then $|\varphi| \le 1$ by the Cauchy–Binet formula (see e.g. [2, Proposition 2.69]). It follows easily that $\mathbf{M}_A(\partial[u]) \le \mathbf{V}_A(u)$ for every open set $A \subset U$. This completes the proof of (1).

Let $T \in \mathbf{N}_{m,\mathrm{loc}}(U)$. Specializing Theorem 6.4(2) to the case where $X = U$, $k = m$, and $\pi\colon U \to \mathbb{R}^k$ is the inclusion map, we get

$$T(f, \pi) = \int_U \langle T, \pi, x\rangle(f)\, dx = \int_U f(x)u(x)\, dx$$

for every $f \in \mathscr{B}^\infty_{\mathrm{c}}(U)$, where $u(x) := \langle T, \pi, x\rangle(\sigma)$ for any $\sigma \in \mathscr{D}(U)$ with $\sigma(x) = 1$ (note that $\mathrm{spt}(\langle T, \pi, x\rangle) \subset \{x\}$). By (2.11) and a smoothing argument it follows that $T = [u]$. Together with (1), this proves (2). □

Next we discuss two auxiliary results for currents $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$. We look at $\pi_\#(T\lfloor f)$ for $f \in \mathscr{B}^\infty_{\mathrm{c}}(X)$ and $\pi \in \mathrm{Lip}(X, \mathbb{R}^m)$. Note that since $\mathrm{spt}(T\lfloor f)$ is compact, the push-forward is defined, moreover $\pi_\#(T\lfloor f)$ is an element of $\mathbf{M}_m(\mathbb{R}^m)$ with compact support. In order to gain information on the local structure of T, we would like to know that $\pi_\#(T\lfloor f)$ is of the standard type $[u]$ for some $u \in L^1(\mathbb{R}^m)$. Except for $m = 1$ and $m = 2$, it is an open problem whether every $S \in \mathbf{M}_m(\mathbb{R}^m)$ is of this form, cf. [3, p. 15 and p. 21]. However, by Theorem 7.2, this is the case if S is normal, in particular $\pi_\#(T\lfloor f)$ is of standard type if T is locally normal and $f \in \mathscr{D}(X)$. Thus, for the purpose of this paper, the following result suffices.

Lemma 7.3. *Let $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$, $m \geq 1$. For $\pi \in \mathrm{Lip}(X, \mathbb{R}^m)$, each of the following two statements implies the other:*

(1) *Whenever $f \in \mathscr{D}(X)$, then $\pi_\#(T\lfloor f) = [u_f]$ for some $u_f \in L^1(\mathbb{R}^m)$.*

(2) *Whenever $B \Subset X$ is a Borel set, then $\pi_\#(T\lfloor B) = [u_B]$ for some $u_B \in L^1(\mathbb{R}^m)$.*

Proof. Suppose first that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $\mathscr{B}^\infty_{\mathrm{c}}(X)$ that converges in $L^1(\|T\|)$ to some $f \in \mathscr{B}^\infty_{\mathrm{c}}(X)$. Suppose further that $\pi \in \mathrm{Lip}_1(X, \mathbb{R}^m)$ and $\pi_\#(T\lfloor f_k) = [u_k]$ for some $u_k \in L^1(\mathbb{R}^m)$. Then $[u_k - u_l] = [u_k] - [u_l] = \pi_\#(T\lfloor f_k) - \pi_\#(T\lfloor f_l) = \pi_\#(T\lfloor (f_k - f_l))$, and

$$\begin{aligned}\int_{\mathbb{R}^m} |u_k - u_l|\, dx &= \mathbf{M}([u_k - u_l]) = \mathbf{M}\big(\pi_\#(T\lfloor (f_k - f_l))\big) \\ &\leq \mathbf{M}(T\lfloor (f_k - f_l)) \leq \int_X |f_k - f_l|\, d\|T\|\end{aligned}$$

by (4.5), Lemma 4.6(2), and (4.10). Since $f_k \to f$ in $L^1(\|T\|)$, by the completeness of $L^1(\mathbb{R}^m)$ there is a function $u \in L^1(\mathbb{R}^m)$ such that $u_k \to u$ in $L^1(\mathbb{R}^m)$. By passing to the $\mathbf{M}$-limit on either side of the identity $\pi_\#(T\lfloor f_k) = [u_k]$, for $k \to \infty$, we conclude that $\pi_\#(T\lfloor f) = [u]$. Now, to see that (1) implies (2), apply this procedure to the characteristic function f of a Borel set $B \Subset X$ and an approximating sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathscr{D}(X)$. To show that (2) implies (1), choose $(f_k)_{k\in\mathbb{N}}$ such that each f_k is a finite linear combination of characteristic functions of compact sets, and such that $f_k \to f \in \mathscr{D}(X)$ in $L^1(\|T\|)$. □

Proposition 7.4 (absolute continuity)**.** *Let $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$, $m \geq 1$.*

(1) *If condition (1) or (2) of Lemma 7.3 holds for some $\pi \in \mathrm{Lip}(X, \mathbb{R}^m)$, then*

$$\|T\lfloor(1,\pi)\|(\pi^{-1}(N) \cap B) = 0$$

whenever $N \subset \mathbb{R}^m$ is a Borel set with $\mathscr{L}^m(N) = 0$ and $B \subset X$ is a Borel set that is σ-finite with respect to $\|T\|$.

(2) *If condition (1) or (2) of Lemma 7.3 holds for all $\pi \in \mathrm{Lip}(X, \mathbb{R}^m)$, then $\|T\|(A) = 0$ for every set $A \subset X$ with $\mathscr{H}^m(A) = 0$.*

Together with Theorem 7.2, this shows that for every $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$, $\|T\|$ is absolutely continuous with respect to $\mathscr{H}^m$ (cf. [3, Theorem 3.9]).

Proof. We assume that condition (1) of Lemma 7.3 holds for some $\pi \in \mathrm{Lip}(X, \mathbb{R}^m)$. Suppose $N \subset \mathbb{R}^m$ is a bounded Borel set with $\mathscr{L}^m(N) = 0$. Let $f \in \mathscr{D}(X)$, and choose $\sigma \in \mathscr{D}(X)$ with $\sigma|_{\mathrm{spt}(f)} = 1$. By assumption, $\pi_\#(T\lfloor f) = [u_f]$ for some $u_f \in L^1(\mathbb{R}^m)$. We abbreviate $T_\pi := T\lfloor(1,\pi)$; note that $T_\pi \in \mathbf{M}_{0,\mathrm{loc}}(X)$. Using Lemma 4.6(1) we obtain

$$\begin{aligned}(T_\pi\lfloor\pi^{-1}(N))(f) &= T((\chi_N \circ \pi)f, \pi) = (T\lfloor f)(\sigma(\chi_N \circ \pi), \pi) \\ &= \pi_\#(T\lfloor f)(\chi_N, \mathrm{id}) = [u_f](\chi_N, \mathrm{id}) = \int_{\mathbb{R}^m} u_f \chi_N \, dx = 0.\end{aligned}$$

As this holds for all $f \in \mathscr{D}(X)$, we have $T_\pi\lfloor\pi^{-1}(N) = 0$. Let $B \subset X$ be a Borel set that is σ-finite with respect to $\|T\|$. By Lemma 4.7, $\|T_\pi\|(\pi^{-1}(N) \cap B) = 0$, and it follows that the same identity holds if N is unbounded.

To prove (2), let $A \subset X$ be a Borel set with $\mathscr{H}^m(A) = 0$. Then A is separable and hence σ-finite with respect to $\|T\|$. Let $\pi \in \mathrm{Lip}(X, \mathbb{R}^m)$. Since $\mathscr{L}^m(\pi(A)) = 0$, there is a Borel set $N \subset \mathbb{R}^m$ with $\mathscr{L}^m(N) = 0$ that contains $\pi(A)$. Then $\|T_\pi\|(A) = \|T_\pi\|(\pi^{-1}(N) \cap A) = 0$ by (1). Hence $T(f, \pi) = T_\pi(f) = 0$ for all $f \in \mathscr{B}^\infty_{\mathrm{c}}(X)$ with $\{f \neq 0\} \subset A$. As this holds for all $\pi \in \mathrm{Lip}(X, \mathbb{R}^m)$, we conclude from Lemma 4.7 that $\|T\|(A) = 0$. □

We conclude this section with two results for locally normal currents that relate the previous discussion to the slicing theory.

Theorem 7.5 (0-dimensional slices)**.** *Suppose $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$, $m \geq 1$, and $\pi \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R}^m)$. Whenever $f \in \mathscr{B}^\infty_{\mathrm{c}}(X)$, then $\pi_\#(T\lfloor f) = [u_f]$ for some unique $u_f \in L^1(\mathbb{R}^m)$, moreover $u_f \in \mathrm{BV}(\mathbb{R}^m)$ if $f \in \mathscr{D}(X)$. If $\mathrm{spt}(T)$ is separable, and if $f \in \mathscr{B}^\infty_{\mathrm{c}}(X)$, then*

$$\langle T, \pi, y\rangle(f) = u_f(y)$$

for $\mathscr{L}^m$-almost every $y \in \mathbb{R}^m$.

Proof. If $f \in \mathscr{D}(X)$, then $\pi_\#(T\lfloor f) \in \mathbf{N}_m(\mathbb{R}^m)$, hence $\pi_\#(T\lfloor f) = [u_f]$ for some $u_f \in \mathrm{BV}(\mathbb{R}^m)$ by Theorem 7.2. The argument of the proof of Lemma 7.3 then shows that $\pi_\#(T\lfloor f) = [u_f]$ for some $u_f \in L^1(\mathbb{R}^m)$ in the general case, when $f \in \mathscr{B}_c^\infty(X)$. Clearly (the equivalence class) $u_f \in L^1(\mathbb{R}^m)$ is uniquely determined.

Now suppose $\mathrm{spt}(T)$ is separable, and $f \in \mathscr{B}_c^\infty(X)$. Choose $\sigma \in \mathscr{D}(X)$ with $\sigma|_{\mathrm{spt}(f)} = 1$. For all $\phi \in \mathscr{D}(\mathbb{R}^m)$, Theorem 6.4(2) (the case $k = m$) gives

$$\begin{aligned}\int_{\mathbb{R}^m} \phi(y)\langle T,\pi,y\rangle(f)\,dy &= (T\lfloor(1,\pi))((\phi\circ\pi)f)\\ &= (T\lfloor f)(\sigma(\phi\circ\pi),\pi)\\ &= \pi_\#(T\lfloor f)(\phi,\mathrm{id})\\ &= [u_f](\phi,\mathrm{id})\\ &= \int_{\mathbb{R}^m}\phi(y)u_f(y)\,dy.\end{aligned}$$

We conclude that $\langle T,\pi,y\rangle(f) = u_f(y)$ for $\mathscr{L}^m$-almost every $y \in \mathbb{R}^m$. □

Theorem 7.6 (partial rectifiability). *Suppose $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$, $m \geq 1$, $\mathrm{spt}(T)$ is separable, and $\pi \in \mathrm{Lip}_{\mathrm{loc}}(X,\mathbb{R}^m)$. Put*

$$A := \bigl\{x \in X : \langle T,\pi,\pi(x)\rangle \in \mathbf{M}_{0,\mathrm{loc}}(X) \text{ and } \|\langle T,\pi,\pi(x)\rangle\|(\{x\}) > 0\bigr\}.$$

Then there exists a countable family of pairwise disjoint compact sets $B_i \subset A$ such that $\|T\lfloor(1,\pi)\|\bigl(A \setminus \bigcup_i B_i\bigr) = 0$ and $\pi|_{B_i}$ is bi-Lipschitz for every i.

Compare [3, Theorem 7.4].

Proof. We assume without loss of generality that X is compact, so that $T \in \mathbf{N}_m(X)$ and $\pi \in \mathrm{Lip}(X,\mathbb{R}^m)$. By Theorem 7.5, for every $f \in \mathscr{D}(X)$ there is a function $u_f \in \mathrm{BV}(\mathbb{R}^m)$ such that $\pi_\#(T\lfloor f) = [u_f]$. Hence, using Theorem 7.2, Lemma 4.7 and Lemma 4.6(2), we obtain

$$\begin{aligned}|Du_f|(B) = \|\partial[u_f]\|(B) &= \|\partial(\pi_\#(T\lfloor f))\|(B) = \|\pi_\#(\partial(T\lfloor f))\|(B)\\ &\leq \mathrm{Lip}(\pi)^{m-1}\|\partial(T\lfloor f)\|(\pi^{-1}(B))\end{aligned}$$

for every Borel set $B \subset \mathbb{R}^m$. Suppose that $|f| \leq 1$ and $\mathrm{Lip}(f) \leq 1$. Since $\partial(T\lfloor f) = (\partial T)\lfloor f - T\lfloor(1,f)$, cf. (3.6), it follows that

$$|Du_f|(B) \leq \mathrm{Lip}(\pi)^{m-1}(\|T\| + \|\partial T\|)(\pi^{-1}(B)) =: \mu(B)$$

for all Borel sets $B \subset \mathbb{R}^m$. Note that the finite Borel measure μ so defined is independent of f. Now we choose a countable subset $\mathscr{F}$ of $\{f \in \mathscr{D}(X) : 0 \leq f \leq 1,\ \mathrm{Lip}(f) \leq 1\}$ with the following property: Whenever $x \in X$ and $0 < r \leq 1$, there is an $f \in \mathscr{F}$ such that

$$f(x) \geq \frac{3}{4}r, \quad f \leq r, \quad \mathrm{spt}(f) \subset \mathrm{U}(x,r). \tag{7.2}$$

By Theorem 6.4, Theorem 7.5, and Lemma 7.1 there exists a Borel set $N \subset \mathbb{R}^m$ with $\mathscr{L}^m(N) = 0$ such that whenever $y, y' \in \mathbb{R}^m \setminus N$ and $f \in \mathscr{F}$, the slice $T_y := \langle T, \pi, y\rangle$ exists as an element of $\mathbf{M}_0(X)$, $T_y(f) = u_f(y)$, $M_\mu(y) < \infty$, and

$$|T_y(f) - T_{y'}(f)| = |u_f(y) - u_f(y')| \le c_m\big(M_\mu(y) + M_\mu(y')\big)|y - y'|.$$

By Proposition 7.4(1),

$$\|T\lfloor(1,\pi)\|(\pi^{-1}(N)) = 0.$$

Let $\epsilon, \delta \in (0,1] \cap \mathbb{Q}$, and recall that $\|T_y\|$ is concentrated on $\pi^{-1}\{y\}$. Denote by $E_{\epsilon,\delta}$ the Borel set of all $x \in X$ such that $y := \pi(x) \notin N$ and

$$M_\mu(y) \le \frac{1}{2\epsilon}, \quad \|T_y\|(\{x\}) \ge \epsilon, \quad \|T_y\|(\mathrm{U}(x, 2\delta) \setminus \{x\}) \le \frac{\epsilon}{4}.$$

We have $A \setminus \pi^{-1}(N) = \bigcup_{\epsilon,\delta} E_{\epsilon,\delta}$. Let $x, x' \in E_{\epsilon,\delta}$ with $0 < r := d(x, x') \le \delta$ and put $y := \pi(x)$, $y' = \pi(x')$. Then choose $f \in \mathscr{F}$ as in (7.2). Since $\mathrm{U}(x, r) \subset \mathrm{U}(x', 2r) \setminus \{x'\}$, it follows that

$$|T_{y'}(f)| \le \int_X f\, d\|T_{y'}\| \le r \cdot \|T_{y'}\|(\mathrm{U}(x', 2\delta) \setminus \{x'\}) \le \frac{\epsilon}{4} r.$$

Similarly, using (4.12), we obtain

$$|T_y(f)| \ge |T_y(\chi_{\{x\}} f)| - |T_y(\chi_{X\setminus\{x\}} f)| \ge \frac{3}{4} r \cdot \epsilon - r \cdot \frac{\epsilon}{4} = \frac{\epsilon}{2} r.$$

We conclude that

$$\begin{aligned} \frac{\epsilon}{4} d(x, x') = \frac{\epsilon}{4} r &\le |T_y(f)| - |T_{y'}(f)| \le |T_y(f) - T_{y'}(f)| \\ &\le c_m\big(M_\mu(y) + M_\mu(y')\big)|y - y'| \le \frac{c_m}{\epsilon}|y - y'| = \frac{c_m}{\epsilon}|\pi(x) - \pi(x')|. \end{aligned}$$

This shows that whenever we restrict π to a subset of $E_{\epsilon,\delta}$ with diameter at most δ, the resulting map is a bi-Lipschitz map into $\mathbb{R}^m \setminus N$. The result follows. $\square$

8 Integer rectifiable currents

We now turn to integer multiplicity rectifiable currents.

We recall that a subset E of a metric space X is *countably m-rectifiable* if there is a countable family of Lipschitz maps $F_i\colon A_i \to X$, $A_i \subset \mathbb{R}^m$, such that $E \subset \bigcup_i F_i(A_i)$. The set $E \subset X$ is *countably $\mathscr{H}^m$-rectifiable* if there is a countably m-rectifiable set $E' \subset X$ with $\mathscr{H}^m(E \setminus E') = 0$ (cf. [14, 3.2.14]).

Now we let again X denote a locally compact metric space. Since X is locally complete, it is not difficult to see that every countably m-rectifiable set $E \subset X$ is contained in a countably m-rectifiable and σ-compact set $\bar{E} \subset X$.

Definition 8.1 (integer rectifiable current)**.** *Let $T \in \mathscr{D}_m(X)$, $m \geq 0$. We call T a* locally integer rectifiable current *if*

(1) $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$,

(2) *whenever $B \Subset X$ is a Borel set and $\pi \in \mathrm{Lip}(X, \mathbb{R}^m)$, then $\pi_\#(T \lfloor B) = [u]$ for some $u = u_{B,\pi} \in L^1(\mathbb{R}^m, \mathbb{Z})$, and*

(3) *$\|T\|$ is concentrated on some countably $\mathscr{H}^m$-rectifiable Borel set $E \subset X$.*

The set of all m-dimensional locally integer rectifiable currents in X is denoted by

$$\mathscr{I}_{m,\mathrm{loc}}(X).$$

An m-dimensional integer rectifiable current *in X is an element of*

$$\mathscr{I}_m(X) := \mathscr{I}_{m,\mathrm{loc}}(X) \cap \mathbf{M}_m(X).$$

Note that if a current $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$ satisfies condition (2), then $\|T\|$ is absolutely continuous with respect to $\mathscr{H}^m$ by Proposition 7.4. Moreover, condition (3) implies that the support of T is separable. Clearly $\mathscr{I}_{m,\mathrm{loc}}(X)$ forms an abelian group, as does $\mathscr{I}_m(X)$. Push-forwards of locally integer rectifiable currents under locally Lipschitz maps are again locally integer rectifiable. If $T \in \mathscr{I}_{m,\mathrm{loc}}(X)$, then $T \lfloor B \in \mathscr{I}_{m,\mathrm{loc}}(X)$ for every Borel set $B \subset X$.

In case $m = 0$, condition (2) for $T \in \mathbf{M}_{0,\mathrm{loc}}(X)$ means that $T(\chi_B) \in \mathbb{Z}$ for every Borel set $B \Subset X$ (cf. the remark following Proposition 2.6). If $T \in \mathscr{I}_{0,\mathrm{loc}}(X)$, then $\|T\|$ is concentrated on some countable set $E \subset X$ consisting of atoms of $\|T\|$, i.e., $\|T\|(\{x\}) > 0$ for $x \in E$. By (4.12) and condition (2), $\|T\|(\{x\}) = |T(\chi_{\{x\}})| \in \mathbb{Z}$ for $x \in E$, so E is discrete, and

$$T(f) = \sum_{x \in E} a_x f(x) \tag{8.1}$$

for all $f \in \mathscr{B}^\infty_{\mathrm{c}}(X)$, where $a_x = T(\chi_{\{x\}}) \in \mathbb{Z} \setminus \{0\}$.

Lemma 8.2 (characterizing $\mathscr{I}_{0,\mathrm{loc}}$)**.** *Suppose $S \in \mathbf{M}_{0,\mathrm{loc}}(X)$, $\mathrm{spt}(S)$ is separable, and $S(\chi_K) \in \mathbb{Z}$ for every compact set $K \subset X$. Then $S \in \mathscr{I}_{0,\mathrm{loc}}(X)$.*

Proof. Let Σ be the set of all $x \in X$ such that $\|S\|(\{x\}) \geq 1$. Since S has locally finite mass, Σ is discrete. Now let $x \in X \setminus \Sigma$. There is an open neighborhood V of x such that $\|S\|(V) < 1$. Then $|S(\chi_K)| \leq \|S\|(V) < 1$ and thus $S(\chi_K) = 0$ for every compact set $K \subset V$. By approximation, this implies that $S(f) = 0$ for every $f \in \mathscr{D}(X)$ with $\mathrm{spt}(f) \subset V$, hence $x \notin \mathrm{spt}(S)$. This shows that $\mathrm{spt}(S) \subset \Sigma$. Since $\mathrm{spt}(S)$ is separable by assumption, it follows that $S \in \mathscr{I}_{0,\mathrm{loc}}(X)$. □

The next theorem corresponds to [3, Theorem 4.5] in the case of finite mass. A similar characterization of classical rectifiable currents is given in [14, Theorem 4.1.28].

Theorem 8.3 (parametric representation)**.** *Let $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$, $m \geq 1$. Then $T \in \mathscr{I}_{m,\mathrm{loc}}(X)$ if and only if there exists a countable family of currents T^i, each of the form $T^i = F_{i\#}[u_i]$ for some function $u_i \in L^1(\mathbb{R}^m, \mathbb{Z})$ and some bi-Lipschitz map $F_i \colon K_i \to X$ defined on a compact set $K_i \subset \mathbb{R}^m$ containing* $\mathrm{spt}(u_i)$*, such that*

$$\|T\|(A) = \sum_i \|T^i\|(A), \quad T(f, \pi) = \sum_i T^i(f, \pi)$$

for all Borel sets $A \subset X$ and for all $(f, \pi) \in \mathscr{D}^m(X)$.

The proof of the 'only if' part uses the following fact: If $E \subset X$ is an $\mathscr{H}^m$-measurable and countably $\mathscr{H}^m$-rectifiable set, then there exists a countable family of bi-Lipschitz maps $F_i \colon K_i \to E$, with $K_i \subset \mathbb{R}^m$ compact, such that the $F_i(K_i)$ are pairwise disjoint and

$$\mathscr{H}^m\big(E \setminus \textstyle\bigcup_i F_i(K_i)\big) = 0. \tag{8.2}$$

For $X = \mathbb{R}^n$, this result is contained in [14, Lemma 3.2.18]. For a general complete metric space X, it is stated in [3, Lemma 4.1]; the argument relies on the metric differentiability theorem of [23] and generalizes to every locally complete metric space, in particular to our locally compact metric space X. We omit the details since the 'only if' part of the theorem will not be used in the sequel. In fact, for locally integer rectifiable currents whose boundary has locally finite mass, the existence of such parametric representations also follows from Theorem 7.6, cf. the proof of Theorem 8.4 together with Theorem 8.5(1) (the case $k = m$).

Proof of Theorem 8.3. Suppose $T \in \mathscr{I}_{m,\mathrm{loc}}(X)$, $m \geq 1$. Then $\|T\|$ is concentrated on some countably $\mathscr{H}^m$-rectifiable Borel set $E \subset X$. Choose a countable family of bi-Lipschitz maps $F_i \colon K_i \to E$, with $K_i \subset \mathbb{R}^m$ compact, such that the sets $B^i := F_i(K_i)$ are pairwise disjoint and $\mathscr{H}^m\big(E \setminus \bigcup_i B^i\big) = 0$, cf. (8.2). Since $\|T\|$ is absolutely continuous with respect to $\mathscr{H}^m$ by Proposition 7.4,

$$\|T\|\big(X \setminus \textstyle\bigcup_i B^i\big) = \|T\|\big(E \setminus \textstyle\bigcup_i B^i\big) = 0.$$

Let $\pi^i \colon X \to \mathbb{R}^m$ be a Lipschitz extension of $F_i^{-1} \colon B^i \to \mathbb{R}^m$, and put $T^i := T \lfloor B^i$. Since $T \in \mathscr{I}_{m,\mathrm{loc}}(X)$, we have $\pi^i_\# T^i = [u_i]$ for some $u_i \in L^1(\mathbb{R}^m, \mathbb{Z})$ with $\mathrm{spt}(u_i) \subset K_i$. It follows that $T^i = F_{i\#}(\pi^i_\# T^i) = F_{i\#}[u_i]$. For every Borel set $A \subset X$,

$$\|T\|(A) = \|T\|\big(A \cap \textstyle\bigcup_i B^i\big) = \displaystyle\sum_i \|T\|(A \cap B^i) = \sum_i \|T^i\|(A).$$

Let $(f, \pi) \in \mathscr{D}^m(X)$, and denote by χ_i and χ the characteristic functions of B^i and $\bigcup_i B^i$, respectively. Using Theorem 4.4(2) we get

$$T(f, \pi) = T(\chi f, \pi) = \sum_i T(\chi_i f, \pi) = \sum_i T^i(f, \pi).$$

Conversely, suppose that $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$ admits a representation as in the theorem. Since $\|T^i\|(X \setminus F_i(K_i)) = 0$, $\|T\|$ is concentrated on the countably m-rectifiable Borel set $\bigcup_i F_i(K_i)$. From Lemma 3.7 it follows that $[u_i] \in \mathscr{I}_m(\mathbb{R}^m)$, hence $T^i = F_{i\#}[u_i] \in \mathscr{I}_m(X)$. By forming partial sums of $\sum_i T^i$ we find a sequence $(S^k)_{k\in\mathbb{N}}$ in $\mathscr{I}_m(X)$ such that for every Borel set $B \Subset X$, $\mathbf{M}(T\lfloor B - S^k\lfloor B) \to 0$ for $k \to \infty$. For $\pi \in \mathrm{Lip}(X, \mathbb{R}^m)$, we have $\pi_\#(S^k\lfloor B) = [u_k]$, $u_k \in L^1(\mathbb{R}^m, \mathbb{Z})$. A similar argument as in the proof of Lemma 7.3 then shows that $\pi_\#(T\lfloor B) = [u]$ for some $u \in L^1(\mathbb{R}^m, \mathbb{Z})$. Thus $T \in \mathscr{I}_{m,\mathrm{loc}}(X)$. □

We now supplement the slicing theory for locally normal currents T with the following two theorems, providing criteria for the rectifiability of T in terms of the rectifiability of slices, and vice-versa. We refer to [38], [20] for similar statements in the context of classical currents, and to [3, Theorem 8.1] for a corresponding result for normal metric currents.

Theorem 8.4 (rectifiability criterion)**.** *Suppose $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$, $m \geq 1$, $\mathrm{spt}(T)$ is separable, and $\mathscr{P}$ is a countable subset of $\mathrm{Lip}_1(X)$ such that for every $\pi_0 \in \mathrm{Lip}_1(X)$ and every compact set $K \subset \mathrm{spt}(T)$ there is a sequence in $\mathscr{P}$ converging uniformly on K to π_0. If for each $\pi = (\pi_1, \dots, \pi_m) \in \mathscr{P}^m$, $\langle T, \pi, y\rangle \in \mathscr{I}_{0,\mathrm{loc}}(X)$ for $\mathscr{L}^m$-almost every $y \in \mathbb{R}^m$, then $T \in \mathscr{I}_{m,\mathrm{loc}}(X)$.*

Proof. Fix $\pi \in \mathscr{P}^m$ for the moment. Put

$$A_\pi := \big\{x \in X : \langle T, \pi, \pi(x)\rangle \in \mathbf{M}_{0,\mathrm{loc}}(X) \text{ and } \|\langle T, \pi, \pi(x)\rangle\|(\{x\}) > 0\big\},$$

and let $E_\pi \subset A_\pi$ be the respective σ-compact set provided by Theorem 7.6, so that $\|T\lfloor(1,\pi)\|(A_\pi \setminus E_\pi) = 0$. By assumption, for $\mathscr{L}^m$-almost every $y \in \mathbb{R}^m$, $\|\langle T, \pi, y\rangle\|$ is concentrated on some countable set contained in A_π. Hence, by Theorem 6.4(3),

$$\|T\lfloor(1,\pi)\|(X \setminus A_\pi) = \int_{\mathbb{R}^m} \|\langle T, \pi, y\rangle\|(X \setminus A_\pi)\, dy = 0.$$

Thus $\|T\lfloor(1,\pi)\|(X \setminus E_\pi) = 0$.

Now let $E := \bigcup_{\pi\in\mathscr{P}^m} E_\pi$. Suppose $(f, \pi) \in \mathscr{B}_c^\infty(X) \times \mathscr{P}^m$, and $\{f \neq 0\} \subset \mathrm{spt}(T) \setminus E$. Since $\{f \neq 0\} \subset X \setminus E_\pi$ and $\|T\lfloor(1,\pi)\|(X \setminus E_\pi) = 0$, we have $T(f,\pi) = (T\lfloor(1,\pi))(f) = 0$. By the choice of the family $\mathscr{P}$, it follows that $T(f,\pi) = 0$ for all $(f,\pi) \in \mathscr{B}_c^\infty(X) \times [\mathrm{Lip}_1(X)]^m$ with $\{f \neq 0\} \subset \mathrm{spt}(T) \setminus E$. Since $\|T\|$ is σ-finite, Lemma 4.7 yields $\|T\|(X \setminus E) = 0$. So $\|T\|$ is concentrated on the countably m-rectifiable Borel set E.

By the respective property of the sets E_π, we find a countable family of pairwise disjoint compact sets $B^k \subset E$ with $\|T\|\big(E \setminus \bigcup_k B^k\big) = 0$ and the property that for every k, there is a $\pi^k \in \mathscr{P}^m$ such that $\pi^k|_{B^k}$ is bi-Lipschitz. By Theorem 7.5, $\pi^k_\#(T\lfloor B^k) = [u_k]$ for some $u_k \in L^1(\mathbb{R}^m)$, and

$$u_k(y) = \langle T, \pi^k, y\rangle(\chi_{B^k})$$

for $\mathscr{L}^m$-almost every y. By assumption, $\langle T, \pi^k, y\rangle(\chi_{B^k}) \in \mathbb{Z}$ for $\mathscr{L}^m$-almost every y, hence $u_k \in L^1(\mathbb{R}^m, \mathbb{Z})$. Now it follows that T admits a parametric representation as in Theorem 8.3. Hence $T \in \mathscr{I}_{m,\mathrm{loc}}(X)$. □

Theorem 8.5 (rectifiable slices)**.** *Suppose* $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$*,* $1 \le k \le m$*, and* $\mathrm{spt}(T)$ *is separable.*

(1) *If T satisfies condition (2) of Definition 8.1, in particular if* $T \in \mathscr{I}_{m,\mathrm{loc}}(X)$*, and if* $\pi \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R}^k)$*, then* $\langle T, \pi, y\rangle \in \mathscr{I}_{m-k,\mathrm{loc}}(X)$ *for* $\mathscr{L}^k$*-almost every* $y \in \mathbb{R}^k$*.*

(2) *Conversely, if for each* $\pi \in \mathrm{Lip}(X, \mathbb{R}^k)$*,* $\langle T, \pi, y\rangle \in \mathscr{I}_{m-k,\mathrm{loc}}(X)$ *for* $\mathscr{L}^k$*-almost every* $y \in \mathbb{R}^k$*, then* $T \in \mathscr{I}_{m,\mathrm{loc}}(X)$*.*

Proof. For the proof of (1), we first consider the case $k = m$, so that $\pi \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R}^m)$. Choose a countable family $\mathscr{U}$ of open sets $U \Subset X$ such that for every $x \in \mathrm{spt}(T)$ and $\epsilon > 0$ there is a $U \in \mathscr{U}$ with $x \in U \subset \mathrm{U}(x, \epsilon)$, and such that the union of finitely many elements of $\mathscr{U}$ is in $\mathscr{U}$. By assumption, for every $U \in \mathscr{U}$ there exists a function $u_U \in L^1(\mathbb{R}^m, \mathbb{Z})$ such that $\pi_\#(T \lfloor U) = [u_U]$. Thus, for $\mathscr{L}^m$-almost every y, $\langle T, \pi, y\rangle \in \mathbf{M}_{0,\mathrm{loc}}(X)$, and, by Theorem 7.5,

$$\langle T, \pi, y\rangle(\chi_U) = u_U(y) \in \mathbb{Z}$$

for all $U \in \mathscr{U}$. It follows that for every such y, $\langle T, \pi, y\rangle(\chi_K) \in \mathbb{Z}$ for all compact sets $K \subset X$. Lemma 8.2 then shows that $\langle T, \pi, y\rangle \in \mathscr{I}_{0,\mathrm{loc}}(X)$.

Now suppose $1 \le k < m$, and let $\pi \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R}^k)$. Choose a countable set $\mathscr{P} \subset \mathrm{Lip}_1(X)$ as in Theorem 8.4. Fix $\rho \in \mathscr{P}^{m-k}$ for the moment. We know from Theorem 6.5 (iterated slices) and the result in the case $k = m$ that

$$\big\langle \langle T, \pi, y\rangle, \rho, z\big\rangle = \langle T, (\pi, \rho), (y, z)\rangle \in \mathscr{I}_{0,\mathrm{loc}}(X)$$

for $\mathscr{L}^m$-almost every $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$. Hence, there is a set $N_\rho \subset \mathbb{R}^k$ with $\mathscr{L}^k(N_\rho) = 0$ such that for every $y \in \mathbb{R}^k \setminus N_\rho$, $\langle T, \pi, y\rangle \in \mathbf{N}_{m-k,\mathrm{loc}}(X)$ and $\big\langle \langle T, \pi, y\rangle, \rho, z\big\rangle \in \mathscr{I}_{0,\mathrm{loc}}(X)$ for all $z \in \mathbb{R}^{m-k} \setminus N'_{y,\rho}$, where $\mathscr{L}^{m-k}(N'_{y,\rho}) = 0$. Suppose $y \in \mathbb{R}^k \setminus \bigcup_{\rho \in \mathscr{P}^{m-k}} N_\rho$, and put $T_y := \langle T, \pi, y\rangle$. For each $\rho \in \mathscr{P}^{m-k}$, $\langle T_y, \rho, z\rangle \in \mathscr{I}_{0,\mathrm{loc}}(X)$ for all $z \in \mathbb{R}^{m-k} \setminus N'_{y,\rho}$. Hence $T_y \in \mathscr{I}_{m-k,\mathrm{loc}}(X)$ by Theorem 8.4. Since $\mathscr{P}$ is countable, this proves (1).

We show (2). In case $k = m$, the result holds by Theorem 8.4.

Now suppose $1 \le k < m$. Let $\pi \in \mathrm{Lip}(X, \mathbb{R}^k)$ and $\rho \in \mathrm{Lip}(X, \mathbb{R}^{m-k})$. Let again $\mathscr{U}$ be given as above. It follows from Theorem 6.4 that the set M of all $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$ such that $\langle T, (\pi, \rho), (y, z)\rangle \in \mathbf{M}_{0,\mathrm{loc}}(X)$ and $\langle T, (\pi, \rho), (y, z)\rangle(\chi_U) \in \mathbb{Z}$ for all $U \in \mathscr{U}$ is $\mathscr{L}^m$-measurable. By assumption (and Theorem 6.4), $\langle T, \pi, y\rangle \in \mathscr{I}_{m-k,\mathrm{loc}}(X) \cap \mathbf{N}_{m-k,\mathrm{loc}}(X)$ for $\mathscr{L}^k$-almost every $y \in \mathbb{R}^k$. Hence, for $\mathscr{L}^k$-almost every $y \in \mathbb{R}^k$, it follows from Theorem 6.5 and the first part of the proof of (1) that for $\mathscr{L}^{m-k}$-almost every $z \in \mathbb{R}^{m-k}$,

$$\langle T, (\pi, \rho), (y, z)\rangle = \big\langle \langle T, \pi, y\rangle, \rho, z\big\rangle \in \mathscr{I}_{0,\mathrm{loc}}(X),$$

thus $\mathscr{L}^{m-k}(\{z \in \mathbb{R}^{m-k} : (y,z) \notin M\}) = 0$. We conclude that $\mathscr{L}^m(\mathbb{R}^m \setminus M) = 0$, and, by virtue of Lemma 8.2, that $\langle T, (\pi,\rho), (y,z)\rangle \in \mathscr{I}_{0,\mathrm{loc}}(X)$ for all $(y,z) \in M$. As π and ρ were arbitrary, we have $T \in \mathscr{I}_{m,\mathrm{loc}}(X)$ by the result in the case $k = m$. □

Finally, we consider the chain complex of integral currents.

Definition 8.6 (integral current)**.** *The abelian group of m-dimensional* locally integral currents *in X is defined by*

$$\mathbf{I}_{m,\mathrm{loc}}(X) := \{T \in \mathscr{I}_{m,\mathrm{loc}}(X) : \partial T \in \mathscr{I}_{m-1,\mathrm{loc}}(X)\}$$

if $m \geq 1$*, and* $\mathbf{I}_{0,\mathrm{loc}}(X) := \mathscr{I}_{0,\mathrm{loc}}(X)$*. An m-dimensional* integral current *in X is an element of*

$$\mathbf{I}_m(X) := \mathbf{I}_{m,\mathrm{loc}}(X) \cap \mathbf{N}_m(X).$$

The following important result yields a simpler characterization of $\mathbf{I}_{m,\mathrm{loc}}(X)$, cf. [29, Theorem 30.3] and [3, Theorem 8.6].

Theorem 8.7 (boundary rectifiability)**.** *For all* $m \geq 0$*,* $\mathscr{I}_{m,\mathrm{loc}}(X) \cap \mathbf{N}_{m,\mathrm{loc}}(X) = \mathbf{I}_{m,\mathrm{loc}}(X)$*.*

In other words, if $T \in \mathscr{I}_{m,\mathrm{loc}}(X)$, $m \geq 1$, and if ∂T has locally finite mass, then $\partial T \in \mathscr{I}_{m-1,\mathrm{loc}}(X)$.

Proof. In case $m = 0$ there is nothing to prove.

Now consider the case $m = 1$. Let $T \in \mathscr{I}_{1,\mathrm{loc}}(X) \cap \mathbf{N}_{1,\mathrm{loc}}(X)$, and let $U \Subset X$ be an open set. If $U = X$, then $\partial T(\chi_U) = \partial T(1) = T(1,1) = 0$. If $U \neq X$, put $\pi(x) := \inf\{d(x,z) : z \in X \setminus U\}$ for $x \in X$. Then for $\mathscr{L}^1$-almost every $s > 0$,

$$\begin{aligned}\partial T(\chi_{\{\pi>s\}}) &= ((\partial T)\lfloor\{\pi > s\})(\chi_U)\\ &= \partial(T\lfloor\{\pi > s\})(\chi_U) + \langle T, \pi, s\rangle(\chi_U)\\ &= \langle T, \pi, s\rangle(\chi_U) \in \mathbb{Z}\end{aligned}$$

by Theorem 8.5(1). By continuity, letting s tend to 0 we conclude that $\partial T(\chi_U)$ is an integer. As this holds for every open set $U \Subset X$, it follows that $\partial T(\chi_K)$ is an integer for every compact set $K \subset X$, and Lemma 8.2 yields $\partial T \in \mathscr{I}_{0,\mathrm{loc}}(X)$.

Now let $m \geq 1$ and suppose the result holds in dimension m. Let $T \in \mathscr{I}_{m+1,\mathrm{loc}}(X) \cap \mathbf{N}_{m+1,\mathrm{loc}}(X)$ and $\pi \in \mathrm{Lip}(X)$. For almost every $y \in \mathbb{R}$, $\langle T, \pi, y\rangle \in \mathscr{I}_{m,\mathrm{loc}}(X) \cap \mathbf{N}_{m,\mathrm{loc}}(X)$ by Theorem 8.5(1), hence

$$\langle \partial T, \pi, y\rangle = -\partial\langle T, \pi, y\rangle \in \mathscr{I}_{m-1,\mathrm{loc}}(X)$$

by (6.3) and the induction hypothesis. As this holds for every $\pi \in \mathrm{Lip}(X)$, Theorem 8.5(2) implies that $\partial T \in \mathscr{I}_{m,\mathrm{loc}}(X)$. □

As a corollary, we obtain another characterization of locally integral currents (cf. [3, Theorem 8.8(ii)]).

Theorem 8.8. *Let $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$, $m \geq 0$. Then $T \in \mathbf{I}_{m,\mathrm{loc}}(X)$ if and only if* $\operatorname{spt}(T)$ *is separable and T satisfies condition (2) of Definition 8.1.*

Proof. One implication is clear from Definition 8.1. The other follows from Lemma 8.2 if $m = 0$ and from Theorem 8.5 (e.g. the case $k = m$) and Theorem 8.7 if $m \geq 1$. □

A similar induction argument as in the proof of Theorem 8.7 also shows the next result, cf. [29, Theorem 32.2], [36, Sect. 3], and [3, Theorem 8.5].

Theorem 8.9 (closure theorem)**.** *If $(T_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbf{I}_{m,\mathrm{loc}}(X)$ converging weakly to $T \in \mathscr{D}_m(X)$, $m \geq 0$, and if* $\sup_n \mathbf{N}_V(T_n) < \infty$ *for every open set $V \Subset X$, then $T \in \mathbf{I}_{m,\mathrm{loc}}(X)$.*

Proof. Suppose first that $m = 0$. Each T_n is a locally finite sum $\sum_i a_{n,i}[x_{n,i}]$ with coefficients $a_{n,i} \in \mathbb{Z} \setminus \{0\}$, where $[x_{n,i}](f) = f(x_{n,i})$ for all $f \in \mathscr{D}(X)$, cf. (8.1). For every open set $V \Subset X$, $\operatorname{spt}(T_n) \cap V$ contains at most $\mathbf{M}_V(T_n)$ points. Since $T_n \to T$, it follows that $\operatorname{spt}(T) \cap V$ consists of at most $\sup_n \mathbf{M}_V(T_n)$ points. Hence, T is a locally finite sum $\sum_j a_j[x_j]$ with coefficients $a_j \in \mathbb{R} \setminus \{0\}$. To see that $a_j \in \mathbb{Z}$, choose $f \in \mathscr{D}(X)$ so that $0 \leq f \leq 1$, $f(x_j) = 1$, $f(x) = 0$ for all $x \in \operatorname{spt}(T) \setminus \{x_j\}$, and such that the set $\bigl\{n : \{0 < f < 1\} \cap \operatorname{spt}(T_n) \neq \emptyset\bigr\}$ is finite. Then, for sufficiently large n, $T_n(f) = \sum_i a_{n,i} f(x_{n,i}) \in \mathbb{Z}$, thus $a_j = T(f) = \lim_{n\to\infty} T_n(f) \in \mathbb{Z}$.

Now let $m \geq 1$ and suppose the result holds in dimension $m - 1$. Let $\pi \in \operatorname{Lip}(X)$. For almost every $y \in \mathbb{R}$, we have $\langle T_n, \pi, y\rangle \in \mathbf{I}_{m-1,\mathrm{loc}}(X)$ for all n by Theorem 8.5(1) and Theorem 8.7. Note that $T \in \mathbf{N}_{m,\mathrm{loc}}(X)$ by the lower semicontinuity of mass. By Proposition 6.6 (slicing convergent sequences), for almost every $y \in \mathbb{R}$, $\langle T, \pi, y\rangle$ is the weak limit of some locally $\mathbf{N}$-bounded subsequence of $(\langle T_n, \pi, y\rangle)_{n\in\mathbb{N}}$. Hence $\langle T, \pi, y\rangle \in \mathbf{I}_{m-1,\mathrm{loc}}(X)$ by the induction hypothesis. As this holds for every $\pi \in \operatorname{Lip}(X)$, Theorem 8.5(2) and Theorem 8.7 imply that $T \in \mathbf{I}_{m,\mathrm{loc}}(X)$. □

The compactness theorem for locally integral currents is now an immediate consequence. See [29, Theorem 27.3] for the classical result.

Theorem 8.10 ($\mathbf{I}_{m,\mathrm{loc}}$ compactness)**.** *Suppose $(T_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbf{I}_{m,\mathrm{loc}}(X)$ such that* $\sup_n \mathbf{N}_V(T_n) < \infty$ *for every open set $V \Subset X$. Then there is a subsequence $(T_{n(i)})_{i\in\mathbb{N}}$ that converges weakly to some $T \in \mathbf{I}_{m,\mathrm{loc}}(X)$.*

Note that X still denotes an arbitrary locally compact metric space. However, the closure of $\bigcup_{n\in\mathbb{N}} \operatorname{spt}(T_n)$, which also contains the support of the limit T, is separable.

Proof. Combine Theorem 5.4 ($\mathbf{N}_{m,\mathrm{loc}}$ compactness) and Theorem 8.9. □

In [24] applications of these results to the asymptotic geometry of spaces of nonpositive curvature will be discussed.

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