Length Spaces

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Abstract

These are the notes to the first chapter (out of four) of a lecture course on Metric Geometry given at ETH Zurich in the summer semester 2004.

1 Basic Definitions

We start with some basic definitions.

1.1 Definition (pseudometric, metric, metric space)
Let X be a set. A function $d\colon X \times X \to [0, \infty)$ is called a *pseudometric* on X if it satisfies

(1) $d(x, x) = 0$ for all $x \in X$,

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$, and

(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$ (*triangle inequality*).

The pseudometric d is called a *metric* if in addition

(4) $d(x, y) > 0$ for all $x, y \in X$ with $x \neq y$;

then the pair (X, d) is called a *metric space*. Sometimes we will also allow the value ∞ for d.

We use the following notation. For $x \in X$, $r \in \mathbb{R}$, and $A, B \subset X$,

$$\begin{aligned}
B(x, r) &:= \{y \in X \,|\, d(x, y) \le r\},\\
U(x, r) &:= \{y \in X \,|\, d(x, y) < r\},\\
S(x, r) &:= \{y \in X \,|\, d(x, y) = r\},\\
d(x, A) &:= \inf\{d(x, p) \,|\, p \in A\},\\
d(A, B) &:= \inf\{d(p, q) \,|\, p \in A,\ q \in B\},\\
B_r(A) &:= \{x \in X \,|\, d(x, A) \le r\},\\
U_r(A) &:= \{x \in X \,|\, d(x, A) < r\},\\
\operatorname{diam}(A) &:= \sup\{d(x, y) \,|\, x, y \in A\}.
\end{aligned}$$

Let (X, d), $(\bar{X}, \bar{d})$ be metric spaces. We call a map $f\colon X \to \bar{X}$ an *isometric embedding* if $\bar{d}(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. If f is surjective in addition, then it is called an *isometry*. We denote the *isometry group* of a metric space X by $\mathrm{Isom}(X)$. The map $f\colon X \to \bar{X}$ is a *local isometry* if for every $x \in X$ there is an $\epsilon > 0$ such that $f|_{U(x,\epsilon)}$ is an isometry onto $U(f(x), \epsilon)$.

1.2 Definition (length of a curve)
Let (X, d) be a metric space, $I \subset \mathbb{R}$ a non-empty interval (i.e. a connected set) and $\sigma\colon I \to X$ a curve (i.e. a continuous map). We define the *length* $L(\sigma) \in [0, \infty]$ of σ by

$$L(\sigma) := \sup \sum_{i=1}^{k} d(\sigma(t_{i-1}), \sigma(t_i)),$$

where the supremum is taken over all $k \in \mathbb{N}$ and all sequences $t_0 \le t_1 \le \ldots \le t_k$ in I. We say that σ is *rectifiable* if $L(\sigma) < \infty$.

Exercise: Let $X = \mathbb{R}^n$ be equipped with the Euclidean metric $d(x, y) = |x - y| = (\sum_{i=1}^{n} |x_i - y_i|^2)^{1/2}$. Show that for every piecewise C^1 curve $\sigma\colon [a, b] \to X$, $L(\sigma)$ agrees with $\int_a^b |\dot{\sigma}(t)|\, dt$.

If $\sigma\colon I \to X$ is a curve, $\tilde{I} \subset \mathbb{R}$ is an interval, and $\varphi\colon \tilde{I} \to I$ is continuous, surjective, and non-decreasing or non-increasing (i.e. $t \le t'$ implies $\varphi(t) \le \varphi(t')$ or $\varphi(t) \ge \varphi(t')$, respectively), then the curve $\tilde{\sigma} := \sigma \circ \varphi\colon \tilde{I} \to X$ satisfies $L(\tilde{\sigma}) = L(\sigma)$. A curve $\sigma\colon I \to X$ is said to have *constant speed* if there exists a constant $\lambda \ge 0$, the *speed* of σ, such that $L(\sigma|_{[t,t']}) = \lambda\,|t - t'|$ for all $t, t' \in I$, $t \le t'$; σ has *unit speed* or is *parametrized by arc length* if $\lambda = 1$. Suppose that a curve $\tilde{\sigma}\colon \tilde{I} \to X$ satisfies $L(\tilde{\sigma}|_{[t,t']}) < \infty$ for all $[t, t'] \subset \tilde{I}$. Pick $t_0 \in \tilde{I}$ and define $\varphi\colon \tilde{I} \to \mathbb{R}$ such that $\varphi(t) = L(\tilde{\sigma}|_{[t_0,t]})$ for $t \ge t_0$ and $\varphi(t) = -L(\tilde{\sigma}|_{[t,t_0]})$ for $t \le t_0$; φ is continuous and non-decreasing. Then the map

$$\sigma\colon \varphi(\tilde{I}) \to X, \quad \sigma(\varphi(t)) := \tilde{\sigma}(t),$$

is well-defined, continuous, and parametrized by arc length since

$$L(\sigma|_{[\varphi(t),\varphi(t')]}) = L(\tilde{\sigma}|_{[t,t']}) = \varphi(t') - \varphi(t)$$

for all $[t, t'] \subset \tilde{I}$.

1.3 Definition (inner metric, length space)
Let (X, d) be a metric space. The *inner metric* or *length metric* associated with d is the function $d_i\colon X \times X \to [0, \infty]$ defined by

$$d_i(x, y) := \inf L(\sigma),$$

where the infimum is taken over all rectifiable curves $\sigma\colon [0, 1] \to X$ from x to y, i.e. $\sigma(0) = x$, $\sigma(1) = y$. (X, d) is called an *inner metric space* or *length space* if $d_i = d$.

The metric d_i is finite if and only if every pair of points in X can be joined by a rectifiable curve. Note that always $d_i \geq d$ and $(d_i)_i = d_i$.

Examples: 1. Let (X, g) be a connected Riemannian manifold with the metric d defined by

$$d(x, y) := \inf\{L(\sigma) \mid \sigma\colon [0, 1] \to X \text{ a piecewise } C^1 \text{ curve from } x \text{ to } y\}.$$

Then $d_i = d$, i.e. (X, d) is a length space.

2. Suppose that $X = \mathbb{R}^n$, $d(x, y) = |x - y|$ for $x, y \in X$, and $X' = S^{n-1}(r) := S(0, r) \subset X$ is equipped with the metric d' induced by d. Let d'_i be the inner metric associated with d'; this will be called the *induced inner metric*. We have $d'(x, y) = |x - y|$ and $d'_i(x, y) = r \arccos(\langle x, y\rangle / r^2) > d'(x, y)$ for $x, y \in X'$, $x \neq y$.

2 Geodesics and the Hopf–Rinow Theorem

2.1 Definition (geodesic, local geodesic, geodesic space)
Let (X, d) be a metric space. A curve $\sigma\colon I \to X$ is called a *geodesic* if σ has constant speed and if $L(\sigma|_{[t,t']}) = d(\sigma(t), \sigma(t'))$ for all $t, t' \in I$, $t \leq t'$. The curve σ is called a *local geodesic* if for all $t \in I$ there exists an $\epsilon > 0$ such that $\sigma|_{I\cap[t-\epsilon,t+\epsilon]}$ is a geodesic. (X, d) is a *geodesic space* if for every pair of points $x, y \in X$ there exists a geodesic $\sigma\colon [0, 1] \to X$ joining x to y. (X, d) is called *uniquely geodesic* if for every pair of points $x, y \in X$ there is a unique geodesic $\sigma\colon [0, 1] \to X$ from x to y.

Every geodesic space is a length space. When is the converse true? Theorem 2.4 (Hopf–Rinow) below gives an answer to this question.

Note that a curve $\sigma\colon I \to X$ is a geodesic if and only if there exists a constant $\lambda \geq 0$ such that $d(\sigma(t), \sigma(t')) = \lambda\,|t - t'|$ for all $t, t' \in I$.

Exercise: Every normed vector space $(V, \|\cdot\|)$ with the metric $d(v, w) = \|v - w\|$ is a geodesic space. V is uniquely geodesic if and only if $B(0, 1) \subset V$ is strictly convex, cf. [BriH, Proposition I.1.6].

2.2 Lemma (midpoints)
Let X be a complete metric space.

(1) *X is a length space if and only if for all $x, y \in X$ and all $\epsilon > 0$ there is a point $z \in X$ such that*

$$d(x, z), d(y, z) \leq \tfrac{1}{2} d(x, y) + \epsilon.$$

(2) *X is a geodesic space if and only if for all $x, y \in X$ there is a point $z \in X$ such that*

$$d(x, z) = d(y, z) = \tfrac{1}{2} d(x, y).$$

Proof: (2): Let $x, y \in X$. To construct a geodesic $\sigma\colon [0,1] \to X$ from x to y, we first define $\sigma(t)$ for all $t \in [0,1]$ of the form $t = k/2^n$, $k, n \in \mathbb{N} \cup \{0\}$, such that $\sigma(0) = x$, $\sigma(1) = y$, $\sigma(\frac{1}{2})$ is a midpoint between $\sigma(0)$ and $\sigma(1)$, $\sigma(\frac{1}{4})$ is a midpoint between $\sigma(0)$ and $\sigma(\frac{1}{2})$, $\sigma(\frac{3}{4})$ is a midpoint between $\sigma(\frac{1}{2})$ and $\sigma(1)$, and so on. The map constructed this way is λ-Lipschitz for $\lambda = d(x,y)$. Since X is complete we can extend σ to the whole interval $[0,1]$: For $t \in [0,1]$ we choose a sequence (t_i) converging to t such that each $\sigma(t_i)$ is already defined. As (t_i) is Cauchy and σ is λ-Lipschitz, $(\sigma(t_i))$ is also Cauchy. Then we put $\sigma(t) := \lim_{i\to\infty} \sigma(t_i)$. It follows that $\sigma\colon [0,1] \to X$ is λ-Lipschitz for $\lambda = d(x,y)$, hence σ must be a geodesic joining x to y. □

2.3 Lemma (Arzelà–Ascoli)
If X is a compact metric space and Y is a separable metric space, then every equicontinuous sequence of maps $f_i\colon Y \to X$ has a subsequence that converges uniformly on compact subsets to a continuous map $f\colon Y \to X$.

Proof: Choose a dense set $Q = \{q_1, q_2, \ldots\}$ in Y. X is compact; pick a subsequence $(f_{1,i})$ of (f_i) such that $(f_{1,i}(q_1))$ converges. Then choose a subsequence $(f_{2,i})$ of $(f_{1,i})$ such that $(f_{1,i}(q_2))$ converges, a subsequence $(f_{3,i})$ of $(f_{2,i})$ such that $(f_{3,i}(q_3))$ converges, and so on. The diagonal sequence $(f_{i,i})$ converges pointwise on Q to a map $f\colon Q \to X$. For every $\epsilon > 0$ there is a $\delta > 0$ such that if $q, q' \in Q$ and $d(q,q') \le \delta$, then $d(f_{i,i}(q), f_{i,i}(q')) \le \epsilon$ for all i, hence $d(f(q), f(q')) \le \epsilon$. Since Q is dense in Y and X is complete, there exists a unique continuous extension of f to Y, $f\colon Y \to X$, and $d(f(y), f(y')) \le \epsilon$ whenever $d(y,y') \le \delta$. To prove the uniform convergence on a compact set $C \subset Y$, given $\epsilon > 0$, choose $\delta > 0$ as above. Pick numbers N, M such that for every $y \in C$ there is a $j(y) \le N$ with $d(y, q_{j(y)}) \le \delta$ and such that $d(f(q_j), f_{i,i}(q_j)) \le \epsilon$ whenever $i \ge M$ and $j \le N$. Then for all $y \in C$ and $i \ge M$, $d(f(y), f_{i,i}(y)) \le d(f(y), f(q_{j(y)})) + d(f(q_{j(y)}), f_{i,i}(q_{j(y)})) + d(f_{i,i}(q_{j(y)}), f_{i,i}(y)) \le 3\epsilon$. □

2.4 Theorem (Hopf–Rinow, Cohn-Vossen 1935)
Let X be a length space. If X is complete and locally compact, then

(1) *X is proper, i.e. every closed bounded subset of X is compact, and*

(2) *X is a geodesic space.*

The theorem is optimal, as the following examples show. The length space $\mathbb{R}^2 \setminus \{0\}$ (with the induced inner metric) is locally compact, but not complete. The length space obtained from a sequence of disjoint segments $[a_i, b_i]$ with $b_i - a_i = 1 + \frac{1}{i}$, $i \in \mathbb{N}$, by gluing each a_i to a_1 and each b_i to b_1 is complete, but not locally compact. (See Example 2 on page 9 for a precise description of this space.) Neither of these length spaces is a geodesic space.

Proof of 2.4: (1): Fix $z \in X$. It suffices to show that $B(z,r)$ is compact for all $r \ge 0$. Let $I := \{r \ge 0 \,|\, B(z,r) \text{ is compact}\}$; I is an interval containing 0. Let

$r \in I$. Use the local compactness of X to cover the compact ball $B(z, r)$ with finitely many balls $U(x_i, \epsilon_i)$ such that the $B(x_i, \epsilon_i)$ are compact. Then $\bigcup B(x_i, \epsilon_i)$ is compact and contains $B(z, r + \delta)$ for some $\delta > 0$, hence $r + \delta \in I$. This shows that I is open relative to $[0, \infty)$.

To prove that I is also closed, suppose that $[0, R) \subset I$, $R > 0$, and let $(y_j)_{j \in \mathbb{N}}$ be a sequence in $B(z, R)$. Choose a decreasing sequence $(\epsilon_i)_{i \in \mathbb{N}}$ converging to 0, with $\epsilon_i < R$. Since X is a length space, for all i, j there exists an $x^i_j \in B(z, R - \frac{\epsilon_i}{2})$ with $d(x^i_j, y_j) \leq \epsilon_i$. The sequence (x^1_j) has a convergent subsequence $(x^1_{j(1,k)})$. Consider the corresponding sequence $(x^2_{j(1,k)})$ and pick a convergent subsequence $(x^2_{j(2,k)})$. Then consider $(x^3_{j(2,k)})$ and select a converging subsequence $(x^3_{j(3,k)})$. Continue in this manner. Finally, put $j(k) := j(k, k)$ for $k \in \mathbb{N}$; the sequence $(x^i_{j(k)})_{k \in \mathbb{N}}$ converges for all $i \in \mathbb{N}$. We claim that the associated sequence $(y_{j(k)})$ is Cauchy. Let $\epsilon > 0$ and choose i with $\epsilon_i \leq \epsilon$. Then $d(x^i_{j(k)}, x^i_{j(l)}) \leq \epsilon$ for k, l sufficiently large. It follows that

$$\begin{aligned} d(y_{j(k)}, y_{j(l)}) &\leq d(y_{j(k)}, x^i_{j(k)}) + d(x^i_{j(k)}, x^i_{j(l)}) + d(x^i_{j(l)}, y_{j(l)}) \\ &\leq \epsilon_i + \epsilon + \epsilon_i \leq 3\epsilon. \end{aligned}$$

Since X is complete, $(y_{j(k)})$ converges. This shows that every sequence $(y_j)_{j \in \mathbb{N}}$ in $B(z, R)$ has a convergent subsequence. Hence $B(z, R)$ is compact, i.e. $[0, R] \subset I$. Thus I is both open and closed in $[0, \infty)$, so $I = [0, \infty)$.

(2): Let $x, y \in X$. Since X is a length space, it follows that for every $j \in \mathbb{N}$ there exists a point $z_j \in X$ such that

$$d(x, z_j), d(y, z_j) \leq \tfrac{1}{2} d(x, y) + \tfrac{1}{j}.$$

The sequence (z_j) lies in $B(x, \frac{1}{2} d(x, y) + 1)$ which is compact by (1). Hence some subsequence converges to a midpoint between x and y. From Lemma 2.2(2) (midpoints) it follows that X is a geodesic space. □

2.5 Lemma (continuously varying geodesics)
Suppose that X is a proper metric space, $\sigma\colon [0, 1] \to X$ is a geodesic, and $\sigma([0, 1])$ is the only geodesic segment connecting $\sigma(0)$ and $\sigma(1)$. Suppose further that there is a sequence of geodesics $\sigma_k\colon [0, 1] \to X$ such that the sequences $(\sigma_k(0))$ and $(\sigma_k(1))$ converge to $\sigma(0)$ and $\sigma(1)$, respectively. Then (σ_k) converges uniformly to σ.

Proof: Fix $R > 0$ such that the compact ball $B(\sigma(0), R)$ contains $\sigma_k([0, 1])$ for all k. If (σ_k) did not converge to σ pointwise, then there would exist $t \in [0, 1]$, $\epsilon > 0$ and an infinite subsequence $(\sigma_{k(l)})$ such that $d(\sigma_{k(l)}(t), \sigma(t)) \geq \epsilon$ for all l. Lemma 2.3 (Arzelà–Ascoli) would then yield a subsequence of $(\sigma_{k(l)})$ converging uniformly to a geodesic $\bar{\sigma}\colon [0, 1] \to X$ from $\sigma(0)$ to $\sigma(1)$ with $d(\bar{\sigma}(t), \sigma(t)) \geq \epsilon$, in contradiction to the uniqueness of σ. Hence (σ_k) converges pointwise to σ. By equicontinuity, the convergence is uniform. □

Lemma 2.5 is no longer valid without the word "proper", even if X is assumed to be uniquely geodesic and contractible, cf. [BriH, Exercise I.3.14].

A curve $\sigma\colon [0,1] \to X$ is called a *loop* or is said to be *closed* if $\sigma(1) = \sigma(0)$; then it may be viewed as a map $\sigma\colon \mathbb{R}/\mathbb{Z} \to X$. If its lift $\tilde{\sigma}\colon \mathbb{R} \to X$ is a local geodesic, then we will call σ a *closed geodesic*. The next result asserts the existence of a closed geodesic in certain metric spaces X. Recall that a path-connected topological space X is called *semi-locally simply connected* if each point $x \in X$ has a neighbourhood U such that each closed curve in U is null-homotopic (i.e. homotopic to a constant map) in X.

2.6 Theorem (closed geodesics)
If X is a compact length space that is semi-locally simply connected, then every closed curve $\sigma\colon \mathbb{R}/\mathbb{Z} \to X$ is homotopic to a closed geodesic or homotopic to a constant map.

Note that X is geodesic by Theorem 2.4 (Hopf–Rinow). Theorem 2.6 is no longer true without the assumption "semi-locally simply connected".

Proof: Since X is compact and semi-locally simply connected it follows that there is an $r > 0$ such that every closed curve of length $< r$ is null-homotopic. Assume that $\sigma\colon \mathbb{R}/\mathbb{Z} \to X$ is not homotopic to a constant map. Then

$$\lambda := \inf\{L(\sigma') \,|\, \sigma'\colon \mathbb{R}/\mathbb{Z} \to X,\ \sigma' \simeq \sigma \text{ (homotopic)}\} > 0.$$

Choose a sequence of closed curves $\sigma_i\colon \mathbb{R}/\mathbb{Z} \to X$, $\sigma_i \simeq \sigma$, such that $L(\sigma_i) \to \lambda$ and each σ_i has constant speed. The sequence is equicontinuous. Lemma 2.3 (Arzelà–Ascoli) yields a subsequence $(\sigma_{i(j)})$ converging uniformly to a λ-Lipschitz curve $\bar{\sigma}\colon \mathbb{R}/\mathbb{Z} \to X$. It remains to prove that $\bar{\sigma} \simeq \sigma$. Choose an index $i = i(j)$ with $d(\sigma_i(t), \bar{\sigma}(t)) < \frac{r}{4}$ for all $t \in \mathbb{R}/\mathbb{Z}$. Then pick $0 = t_0 < t_1 < \ldots < t_m = 1$ such that $L(\sigma_i|_{[t_{k-1},t_k]}) < \frac{r}{4}$ and $L(\bar{\sigma}|_{[t_{k-1},t_k]}) < \frac{r}{4}$ for $k = 1, \ldots, m$. Choose curves γ_k from $\sigma_i(t_k)$ to $\bar{\sigma}(t_k)$ of length $< \frac{r}{4}$. This gives m closed curves of length $< r$, hence they are all null-homotopic. Using this fact one can see that $\bar{\sigma} \simeq \sigma_i \simeq \sigma$. □

3 Constructions of Metric Spaces

In this section we discuss various constructions of metric spaces.

Products. Let X_1, X_2 be metric spaces. The (usual l_2) *product* of X_1 and X_2 is the metric space $X_1 \times X_2$ with the metric

$$d((x_1,x_2),(y_1,y_2)) := \big(d(x_1,y_1)^2 + d(x_2,y_2)^2\big)^{1/2}. \tag{3.1}$$

3.1 Proposition (product)
Suppose that X_1, X_2 are metric spaces, and $X = X_1 \times X_2$.

(1) *X is a length space if and only if X_1, X_2 are length spaces.*

(2) *X is a geodesic space if and only if X_1, X_2 are geodesic spaces; a curve $\sigma = (\sigma_1, \sigma_2)\colon I \to X$ is a geodesic if and only if both $\sigma_1\colon I \to X_1$ and $\sigma_2\colon I \to X_2$ are geodesics.*

(3) *An isometry $g\colon X \to X$ is of the form $g(x_1, x_2) = (g_1(x_1), g_2(x_2))$ for isometries $g_1\colon X_1 \to X_1$, $g_2\colon X_2 \to X_2$ if and only if for every $x_1 \in X_1$ there exists a point denoted $g_1(x_1) \in X_1$ such that $g(\{x_1\} \times X_2) = \{g_1(x_1)\} \times X_2$.*

More generally, the l_p *product* of X_1 and X_2 carries the metric

$$d((x_1, x_2), (y_1, y_2)) := \big(d(x_1, y_1)^p + d(x_2, y_2)^p\big)^{1/p}.$$

The analogues of (1) and (2) are valid for $1 < p < \infty$.

Proof of 3.1: (1): Suppose that X is a length space. Since the canonical projection $\pi_i\colon X \to X_i$ is 1-Lipschitz it follows easily that X_i is a length space. Conversely, suppose that X_1, X_2 are length spaces. Let $(x_1, x_2), (y_1, y_2) \in X$. Given unit speed curves $\sigma_i\colon [0, l_i] \to X_i$ from x_i to y_i, $i = 1, 2$, we obtain a 1-Lipschitz map $(s_1, s_2) \mapsto (\sigma_1(s_1), \sigma_2(s_2))$ from the product $[0, l_1] \times [0, l_2]$ into X. Restricting this map to the diagonal $[(0, 0), (l_1, l_2)]$ we get a curve from (x_1, x_2) to (y_1, y_2) of length at most $l = ({l_1}^2 + {l_2}^2)^{1/2}$. As $l_i \to d(x_i, y_i)$, $i = 1, 2$, $l \to d((x_1, x_2), (y_1, y_2))$.

(2): It suffices to show that (z_1, z_2) is a midpoint between (x_1, x_2) and (y_1, y_2) if and only if for $i = 1, 2$, z_i is a midpoint between x_i and y_i. Setting $a_i := d(x_i, z_i)$, $b_i := d(y_i, z_i)$, and $c_i := d(x_i, y_i)$, we must show that

$$ {a_1}^2 + {a_2}^2 = {b_1}^2 + {b_2}^2 = \tfrac{1}{4}({c_1}^2 + {c_2}^2) \tag{3.2}$$

if and only if $a_1 = b_1 = \frac{1}{2}c_1$ and $a_2 = b_2 = \frac{1}{2}c_2$. One implication is clear. For the other, (3.2) yields

$$ {a_1}^2 + {b_1}^2 + {a_2}^2 + {b_2}^2 = \tfrac{1}{2}({c_1}^2 + {c_2}^2).$$

Moreover, ${a_i}^2 + {b_i}^2 \geq \frac{1}{2}(a_i + b_i)^2 \geq \frac{1}{2}{c_i}^2$ by the triangle inequality, with ${a_i}^2 + {b_i}^2 = \frac{1}{2}{c_i}^2$ if and only if $a_i = b_i = \frac{1}{2}c_i$. The result follows.

(3): See [BriH, Proposition I.5.3]. □

Disjoint union. Suppose $(X_\alpha, d_\alpha)_{\alpha \in A}$ is a family of metric spaces. The *disjoint union* of this family is the set

$$\coprod_{\alpha \in A} X_\alpha = \bigcup_{\alpha \in A} X_\alpha \times \{\alpha\},$$

equipped with the metric

$$d((x, \alpha), (x', \alpha')) := \begin{cases} d_\alpha(x, x') & \text{if } \alpha = \alpha', \\ \infty & \text{otherwise.} \end{cases} \tag{3.3}$$

Quotient pseudometrics. Let X be a metric space with a possibly infinite metric, $\sim$ an equivalence relation on X, and $\bar{X} = X/{\sim}$ the set of equivalence classes. For $\bar{x}, \bar{y} \in \bar{X}$, we define

$$\bar{d}(\bar{x}, \bar{y}) := \inf \sum_{j=1}^{k} d(x_j, y_j), \tag{3.4}$$

where the infimum is taken over all $k \in \mathbb{N}$ and sequences $x_1, y_1, x_2, y_2, \dots, x_k, y_k$ with $x_1 \in \bar{x}$, $y_k \in \bar{y}$, and $y_j \sim x_{j+1}$ for $j = 1, 2, \dots, k-1$. Clearly $\bar{d}$ is a pseudometric on $\bar{X}$, the *quotient pseudometric* on $X/{\sim} = \bar{X}$. We have $\bar{d}(\bar{x}, \bar{y}) \le d(x, y)$ for all $x, y \in X$, and $\bar{d}$ is exactly the biggest pseudometric on $\bar{X}$ with that property.

3.2 Proposition (quotient pseudometric)
Suppose that (X, d), $\sim$, $\bar{X} = X/{\sim}$, and $\bar{d}$ are given as above.

(1) *Suppose that for every equivalence class $\bar{x} \subset X$ there exists an $\epsilon(\bar{x}) > 0$ such that $U_\delta(\bar{x})$ is a union of equivalence classes for $0 < \delta \le \epsilon(\bar{x})$. Then*

$$\bar{d}(\bar{x}, \bar{y}) = d(\bar{x}, \bar{y}) \quad \textit{whenever } \bar{x}, \bar{y} \in \bar{X} \textit{ and } \bar{d}(\bar{x}, \bar{y}) < \epsilon(\bar{x}). \tag{3.5}$$

If in addition every equivalence class $\bar{x} \subset X$ is closed, then $\bar{d}$ is a metric on $\bar{X}$.

(2) *If $\bar{d}$ is a metric and (X, d) is a length space, then $(\bar{X}, \bar{d})$ is a length space.*

Note that in (3.5), $d(\bar{x}, \bar{y})$ stands for $\inf\{d(x', y') \mid x' \in \bar{x},\ y' \in \bar{y}\}$. Clearly $\bar{d}(\bar{x}, \bar{y}) \le d(\bar{x}, \bar{y})$ for all $\bar{x}, \bar{y} \in \bar{X}$.

Proof: (1): Suppose that $\bar{x}, \bar{y} \in \bar{X}$, $\bar{x} \ne \bar{y}$, and $\bar{d}(\bar{x}, \bar{y}) < \epsilon(\bar{x})$. Since $U_\delta(\bar{x})$ is a union of equivalence classes for $0 < \delta \le \epsilon(\bar{x})$, it follows that

$$d(z, \bar{x}) = d(z', \bar{x}) \quad \text{whenever } z \sim z' \text{ and } d(\bar{x}, \bar{z}) < \epsilon(\bar{x}). \tag{3.6}$$

Choose $x_1, y_1, \dots, x_k, y_k \in X$ with $x_1 \in \bar{x}$, $y_k \in \bar{y}$, $y_i \sim x_{i+1}$ for $i = 1, \dots, i-1$, and $\sum_{i=1}^{k} d(x_i, y_i) < \epsilon(\bar{x})$. We show by induction that for $j = 1, \dots, k$, we have

$$d(y_j, \bar{x}) \le \sum_{i=1}^{j} d(x_i, y_i). \tag{3.7}$$

This is obvious for $j = 1$ as $x_1 \in \bar{x}$. Now let $j \in \{2, \dots, k\}$ and suppose that (3.7) holds with $j-1$ in place of j. Then $d(y_{j-1}, \bar{x}) \le \sum_{i=1}^{j-1} d(x_i, y_i) < \epsilon(\bar{x})$. Since $y_{j-1} \sim x_j$, using (3.6) we get

$$d(y_j, \bar{x}) \le d(x_j, y_j) + d(x_j, \bar{x}) = d(x_j, y_j) + d(y_{j-1}, \bar{x}) \le \sum_{i=1}^{j} d(x_i, y_i),$$

hence (3.7). Consequently, $\bar{d}(\bar{x}, \bar{y}) \leq d(\bar{x}, \bar{y}) \leq d(y_k, \bar{x}) \leq \sum_{i=1}^{k} d(x_i, y_i)$. As this sum can be chosen arbitrarily close to $\bar{d}(\bar{x}, \bar{y})$ we have $\bar{d}(\bar{x}, \bar{y}) = d(\bar{x}, \bar{y})$.

If $\bar{x}$ is a closed subset of X and $y \notin \bar{x}$, then $d(y, \bar{x}) > 0$. Hence, if $d(\bar{x}, \bar{y}) = \bar{d}(\bar{x}, \bar{y}) < \epsilon(\bar{x})$, using (3.6) again we see that

$$d(\bar{x}, \bar{y}) = \inf_{y' \in \bar{y}} d(y', \bar{x}) = d(y, \bar{x}) > 0.$$

So $\bar{d}$ is a metric on $\bar{X}$.

(2): Suppose that $0 < \bar{d}(\bar{x}, \bar{y}) < \infty$. Let $\epsilon > 0$, and choose $x_1, y_1, \ldots, x_k, y_k \in X$ with $x_1 \in \bar{x}$, $y_k \in \bar{y}$ and $y_j \sim x_{j+1}$ for $j = 1, \ldots, k-1$ such that

$$\sum_{j=1}^{k} d(x_j, y_j) < \bar{d}(\bar{x}, \bar{y}) + \frac{\epsilon}{2}.$$

Now pick curves $\sigma_j \colon [0,1] \to X$ joining x_j to y_j, $j = 1, \ldots, k$, with $L(\sigma_j) < d(x_j, y_j) + \frac{\epsilon}{2k}$. Let $\pi \colon X \to \bar{X}$ be the canonical projection and $\bar{\sigma} \colon [0, k] \to \bar{X}$ the concatenation of the curves $\pi \circ \sigma_1, \ldots, \pi \circ \sigma_k$. Since $\bar{d}(\pi(x'), \pi(y')) \leq d(x', y')$ for all $x', y' \in X$, we have

$$L(\bar{\sigma}) = \sum_{j=1}^{k} L(\pi \circ \sigma_j) \leq \sum_{j=1}^{k} L(\sigma_j) < \sum_{j=1}^{k} d(x_j, y_j) + \frac{\epsilon}{2} < \bar{d}(\bar{x}, \bar{y}) + \epsilon.$$

Hence $(\bar{X}, \bar{d})$ is a length space. □

Examples: 1. Let $\bar{X} = \mathbb{R}/\!\sim$, where $x \sim y$ if and only if $y - x \in \mathbb{Z}$, i.e. $\bar{X} = \mathbb{R}/\mathbb{Z}$. Then we have $\bar{d}(\bar{x}, \bar{y}) = d(\bar{x}, \bar{y})$, and $(\bar{X}, \bar{d})$ is a geodesic space isometric to circle of length 1.

2. Let $X = \coprod_{j \in \mathbb{N}} [0, 1 + \frac{1}{j}] = \bigcup_{j \in \mathbb{N}} [0, 1 + \frac{1}{j}] \times \{j\}$. Consider the equivalence relation on X generated by the relations $(0, j) \sim (0, k)$ and $(1 + \frac{1}{j}, j) \sim (1 + \frac{1}{k}, k)$, $j, k \in \mathbb{N}$. Then $\bar{X} = X/\!\sim$ is the complete length space from the remark following Theorem 2.4 (Hopf–Rinow).

3. *Metric graphs.* Let G be a combinatorial graph with vertex set V, edge set E, and endpoint maps $\partial, \partial' \colon E \to V$, where $\partial(E) \cup \partial'(E) = V$. Choose a function $l \colon E \to (0, \infty)$. Let (X, d) be the disjoint union $\coprod_{e \in E} [0, l(e)] = \bigcup_{e \in E} ([0, l(e)] \times \{e\})$ of the intervals $[0, l(e)] \subset \mathbb{R}$, cf. (3.3). Consider the equivalence relation on X generated by the classes $\{(0, e) \mid \partial(e) = v\} \cup \{(l(e), e) \mid \partial'(e) = v\}$ for $v \in V$, and let $\bar{X} = X/\!\sim$. If

$$\epsilon(v) := \inf\{l(e) \mid e \in E,\, \partial(e) = v \text{ or } \partial'(e) = v\} > 0$$

for all $v \in V$, then the quotient pseudometric $\bar{d}$ is a metric, and $\bar{X}$ is a length space.

Isometric gluing. Let $(X_\alpha, d_\alpha)_{\alpha\in A}$ be a family of metric spaces. Suppose that Z is a metric space, and $g_\alpha\colon Z \to Z_\alpha$ is an isometry onto a closed set $Z_\alpha \subset X_\alpha$ for every $\alpha \in A$. Consider the equivalence relation on the disjoint union $\big(\coprod_{\alpha\in A} X_\alpha, d\big)$ generated by the relations $g_\alpha(z) \sim g_\beta(z)$ for all $z \in Z$, $\alpha, \beta \in A$. The quotient space $\bar{X} = \coprod_{\alpha\in A} X_\alpha/{\sim}$ with the quotient (pseudo-)metric $\bar{d}$ is called the *isometric gluing* of the spaces X_α along the g_α.

3.3 Proposition (isometric gluing)
Let $(X_\alpha, d_\alpha)_{\alpha\in A}$, Z, $g_\alpha\colon Z \to Z_\alpha$, and $(\bar{X}, \bar{d})$ be given as above.

(1) *$\bar{d}$ is a metric on $\bar{X}$, and for all $x \in X_\alpha$, $y \in X_\beta$,*

$$\bar{d}(\bar{x}, \bar{y}) = \begin{cases} d_\alpha(x, y) & \text{if } \alpha = \beta, \\ \inf_{z\in Z}\big(d_\alpha(x, g_\alpha(z)) + d_\beta(y, g_\beta(z))\big) & \text{if } \alpha \neq \beta. \end{cases}$$

(2) *If every X_α is a length space, then $\bar{X}$ is a length space.*

(3) *If every X_α is a geodesic space and Z is proper, then $\bar{X}$ is a geodesic space.*

Proof: Since the gluing maps g_α are isometries and the sets Z_α are closed, it follows from Proposition 3.2(1) that $\bar{d}$ is a metric. Moreover, 3.2(2) implies (2). To prove the claimed identity for $\bar{d}(\bar{x}, \bar{y})$, consider points $x_1, y_1, x_2, y_2, x_3, y_3$ such that $y_1 \sim x_2$, $y_2 \sim x_3$, and $\sum_{i=1}^3 d(x_i, y_i) < \infty$, i.e. $x_i, y_i \in X_{\alpha_i}$ for $i = 1, 2, 3$. Since $y_2 \sim x_3$, there is a $y_1' \in X_{\alpha_1}$ such that $y_1' \sim x_3$. Then

$$\sum_{i=1}^3 d(x_i, y_i) = d(x_1, y_1) + d(y_1, y_1') + d(x_3, y_3) \geq d(x_1, y_1') + d(x_3, y_3),$$

and in view of (3.4) the identity follows. Now suppose that every X_α is a geodesic space and Z is proper. Using (1) we see that for all $x \in X_\alpha$ and $y \in X_\beta$, $\alpha \neq \beta$, there exists a point $z \in Z$ such that $\bar{d}(\bar{x}, \bar{y}) = d_\alpha(x, g_\alpha(z)) + d_\beta(y, g_\beta(z))$. This yields (3). □

Euclidean polyhedral complexes. By a *convex euclidean polyhedral cell* $C = (C, d)$ we mean a (compact, geodesic) metric space isometric to the convex hull of a positive finite number of points in a euclidean space. *Vertices*, *edges*, *faces*, the *interior* $\operatorname{int} C$ and the *dimension* of C are defined in an obvious way. An n-dimensional convex euclidean polyhedral cell with exactly $n + 1$ vertices is called a *euclidean simplex*.

3.4 Definition (euclidean polyhedral/simplicial complex)
Let X be a set and $\mathcal{C} = (C_\alpha)_{\alpha\in A}$ a family of convex euclidean polyhedral cells with the following properties:

(1) $X = \bigcup_{\alpha\in A} C_\alpha$, and $C_\alpha \neq C_\beta$ for $\alpha \neq \beta$.

(2) If $C_\alpha \cap C_\beta \neq \emptyset$, then the respective metrics d_α and d_β coincide on $C_\alpha \cap C_\beta$, and $C_\alpha \cap C_\beta$ is a common face of C_α and C_β.

The pair $(X, \mathcal{C})$, where X is equipped with the maximal pseudometric d such that $d \leq d_\alpha$ on C_α for every $\alpha \in A$, is called a *euclidean polyhedral complex*. If each cell C_α is a euclidean simplex, then $(X, \mathcal{C})$ is a *euclidean simplicial complex*.

The pseudometric d can be described in a different way. Consider the equivalence relation on the disjoint union $\big(\coprod_{\alpha\in A} C_\alpha, \tilde{d}\big)$ generated by the relations $(x,\alpha) \sim (x,\beta)$ for $x \in X$, $\alpha,\beta \in A$. Then $\coprod_{\alpha\in A} C_\alpha/{\sim} = X$, and d equals the quotient pseudometric. More explicitly,

$$d(x,y) = \inf\big\{\textstyle\sum_{j=1}^k d_{\alpha_j}(x_{j-1},x_j) \,\big|\, x_0 = x,\, x_k = y,\, x_{j-1}, x_j \in C_{\alpha_j}\big\}.$$

3.5 Lemma (euclidean polyhedral complexes)
Suppose that for all $x \in X$,

$$\epsilon(x) := \inf\{d_\alpha(x,F) \,|\, x \in C_\alpha,\ F \text{ a face of } C_\alpha,\ x \notin F\} > 0.$$

Then d is a metric and (X,d) is a length space.

Proof: This follows directly from Proposition 3.2. □

In [BriH, Theorem I.7.19] it is shown that if X is connected and $\mathcal{C}$ contains only finitely many isometry types of cells, then (X,d) is a complete geodesic space.

4 Group Actions and Coverings

By an *action* of a group G on a set X we mean a map $G \times X \to X$, $(g,x) \mapsto g(x)$, such that $g(h(x)) = gh(x)$ and $e(x) = x$ for all $g,h \in G$ and $x \in X$. The action is called *free* if $g(x) \neq x$ for all $g \neq e$ and all $x \in X$.

Let G be a subgroup of the isometry group of a metric space X. The action of G on X is said to be *proper* if for each $x \in X$ there exists an $\epsilon > 0$ such that the set $\{g \in G \,|\, g(U(x,\epsilon)) \cap U(x,\epsilon) \neq \emptyset\}$ is finite. Regarding this terminology, see [BriH, I.8.2 and I.8.3]. Define an equivalence relation on X such that $x \sim y$ if and only if $y = g(x)$ for some $g \in G$. The quotient space $\bar{X} := X/G = X/{\sim}$ is the set of all *G-orbits* $\bar{x} := G(x) := \{g(x) \,|\, g \in G\}$.

4.1 Proposition (X/G)
Let G be a group of isometries of X, and let $\bar{d}$ be the quotient pseudometric on $\bar{X} = X/G$.

(1) *For all $\bar{x}, \bar{y} \in \bar{X}$, $\bar{d}(\bar{x},\bar{y}) = d(\bar{x},\bar{y})$.*

(2) *Suppose that the action is proper. Then $\bar{d}$ is a metric on $\bar{X}$, and $(\bar{X},\bar{d})$ is a length space if (X,d) is a length space.*

(3) *Suppose that the action is proper and free. Then the canonical projection* $\pi\colon X \to \bar{X}$ *is a covering map and a local isometry.*

Recall that a continuous map $\pi\colon X \to Y$ between topological spaces is a *covering map* if π is surjective and every point $y \in Y$ has an open neighbourhood V such that $\pi^{-1}(V)$ is a disjoint union of open sets U_α and $\pi|_{U_\alpha}$ is a homeomorphism onto V for every α.

Proof: (1): If $\bar{x} \in \bar{X}$ and $y \in U_\delta(\bar{x})$ for a $\delta > 0$, then $\bar{y} \subset U_\delta(\bar{x})$. The claim follows from Proposition 3.2(1).

(2): Since G acts properly on X, each orbit $\bar{x} \subset X$ is closed. Hence Proposition 3.2 yields the result.

(3): Since G acts properly and freely, for each $x \in X$ there is an $\epsilon > 0$ such that $g(U(x,\epsilon)) \cap U(x,\epsilon) = \emptyset$ for all $g \neq e$. Using (1) we see that $\bar{d}(\bar{x},\bar{y}) = d(\bar{x},\bar{y}) = d(x,y)$ for all $y \in U(x,\epsilon)$. This implies (3). □

Suppose now that X is a length space, $\tilde{X}$ is a topological space, and $\pi\colon \tilde{X} \to X$ is a *local homeomorphism*, i.e., a map with the property that every $\tilde{x} \in \tilde{X}$ has an open neighbourhood $\tilde{U}$ such that $\pi|_{\tilde{U}}$ is a homeomorphism onto an open subset of X. In particular, π is continuous. For $\tilde{x}, \tilde{y} \in \tilde{X}$ put

$$\tilde{d}(\tilde{x},\tilde{y}) := \inf\{L(\pi \circ \tilde{\sigma}) \mid \tilde{\sigma}\colon [0,1] \to \tilde{X} \text{ a curve from } \tilde{x} \text{ to } \tilde{y}\}.$$

This defines a pseudometric $\tilde{d}$ on $\tilde{X}$, and $d(\pi(\tilde{x}),\pi(\tilde{y})) \le \tilde{d}(\tilde{x},\tilde{y})$ for all $\tilde{x},\tilde{y} \in \tilde{X}$. If $\tilde{X}$ is a Hausdorff space, then $\tilde{d}$ is a metric.

4.2 Lemma (pull-back length metric)
Suppose that (X,d) *is a length space,* $\tilde{X}$ *is a Hausdorff space,* $\pi\colon \tilde{X} \to X$ *is a local homeomorphism, and* $\tilde{d}$ *is defined as above. Then*

(1) π *is a local isometry, and*

(2) $(\tilde{X},\tilde{d})$ *is a length space.*

Moreover, $\tilde{d}$ *is the only metric on* $\tilde{X}$ *with these two properties.*

Proof: (1): Let $\tilde{x} \in \tilde{X}$, and put $x := \pi(\tilde{x})$. We must show that there is an $\epsilon > 0$ such that $\pi|_{U(\tilde{x},\epsilon)}$ is an isometry onto $U(x,\epsilon)$. Choose an open neighborhood $\tilde{U}$ of $\tilde{x}$ such that $\pi|_{\tilde{U}}$ is a homeomorphism onto the open set $U = \pi(\tilde{U})$. Put $\varrho := (\pi|_{\tilde{U}})^{-1}\colon U \to \tilde{U}$. Pick $\epsilon > 0$ such that $U(x,2\epsilon) \subset U$. Let $y,z \in U(x,\epsilon)$. For all $\delta > 0$ there exists a curve $\sigma\colon [0,1] \to U(x,2\epsilon)$ from y to z with $L(\sigma) < d(y,z) + \delta$. Then $\tilde{\sigma} := \varrho \circ \sigma$ is a curve joining $\varrho(y)$ to $\varrho(z)$ with $\pi \circ \tilde{\sigma} = \sigma$, hence $\tilde{d}(\varrho(y),\varrho(z)) \le L(\sigma) < d(y,z) + \delta$. On the other hand we have $\tilde{d}(\tilde{y},\tilde{z}) \ge d(\pi(\tilde{y}),\pi(\tilde{z}))$ for all $\tilde{y},\tilde{z} \in \tilde{X}$. Hence $\tilde{d}(\varrho(y),\varrho(z)) = d(y,z)$. Finally, we observe that $\varrho|_{U(x,\epsilon)}\colon U(x,\epsilon) \to U(\tilde{x},\epsilon)$ is surjective since $\varrho \circ \pi|_{U(\tilde{x},\epsilon)}$ is the identity on $U(\tilde{x},\epsilon)$ and π is 1-Lipschitz.

(2): Since π is a local isometry, we have $L(\tilde{\sigma}) = L(\pi \circ \tilde{\sigma})$ for all curves $\tilde{\sigma}$. The result follows with the definition of $\tilde{d}$. □

4.3 Proposition
Let $\pi\colon \tilde{X} \to X$ be a map between two length spaces such that

(1) *$\tilde{X}$ is complete,*

(2) *X is connected,*

(3) *for every $x \in X$ there exists an $\epsilon > 0$ such that for all $y \in U(x,\epsilon)$ there is a unique geodesic $\sigma_y\colon [0,1] \to U(x,\epsilon)$ joining x to y, and σ_y varies continuously with y,*

(4) *π is a local homeomorphism, and*

(5) *$L(\tilde{\sigma}) \le L(\pi \circ \tilde{\sigma})$ for every rectifiable curve $\tilde{\sigma}\colon [0,1] \to \tilde{X}$.*

Then π is a covering map.

Note that (4) and (5) hold in particular if π is a local isometry. (See also Olivia Gutú and Jesús A. Jaramillo, Global homeomorphisms and covering projections on metric spaces, Math. Ann. 338 (2007), 75–95.)

Proof: We first show that for every rectifiable curve $\sigma\colon [0,1] \to X$ and every $\tilde{x} \in \tilde{X}$ with $\pi(\tilde{x}) = \sigma(0)$ there exists a unique curve $\tilde{\sigma}\colon [0,1] \to \tilde{X}$ such that $\tilde{\sigma}(0) = \tilde{x}$ and $\pi \circ \tilde{\sigma} = \sigma$. Suppose that $\tilde{\sigma}$ is already constructed on $[0,a)$ for some $0 < a \le 1$. Choose a sequence $0 < t_1 < t_2 < \ldots$ converging to a. Using (5) we get

$$d(\tilde{\sigma}(t_i), \tilde{\sigma}(t_j)) \le L(\tilde{\sigma}|_{[t_i,t_j]}) \le L(\pi \circ \tilde{\sigma}|_{[t_i,t_j]}) = L(\sigma|_{[t_i,t_j]})$$

for $i < j$. Since $(L(\sigma|_{[0,t_i]}))_{i\in\mathbb{N}}$ is Cauchy it follows that $(\tilde{\sigma}(t_i))_{i\in\mathbb{N}}$ is Cauchy and hence convergent by (1); define $\tilde{\sigma}(a)$ to be the limit point. This shows that the maximal subinterval of $[0,1]$ that contains 0 and on which the lift $\tilde{\sigma}$ exists is closed. Due to (4) it is also open, so it is equal to $[0,1]$.

Using (2) and (3) we see that for every pair of points in X, there is a rectifiable path joining them. With the existence of the lifts it follows that the restriction of π to every connected component of $\tilde{X}$ is a surjection onto X.

Let $x \in X$, and let $\epsilon > 0$ be given as in (3). Fix $\tilde{x} \in \pi^{-1}\{x\}$. For $y \in U(x,\epsilon)$ let $\tilde{\sigma}_y\colon [0,1] \to \tilde{X}$ be the unique lift of $\sigma_y\colon [0,1] \to X$ with $\tilde{\sigma}_y(0) = \tilde{x}$. Define $\varrho_{\tilde{x}}\colon U(x,\epsilon) \to \tilde{X}$ by $\varrho_{\tilde{x}}(y) = \tilde{\sigma}_y(1)$. We claim that $\varrho_{\tilde{x}}$ is a homeomorphism onto an open subset of $\tilde{X}$. By (4) and the fact that $\pi \circ \varrho_{\tilde{x}} = \mathrm{id}_{U(x,\epsilon)}$ it suffices to show that $\varrho_{\tilde{x}}$ is continuous in y. Cover $\sigma_y([0,1])$ with open balls $U_1, \ldots, U_k \subset U(x,\epsilon)$ such that

$$\sigma_y\bigl(\bigl[\tfrac{j-1}{k}, \tfrac{j}{k}\bigr]\bigr) \subset U_j \quad \text{for } j = 1, \ldots, k$$

and such that there exist continuous maps $\varrho^j\colon U_j \to \tilde{X}$ with $\pi \circ \varrho^j = \mathrm{id}_{U_j}$ and $\varrho^j \circ \sigma_y = \varrho_{\tilde{x}} \circ \sigma_y$ on $[\frac{j-1}{k}, \frac{j}{k}]$. This is possible due to (4). By (3), for $\delta > 0$ small enough and $z \in U(y,\delta) \subset U(x,\epsilon)$ we have $\sigma_z([\frac{j-1}{k}, \frac{j}{k}]) \subset U_j$. Define a continuous map $\tilde{\varrho}\colon U(y,\delta) \times [0,1] \to \tilde{X}$ such that

$$\tilde{\varrho}(z,t) = \varrho^j(\sigma_z(t)) \quad \text{whenever } (z,t) \in U(y,\delta) \times \left[\tfrac{j-1}{k}, \tfrac{j}{k}\right].$$

Then $t \mapsto \tilde{\varrho}(z,t)$ is a lift of σ_z starting at $\tilde{x}$, so $\tilde{\varrho}(z,t) = \tilde{\sigma}_z(t)$. In particular, $\tilde{\varrho}(z,1) = \tilde{\sigma}_z(1) = \varrho_{\tilde{x}}(z)$ for all $z \in U(y,\delta)$, hence $\varrho_{\tilde{x}}$ is continuous in y.

We have shown that $\pi^{-1}(U(x,\epsilon))$ is the union of the open sets $\varrho_{\tilde{x}}(U(x,\epsilon))$ with $\tilde{x} \in \pi^{-1}\{x\}$, and $\pi|_{\varrho_{\tilde{x}}(U(x,\epsilon))}$ is a homeomorphism onto $U(x,\epsilon)$. The sets $\varrho_{\tilde{x}}(U(x,\epsilon))$ are pairwise disjoint: If $\tilde{y} \in \varrho_{\tilde{x}}(U(x,\epsilon)) \cap \varrho_{\tilde{x}'}(U(x,\epsilon))$, then the two lifts of $c_{\pi(\tilde{y})}$ both end at $\tilde{y}$ (by the definition of $\varrho_{\tilde{x}}$ and $\varrho_{\tilde{x}'}$), thus $\tilde{x} = \tilde{x}'$. Hence π is a covering map. □

5 Quasi-Isometries

5.1 Definition (quasi-isometric embedding, quasi-isometry)
Let (X,d), $(\bar{X},\bar{d})$ be metric spaces. A (not necessarily continuous) map $f\colon X \to \bar{X}$ is called *quasi-isometric embedding* if there exist constants $\lambda \ge 1$ and $\epsilon \ge 0$ such that

$$\frac{1}{\lambda}\, d(x,y) - \epsilon \le \bar{d}(f(x), f(y)) \le \lambda\, d(x,y) + \epsilon \quad \text{for all } x, y \in X.$$

If in addition $\sup_{\bar{x}\in\bar{X}} \bar{d}(\bar{x}, f(X)) < \infty$, then f is called a *quasi-isometry*. X and $\bar{X}$ are called *quasi-isometric* if there exists a quasi-isometry $f\colon X \to \bar{X}$.

If $f\colon X \to \bar{X}$ is a quasi-isometry, then there exists a quasi-isometry $\bar{f}\colon \bar{X} \to X$ such that $\sup_{x\in X} d(x, \bar{f} \circ f(x)) < \infty$ and $\sup_{\bar{x}\in\bar{X}} \bar{d}(\bar{x}, f \circ \bar{f}(\bar{x})) < \infty$; $\bar{f}$ is called a *quasi-inverse* of f. X is quasi-isometric to $\{0\}$ if and only if $\mathrm{diam}(X) < \infty$. The composition of quasi-isometric embeddings/quasi-isometries is again a quasi-isometric embedding/quasi-isometry. We call two maps $f, g\colon X \to X$ equivalent if $\sup_{x\in X} d(f(x), g(x)) < \infty$; the *quasi-isometry group* Q-Isom(X) of X is the set of all equivalence classes, endowed with the multiplication defined by $[f]\cdot[g] := [f \circ g]$.

Exercise: Give examples of a metric space X such that (a) X is unbounded and the natural homomorphism Isom$(X) \to$ Q-Isom(X) is an isomorphism, (b) Isom(X) is trivial and Q-Isom(X) is infinite, (c) Isom(X) is infinite and Q-Isom(X) is trivial.

Let A be a finite set. We denote by A^{-1} a disjoint set of the same cardinality, and we fix an involution $\iota\colon A \cup A^{-1} \to A \cup A^{-1}$ that interchanges A and A^{-1}. We use a^{-1} as a shorthand for $\iota(a)$. A *word* over the alphabet A is a finite string $a_1 \dots a_n$ where $a_i \in A \cup A^{-1}$ and $n \in \mathbb{N} \cup \{0\}$; for $n = 0$ we have the empty word $\emptyset$. A word is called *reduced* if it has no substring of the form aa^{-1}. The set $F(A)$ of reduced words over A has a natural group structure: The product of two elements is obtained by concatenating them and cancelling out substrings of the form aa^{-1}, and the neutral element is the empty word. We call $F(A)$ the *free group* on the alphabet A.

Now let G be a group with a finite generating set $A \subset G$, $e \notin A$. Every element of G can be written as a product of elements of $A \cup A^{-1}$ (now $A^{-1} = \{a^{-1} \mid a \in A\}$ need not be disjoint from A), thus there is a natural group epimorphism $\pi\colon F(A) \to G$ with $\pi(a) = a$ and $\pi(\iota(a)) = a^{-1}$ for all $a \in A \subset F(A)$. If $R \subset F(A)$ is a set of reduced words such that the kernel of π is the smallest normal subgroup of $F(A)$ containing R, we say that R is a set of *relators* for G. Then A and R together form a *group presentation* for G; this is denoted by $G = \langle A \mid R \rangle$. The presentation is *finite* if R is finite.

The *word metric* d_A on G with respect to the finite generating set $A \subset G \setminus \{e\}$ is defined so that $d_A(g,h)$ is the length of the shortest word in $\pi^{-1}(g^{-1}h) \subset F(A)$. The metric d_A is invariant under left multiplication: $d_A(g'g, g'h) = d_A(g,h)$ since $(g'g)^{-1}(g'h) = g^{-1}h$. If $A' \subset G \setminus \{e\}$ is another finite generating set for G, then there is a constant $\lambda \in \mathbb{N}$ such that

$$\frac{1}{\lambda}\, d_A(g,h) \le d_{A'}(g,h) \le \lambda\, d_A(g,h) \quad \text{for all } g,h \in G,$$

i.e. the identity map from (G, d_A) to $(G, d_{A'})$ is a bi-Lipschitz equivalence, in particular a quasi-isometry.

An illustrative picture for the word metric is given by the *Cayley graph*. Let G and A be given as above; $\mathcal{C}_A(G)$ is the metric graph with vertex set $V = G$, edge set $E = \{(g,a) \mid g \in G,\ a \in A\}$, endpoint maps $\partial(g,a) = g$, $\partial'(g,a) = ga$, and length function $l\colon E \to \{1\}$ (see Example 3 on page 9). The length space $\mathcal{C}_A(G)$ is complete and locally compact, hence it is a proper geodesic space. We denote the metric on $\mathcal{C}_A(G)$ by d_A; this is justified by the fact that the metric induced on $G = V \subset \mathcal{C}_A(G)$ agrees with the word metric d_A.

Examples: 1. The Cayley graph of the free group $G = \langle a,b \mid \emptyset \rangle = F(a,b)$ on two generators is a regular tree with valence 4.

2. The Cayley graph of the free abelian group $G = \langle a,b \mid aba^{-1}b^{-1} \rangle = \mathbb{Z}^2$ on two generators is the familiar square grid.

3. The group $G = \langle a,b,c \mid a^2, b^2, c^2, (ab)^2, (bc)^3, (ca)^6 \rangle$ is isomorphic to a subgroup of $\mathrm{Isom}(\mathbb{R}^2)$ generated by the reflections in the sides of a fixed triangle with angles $\pi/2$, $\pi/3$ and $\pi/6$. To obtain a picture of its Cayley graph, draw the resulting triangular tesselation of $\mathbb{R}^2$, mark the center of the inscribed circle in each triangle, and connect every pair of centers belonging to two triangles with a common side by a pair of edges.

For two different generating sets A and A' of G, the spaces $(\mathcal{C}_A(G), d_A)$ and $(\mathcal{C}_{A'}(G), d_{A'})$ are still quasi-isometric, but rarely bi-Lipschitz equivalent.

5.2 Theorem (Švarc–Milnor)
Let X be a length space and G a group of isometries of X. If G acts properly and cocompactly on X, then G is finitely generated and the map $g \mapsto g(z)$ is a quasi-isometry for every basepoint $z \in X$.

Recall that we defined the action of G on X to be proper if for all $x \in X$ there is an $\epsilon > 0$ such that the set $\{g \in G \,|\, g(U(x,\epsilon)) \cap U(x,\epsilon) \neq \emptyset\}$ is finite. The action is said to be *cocompact* if there is a compact set $C \subset X$ such that $\bigcup_{g \in G} g(C) = X$.

Proof: Choose a compact set $C \subset X$ such that $\bigcup_{g \in G} g(C) = X$. Since G acts properly, it follows that there exists an open neighbourhood V of C such that the set $\{g \in G \,|\, g(V) \cap V \neq \emptyset\}$ is finite. Hence, only finitely many sets $g(C)$ with $g \in G$ meet V; their union is compact and contains a tubular neighborhood of V. This shows that X is locally compact and complete. Since X is a length space, Theorem 2.4 (Hopf–Rinow) tells us that X is a proper geodesic space.
Now choose $z \in C$ and $r > 0$ such that $C \subset B(z,r)$. Let $A' := \{g \in G \,|\, d(z,g(z)) \leq 3r\}$. Because X is proper and G acts properly, A' is a finite set. Let $g \in G \backslash \{e\}$, and let $\sigma\colon [0,1] \to X$ be a geodesic from z to $g(z)$. Choose $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $d(\sigma(t_{i-1}), \sigma(t_i)) \leq r$ and $k \leq \frac{1}{r}\, d(z,g(z)) + 1$. For every i there exists a $g_i \in G$ such that $d(\sigma(t_i), g_i(z)) \leq r$, where we take $g_0 = e$ and $g_k = g$. We have $d(g_{i-1}(z), g_i(z)) \leq 3r$ and therefore $d(z, g_{i-1}^{-1} g_i(z)) \leq 3r$, so $a_i := g_{i-1}^{-1} g_i \in A'$. If follows that

$$g = g_0(g_0^{-1}g_1) \cdots (g_{k-2}^{-1}g_{k-1})(g_{k-1}^{-1}g_k) = a_1 \cdots a_k.$$

Hence A' is a generating set; let $A := A' \backslash \{e\}$. Then $d_A(e,g) \leq k \leq \frac{1}{r}\, d(z,g(z)) + 1$.
Conversely, if $d_A(e,g) = k \in \mathbb{N}$, we can write g as a product $a_1 \cdots a_k$ with $a_i \in A \cup A^{-1}$. Put $g_0 := e$ and $g_i := a_1 \cdots a_i$. Then we have $a_i = g_{i-1}^{-1} g_i$ and

$$d(z,g(z)) \leq \sum_{i=1}^{k} d(g_{i-1}(z), g_i(z)) = \sum_{i=1}^{k} d(z, a_i(z)) \leq k\lambda = \lambda\, d_A(e,g)$$

for $\lambda := \max_{a \in A \cup A^{-1}} d(z, a(z))$. □

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