Minimum Degree Threshold for H-factors with High Discrepancy

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February 2023

Abstract

Given a graph H, a perfect H-factor in a graph G is a collection of vertex-disjoint copies of H spanning G. Kühn and Osthus showed that the minimum degree threshold for a graph G to contain a perfect H-factor is either given by $1-1/\chi(H)$ or by $1-1/\chi_{cr}(H)$ depending on certain natural divisibility considerations. Given a graph G of order n, a 2-edge-coloring of G and a subgraph G' of G, we say that G' has high discrepancy if it contains significantly (linear in n) more edges of one color than the other. Balogh, Csaba, Pluhár and Treglown asked for the minimum degree threshold guaranteeing that every 2-edge-coloring of G has an H-factor with high discrepancy and they settled the case where H is a clique. Here we completely resolve this question by determining the minimum degree threshold for high discrepancy of H-factors for every graph H.

1 Introduction

Combinatorial discrepancy concerns itself with problems of the following form: Given a ground set S and a family of subsets $\mathcal{H}$ of S, does there exist a 2-coloring (or k-coloring) of S such that each set in $\mathcal{H}$ contains roughly the same number of elements from each of the colors? The theory studies conditions guaranteeing that such a coloring does or does not exist. We refer the reader to [19, Chapter 4] for an overview. In recent years there has been considerable interest in discrepancy-type problems on graphs. Here, S is the set of edges of a graph G, and $\mathcal{H}$ is a family of subgraphs of G (e.g. Hamilton cycles, perfect matching, clique-factors). Thus, the goal is to find conditions on G which guarantee that in every 2-coloring of the edges of G, there exists a subgraph of a certain type whose coloring is unbalanced, namely one color appears significantly more than the other. One of the first investigations of this type is by Erdős, Füredi, Loebl and Sós [8], who studied the discrepancy of bounded-degree spanning trees in 2-colorings of the complete graph. In recent years the subject was revived and there are many new works: subgraph discrepancy problems have been studied for Hamilton cycles [3, 10, 11, 9], spanning trees [11], clique-factors [4] and powers of Hamilton cycles [5], among others. In this paper we settle the problem of minimum degree thresholds for the discrepancy of H-factors, resolving a question of Balogh, Csaba, Pluhár and Treglown [4]. Let us give the precise definitions.

Definition 1.1. *For a graph G, a 2-edge-coloring (or just 2-coloring) of G is a function $f : E(G) \to \{-1, 1\}$. For a 2-coloring f and a subgraph G' of G, the* discrepancy *of G' is defined as*

$$f(G') = \sum_{e \in E(G')} f(e).$$

Given a graph H, an *H-factor* is a graph consisting of vertex-disjoint copies of H. A *perfect H-factor* of a graph G is an H-factor which is a spanning (i.e. covering all vertices) subgraph of G. Clearly, this is only possible if $|G|$, the number of vertices in G, is divisible by $|H|$. Our main result determines the minimum degree threshold guaranteeing that in every 2-edge-coloring of G, there is a perfect H-factor with discrepancy linear in n. Before stating this result, we give some background.

The study of perfect H-factors of graphs has a long and rich history. Tutte's famous theorem gives a necessary and sufficient condition for a graph to have a perfect K_2-factor, namely a perfect matching. On the computational side, Kirkpatrick and Hell [13] showed that for a fixed graph H, finding a perfect H-factor in an input graph G is NP-complete whenever H has a connected component of size at least three. It is therefore desirable to find sufficient conditions ensuring that a graph G has a perfect H-factor. One such direction of research is the study of minimum degree conditions. The fundamental Hajnal-Szemerédi[12] theorem states that for every $r \geq 2$, every graph G with order n divisible by r and with minimum $\delta(G) \geq (1 - 1/r)n$ has a perfect K_r-factor. This bound is tight, as can be seen by taking a balanced complete r-partite graph and moving one vertex from one part to another. Indeed, the resulting graph has minimum degree $(1-1/r)n-1$ but no perfect K_r-factor. Alon and Yuster [2] proved an asymptotic generalization of the Hajnal-Szemerédi theorem to all graphs, by showing that for every graph H, if G is an n-vertex graph with n divisible by $|H|$ and with $\delta(G) \geq (1 - \frac{1}{\chi(H)} + \varepsilon)n$, then G has a perfect H factor (where $\varepsilon > 0$ is arbitrary and n is large enough in terms of ε). Later, using their celebrated blow-up lemma, Komlós, Sárközy and Szemerédi [15] improved the error term εn to a constant depending on H. It turns out, however, that $1 - \frac{1}{\chi(H)}$ is not always the correct threshold for forcing a perfect H-factor. Komlós [14] (see also [1, 20]) introduced the so-called *critical chromatic number* $\chi_{cr}(H)$ and showed that having minimum degree $(1 - \frac{1}{\chi_{cr}(H)} + \varepsilon)n$ already suffices for guaranteeing an H-factor that covers *almost all* vertices of G (we give the precise definition of χ_{cr} shortly). Finally, the ultimate result in this direction was obtained by Kühn and Osthus [18], who determined the minimum degree threshold for the existence of a perfect H-factor for every graph H, showing that this threshold is either $1 - \frac{1}{\chi(H)}$ or $1 - \frac{1}{\chi_{cr}(H)}$, depending on certain divisibility conditions. To state this result, we need to introduce the following definitions.

Given a graph H, let $r = \chi(H)$ be the chromatic number of H. Let $\mathcal{C}$ be the class of all r-vertex-colorings of H. For $c \in \mathcal{C}$, let $\sigma(c)$ denote the size of the smallest color class in c. Let $\sigma(H) = \min_{c \in \mathcal{C}} \sigma(c)$. The following is the definition of the critical chromatic number:

$$\chi_{cr}(H) := \frac{(\chi(H) - 1)|H|}{|H| - \sigma(H)}.$$

For each $c \in \mathcal{C}$ with color classes of size $s_1 \leq s_2 \leq ... \leq s_r$, let

$$D(c) := \{s_{i+1} - s_i : 1 \leq i \leq r - 1\}.$$

Let $D(\mathcal{C})$ be the union of $D(c)$ over all $c \in \mathcal{C}$ and let $hcf_\chi(H)$ be the greatest common divisor of the elements in $D(\mathcal{C})$. Let $hcf_c(H)$ denote the largest common divisor of the orders of the connected components of H. Define a parameter $hcf(H)$ as follows: If $r \geq 3$, then set $hcf(H) = 1$ if $hcf_\chi(H) = 1$, and if $r = 2$, then set $hcf(H) = 1$ if $hcf_\chi(H) \leq 2$ and $hcf_c(H) = 1$. In all other cases, $hcf(H) \neq 1$. Now define

$$\chi^*(H) = \begin{cases} \chi(H) & \text{if } hcf(H) \neq 1, \\ \chi_{cr}(H) & \text{otherwise.} \end{cases}$$

Note that $r - 1 \leq \chi_{cr}(H) \leq \chi^*(H) \leq r$ for every H with $\chi(H) = r$. Also, if H has only balanced r-colorings (i.e. if in every r-coloring of H, all color-classes have the same size), then $hcf(H) \neq 1$, hence $\chi^*(H) = \chi(H) = r$.

The aforementioned result of Kühn and Osthus [18] states that $1 - 1/\chi^*(H)$ is the minimum degree threshold for the existence of a perfect H-factor. More precisely, they prove the following:

Theorem 1.2 ([18]). *For every graph H there exists a constant C such that every graph G whose order n is divisible by $|H|$ with*

$$\delta(G) \geq (1 - 1/\chi^*(H))n + C$$

contains a perfect H-factor. Moreover, for every m_0 there exists a graph J of order $m \geq m_0$ such that m is divisible by H with

$$\delta(J) = (1 - 1/\chi^*(H))m - 1$$

such that J does not contain a perfect H-factor.

We now move on to discrepancy of H-factors. For a graph H, the *H-factor discrepancy threshold* for H, denoted by $\delta^*(H)$, is defined as the infimum δ which satisfies the following: for every $\eta > 0$ there exists $\gamma > 0$ and n_0, such that for every graph G of order $n \geq n_0$ and $\delta(G) \geq (\delta + \eta)n$, with $|H|$ dividing n and for every 2-edge-coloring f of G there exists a perfect H-factor F in G with $|f(F)| \geq \gamma n$. In other words, $\delta^*(H)$ is the (normalized) minimum degree threshold guaranteeing an H-factor with linear discrepancy. Trivially, $\delta^*(H) \geq 1 - 1/\chi^*(H)$, because $1 - 1/\chi^*(H)$ is the minimum degree threshold for the existence of a perfect H-factor.

The study of minimum degree discrepancy thresholds for H-factors was initiated by Balogh, Csaba, Pluhár and Treglown [4], who determined $\delta^*(K_r)$.

Theorem 1.3 ([4])**.** $\delta^*(K_r) = \max\{3/4, 1 - 1/(r+1)\}$

Balogh et al. [4] further asked for the discrepancy threshold of other graphs H. Our main result completely settles this problem, determining the value of $\delta^*(H)$ for every graph H. We split the statement into three cases: $\chi(H) = 2$, $\chi(H) = 3$ and $\chi(H) \geq 4$. First, for bipartite H, we have the following:

Theorem 1.4. *For every graph H with $\chi(H) = 2$, it holds that*

$$\delta^*(H) = \begin{cases} \frac{3}{4} & \text{if } H \text{ is regular,} \\ 1/2 & \text{if } H \text{ is non-regular and there exists } \rho > 0 \text{ such that for every connected} \\ & \text{component } U \text{ of } H \text{ it holds that } e_H(U) = \rho|U|, \\ 1 - 1/\chi^*(H) & \text{otherwise.} \end{cases}$$

To state our results for r-chromatic graphs, $r \geq 3$, we first need to introduce some definitions. Given a graph G, a *blowup* of G is any graph obtained from G by replacing each vertex $x \in V(G)$ with a vertex-set V_x and replacing edges $xy \in E(G)$ with complete bipartite graphs (V_x, V_y). The *b-blowup* of G is the blowup where $|V_x| = b$ for every $x \in V(G)$. Given a 2-edge-coloring c of G, a *blowup* of (G, c) is a blowup of G whose edges are colored according to c, namely, where for $xy \in E(G)$, all the edges in the complete bipartite graph (V_x, V_y) have color $c(xy)$. We denote the coloring of this blowup also by c. A central strategy of our argument is to find so-called *templates*, defined as follows:

Definition 1.5 (Template)**.** *Given graphs F, H and a 2-edge-coloring c of F, we say that (F, c) is a* template *for H if there exists a blowup B of (F, c) and two perfect H-factors of B with different discrepancies.*

The *size* of the template (F, c) is simply $|F|$. Next, we introduce the following important parameters of a graph H.

Definition 1.6 ($\mathcal{K}(H)$, $\delta_0(H)$)**.** *Let H be an r-chromatic graph. The set of* non-template colorings *of H, denoted $\mathcal{K}(H)$, is the set of all 2-edge-colorings c of K_r such that (K_r, c) is not a template for H.*

Let $\delta_0(H)$ be the maximum over all δ such that there exists a coloring $c \in \mathcal{K}(H)$ and a blowup B of (K_r, c), such that $\delta(B) = \delta \cdot |B|$ and B has a perfect H-factor F with $c(F) = 0$ (by the definition of $\mathcal{K}(H)$, this implies that $c(F) = 0$ for every perfect H-factor F in B). If there exists no such $c \in \mathcal{K}(H)$ then let $\delta_0(H) = 0$.

Note that $\delta_0(H) \leq 1 - 1/r$ because every r-partite graph B has minimum degree at most $(1 - 1/r)|B|$. In Section 5 we show that the maximum in Definition 1.6 is attained and also provide an algorithm which computes $\delta_0(H)$. Note that if $r = 2$ then $\delta_0(H) = 0$, because every blowup of K_2 is monochromatic so all its perfect H-factors have non-zero discrepancy.

Observe that $\delta^*(H) \geq \delta_0(H)$. Indeed, by the definition of $\delta_0(H)$, there exist $c \in \mathcal{K}(H)$ and a blowup B of (K_r, c) with $\delta(B) = \delta_0(H)|B|$ such that every perfect H-factor of B (and there exists one) has discrepancy

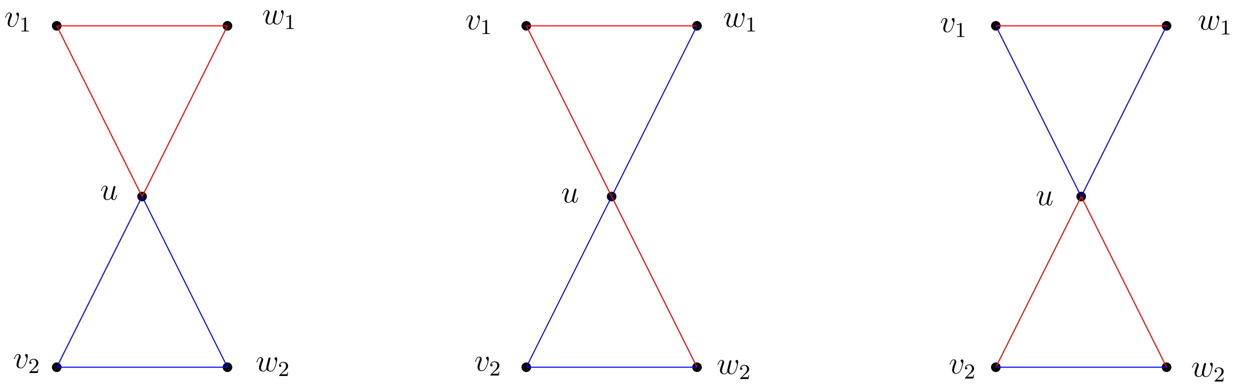

Figure 1: The three different types of butterflies up to isomorphism.

zero. Then, for every $b \in \mathbb{N}$, the b-blowup B' of (B, c) has a perfect H-factor with discrepancy zero. Note that B' is also a blowup of (K_r, c) and since $c \in \mathcal{K}(H)$, every perfect H-factor of B' must have zero discrepancy. As we can choose b arbitrarily large, we get that $\delta^*(H) \geq \delta(B)/|B| = \delta_0(H)$.

Next we state our result for 3-chromatic graphs. Here the following graphs, called *butterflies*, play an important role. A butterfly is a 2-edge-colored graph (L, c), where L consists of two triangles u, v_1, w_1 and u, v_2, w_2 intersecting in a single vertex u, and the coloring c is "antisymmetric" in the sense that $c(uv_1) = -c(uv_2)$, $c(uw_1) = -c(uw_2)$ and $c(v_1w_1) = -c(v_2w_2)$ (see Figure 1).

Theorem 1.7. *For every graph H with $\chi(H) = 3$, it holds that*

$$\delta^*(H) = \begin{cases} \frac{3}{4} & \textit{if } H \textit{ is regular,} \\ \max\{1 - 1/\chi^*(H), \delta_0(H), 4/7\} & \textit{if } H \textit{ is non-regular and some butterfly is not a template for } H, \\ \max\{1 - 1/\chi^*(H), \delta_0(H)\} & \textit{otherwise.} \end{cases}$$

For $k \geq 4$, the following definition, which we call the *k-wise C_4-condition*, plays an essential role in determining the value of $\delta^*(H)$.

Definition 1.8 (C_4-Condition)**.** *For an integer $k \geq 4$, we say that a graph H fulfills the* k-wise C_4-condition *if for every (proper) k-vertex-coloring of H with parts $A_1, A_2, \dots, A_k$, we have that*

$$e_H(A_1, A_2) + e_H(A_3, A_4) = e_H(A_1, A_3) + e_H(A_2, A_4).$$

Note that if H satisfies the k-wise C_4-condition for some $k \geq 5$, then H also satisfies the $(k-1)$-wise C_4-condition, as each proper $(k-1)$-coloring of H is also a proper k-coloring (by taking the last color-class to be empty). Observe also that if H satisfies the k-wise C_4-condition then so does every H-factor. To state our result for r-chromatic graphs with $r \geq 4$, it is convenient to state the following two conditions. In the entire paper, we write $a \equiv_m b$ for $a \equiv b \pmod{m}$.

Condition 1.9. *H fulfills the $(r+1)$-wise C_4-condition and additionally, $r \equiv_4 0$ or H is regular.*

Condition 1.10. *H fulfills the r-wise C_4-condition and is regular.*

The following result determines $\delta^*(H)$ for H with $\chi(H) \geq 4$.

Theorem 1.11. *For every graph H with $r = \chi(H) \geq 4$, it holds that*

$$\delta^*(H) = \begin{cases} 1 - 1/(r+1) & H \textit{ fulfills Condition 1.9,} \\ 1 - 1/r & H \textit{ fulfills Condition 1.10 but not Condition 1.9,} \\ \max\{1 - 1/\chi^*(H), \delta_0(H)\} & H \textit{ violates both Conditions 1.9 and 1.10.} \end{cases}$$

In the next section, we explain the notation that we use throughout the paper. Then, in Section 3, we give a short overview of the proof and explain the main ideas. In Section 4 we prove two general lemmas that will play an important role in our proofs. In Section 5, we establish properties of the parameter $\delta_0(H)$, and describe a construction of a graph H^* (depending on H) that is important in some of our arguments. Section 6 is split into two subsections. First, we recall the notions related to Szemerédi's regularity lemma and the blowup lemma. And second, we introduce the general setup of how we use the regularity lemma in our proofs. This setup is used throughout the rest of the paper. Section 7 deals with templates, giving conditions on H that guarantee that various colored graphs are templates for H. In Section 8 we prove the lower bounds on $\delta^*(H)$ in Theorems 1.4, 1.7 and 1.11. Sections 9, 10 and 11 contain key lemmas that are used in the proofs of the main result. More specifically, in Section 9 we show how to find perfect H-factors of high discrepancy if the coloring of G is unbalanced in a specific way. Section 10 covers graphs H which violate the C_4-condition, and Section 11 covers graphs H which are non-regular. Using the tools from Sections 9-11, we then easily derive the main results (Theorems 1.4, 1.7 and 1.11) in Section 12. To end the paper, in Section 13 we give examples of graphs H which fall into different cases of the three main theorems. The purpose of these examples is to show that all cases in Theorems 1.4, 1.7 and 1.11 are necessary.

2 Notation

Given a graph G, let $V(G)$ denote the set of vertices of G, $E(G)$ the set of edges and $e(G) = |E(G)|$. Let $|G|$ denote the number of vertices in G. Given two sets $U, V \in V(G)$, we write $G[U]$ for the graph induced by U on G and $G[U, V]$ for the graph with edges with one endpoint in U and the other in V. Also, let $e_G(U, V) = e(G[U, V])$, and let $e_G(U)$ denote the number of edges in $G[U]$. For $v \in V(G)$, let $d_G(v)$ denote the number of edges incident to v in G.

For a 2-edge-coloring c of a graph G, we write G^+, G^- for the subgraph of G consisting of the edges of color $+1$ and -1, respectively. Note that $c(G) = e(G^+) - e(G^-) = 2e(G^+) - e(G)$. For a color $c \in \{1, -1\}$, we use G^c for G^+ if $c = 1$ and for G^- if $c = -1$.

Given a blowup B of G and a set of vertices $U \subseteq V(G)$, we write V_U for $\cup_{u \in U} V_u$ and given a vertex $u \in V(B)$, we refer to the vertex in $V(G)$ corresponding to the cluster of u by V_u^G.

Throughout the paper, H is a fixed graph and r is the chromatic number of H. An r-coloring of H always means a proper r-vertex-coloring. We identify an r-coloring with its set of color classes, usually denoted $A_1, \dots, A_r$. We think of the parts as ordered, namely permuting them gives a different r-coloring. An r-coloring $A_1, \dots, A_r$ is called *balanced* if $|A_1| = \cdots = |A_r|$, and *unbalaned* otherwise.

3 Proof Overview

In this section we give a high level overview of our proofs. Some of our arguments apply to any graph H, while some require H to have certain properties. We start by explaining the general setup.

We employ a similar strategy to that used by Balogh, Csaba, Pluhár and Treglown [4] to determine $\delta^*(K_r)$. Given a 2-edge-coloring of the graph G, we apply a colored version of Szemerédi's regularity lemma and consider the corresponding reduced graph R which, by standard techniques, naturally inherits a 2-edge coloring f_R from G and has essentially the same minimum degree relative to its number of vertices. A crucial ingredient of our proof is the notion of a template (Definition 1.5) which has been introduced in a slightly different form in [4]. The importance of this notion is that if there is a subgraph $F \subseteq R$ of size independent of n such that (F, f_R) is a template for H, then, by definition, there is a blowup of F such that there are two H-factors of (F, f_R) with different discrepancies. A standard application of the blowup lemma then implies that we can tile a set U of $\Omega(n)$ vertices of G in the clusters of the regular partition corresponding to $V(F)$ with two different H-factors whose discrepancies differ by $\Omega(n)$. Taking U to be of small linear size, the graph $G \setminus U$ still has high enough minimum degree to force a perfect H-factor by Theorem 1.2. It is then easy to see that by adding the two H-factors of U to this perfect H-factor of $G \setminus U$, we obtain two perfect H-factors of G whose discrepancies differ by $\Omega(n)$, hence one of them must have absolute discrepancy $\Omega(n)$, as needed. This shows that finding small templates for H in the reduced graph suffices for finding an H-factor of high discrepancy.

Let us explain another important aspect of the notion of a template. If a certain coloured graph (F, c) is not a template for H, then by definition, for every blowup B of (F, c), all perfect H-factors of B have the same discrepancy. If this discrepancy equals 0 (e.g. this happens if (F, c) is symmetric with respect to the two colours), then this provides us with a lower bound construction for $\delta^*(H)$. The most important special case is when $F = K_r$ (where $r = \chi(H)$) which leads us to the definition of $\delta_0(H)$. Indeed, $\delta_0(H)$ is the best lower bound on $\delta^*(H)$ that one can obtain by considering blowups of a coloured K_r (apart from the potential divisibility constraints which are encapsulated by the parameter $\chi^*(H)$.)

By the above discussion, we may assume that the reduced graph R has no subgraph on $O(1)$ vertices which is a template for H. This can be exploited from two angles. Taking a fixed colored graph (F, c) which is a template for H, we obtain structural information about R as it must be (F, c)-free. On the other hand, the fact that a certain coloured graph F is not a template gives us structural information about H. More precisely, we obtain that for every r-coloring of H, the sizes of the color classes and the number of edges between the pairs of them must satisfy certain linear equations. A typical example is Lemma 7.18. (We sometimes also have constraints in terms of k-colorings of H for $k = r + 1$ or $r + 2$. An example is the C_4-condition, see Definition 1.8.)

It is possible that there is no small subgraph of R forming a template for H, e.g. if all edges in R have color 1. However, if the coloring of R is so unbalanced, we can find a perfect H-factor in G with high discrepancy. So, roughly speaking, our strategy is to show that either R has a small template for H, or the coloring of R must be in some sense unbalanced, allowing us to find a perfect H-factor with high discrepancy by other means. A concrete example is Lemma 9.1, which shows that if all r-cliques in R have positive discrepancy, then we can indeed find a perfect H-factor of high discrepancy, provided we assume additionally that H is not regular. So to illustrate our strategy in more detail, let us consider the case when H is not regular, so that Lemma 9.1 applies and we may assume that not all r-cliques in R have positive discrepancy, and by symmetry not all r-cliques have negative discrepancy. Then, since R has minimum degree larger than $1 - 1/(r-1)$, it is not difficult to show that there are two r-cliques L_1 and L_2 sharing $r - 2$ vertices, where one of them has positive discrepancy, while the other has negative discrepancy. If the coloring on $L_1 \cup L_2$ is a template for H, we are done, and otherwise H must have a certain structure. Now, by the minimum degree condition on R, for $v \in L_1 \setminus L_2$, there are many vertices u such that $L_1 \cup \{u\} \setminus \{v\}$ forms an r-clique. We show that, essentially, the edges from u to $L_1 \setminus \{v\}$ must be colored in the same way as those from v to $L_1 \setminus \{v\}$ (or else R contains a template for H). Such arguments eventually lead to showing that one of the colors is represented significantly more in R than the other color. For example, in one of the cases in the proof of Lemma 11.4, we show that there is a set of size more than $3n/4$ which is entirely monochromatic. This allows us to find a perfect H-factor with high discrepancy.

Another ingredient in our proof is the idea of using certain complete r-partite graphs (where $r = \chi(H)$). More precisely, in order to find a perfect H-factor, we sometimes instead find a perfect H^*-factor for a certain complete r-partite graph H^*, and then tile each copy of H^* with copies of H. The advantage of working with complete r-partite graphs (rather than with general r-partite graphs) is that they consist of r-cliques, and our templates for H also consist of r-cliques. Thus, assuming that there are no small templates for H allows us to deduce things about the colors of the edges of copies of H^*. Typically, we show that H^* is colored as a blowup of K_r, namely, that all bipartite graphs between color classes are monochromatic. In order to use this approach, H^* must satisfy certain properties. First, it must contain a perfect H-factor. And second, the minimum degree threshold for the existence of an H^*-factor must be only slightly larger than that of H, so that our minimum degree assumptions guarantee the existence of an H^*-factor. Such a graph H^* is constructed in Lemma 5.4.

4 General Lemmas

In this section we give two general lemmas which are essential to the proofs of our main results. The following simple lemma allows us to find a "chain" of r-cliques connecting two given r-cliques in a graph of sufficiently high minimum degree. Such a lemma has already appeared in previous works, see e.g. [5]. For completeness, we include a proof.

Lemma 4.1. *Let $k \in \mathbb{N}$, let J be an m-vertex graph, and let $L, L' \subseteq J$ be two copies of K_k.*

1. *If $\delta(J) > \frac{k-1}{k}m$, then there is a sequence $L_1, L_2, ..., L_\ell \subseteq J$ of k-cliques with $L_1 = L$, $L_\ell = L'$ and $|L_i \cap L_{i+1}| = k-1$ for each $1 \leq i < \ell$.*
2. *If $\delta(J) > \frac{k-2}{k-1}m$, then there is a sequence $L_1, L_2, ..., L_\ell \subseteq J$ of k-cliques with $L_1 = L$, $L_\ell = L'$ and $|L_i \cap L_{i+1}| \geq k-2$ for each $1 \leq i < \ell$.*

Proof. We start with the first item. Here we assume that $\delta(J) > \frac{k-1}{k}m$, which implies that every k vertices have a common neighbour. It is enough to find a sequence $L = L_1, \dots, L_\ell = L'$ with $|L_i \cap L_{i+1}| \geq k-1$ (i.e. we can repeat cliques). We prove the claim by reverse induction on $t := |L \cap L'|$. If $t \geq k-1$ then there is nothing to prove. Suppose then that $t \leq k-2$. Write $L \cap L' = \{s_1, \dots, s_t\}$, $L \setminus L' = \{x_1, \dots, x_{k-t}\}$, $L' \setminus L = \{y_1, \dots, y_{k-t}\}$. We define vertices $z_1, \dots, z_{k-t-1}$ inductively as follows. Let z_1 be a common neighbour of $s_1, \dots, s_t, x_1, \dots, x_{k-t-1}, y_1$. For $2 \leq i \leq k-t-1$, let z_i be a common neighbour of $s_1, \dots, s_t, x_i, \dots, x_{k-t-1}, z_1, \dots, z_{i-1}, y_1$. Write $M_i = \{s_1, \dots, s_t, x_i, \dots, x_{k-t-1}, z_1, \dots, z_i\}$ for $1 \leq i \leq k-t-1$. Then $M_1, \dots, M_{k-t-1}$ are k-cliques, $|M_1 \cap L| \geq k-1$, and $|M_i \cap M_{i+1}| \geq k-1$ for $1 \leq i \leq k-t-2$. Also, $L'' := \{s_1, \dots, s_t, z_1, \dots, z_{k-t-1}, y_1\}$ is a k-clique, $|L'' \cap M_{k-t-1}| \geq k-1$ and $|L'' \cap L'| \geq t+1$. By the induction hypothesis, there is a chain $L'' = N_1, \dots, N_\ell = L'$ with $|N_i \cap N_{i+1}| \geq k-1$ for $1 \leq i \leq \ell-1$. Now $L, M_1, \dots, M_{k-t-1}, N_1, \dots, N_\ell = L'$ is the required chain.

Next, we prove Item 2 by reducing to Item 1. Here we assume that $\delta(J) > \frac{k-2}{k-1}m$, which implies that every $k-1$ vertices have a common neighbour, and hence every $(k-1)$-clique is contained in a k-clique. Take $M \subseteq L, M' \subseteq L'$ of size $k-1$ each. By Item 1 with parameter $k-1$, there are $(k-1)$-cliques $M = M_1, \dots, M_\ell = M'$ with $|M_i \cap M_{i+1}| \geq k-2$ for each $1 \leq i \leq \ell-1$. Let L_i be a k-clique containing M_i, where $L_1 = L$ and $L_\ell = L'$. Then $L_1, \dots, L_\ell$ is the required sequence. □

The next lemma is a key reason why the C_4-condition is one of the determining factors for the value of $\delta^*(H)$. The lemma allows us to control the discrepancy of subgraphs fulfilling the C_4-condition in blowups of regular colorings of K_k (i.e., 2-edge-colorings in which the color-classes form regular graphs). We will later apply this lemma to H-factors (using that an H-factor satisfies the C_4-condition if H does), to deduce that a regular coloring of K_k is not a template for H.

Lemma 4.2. *Let c be a 2-edge-coloring of K_k, $k \geq 2$, and suppose that K_k^+ is d-regular for some $d \in \mathbb{N}$. Let B be a blowup of (K_k, c) and J an arbitrary subgraph of B. If J fulfills the k-wise C_4-condition, then*

$$c(J) = \frac{2d-k+1}{k-1} e(J).$$

Proof. First, we estimate the number of edges of J contained in the blowup of a given 2-factor of K_k. Here, by "2-factor" we mean a disjoint union of cycles covering $V(K_k)$.

Claim 4.3. *Let C be a 2-factor in K_k and C' the corresponding graph in B (i.e., the blowup of C). Then*

$$e(J \cap C') = \frac{2}{k-1} e(J).$$

Proof. Let $X = x_1x_2,...,x_\ell$ be an arbitrary cycle in C of length ℓ. Consider any pair $i < j$ such that the vertices x_i, x_j are not adjacent on X. Note that for each i there are $\ell - 3$ such j. Since J satisfies the k-wise C_4-condition, we have

$$e_J(V_{x_i}, V_{x_{i+1}}) + e_J(V_{x_j}, V_{x_{j+1}}) - e_J(V_{x_i}, V_{x_j}) - e_J(V_{x_{i+1}}, V_{x_{j+1}}) = 0,$$

where indices are taken modulo ℓ. Summing over all such pairs $i < j$, we obtain

$$(\ell - 3) \sum_{1 \le i \le \ell} e_J(V_{x_i}, V_{x_{i+1}}) - 2 \sum_{i,j \,:\, |i-j| \not\equiv \pm 1 \pmod{\ell}} e_J(V_{x_i}, V_{x_j}) = 0. \tag{1}$$

If X is the only cycle in C, then $\ell = k$, and by (1),

$$e(J \cap C') = \sum_{1 \le i \le \ell} e_J(V_{x_i}, V_{x_{i+1}}) = e(J) - \sum_{i,j \,:\, |i-j| \not\equiv \pm 1 \pmod{\ell}} e_J(V_{x_i}, V_{x_j}) = \frac{2}{k-1} e(J),$$

as required. Therefore, let us assume that there is a second cycle $Y = y_1, y, ..., y_h$ in C. By the C_4-condition,

$$\sum_{1 \le i \le \ell} \sum_{1 \le j \le h} \left[e_J(V_{x_i}, V_{x_{i+1}}) + e_J(V_{y_j}, V_{y_{j+1}}) - e_J(V_{x_i}, V_{y_j}) - e_J(V_{x_{i+1}}, V_{y_{j+1}}) \right] = 0.$$

Reordering the terms in the above equality, we get

$$h \sum_{1 \le i \le \ell} e_J(V_{x_i}, V_{x_{i+1}}) + \ell \sum_{1 \le j \le h} e_J(V_{y_j}, V_{y_{j+1}}) - 2 \sum_{1 \le i \le \ell} \sum_{1 \le j \le h} e_J(V_{x_i}, V_{y_j}) = 0. \tag{2}$$

Now, we take the sum of (1) over all cycles in C and the sum of (2) over all pairs of cycles in C. For each cycle X in C, each edge of X is counted $k - |X|$ times when summing (2) over pairs X, Y with $Y \in C \setminus \{X\}$, and is counted $|X| - 3$ times in (1). Therefore, each edge of C is counted exactly $k - 3$ times. Also, each edge $e \in K_k \setminus C$ is counted twice (with a negative sign) when summing (1) and (2); indeed, if e goes between vertices of the same cycle X, then e is counted twice in (1), and if e goes between vertices of two different cycles X, Y, then e is counted twice in (2). All in all, we get that

$$(k - 3) \sum_{uv \in C} e_J(V_u, V_v) - 2 \sum_{uv \in K_k \setminus C} e_J(V_u, V_v) = 0.$$

It follows that

$$e(J \cap C') = \sum_{uv \in C} e_J(V_u, V_v) = \frac{2}{(k-1)} e(J).$$

□

We now complete the proof of Lemma 4.2 using Claim 4.3. Note that either d or $k-1-d$ is even because k and d cannot both be odd. Without loss of generality, let us assume that d is even (else apply the same argument to the complement coloring). By Petersen's theorem (see e.g. [6, Corollary 2.1.5]), the edges of K_k^+ can be decomposed into $\frac{d}{2}$ 2-factors. By applying Claim 4.3 to each of these 2-factors, we get that

$$e(J^+) = \frac{d}{2} \cdot \frac{2}{(k-1)} e(J) = \frac{d}{(k-1)} e(J).$$

The result follows since

$$c(J) = e(J^+) - e(J^-) = 2e(J^+) - e(J).$$

□

5 The parameters δ_0 and H^*

The goal of this section is twofold. First, we consider the parameter $\delta_0(H)$; we show that it is well-defined, i.e. that the maximum in Definition 1.6 is attained, prove some useful properties of $\delta_0(H)$ and give an algorithm that computes $\delta_0(H)$ in finite time. And second, we describe a construction of a certain complete r-partite graph H^* that will play an important role in our proofs.

Proposition 5.1. *Let H be a graph. The maximum in the definition of $\delta_0(H)$ (see Definition 1.6) is attained and $\delta_0(H) \in \mathbb{Q}$. Moreover, there is an algorithm which, given a graph H, computes $\delta_0(H)$.*

Proof. We present an algorithm for computing $\delta_0(H)$. From the algorithm, it will be clear that the maximum in Definition 1.6 is attained.

The algorithm is as follows. We iterate over all possible $2^{\binom{r}{2}}$ 2-edge-colorings of K_r. For each coloring c we need to check whether it is a template with respect to H and if it is not, to find the maximum value of δ such that there is a blowup B of (K_r, c) with $\delta(B) = \delta|B|$ and B has a perfect H-factor with $c(F) = 0$.

Fix a 2-edge-coloring c of K_r. Let $\mathcal{C} \subseteq [r]^{V(H)}$ denote the set of all proper r-vertex-colorings of H. Consider a blowup B of (K_r, c) with parts $A_1, A_2, \ldots, A_r$ of sizes $|A_i| = a_i, i \in [r]$. For $f \in \mathcal{C}$, we define $a_i(f) = |\{v \in V(H) | f(v) = i\}|$, for $i \in [r]$, and $g(f) = \sum_{uv \in E(H)} c(f(u)f(v))$, where we denote each vertex in K_r by a color of f. We think of f as an embedding of H into B. Then, $a_i(f)$ counts the number of vertices embedded into A_i, while $g(f)$ denotes the discrepancy of the embedding.

Now, checking whether (K_r, c) is a template for H can be done using the following linear program.

$$
\begin{aligned}
\text{maximize } & \sum_{f \in \mathcal{C}} (x_f - y_f) \cdot g(f) \\
\text{subject to } & \sum_{f \in \mathcal{C}} x_f \cdot a_i(f) = a_i, \forall i \in [r], \\
& \sum_{f \in \mathcal{C}} y_f \cdot a_i(f) = a_i, \forall i \in [r], \\
& \sum_{i=1}^{r} a_i = 1, \\
& a_i \geq 0, \forall i \in [r] \\
& x_f, y_f \geq 0, \forall f \in \mathcal{C}.
\end{aligned}
$$

We claim that if the maximum in the above linear program is 0, then (K_r, c) is not a template, otherwise it is. Indeed, there exists an optimal feasible solution for which the vectors x, y are fractional H-factors of a blowup of (K_r, c) with parts of relative sizes $a_1, \ldots, a_r$, whereas the objective function corresponds to the difference in the discrepancies of the two fractional H-factors. Hence, if the maximum is 0, no two H-factors can have different discrepancies. On the other hand, if the maximum is nonzero, since the optimum is attained by some solution vector with rational entries, we may multiply it by a large number to get a solution with integer entries. It is not difficult to see that this corresponds to a blowup of (K_r, c) and two H-factors of it with different discrepancies.

Now, suppose we are given a coloring c such that (K_r, c) is not a template for H. We wish to find a maximum δ such that there is a blowup B of (K_r, c) with $\delta(B) = \delta|B|$ for which there is an H-factor with discrepancy 0.

This can be found with the following linear program:

$$\begin{aligned} &\text{maximize } 1 - a_r \\ &\text{subject to } 0 \leq a_1 \leq a_2 \leq \dots a_r, \\ &\sum_{i=1}^{r} a_i = 1, \\ &\sum_{f \in \mathcal{C}} x_f \cdot g(f) = 0, \\ &x_f \geq 0, \forall f \in \mathcal{C}. \end{aligned}$$

Again, there exists an optimal feasible solution to the above linear program corresponding to a blowup B' of (K_r, c) with relative part sizes $a_1 \leq a_2 \leq \dots a_r$ and a fractional H-factor x of B' with discrepancy 0. Multiplying this optimal vector with an appropriate integer, we obtain an integral vector which corresponds to a blowup B of (K_r, c) and an H-factor with discrepancy 0 with respect to c. By the ordering of the a_i's it follows that $\delta(B) = (1 - a_r)|B|$, as needed. Finally, since the above linear program has integer coefficients, it has a rational solution, giving that $\delta_0(H) \in \mathbb{Q}$. □

Next, we prove some useful facts related to $\delta_0(H)$. We begin with the following simple claim, stating that the b-blowup of K_r with $b = (r-1)! \cdot |H|$ can be tiled by copies of H in a uniform manner.

Lemma 5.2. *Let B be the $(r-1)!|H|$-blowup of K_r with parts $B_1, \dots, B_r$. Then, there exists a perfect H-factor F in B such that $e_F(B_i, B_j) = 2(r-2)!e(H)$ for every pair $1 \leq i < j \leq r$. Therefore, if B is colored such that B is the blowup of (K_r, c) for a coloring c of K_r, then $c(F) = c(K_r) \cdot 2(r-2)!e(H)$.*

Proof. Let $A_1, A_2, \dots, A_r$ be the parts of an r-vertex-coloring of H. Then, there exists a perfect H-factor F of B that contains for every permutation $\sigma : [r] \to [r]$ a copy of H with the vertices of A_i in cluster $B_{\sigma(i)}$ for every $1 \leq i \leq r$. Note that for every $1 \leq i, j \leq r$, by the symmetry of F we have that

$$e_F(B_i, B_j) = \frac{e(H)|B|}{\binom{r}{2}|H|}.$$

Using that $|B| = r!|H|$, the statement follows. □

Lemma 5.2 implies that if $\mathcal{K}(H)$ contains a coloring c of K_r with $c(K_r) = 0$, then $\delta_0(H) = 1 - 1/r$. Indeed, by taking B to be the $(r-1)!|H|$-blowup of (K_r, c), we get by Lemma 5.2 that B has a perfect H-factor F with $c(F) = c(K_r) \cdot 2(r-2)!e(H) = 0$. This implies that $\delta_0(H) \geq \delta(B)/|B| = 1 - 1/r$ by the definition of δ_0.

The next lemma gives an important property of $\delta_0(H)$.

Lemma 5.3. *For every 2-edge-coloring $c \in \mathcal{K}(H)$ with $c(K_r) > 0$ the following holds. Let B be a blowup of (K_r, c) with $\delta(B) > \delta_0(H)|B|$. Then for every H-factor F of B, $c(F) > 0$.*

Proof. Fix $c \in \mathcal{K}(H)$ with $c(K_r) > 0$. Let us assume towards a contradiction that there exists a blowup B of (K_r, c) with $\delta(B) > \delta_0(H)|B|$ and a perfect H-factor F of B such that $c(F) \leq 0$. Note that if $c(F) = 0$ then $\delta_0(H) \geq \delta(B)/|B| > \delta_0(H)$ which is a contradiction. Therefore, let us assume that $c(F) < 0$. We use an intermediate-value argument: we will take a "union" of B with a balanced blowup of (K_r, c), which has positive discrepancy, choosing the parameters in such a way that this union B_2 will have a perfect H-factor with discrepancy 0. However, B_2 will also have normalized minimum degree as large as that of B, and this would contradict the definition of $\delta_0(H)$. The details follow. Let B_1 be an $(r-1)!|H|$-blowup of (K_r, c). By Lemma 5.2, there exists a perfect H-factor F_1 in B_1 such that

$$c(F_1) = c(K_r) \cdot 2(r-2)!e(H) > 0.$$

Let $a_1 \le a_2 \le \cdots \le a_r$ be the sizes of the parts of B. Let B_2 be a blowup of (K_r, c) with parts of sizes $b_1, b_2, \ldots b_r$ where $b_i = c(F_1) \cdot a_i - c(F)(r-1)!|H|$. Note that $|B_2| = c(F_1)|B| - r \cdot c(F)(r-1)!|H|$, $\delta(B) = |B| - a_r$ and $\delta(B_2) = |B_2| - b_r$. It follows that $\frac{\delta(B_2)}{|B_2|} \ge \frac{\delta(B)}{|B|}$. Indeed, this inequality is equivalent to $a_r \ge |B|/r$, which clearly holds. Let F_2 be a perfect H-factor in B_2 consisting of $c(F_1)$ copies of F and $-c(F)$ copies of F_1. It follows that

$$c(F_2) = c(F_1)c(F) - c(F)c(F_1) = 0.$$

By the definition of $\delta_0(H)$ and since $c \in \mathcal{K}(H)$, we get that

$$\delta_0(H) \ge \frac{\delta(B_2)}{|B_2|} \ge \frac{\delta(B)}{|B|} > \delta_0(H).$$

As this is a contradiction, the statement must hold. □

We now move to define, for every r-chromatic H, a certain r-partite graph H^* having several useful properties. Later on, when trying to find an H-factor with high discrepancy, we often do this by finding an H^*-factor and tiling each copy of H^* with copies of H. For $r = 2$ we simply set $H^* = H$. The key case is $r \ge 3$, handled by the following lemma. Note that if $r \ge 3$, then H^* is complete r-partite.

Lemma 5.4. *Let $r \ge 3$ and $\eta > 0$. For convenience, put $\alpha(H) := \max\{\delta_0(H), 1 - 1/\chi^*(H)\}$. There exists a graph $H^* = H^*(H, \eta)$ such that:*

- *H^* is a complete r-partite graph;*
- *H^* has a perfect H-factor;*
- *$\alpha(H) \le 1 - 1/\chi_{cr}(H^*) \le \alpha(H) + \eta/4$.*
- *If $\alpha(H) + \eta/4 < (r-1)/r$, then $hcf(H^*) = 1$.*
- *If $\delta_0(H) < 1 - 1/r$ then $\delta(H^*)/|H^*| > \delta_0(H)$.*

Definition 5.5 (The graph H^*)**.** *Let H be an r-chromatic graph and let $\eta > 0$. If $r = 2$ then define $H^* = H$, and else define H^* to be the graph given by Lemma 5.4.*

Proof of Lemma 5.4. Let $A_1, A_2, \ldots, A_r$ be an r-coloring of H with $|A_r| = \sigma(H)$. Let B_1 be a blowup of K_r with parts $B_1^1, B_1^2, \ldots, B_1^r$, where the first $r-1$ parts have size $|H| - \sigma(H)$, and B_1^r has size $(r-1)\sigma(H)$. Let $F_1 \subseteq B_1$ be an H-factor containing, for each $1 \le i \le r-1$, a copy of H in which the vertices of A_j are mapped into $B_1^{(j+i) \pmod{r-1}}$ for $1 \le j \le r-1$, and the vertices of A_r are mapped into B_1^r. Note that the sizes of $B_1^1, B_1^2, \ldots, B_1^r$ are precisely chosen to accommodate these (vertex-disjoint) copies of H. So F_1 is a perfect H-factor of B_1.

Let B_2 be the $|H|$-blowup of K_r and let $B_2^1, B_2^2, \ldots, B_2^r$ be the parts of B_2. Similarly as before, let F_2 be a perfect H-factor of B_2 containing, for every $1 \le i \le r$, a copy of H in which A_j is mapped into $B_2^{(i+j) \pmod{r}}$ for every $1 \le j \le r$. Again, the sizes of $B_2^1, B_2^2, \ldots, B_2^r$ are precisely chosen to fit these copies of H, as $|H| = |A_1| + \cdots + |A_r|$.

Note that if $\alpha(H) + \eta/4 \ge (r-1)/r$, then we can take $H^* = B_2$, trivially fulfilling all the necessary conditions. Indeed, the first two items in Lemma 5.4 are immediate, the third item holds because $\chi_{cr}(B_2) = r$ and $\alpha(H) \le 1 - 1/r$, the fourth item holds vacuously, and the fifth item holds because $\delta(H^*)/|H^*| = 1 - 1/r$. Let us therefore assume that $\alpha(H) + \eta/4 < (r-1)/r$. Hence,

$$\min\{1 - \delta_0(H), 1/\chi^*(H)\} - \eta/4 = 1 - \alpha(H) - \eta/4 > 1/r.$$

In particular, $\chi^*(H) < r$, which implies that $\chi^*(H) = \chi_{cr}(H)$ and $hcf(H) = 1$ by the definition of $\chi^*(H)$. Fix a rational number β in the range $1 - \alpha(H) - 0.2\eta \leq \beta \leq 1 - \alpha(H) - 0.1\eta$, and note that

$$1/r < \beta < 1 - \alpha(H) \leq 1/\chi_{cr}(H).$$

We now define a complete r-partite graph B_3 with $r-1$ equal parts and an rth smaller part, such that $1/\chi_{cr}(B_3) = \beta$. Indeed, let B_3 be a blowup of K_r with $(r-1)$ parts $B_3^1, \ldots, B_3^{r-1}$ of size $k|H| + \ell(|H| - \sigma(H))$ each, and one (smaller) part B_3^r of size $k|H| + \ell(r-1)\sigma(H)$, where $k, \ell \in \mathbb{N}$ are determined later. Note that B_3 is essentially a "linear combination" of B_1 and B_2, namely, B_3 can be partitioned into k copies of B_2 and ℓ copies of B_1. Since B_1 and B_2 each have a perfect H-factor, so does B_3. Note that

$$1/\chi_{cr}(B_3) = \frac{|B_3| - |B_3^r|}{(r-1)|B_3|} = \frac{k|H| + \ell(|H| - \sigma(H))}{kr|H| + \ell(r-1)|H|}. \tag{3}$$

We now show that there exist k, ℓ such that $1/\chi_{cr}(B_3) = \beta$. For this, we need the right-hand side in (3) to equal β. Reordering this equation, we get

$$k = \frac{\ell(|H| - \sigma(H)) - \ell(r-1)|H|\beta}{r|H|\beta - |H|}.$$

Note that the term above is of the form $\ell \cdot q$ for some suitable $q \in \mathbb{Q}$. Therefore, there exists $\ell \in \mathbb{N}$ such that $k \in \mathbb{Z}$. From now on, fix such k and ℓ. Using $1/r < \beta < 1/\chi_{cr}(H)$, we get

$$(r-1)|H|\beta < |H| - \sigma(H)$$

and

$$|H| < r|H|\beta.$$

Therefore, $k > 0$.

Our final graph H^* will be obtained by blowing up B_3 by a large integer, and then changing the sizes of the parts by a small amount to make sure that $hcf(H^*) = 1$. We now define this small perturbation. Recall the set $\mathcal{D}(\mathcal{C})$ defined in the introduction. For each $s \in \mathcal{D}(\mathcal{C})$, let $A_s^1, A_s^2, \ldots, A_s^r$ be the parts of an r-vertex-coloring of H with $A_s^1 - A_s^2 = s$. Since $hcf(H) = 1$, it follows by Bézout's Identity (see e.g. Theorem 2.3. in [7]) that there exist integers x_s for each $s \in \mathcal{D}(\mathcal{C})$ such that

$$\sum_{s \in \mathcal{D}(\mathcal{C})} x_s s = 1.$$

Let B_4 be a blowup of K_r with parts $B_4^1, B_4^2, \ldots, B_4^r$ of size $a_1, a_2, \ldots, a_r$ respectively, where

$$a_1 = \sum_{\substack{s \in \mathcal{D}(\mathcal{C}) \\ x_s > 0}} x_s \cdot A_s^1 - \sum_{\substack{s \in \mathcal{D}(\mathcal{C}) \\ x_s < 0}} x_s \cdot A_s^2,$$

$$a_2 = \sum_{\substack{s \in \mathcal{D}(\mathcal{C}) \\ x_s > 0}} x_s \cdot A_s^2 - \sum_{\substack{s \in \mathcal{D}(\mathcal{C}) \\ x_s < 0}} x_s \cdot A_s^1,$$

$$a_i = \sum_{s \in \mathcal{D}(\mathcal{C})} |x_s| \cdot A_s^i \text{ for } 3 \leq i \leq r.$$

Note that $a_1 - a_2 = \sum_{s \in \mathcal{D}(\mathcal{C})} x_s s = 1$. Additionally, there exists a perfect H-factor of B_4 containing for each $s \in \mathcal{D}(\mathcal{C})$ with $x_s > 0$, x_s copies of H with A_s^i on B_4^i for $1 \leq i \leq r$; and for each $s \in \mathcal{D}(\mathcal{C})$ with $x_s < 0$, $-x_s$ copies of H with A_s^1 on B_4^2, A_s^2 on B_4^1 and A_s^i on B_4^i for $3 \leq i \leq r$ if . Let $a = \sum_{1 \leq i \leq r} a_i$. Fix a large integer $M = M(H, \eta)$, to be chosen later. Let B_5 be a blowup of K_r with parts of size $b_1, b_2, \dots, b_r$, where for $1 \le i \le r$,

$$b_i = a_i + aM \cdot |B_3^i|.$$

This immediately implies that $hcf(B_5) = 1$, since, using $r \ge 3$, $b_1 - b_2 = a_1 - a_2 = 1$ (recall that $|B_3^1| = \dots = |B_3^{r-1}|$). Note that the vertices of B_5 can be partitioned into a copy of B_4 and aM copies of B_3. As both B_3 and B_4 have a perfect H-factor, so does B_5. As $\chi_{cr}(H) < r$, we have $(r-1)\sigma(H) < |H| - \sigma(H)$. This implies that $b_r \le b_i$ for all $1 \le i \le r-1$, and hence $\sigma(B_5) = b_r = a_r + aM|B_3^r|$. Also, $|B_5| = a + aM|B_3|$. Now we get that

$$\frac{1}{\chi_{cr}(B_5)} = \frac{a + aM|B_3| - a_r - aM|B_3^r|}{(r-1)(a + aM|B_3|)} = \frac{|B_3| - |B_3^r|}{(r-1)|B_3|} - \frac{a|B_3^r| - a_r|B_3|}{(r-1)|B_3|(a + aM|B_3|)},$$

where the second equality above is a simple calculation. Recall that $1/\chi_{cr}(B_3) = \frac{|B_3| - |B_3^r|}{(r-1)|B_3|}$. Choose M large enough so that the second term above is at most 0.05η in absolute value. Then we have

$$|1/\chi_{cr}(B_3) - 1/\chi_{cr}(B_5)| \le 0.05\eta.$$

Recalling that $1/\chi_{cr}(B_3) = \beta$ and using $1 - \alpha(H) - 0.2\eta \le \beta \le 1 - \alpha(H) - 0.1\eta$, we get

$$1 - \alpha(H) - \eta/4 \le 1/\chi_{cr}(B_5) \le 1 - \alpha(H).$$

This proves the third item in the lemma. It remains to prove the last item. Note that the largest part of B_5 has size

$$\max_{1 \le i \le r-1} b_i \le a + aM \cdot (|B_3| - |B_3^r|)/(r-1) = a + aM \cdot \beta|B_3|,$$

where the two equalities use that $|B_3^1| = \dots = |B_3^{r-1}|$ and that $\beta = 1/\chi_{cr}(B_3)$. The minimum degree of B_5 is $|B_5| - \max_i b_i$. Hence,

$$\delta(B_5)/|B_5| \ge 1 - \frac{a + aM \cdot \beta|B_3|}{a + aM|B_3|} = \frac{aM(1-\beta)|B_3|}{a + aM|B_3|} = 1 - \beta - \frac{(1-\beta)a}{a + aM|B_3|} > 1 - \beta - 0.1\eta$$
$$\ge \alpha(H) \ge \delta_0(H),$$

where the strict inequality holds for large enough M. We see that $H^* = B_5$ fulfills all necessary conditions. □

We end this section with the following important property of H^*, allowing us to control the discrepancy of H-factors of H^* under certain assumptions.

Lemma 5.6. *Let $c \in \mathcal{K}(H)$ with $c(K_r) > 0$. Let J be a colored copy of H^* and suppose that J is a blowup of (K_r, c). Then for every perfect H-factor F of J it holds that $c(F) > 0$.*

Proof. The statement holds trivially if $r = 2$ as then, K_2 must be monochromatic. Therefore, let us assume that $r \ge 3$. Recall that H^* is a complete r-partite graph. By Lemma 5.3, we may assume that $\delta(H^*) \le \delta_0(H)|H^*|$. Then, by the definition of H^* in Lemma 5.4, we have $\delta_0(H) = \delta(H^*)/|H^*| = 1 - 1/r$. So H^* is a balanced r-partite graph. Now, let $h \in \mathbb{N}$ and J a colored copy of H^* such that J is an h-blowup of (K_r, c) for some $c \in \mathcal{K}(H)$ with $c(K_r) > 0$. Let F be a perfect H-factor of J and let us assume towards a contradiction that $c(F) \le 0$.

Let B be the $(r-1)!|H|$-blowup of (K_r, c). By Lemma 5.2, there exists a perfect H-factor F' in B with

$$c(F') = c(K_r) \cdot 2(r-2)!e(H) > 0.$$

Note that since $c \in \mathcal{K}(H)$, we get that every perfect H-factor of J must have discrepancy $c(F)$ and every perfect H-factor of B must have discrepancy $c(F')$.

Next, consider an $h \cdot (r-1)!|H|$-blowup B' of (K_r, c). Clearly, B' has a perfect J-factor F_J containing $(r-1)!|H|$ copies of J and a perfect B-factor F_B containing h copies of B. Let F_1 be a perfect H-factor of B' obtained by taking the union of a perfect H-factor of each copy of J in F_J and F_2 similarly by taking the union of a perfect H-factor of each copy of B in F_B. As we showed above, we get $c(F_1) = (r-1)!|H| \cdot c(F) \leq 0$ and $c(F_2) = h \cdot c(F') > 0$. It follows that $c(F_1) \neq c(F_2)$ and therefore, (K_r, c) is a template for H. This contradicts the assumption $c \in \mathcal{K}(H)$. □

6 Regularity and its application

The goal of this section is twofold. First, we recall the well-known Szemerédi's regularity lemma and the blowup lemma, which play a key role in our proofs. And second, we introduce the general setup in which we shall apply these tools. This setup will be used throughout the rest of the paper.

6.1 Background on regularity

Let us recall the basic definitions and notation related to the regularity lemma. Given a bipartite graph G with vertex-classes $A, B \subseteq V(G)$, the density of G is defined as

$$d_G(A, B) = \frac{e_G(A, B)}{|A||B|}.$$

Given $\varepsilon, d > 0$, we say that G is (ε, d)-regular if

$$d_G(A, B) \geq d$$

and for every $X \subseteq A$ with $|X| \geq \varepsilon|A|$ and $Y \subseteq B$ with $|Y| \geq \varepsilon|B|$, we have that

$$|d_G(A, B) - d_G(X, Y)| < \varepsilon.$$

Additionally, we say that G is (ε, d)-superregular if $d_G(a) > d|B|$ for every $a \in A$, $d_G(b) > d|A|$ for every $b \in B$, and for every $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have

$$d_G(X, Y) > d.$$

We shall use the following two color version of the regularity lemma.

Lemma 6.1 ([17]). *For every $\varepsilon > 0$ and $\ell_0 \in \mathbb{N}$ there exists $L_0 = L_0(\varepsilon, \ell_0)$ so that the following holds. Let $d \in [0, 1]$ and let G be a graph on $n \geq L_0$ vertices with 2-edge-coloring f. Then there exists a partition $V_0, V_1, ...V_\ell$ of $V(G)$ and a spanning subgraph G' of G, such that the following conditions hold:*

1. *$\ell_0 \leq \ell \leq L_0$;*
2. *$d_{G'}(x) \geq d_G(x) - (2d + \varepsilon)n$ for every $x \in V(G)$;*
3. *the subgraph $G'[V_i]$ is empty for all $1 \leq i \leq \ell$;*
4. *$|V_0| \leq \varepsilon n$;*
5. *$|V_1| = |V_2| = ... = |V_\ell|$;*
6. *for all $1 \leq i < j \leq \ell$, $G'[V_i, V_j]^+$ is either an (ε, d)-regular pair or empty.*
7. *for all $1 \leq i < j \leq \ell$, $G'[V_i, V_j]^-$ is either an (ε, d)-regular pair or empty.*

We call G' the pure graph of G (for the parameters ε, ℓ_0, d). Given a graph G with 2-edge-coloring f and a pure graph G' of G, the *reduced graph* R is defined as the graph on vertices $V_1, V_2, ...V_\ell$, where V_i and V_j are connected if at least one of $G'[V_i, V_j]_+$ or $G'[V_i, V_j]^-$ is non-empty. Additionally, we associate with R a 2-edge-coloring f_R, where for $V_iV_j \in R$, $f_R(V_iV_j) = 1$ if $G'[V_i, V_j]^+$ is non-empty and $f_R(V_iV_j) = -1$ otherwise. The following is a useful, well-known fact about the reduced graph.

Lemma 6.2. *Given $c > 0$, let G be a graph on n vertices with $\delta(G) \geq cn$ and let R be the reduced graph obtained by applying Lemma 6.1 to it with some parameters ε, d, ℓ_0. Then, $\delta(R) \geq (c - 2d - 2\varepsilon)|R|$.*

We will also use the well-known Blow-up lemma of Komlós, Sárközy and Szemerédi [16].

Lemma 6.3 (Blow-up lemma[16])**.** *Given a graph K on vertices $1, ..., k$ and $d, \Delta > 0$, there exists $\varepsilon_0 = \varepsilon(d, \Delta, k) > 0$ such that the following holds. Given $L_1, ..., L_k \in \mathbb{N}$ and $\varepsilon < \varepsilon_0$, let F^* be the graph obtained from K by replacing each vertex $i \in F$ with a set V_i of L_i new vertices and joining all vertices in V_i to all vertices in V_j whenever ij is an edge in K. Let G be a spanning subgraph of F^* such that for every edge $ij \in F$ the pair $(V_i, V_j)_G$ is (ε, d)-superregular. Then, for every spanning subgraph F of F^* with $\Delta(H) \leq \Delta$, G contains a copy of F in which the vertices playing the role of V_i are mapped to V_i.*

To apply the Blow-up lemma, we will need the following simple lemma.

Lemma 6.4. *Let $1 \leq b_1 \leq \cdots \leq b_t =: b$ be integers. Let G be a graph and let f be a 2-edge-coloring of $E(G)$. Let $W_1, \dots, W_t \subseteq V(G)$ be pairwise-disjoint with $|W_1| = \cdots = |W_t| =: m$. Let $d, \varepsilon > 0$, and suppose that $m \gg 1/\varepsilon \gg b, t, 1/d$. Then there are subsets $W_i' \subseteq W_i$, $i = 1, \dots, t$, such that:*

1. *$|W_i'| = b_i s$ for each $i = 1, \dots, t$, where $s := \lfloor m/b \rfloor - \lfloor 2t\varepsilon m \rfloor$.*
2. *For every $1 \leq i < j \leq t$ and color $c \in \{\pm 1\}$, if $G[W_i, W_j]^c$ is (ε, d)-regular then $G[W_i', W_j']^c$ is $(2b\varepsilon/b_1, d/2)$-superregular.*

The second item of Lemma 6.4 uses the standard fact that regular pairs can be made superregular by deleting a small number of vertices. The goal of the first item is to make it possible to tile $W_1', \dots, W_t'$ with a graph having a t-coloring with color-classes of size $b_1, \dots, b_t$.

Proof of Lemma 6.4. First, for each $i \in [t]$, take an arbitrary $U_i \subseteq W_i$ of size $b_i \lfloor m/b \rfloor \leq m$. Now fix any $1 \leq i \leq t$, and let I_i be the set of pairs (j, c) such that $G[W_i, W_j]^c$ is (ε, d)-regular (where $j \in [t] \setminus \{i\}$ and $c = \pm 1$). For $(j, c) \in I_i$, let $B_{i,j}^c$ be the set of vertices $u_i \in U_i$ which have less than $(d - \varepsilon)|U_j|$ neighbours in color c in U_j. The (ε, d)-regularity of $G[W_i, W_j]^c$ and the fact that $|U_j| \geq \frac{m}{2b} \geq \varepsilon |W_j|$ imply that $|B_{i,j}^c| \leq \varepsilon m$. Therefore, $B_i := \bigcup_{(j,c) \in I_i} B_{i,j}^c$ satisfies $|B_i| \leq 2(t-1)\varepsilon m$. Hence, $|U_i \setminus B_i| \geq b_i \cdot \lfloor m/b \rfloor - 2(t-1)\varepsilon m \geq b_i s$. So take $W_i' \subseteq U_i \setminus B_i$ of size $b_i s$ (for $i \in [t]$). Note that $|W_i'| \geq \frac{b_1 m}{2b}$, using that $m \gg 1/\varepsilon \gg b, t$. Also, $|U_i \setminus W_i'| = b_i \cdot (\lfloor m/b \rfloor - s) \leq 2tb\varepsilon m$.

Now let $1 \leq i < j \leq t$ and $c \in \{\pm 1\}$ such that $G[W_i, W_j]^c$ is (ε, d)-regular. As $W_i' \subseteq U_i \setminus B_i$, all vertices in W_i' have at least $(d - \varepsilon)|U_j|$ color-c neighbours in U_j, and hence at least

$$(d - \varepsilon)|U_j| - |U_j \setminus W_j'| \geq (d - \varepsilon)|U_j| - 2tb\varepsilon m > d/2 \cdot |U_j|$$

neighbours in W_j', where the last inequality uses that $|U_j| \geq \frac{m}{2b}$ and $1/\varepsilon \gg b, t, 1/d$.

Also, for $X \subseteq W_i', Y \subseteq W_j'$ with $|X| \geq \frac{2b\varepsilon}{b_1}|W_i'|$, $|Y| \geq \frac{2b\varepsilon}{b_1}|W_j'|$, we have $|X|, |Y| \geq \varepsilon m$, and therefore, by the (ε, d)-regularity of $G[W_i, W_j]^c$, the color-c density between X, Y is at least $d - \varepsilon > d/2$. This shows that $G[W_i', W_j']^c$ is $(2b\varepsilon/b_1, d/2)$-superregular, as required. □

6.2 Applying the regularity lemma: the general setup

Here we explain the general setup which we use throughout the rest of the paper. Let $\eta > 0$ be fixed and let $\gamma \ll \eta$ be small enough (depending on η). To prove Theorems 1.4, 1.7 and 1.11, we shall consider a graph G with $n \geq n_0$ vertices, n divisible by $|H|$, and minimum degree $\delta(G) \geq (\delta + \eta)n$, for δ corresponding to the particular case of the above theorems that we are proving, and show that in every 2-edge-coloring of G, there must exist a perfect H-factor with discrepancy at least γn in absolute value. We say that such an H-factor has *high discrepancy*. Proving this statement (for all η) would establish that $\delta^*(H) \leq \delta$. Note that we may always assume that η is small enough with respect to H. From now on, fix $n_0, \varepsilon, \ell_0, d_0, L_0 > 0$ such that

$$1/n_0 \ll \gamma \ll 1/L_0 \leq 1/\ell_0 \ll \varepsilon \ll d_0 \ll \eta \ll 1/|H|,$$

where L_0 is the constant obtained by applying Lemma 6.1 with ε, ℓ_0.

Recall that δ_0 and $1 - 1/\chi^*(H)$ are (natural) lower-bounds for $\delta^*(H)$ for every graph H. Therefore, throughout this paper, we shall always assume that our target parameter δ satisfies

$$\delta \geq \max\{\delta_0(H), 1 - 1/\chi^*(H)\}. \tag{4}$$

So let G be a graph with $n \geq n_0$ vertices, where n is divisible by $|H|$, and with $\delta(G) \geq (\delta + \eta)n$. Our strategy for finding a perfect H-factor of high discrepancy sometimes requires us to first find a perfect H^*-factor F^* (and then tile each H^*-copy with copies of H). To this end, we need that $|G|$ is divisible by H^*. Therefore, we shall put aside a small number of vertices to make the number of remaining vertices divisible by $|H^*|$. Indeed, let F_{Rest} be a collection of vertex-disjoint copies of H in G, such that $|V(F_{Rest})| < |H^*|$ and $n - |V(F_{Rest})|$ is divisible by $|H^*|$. Such a collection exists because G even has a perfect H-factor, by Theorem 1.2 and by (4). Set $G^* := G[V(G) \setminus V(F_{Rest})]$. Recall that H^* depends only on H and η, so $|H^*| \ll \gamma n$. Therefore, it will suffice to find a perfect H-factor of G^* with high discrepancy (as this will give a perfect H-factor of G with absolute value discrepancy at least $\frac{\gamma}{2}n$, say). Hence, we concentrate on G^* from now. With a slight abuse of notation, we shall use n to denote $|G^*|$. Note that

$$\delta(G^*) \geq (\delta + 3\eta/4)n.$$

Fix an arbitrary 2-edge-coloring f of G^*. Let G' be the pure graph obtained by applying Lemma 6.1 with parameters ε, ℓ_0, d_0 to G^* and f. Let R be the corresponding reduced graph with 2-edge-coloring f_R. Using $\eta \gg d_0, \varepsilon$ and Lemma 6.2, we get that

$$\delta(G')/n,\ \delta(R)/|R| \geq \delta(G^*)/n - \eta/4 \geq \delta + \eta/2,$$

and by (4), we get that

$$\delta(G')/n,\ \delta(R)/|R| \geq \max\{\delta_0(H), 1 - 1/\chi^*(H)\} + \eta/2. \tag{5}$$

We shall assume throughout the paper that (5) holds. For a vertex $u \in V(G') \setminus V_0$, we denote by $V_u^R \in V(R)$ the vertex of R corresponding to the part of the regular partition containing u.

We now show that G' contains a perfect H^*-factor.

Lemma 6.5. *G' has a perfect H^*-factor, and hence a perfect H-factor.*

Proof. For convenience, set $\alpha(H) := \max\{\delta_0(H), 1 - 1/\chi^*(H)\}$. In this notation, we have

$$\delta(G') \geq (\alpha(H) + \eta/2)n.$$

We claim that

$$\delta(G') \geq (1 - 1/\chi^*(H^*) + \eta/4)n.$$

If $\alpha(H) \geq 1-1/r-\eta/4$ then $\delta(G') \geq (1-1/r+\eta/4)$, which suffices as $\chi^*(H^*) \leq r$. And if $\alpha(H) \leq 1-1/r-\eta/4$, then Lemma 5.4 guarantees that $hcf(H^*) = 1$ and hence

$$1 - 1/\chi^*(H^*) = 1 - 1/\chi_{cr}(H^*) \leq \alpha(H) + \eta/4 \leq \delta(G')/n - \eta/4,$$

as claimed. Now, Theorem 1.2 guarantees that G' has a perfect H^*-factor. This also implies that G' has a perfect H-factor, because H^* has a perfect H-factor by Lemma 5.4. □

The next lemma allows us to assume that for each pair of clusters V_i, V_j in the regular partition, all edges in $G'[V_i, V_j]$ have the same color (namely the color $f_R(V_iV_j)$). Equivalently, for every edge $xy \in E(G')$ with $x, y \notin V_0$, it holds that

$$f_R(V_x^R V_y^R) = f(xy). \tag{6}$$

We shall assume this throughout the rest of the paper.

Lemma 6.6. *If there exist $1 \leq i < j \leq \ell = |R|$ such that both $G'[V_i, V_j]^+$ and $G'[V_i, V_j]^-$ are (ε, d_0)-regular, then there exists a perfect H-factor in G' with high discrepancy.*

Proof. For convenience, put $U := V_i, V := V_j$, and assume that $G'[U, V]^+$ and $G'[U, V]^-$ are both (ε, d_0)-regular. Then $UV \in R$ and $f_R(UV) = 1$ by the definition of R. By (5),

$$\delta(R) > (1 - 1/\chi^*(H))|R| \geq \frac{r-2}{r-1}|R|$$

and therefore, there exists a copy $L \subseteq R$ of K_r containing the edge UV. Let $W_1, W_2, \dots, W_r$ be the clusters of L with $W_1 = U$ and $W_2 = V$ and let $m = (n - |V_0|)/|R|$ denote the size of each cluster W_i. By definition, all pairs $G'[W_i, W_j]^{f_R(W_i,W_j)}$ are (ε, d_0)-regular. By Lemma 6.4 with $b_1 = \cdots = b_t = (r-1)!|H|$, there are $W_i' \subseteq W_i$, $i = 1, \dots, r$, with $|W_1'| = |W_2'| = \cdots = |W_r'| \geq 0.9m$ and $|W_i'|$ divisible by $(r-1)!|H|$, such that $G'[W_i', W_j']^{f_R(W_i,W_j)}$ is $(2\varepsilon, d_0/2)$-superregular for each $1 \leq i < j \leq r$, and $G'[W_1', W_2']^-$ is also $(2\varepsilon, d_0/2)$-superregular. Let $G_1 \subseteq G'$ be the graph on $\bigcup_{1 \leq i \leq r} W_i'$ with edges $\bigcup_{1 \leq i < j \leq r} G'[W_i', W_j']^{f_R(W_iW_j)}$. Let $G_2 \subseteq G'$ be the same graph but with $G'[W_1', W_2']^-$ in place of $G'[W_1', W_2']^+$. Note that $|W_i'| \leq |W_i| \leq \frac{n}{\ell_0}$ and $|W_i'| \geq 0.9|W_i| \geq 0.9\frac{n-\varepsilon n}{L_0} \geq \frac{n}{2L_0}$ (for each $1 \leq i \leq r$). Therefore,

$$\frac{rn}{2L_0} \leq |G_1| = |G_2| \leq \frac{rn}{\ell_0}.$$

Let B be the $(r-1)!|H|$-blowup of K_r with clusters $B_1, B_2, \dots, B_r$. By Lemma 5.2, B has a perfect H-factor F_B such that $e_{F_B}(B_i, B_j) = 2(r-2)!e(H)$ for every pair $1 \leq i < j \leq r$. As $|W_1'| = \cdots = |W_r'|$ and each $|W_i'|$ is divisible by $(r-1)!|H|$, we can apply Lemma 6.3 to deduce that there exist perfect B-factors F_1'', F_2'' of G_1 and G_2 respectively. Taking the H-factor F_B of every copy of B in F_1'', F_2'' gives us perfect H-factors F_1', F_2' of G_1, G_2, respectively. Moreover, for every $1 \leq i, j \leq r$, we have

$$e_{F_1'}(W_i', W_j') = e_{F_2'}(W_i', W_j') = \frac{|G_1|}{|B|} \cdot 2(r-2)!e(H).$$

It follows that

$$f(F_1') - f(F_2') = e_{F_1'}(W_1', W_2') - e_{F_2'}(W_1', W_2') = \frac{|G_1|}{|B|} \cdot 4(r-2)!e(H).$$

Let $G_0 = G' - \bigcup_{1 \leq i \leq r} W_i'$, and note that

$$\delta(G_0) \geq (1 - 1/\chi^*(H) + \eta/4)n,$$

using (5) and that $|G_1| \leq \frac{rn}{\ell_0}$ and $1/\ell_0 \ll \eta$. Also, $|V(G_0)|$ is divisible by $|H|$ because $|V(G')|$ and $\sum_{i=1}^{t} |W_i'|$ are. Thus, G_0 contains a perfect H-factor F' by Theorem 1.2. Let $F_i := F_i' \cup F'$, $i = 1, 2$. Then F_1, F_2 are perfect H-factors of G', and

$$f(F_1) - f(F_2) = f(F_1') - f(F_2') = \frac{|G_1|}{|B|} \cdot 4(r-2)!e(H) \geq 2\gamma n,$$

where the last inequality uses that $|B| = r!|H|$, $|G_1| \geq \frac{rn}{2L_0}$ and $\gamma \ll \frac{1}{|H|}, \frac{1}{L_0}$. Therefore, at least one of F_1, F_2 has absolute discrepancy at least γn, as required. □

From now on, we shall work under the above setup and repeatedly use (5) and (6). Recall that our ultimate goal is to find a perfect H-factor of G' with high discrepancy. Very roughly, our argument works by showing that either R contains a template for H, or else R is colored by f_R in such an imbalanced way that we can directly find an H-factor in G' with high discrepancy. The next lemma handles the case that R contains a template for H, showing that in that case G' indeed contains a perfect H-factor with high discrepancy. The proof of this lemma is similar to that of [4, Claim 6.2] (and also uses some ideas from the proof of Lemma 6.6).

Lemma 6.7. *Let $t \in \mathbb{N}$ depending only on H, and let $T \subseteq R$ be a subgraph of R of order t. If (T, f_R) is a template for H, then there exists a perfect H-factor in G' with high discrepancy.*

Proof. Since (T, f_R) is a template for H, there exists a blowup B of (T, f_R) and two perfect H-factors F_1, F_2 of B with $f_R(F_1) > f_R(F_2)$. Write $V(T) = \{w_1, w_2, \ldots, w_t\}$, let B_i be the part of B corresponding to w_i, and put $b_i = |B_i|$. Suppose that $b_1 \leq \cdots \leq b_t =: b$. Also, let W_i be the cluster of G' corresponding to $w_i \in V(R)$, and recall that $|W_i| = (n - |V_0|)/|R| =: m$ for $i = 1, \ldots, t$. Also, recall that if for some $1 \leq i < j \leq t$, $w_i w_j \in R$, then $G'[W_i, W_j]^{f_R(w_i w_j)}$ is (ε, d_0)-regular. By Lemma 6.4, there is an integer $s \geq 0.9m/b$ and subsets $W_i' \subseteq W_i$ $(i = 1, \ldots, t)$ such that $|W_i'| = b_i s$, and such that for every pair $1 \leq i < j \leq t$, if $w_i w_j \in R$ then $G'[W_i', W_j']^{f_R(w_i w_j)}$ is $(\frac{2b_t}{b_1}\varepsilon, d_0/2)$-superregular.

Let B' be a blowup of T with parts $B_1', \ldots, B_t'$, where B_i' is the cluster corresponding to w_i, and $|B_i'| = |W_i'| = b_i s$. So B' is the s-blowup of B. Note that we may apply Lemma 6.3 with B' in the role of F^* and with $\bigcup_{w_i w_j \in T} G'[W_i', W_j']^{f_R(w_i w_j)}$ in the role of G.

Clearly, B' has a perfect B-factor F_B consisting of s copies of B (where each copy of B places the part B_i of B inside the part B_i' of B'). Let F_1', F_2' be the perfect H-factors of B', obtained by taking the perfect H-factor F_1 or F_2 respectively of each copy of B in F_B. By Lemma 6.3, there exist copies F_1'', F_2'' of F_1', F_2' (respectively) in $\bigcup_{w_i w_j \in T} G'[W_i', W_j']^{f_R(w_i w_j)}$, with all the vertices on the same corresponding parts (i.e., with B_i' embedded to W_i' for $i = 1, \ldots, t$). Now, using that $f_R(F_1) - f_R(F_2) \geq 1$, we get that

$$f(F_1'') - f(F_2'') \geq s.$$

Now let $G_0 = G' - \bigcup_{1 \leq i \leq t} W_i'$, and note that $\delta(G_0) \geq (1 - 1/\chi^*(H) + \eta/4)n$, using (5) and

$$\sum_{i=1}^{t} |W_i'| \leq \sum_{i=1}^{t} |W_i| \leq tm \leq tn/\ell_0 \ll \eta n,$$

as $\frac{1}{\ell_0} \ll \frac{1}{|H|}, \eta$, and t depends only on H. Also, $|V(G_0)|$ is divisible by $|H|$ because $|V(G')|$ and $\sum_{i=1}^{t} |W_i'|$ are. Now, by Theorem 1.2, G_0 has a perfect H-factor F'. Thus, both $F' \cup F_1''$ and $F' \cup F_2''$ are perfect H-factors of G', and

$$f(F' \cup F_1'') - f(F' \cup F_2'') = f(F_1'') - f(F_2'') \geq s \geq \frac{0.9m}{b} \geq \frac{0.9(1-\varepsilon)n}{L_0 b} \geq \gamma n,$$

using that $m = (n - |V_0|)/|R| \geq (1 - \varepsilon)n/L_0$, and that $\gamma \ll \varepsilon, 1/L_0 \ll 1/b$, as b depends only on H. Now we see that $F' \cup F_1''$ or $F' \cup F_2''$ has discrepancy at least γn, as required. □

7 Templates

Lemma 6.7 states that in order to find an H-factor with high discrepancy, it suffices to show that the reduced graph R contains a template for H. The present section is thus dedicated to various constructions of such templates. These constructions form a substantial and crucial part of our proofs. In most cases, the templates consist of either a single K_r or two copies of K_r sharing $r-1$ or $r-2$ vertices. Our results typically state that either a certain colored graph is a template for H, or else H has a certain "uniformity property", e.g. H is regular, only has balanced r-colorings, or has the same number of edges between any two parts in any r-coloring. Each of the following subsections deals with one such uniformity property and gives a suitable template for graphs violating the property. The basic idea for proving that a certain configuration L is a template for H is usually the same: we define a (carefully chosen) perfect H-factor F_1 of a (carefully chosen) blowup of L, and then modify F_1 by moving some vertices to different blowup-clusters, thus obtaining a second perfect H-factor F_2. We then observe that if this modification did not change the discrepancy (i.e. if F_1, F_2 have the same discrepancy), then H must have the relevant uniformity property. The implementation of this rough idea for the various uniformity properties can be quite involved.

7.1 Disconnected bipartite graphs

In this section, we consider disconnected bipartite graphs H. We show that if H has two connected components with different average degrees then the colored graph consisting of two disjoint edges of different color is a template for H.

Lemma 7.1. *Suppose that H is bipartite and there exist two connected components $U, W \subseteq V(H)$ of H such that $e_H(U)/|U| \neq e_H(W)/|W|$. Let e_1, e_2 be two disjoint edges and c be a 2-coloring of e_1, e_2 with $c(e_1) \neq c(e_2)$. Then $(e_1 \cup e_2, c)$ is a template for H.*

Proof. Write $e_1 = x_1y_1, e_2 = x_2y_2$. without loss of generality, suppose that $c(e_1) = 1$, $c(e_2) = -1$. Fix a 2-coloring A_1, A_2 of H, and let $U_i = U \cap A_i$, $W_i = W \cap A_i$, $i = 1, 2$. Let B be a blowup of $(e_1 \cup e_2, c)$ with

$$|V_{x_1}|, |V_{y_1}| = 2|U||W|,$$
$$|V_{x_2}|, |V_{y_2}| = 2|U||H| + 2|W|(|H| - |U|).$$

Our goal is to find two perfect H-factors F_1, F_2 of B with different discrepancies. The idea is simple: F_1 will contain copies of H in which $U = U_1 \cup U_2$ is mapped onto V_{x_1}, V_{y_1} and $H - U$ is mapped onto V_{x_2}, V_{y_2}, while F_2 will contain copies of H in which $W = W_1 \cup W_2$ is mapped onto V_{x_1}, V_{y_1} and $H - W$ is mapped onto V_{x_2}, V_{y_2}. The fact that $H[U]$ and $H[W]$ have different average degrees will imply that F_1 and F_2 have a different number of edges of color 1 (since all edges between V_{x_1}, V_{y_1} have color 1, and all edges between V_{x_2}, V_{y_2} have color -1). This will imply that $c(F_1) \neq c(F_2)$. To make this scheme work, we need two additional ideas. First, F_1 will have H-copies mapping U_1 to V_{x_1} and U_2 to V_{y_1}, as well as H-copies mapping U_2 to V_{x_1} and U_1 to V_{y_1}; there will be the same number of copies of the two types. This "symmetrization" allows us to take V_{x_1}, V_{y_1} to be of the same size, as each H-copy adds $|U|/2$ vertices on average to V_{x_1} and V_{y_1}. When describing this construction, we will say that we take copies of H with each permutation of U_1, U_2 on V_{x_1}, V_{y_1}. (This language is also used later on in this section.) We will do the same for the H-copies in F_2 with respect to W_1, W_2.

The above scheme tiles $V_{x_1} \cup V_{y_1}$ with $(|V_{x_1}| + |V_{y_1}|)/|U|$ copies of H in F_1, and with $(|V_{x_1}| + |V_{y_1}|)/|W|$ copies of H in F_2. A problem that might occur is that F_1, F_2 use a different number of vertices in $V_{x_2} \cup V_{y_2}$. This must be avoided because F_1, F_2 need to be perfect H-factors of B. To remedy this, we add additional H-copies to F_1, F_2 which only use vertices from $V_{x_2} \cup V_{y_2}$. By appropriately choosing the number of these copies, as well as the size of V_{x_2}, V_{y_2}, we can make sure that F_1, F_2 tile B. The details follow.

Define two perfect H-factors F_1, F_2 of B, each containing $4(|U| + |W|)$ copies of H, as follows:

- F_1 contains $|W|$ copies of H for each permutation of U_1, U_2 on V_{x_1}, V_{y_1} and each permutation of $A_1 \backslash U_1, A_2 \backslash U_2$ on V_{x_2}, V_{y_2}. Additionally, F_1 contains $2|U|$ copies of H for each permutation of A_1, A_2 on V_{x_2}, V_{y_2}.
- F_2 contains $|U|$ copies of H for each permutation of W_1, W_2 on V_{x_1}, V_{y_1} and each permutation of $A_1 \backslash W_1, A_2 \backslash W_2$ on V_{x_2}, V_{y_2}. Additionally, F_2 contains $2|W|$ copies of H for each permutation of A_1, A_2 on V_{x_2}, V_{y_2}.

Note that the choice of $|V_{x_1}|, |V_{y_1}|, |V_{x_2}|, |V_{y_2}|$ exactly corresponds to the definition of F_1, F_2. For example, the number of vertices of F_1 in V_{x_2} is exactly $2 \cdot |W| \cdot (|A_1 \setminus U_1| + |A_2 \setminus U_2|) + 2|U| \cdot (|A_1| + |A_2|) = 2|W|(|H| - |U|) + 2|U||H| = |V_{x_2}|$, and similarly for F_2 and the other three clusters $V_{y_2}, V_{x_1}, V_{y_1}$.

Next, observe that

$$e(F_1^+) = e_{F_1}(V_{x_1}, V_{y_1}) = 4|W|e_H(U)$$

and

$$e(F_2^+) = e_{F_2}(V_{x_1}, V_{y_1}) = 4|U|e_H(W).$$

As $|W|e_H(U) \neq |U|e_H(W)$, we have that $e(F_1^+) \neq e(F_2^+)$. Also, $e(F_1) = e(F_2)$ because F_1, F_2 are both perfect H-factors of B. Hence,

$$c(F_1) = 2e(F_1^+) - e(F_1) \neq 2e(F_2^+) - e(F_2) = c(F_2).$$

This implies that $(e_1 \cup e_2, c)$ is a template for H, as required. □

7.2 Non-regular graphs

In this section we consider non-regular graphs H. As always, r denotes the chromatic number of H.

Lemma 7.2. *Let L_1, L_2 be two copies of K_r sharing $r-1$ vertices with 2-edge-coloring c such that $c(L_1) \neq c(L_2)$. If H is non-regular then $(L_1 \cup L_2, c)$ is a template for H.*

Proof. Let $u, v \in V(H)$ be two vertices with $d_H(u) \neq d_H(v)$. Fix an r-vertex-coloring of H with parts $A_1, A_2, \ldots, A_r$. Let A_{i_u} be the part containing u and A_{i_v} the one containing v (possibly $i_u = i_v$). Write $L_1 \cap L_2 = \{q_2, q_3, \ldots, q_r\}$, $L_1 \backslash L_2 = \{s\}$ and $L_2 \backslash L_1 = \{t\}$. Let B be a blowup of $(L_1 \cup L_2, c)$ with

- $|V_s| = (r-1)!$,
- $|V_t| = (r-1)!(|A_{i_u}| + |A_{i_v}| - 1)$ and
- $|V_{q_i}| = (r-2)!(2|H| - |A_{i_u}| - |A_{i_v}|)$ for $2 \leq i \leq r$.

We now define certain H-copies in B. For each permutation $\sigma : \{2, \ldots, r\} \to [r] \setminus \{i_v\}$, let X_σ be a copy of H in which A_{i_v} is embedded into V_t and $A_{\sigma(i)}$ is embedded into V_{q_i} for all $2 \leq i \leq r$. Let X'_σ be the copy of H obtained from X_σ by moving v from V_t to V_s. Similarly, for each permutation $\tau : \{2, \ldots, r\} \to [r] \setminus \{i_u\}$, let Y_τ be a copy of H in which A_{i_u} is embedded into V_t and $A_{\tau(i)}$ is embedded into V_{q_i} for all $2 \leq i \leq r$. Let Y'_τ be the copy of H obtained from Y_τ by moving u from V_t to V_s. We define all these H-copies such that the copies in each of the sets $F_1 := \{X_\sigma, Y'_\tau\}_{\sigma,\tau}$ and $F_2 := \{Y_\tau, X'_\sigma\}_{\sigma,\tau}$ are pairwise-disjoint and partition $V(B)$; note that the sizes of $V_s, V_t, V_{q_2}, \ldots, V_{q_r}$ are precisely chosen to allow this. In other words, F_1, F_2 are perfect H-factors of B. We now calculate $c(F_1) - c(F_2)$. By definition,

$$c(X_\sigma) - c(X'_\sigma) = \sum_{i=2}^{r} (c(tq_i) - c(sq_i)) \cdot e_H(v, A_{\sigma(i)}).$$

For each $j \in [r] \setminus \{i_v\}$ and $2 \leq i \leq r$, there are exactly $(r-2)!$ permutations σ with $\sigma(i) = j$. Hence, summing over all σ, we get

$$\sum_{\sigma}(c(X_\sigma) - c(X'_\sigma)) = \sum_{i=2}^{r}(c(tq_i) - c(sq_i)) \sum_{j \in [r]\setminus\{i_v\}} (r-2)! \cdot e_H(v, A_j)$$
$$= \sum_{i=2}^{r}(c(tq_i) - c(sq_i)) \cdot (r-2)! \cdot d_H(v) = (c(L_2) - c(L_1)) \cdot (r-2)! \cdot d_H(v).$$

Similarly,

$$\sum_{\tau}(c(Y_\tau) - c(Y'_\tau)) = (c(L_2) - c(L_1)) \cdot (r-2)! \cdot d_H(v).$$

We get that

$$c(F_1) - c(F_2) = \sum_{\sigma}(c(X_\sigma) - c(X'_\sigma)) - \sum_{\tau}(c(Y_\tau) - c(Y'_\tau)) = (c(L_2) - c(L_1)) \cdot (r-2)! \cdot (d_H(v) - d_H(u)) \neq 0,$$

using that $c(L_1) \neq c(L_2)$ by assumption and $d_H(u) \neq d_H(v)$. The fact that $c(F_1) \neq c(F_2)$ means that $(L_1 \cup L_2, c)$ is a template with respect to H. □

We now use Lemma 7.2 to show that certain colorings of K_{r+1} are templates for H.

Corollary 7.3. *Let L be a copy of K_{r+1} with 2-edge-coloring c such that L^+ is non-regular. If H is non-regular then (L, c) is a template for H.*

Proof. Since L^+ is non-regular, there exist $u, v \in V(L)$ such that $c(u, L) \neq c(v, L)$. Let $U = V(L)\setminus\{v\}$ and $V = V(L)\setminus\{u\}$. It is not hard to see that $|U \cap V| = r - 1$ and

$$c(L[U]) - c(L[V]) = c(u, L) - c(uv) - (c(v, L) - c(uv)) \neq 0.$$

The statement follows by Lemma 7.2. □

7.3 Different degrees to different parts

In this section we consider another "uniformity property" defined in terms of vertex-degrees. Here, we assume that there is an r-coloring of H such that some vertex has different degrees to two color-classes. We show that in this case, a certain configuration is a template for H.

Lemma 7.4. *Let L_1, L_2 be two copies of K_r sharing $r-1$ vertices with $\{x\} = L_1\setminus L_2$ and $\{y\} = L_2\setminus L_1$. Let c be a 2-edge-coloring of $L_1 \cup L_2$ with $c(L_1) = c(L_2)$ such that there exists a vertex $z \in L_1 \cap L_2$ with $c(xz) = -c(yz) = 1$. If H has an r-coloring $A_1, A_2, \dots, A_r$ and a vertex $a_1 \in A_1$ such that $d_H(a_1, A_2) \neq d_H(a_1, A_3)$, then $(L_1 \cup L_2, c)$ is a template for H.*

Proof. We have

$$0 = c(L_1) - c(L_2) = \sum_{w \in L_1 \cap L_2}(c(xw) - c(yw)).$$

Since $z \in L_1 \cap L_2$ satisfies $c(yz) = -c(xz) = 1$, there must exist $w \in (L_1 \cap L_2) \setminus \{z\}$ such that $c(yw) = -c(xw) = -1$. Note that this implies that $r \geq 3$. Write $L_1 \cap L_2 = \{z, w, v_4, \dots, v_r\}$. Let $A_1, A_2, \dots, A_r$ be an r-vertex-coloring of H such that there exists a vertex $a_1 \in A_1$ with $d_H(a_1, A_2) \neq d_H(a_1, A_3)$. Let B be a blowup of $(L_1 \cup L_2, c)$ with

$$|V_x| = 1,$$
$$|V_y| = 2|A_1| - 1,$$
$$|V_z| = |V_w| = |A_2| + |A_3|,$$
$$|V_{v_i}| = 2|A_i| \text{ for } 4 \leq i \leq r.$$

We now define certain H-copies in B. For a permutation $\sigma : \{4, \dots, r\} \to \{4, \dots, r\}$, let X_σ be a copy of H in which A_1 is embedded into V_y, A_2 is embedded into V_z, A_3 is embedded into V_w and $A_{\sigma(i)}$ is embedded into V_{v_i} for every $4 \le i \le r$. Let Y_σ be the H-copy obtained from X_σ by swapping A_2 and A_3, i.e. embedding A_2 into A_w and A_3 into A_z. Let X'_σ (resp. Y'_σ) be the H-copy obtained from X_σ (resp. Y_σ) by moving a_1 from V_y to V_x. We define all these H-copies such that the copies in each of the sets $F_1 := \{X_\sigma, Y'_\sigma\}_\sigma$ and $F_2 := \{Y_\sigma, X'_\sigma\}_\sigma$ are pairwise-disjoint and partition $V(B)$; note that the sizes of $V_x, V_y, V_z, V_w, V_{v_4}, \dots, V_{v_r}$ are precisely chosen to allow this. In other words, F_1, F_2 are perfect H-factors of B. We now calculate $c(F_1) - c(F_2)$. By definition,

$$c(X_\sigma)-c(X'_\sigma) = (c(yz)-c(xz))\cdot d_H(a_1,A_2)+(c(yw)-c(xw))\cdot d_H(a_1,A_3)+\sum_{i=4}^{r}(c(yv_i)-c(xv_i))\cdot d_H(a_1,A_{\sigma(i)}).$$

For every two indices $4 \le i,j \le r$, there are exactly $(r-4)!$ permutations σ with $\sigma(i) = j$. Hence, $\sum_\sigma d_H(a_1, A_{\sigma(i)}) = (r-4)! \cdot d_H(a_1, A_4 \cup \dots \cup A_r)$. We get that

$$\begin{aligned}\sum_\sigma (c(X_\sigma) - c(X'_\sigma)) &= (r-3)! \cdot (c(yz) - c(xz)) \cdot d_H(a_1, A_2) + (r-3)! \cdot (c(yw) - c(xw)) \cdot d_H(a_1, A_3) \\ &\quad + (r-4)! \cdot \sum_{i=4}^{r} (c(yv_i) - c(xv_i)) \cdot d_H(a_1, A_4 \cup \dots \cup A_r).\end{aligned}$$

By the same argument,

$$\begin{aligned}\sum_\sigma (c(Y_\sigma) - c(Y'_\sigma)) &= (r-3)! \cdot (c(yz) - c(xz)) \cdot d_H(a_1, A_3) + (r-3)! \cdot (c(yw) - c(xw)) \cdot d_H(a_1, A_2) \\ &\quad + (r-4)! \cdot \sum_{i=4}^{r} (c(yv_i) - c(xv_i)) \cdot d_H(a_1, A_4 \cup \dots \cup A_r).\end{aligned}$$

Hence,

$$\begin{aligned}c(F_1) - c(F_2) &= \sum_\sigma (c(X_\sigma) - c(X'_\sigma)) - \sum_\sigma (c(Y_\sigma) - c(Y'_\sigma)) \\ &= (r-3)! \cdot \big(d_H(a_1, A_2) - d_H(a_1, A_3)\big) \cdot \big(c(yz) - c(xz) - c(yw) + c(xw)\big) \neq 0,\end{aligned}$$

using that $d_H(a_1, A_2) \neq d_H(a_1, A_3)$ by assumption and that $c(yz) - c(xz) = 2$ and $c(yw) - c(xw) = -2$. As $c(F_1) \neq c(F_2)$, it follows that $(L_1 \cup L_2, c)$ is a template for H. □

7.4 Unbalanced r-colorings

In this section we consider graphs H having an unbalanced r-coloring (recall that an r-coloring $A_1, \dots, A_r$ is called balanced if $|A_1| = \dots = |A_r|$). We shall prove the following.

Lemma 7.5. *Suppose that $r \ge 4$. Let L_1, L_2 be two copies of K_r sharing $r-2$ vertices, and let c be a 2-edge-coloring of $L_1 \cup L_2$ such that L_1^+ is d-regular and L_2^+ is d'-regular for some $d \neq d'$. If H fulfills the r-wise C_4-condition and has an unbalanced r-coloring, then $(L_1 \cup L_2, c)$ is a template for H.*

Let us give an overview of the proof of Lemma 7.5. Assuming that H satisfies the r-wise C_4-condition allows us to control the discrepancy of H-factors via Lemma 4.2. Indeed, this lemma implies that the discrepancy of an H-factor in any blowup of K_r with a 2-edge-coloring for which K_r^+ is d-regular, is simply determined by d and the number of copies of H in the factor. We will make use of this observation by considering two different H-factors of a carefully chosen blowup of $L_1 \cup L_2$. Each H-copy in both H-factors will be contained in the blowup of L_1 or L_2, and the two H-factors will differ on the number of H-copies of each type (here we will use that H has an unbalanced r-coloring). This will guarantee that the two H-factors have different discrepancies. The detailed proof follows.

Proof of Lemma 7.5. Let $A_1, A_2, \dots, A_r$ be the parts of an unbalanced r-coloring of H with $|A_1| \le |A_2| \le \cdots \le |A_r|$ and $|A_1| < |A_r|$. Write $L_1 \cap L_2 = \{v_3, v_4, \dots, v_r\}$ and $\{x_i, y_i\} = L_i \setminus L_{3-i}$ for $i = 1, 2$. Let B be the blowup of $(L_1 \cup L_2, c)$ with

- $|V_{x_1}| = |V_{x_2}| = |V_{y_1}| = |V_{y_2}| = (|A_1| + |A_2|)(|A_{r-1}| + |A_r|)$ and
- for each $1 \le i \le r-2$, $|V_{v_i}| = 2\big(|A_1| + |A_2|\big)|A_{i+2}| + 2\big(|A_{r-1}| + |A_r|\big)|A_i|$.

Let F_1, F_2 be the following two perfect H-factors of B.

- F_1 contains $(|A_{r-1}| + |A_r|)$ copies of H with each permutation of A_1, A_2 on V_{x_1}, V_{y_1} and for each $3 \le i \le r$, A_i on V_{v_i}. Additionally, F_1 contains $(|A_1| + |A_2|)$ copies of H with each permutation of A_{r-1}, A_r on V_{x_2}, V_{y_2} and for each $1 \le i \le r-2$, A_i on $V_{v_{i+2}}$.
- F_2 contains $(|A_1| + |A_2|)$ copies of H with each permutation of A_{r-1}, A_r on V_{x_1}, V_{y_1} and for each $1 \le i \le r-2$, A_i on $V_{v_{i+2}}$. Additionally, F_2 contains $(|A_{r-1}| + |A_r|)$ copies of H with each permutation of A_1, A_2 on V_{x_2}, V_{y_2} and for each $3 \le i \le r$, A_i on V_{v_i}.

By definition, each copy of H in F is contained either in the blowup of L_1 or in the blowup of L_2. Let $F_{1|L_1}, F_{1|L_2}$ denote the copies of H in F_1 which are contained in the blowup of L_1, L_2, respectively, and define $F_{2|L_1}, F_{2|L_2}$ similarly. Since $F_{1|L_1}$ satisfies the r-wise C_4-condition, we can apply Lemma 4.2 to the blowup $B[V(F_{1|L_1})]$ of the r-clique (L_1, c). As L_1^+ is d-regular, Lemma 4.2 gives

$$c(F_{1|L_1}) = \frac{2d - r + 1}{r - 1} e(F_{1|L_1}).$$

Similarly, we get

$$\begin{aligned} c(F_{1|L_2}) &= \frac{2d' - r + 1}{r - 1} e(F_{1|L_2}), \\ c(F_{2|L_1}) &= \frac{2d - r + 1}{r - 1} e(F_{2|L_1}), \\ c(F_{2|L_2}) &= \frac{2d' - r + 1}{r - 1} e(F_{2|L_2}). \end{aligned}$$

Additionally, by the definition of F_1, F_2 we have that $e(F_{1|L_1}) = e(F_{2|L_2}) = 2(|A_{r-1}| + |A_r|)e(H)$ and $e(F_{1|L_2}) = e(F_{2|L_1}) = 2(|A_1| + |A_2|)e(H)$. We get that

$$c(F_1) - c(F_2) = c(F_{1|L_1}) - c(F_{2|L_2}) + c(F_{1|L_2}) - c(F_{2|L_1}) = \frac{4(d - d')}{r - 1}(|A_{r-1}| + |A_r| - |A_1| - |A_2|)e(H) \ne 0,$$

using that $d \ne d'$ and that $|A_{r-1}| + |A_r| - |A_1| - |A_2| > 0$. so we see that $c(F_1) - c(F_2) \ne 0$, implying that $(L_1 \cup L_2, c)$ is a template for H. □

7.5 Non-uniform r-colorings

An r-coloring of H is called *uniform* if the number of edges between any two color-classes is the same. We will say that H is uniform if it only has uniform colorings:

Definition 7.6. *A graph H is called* uniform *if for every r-coloring of H with parts $A_1, A_2, ..., A_r$, it holds that $e_H(A_i, A_j) = e(H)/\binom{r}{2}$ for all $1 \le i < j \le r$.*

The following lemma gives a template for non-uniform graphs H under some additional conditions.

Lemma 7.7. *Let L_1, L_2 be two copies of K_r sharing $r-2$ or $r-1$ vertices, let $V \subseteq L_1 \cap L_2$ of size $r-2$ and let $e_i := L_i \setminus V$. Let c be a 2-edge-coloring of $L_1 \cup L_2$ such that*

$$c(L_1) - c(e_1) - c(L_2) + c(e_2) \notin \{-4(r-2), -2(r-2), 0, 2(r-2), 4(r-2)\}.$$

If H fulfills the r-wise C_4-condition and is non-uniform, then $(L_1 \cup L_2, c)$ is a template for H.

Before proving Lemma 7.7, let us note some simple facts. Clearly, a non-uniform coloring must have at least three parts, so $r \geq 3$. We also need the following easy claim.

Claim 7.8. *if H is non-uniform, then there exists an r-coloring of H with parts $A_1, A_2 \ldots, A_r$ such that $e_H(A_1, A_2) \neq e_H(A_1, A_3)$.*

Proof. Fix any non-uniform r-coloring $B_1, \ldots, B_r$ of H, and let $1 \leq i < j \leq r$ and $1 \leq k < \ell \leq r$ such that $e_H(B_i, B_j) \neq e_H(B_k, B_\ell)$. Then $i = k$ or $e_H(B_i, B_j) \neq e_H(B_i, B_k)$ or $e_H(B_i, B_k) \neq e_H(B_k, B_\ell)$, and by renaming the parts we get the desired r-coloring $A_1, \ldots, A_r$ with $e_H(A_1, A_2) \neq e_H(A_1, A_3)$. □

Proof of Lemma 7.7. Write $e_i = x_i y_i$, $i = 1, 2$. Note that $\{x_1, y_1\}$ may intersect $\{x_2, y_2\}$ (namely, if $|L_1 \cap L_2| = r - 1$). Without loss of generality, let us assume that if these sets intersect then $x_1 = x_2$, so that $y_1 \neq y_2$ always holds. Let S be the bipartite graph between $\{x_1, y_1\}$ and V (so $S \subseteq L_1$), and let T be the bipartite graph between $\{x_2, y_2\}$ and V (so $T \subseteq L_2$). Observe that

$$c(S) - c(T) = c(L_1) - c(e_1) - c(L_2) + c(e_2),$$

so

$$c(S) - c(T) \notin \{-4(r-2), -2(r-2), 0, 2(r-2), 4(r-2)\}. \tag{7}$$

by assumption. Also, as S and T contain $2(r-2)$ edges each, we have $|c(S) - c(T)| \leq 4(r-2)$. Thus, (7) implies that $c(S) - c(T)$ is not a multiple of $2(r-2)$. For $v \in V$, define $g(v) := c(vx_1) + c(vy_1) - c(vx_2) - c(vy_2)$. We claim that there are $u, v \in V$ with $g(u) \neq g(v)$. Indeed, suppose otherwise. Then there exists $g \in \mathbb{N}$ such that each vertex $v \in V$ has $g(v) = g$. Then

$$c(S) - c(T) = \sum_{v \in V} g(v) = g(r-2).$$

Note also that g is even since $g(v)$ is an even number for every $v \in V$. But now we see that $c(S) - c(T)$ is a multiple of $2(r-2)$, a contradiction. We conclude that there exist vertices $u, v \in V$ with $g(u) \neq g(v)$. Note that this also implies that $r \geq 4$. Write $V = \{u, v, v_5, v_6, \ldots, v_r\}$. By Claim 7.8, there is an r-coloring $A_1, A_2, \ldots, A_r$ of H with $e_H(A_1, A_2) \neq e_H(A_1, A_3)$. Let B be a blowup of $(L_1 \cup L_2, c)$ with

- $|V_{x_1}| = |V_{x_2}| = |A_1|$,
- $|V_{y_1}| = |V_{y_2}| = |A_4|$,
- $|V_u| = |V_v| = |A_2| + |A_3|$,
- $|V_{v_i}| = 2|A_i|$ for every $5 \leq i \leq r$.

Here, V_{x_1}, V_{x_2} are distinct parts even if $x_1 = x_2$. Note that B is a blowup of H also in the case $x_1 = x_2$, as then $V_{x_1} \cup V_{x_2}$ is the part corresponding to the vertex $x_1 = x_2$.

We now define four H-copies in B, denoted $F_{1,1}, F_{1,2}, F_{2,1}, F_{2,2}$. All four copies embed A_i into V_{v_i} for every $5 \leq i \leq r$. Also, $F_{1,1}$ and $F_{2,1}$ embed A_1 into V_{x_1} and A_4 into V_{y_1}, while $F_{1,2}$ and $F_{2,2}$ embed A_1 into V_{x_2} and A_4 into V_{y_2}. Finally, $F_{1,1}$ and $F_{2,2}$ embed A_2 into V_u and A_3 into V_v, while $F_{1,2}$ and $F_{2,1}$ embed A_2 into V_v and A_3 into V_u. We define these copies such that $F_i := \{F_{i,1}, F_{i,2}\}$ forms a perfect H-factor of B for every $i = 1, 2$; this is possible due to our choice of the cluster sizes of B.

We now show that $c(F_1) - c(F_2) \neq 0$. First observe that for an edge $zw \in E(L_1 \cup L_2)$, if $zw \notin \{x_1, y_1, x_2, y_2\} \times \{u, v\}$ then $e_{F_1}(V_z, V_w) = e_{F_2}(V_z, V_w)$. Indeed, if $z, w \in \{v_5, \dots, v_r\}$ then this is immediate. If $z \in \{x_1, y_1, x_2, y_2\}$ and $w \in \{v_5, \dots, v_r\}$ then $e_{F_{1,1}}(V_z, V_w) = e_{F_{2,1}}(V_z, V_w)$ and $e_{F_{1,2}}(V_z, V_w) = e_{F_{2,2}}(V_z, V_w)$, so the assertion holds. If $z \in \{u, v\}$ and $w \in \{v_5, \dots, v_r\}$ then $e_{F_{1,1}}(V_z, V_w) = e_{F_{2,2}}(V_z, V_w)$ and $e_{F_{1,2}}(V_z, V_w) = e_{F_{2,1}}(V_z, V_w)$, so again the assertion holds. The case $\{z, w\} = \{u, v\}$ is also immediate. We see that the bipartite graph (V_z, V_w) does not contribute to $c(F_1) - c(F_2)$ unless $zw \in \{x_1, y_1, x_2, y_2\} \times \{u, v\}$. By definition, we have

$$\begin{aligned} c(F_1[V_{x_1} \cup V_{x_2} \cup V_{y_1} \cup V_{y_2}, V_u \cup V_v]) =& (c(ux_1) + c(vx_2)) \cdot e_H(A_1, A_2) + (c(uy_1) + c(vy_2)) \cdot e_H(A_4, A_2) \\ &+ (c(vx_1) + c(ux_2)) \cdot e_H(A_1, A_3) + (c(vy_1) + c(uy_2)) \cdot e_H(A_4, A_3), \end{aligned}$$

and similarly,

$$\begin{aligned} c(F_2[V_{x_1} \cup V_{x_2} \cup V_{y_1} \cup V_{y_2}, V_u \cup V_v]) =& (c(vx_1) + c(ux_2)) \cdot e_H(A_1, A_2) + (c(vy_1) + c(uy_2)) \cdot e_H(A_4, A_2) \\ &+ (c(ux_1) + c(vx_2)) \cdot e_H(A_1, A_3) + (c(uy_1) + c(vy_2)) \cdot e_H(A_4, A_3). \end{aligned}$$

It follows that

$$\begin{aligned} c(F_1) - c(F_2) =& (c(ux_1) + c(vx_2) - c(vx_1) - c(ux_2)) \cdot (e_H(A_1, A_2) - e_H(A_1, A_3)) \\ &+ (c(uy_1) + c(vy_2) - c(vy_1) - c(uy_2)) \cdot (e_H(A_4, A_2) - e_H(A_4, A_3)). \end{aligned}$$

By the C_4-condition, we also have that

$$e_H(A_1, A_2) + e_H(A_4, A_3) - e_H(A_1, A_3) - e_H(A_4, A_2) = 0.$$

Adding this multiplied by $(c(uy_1) + c(vy_2) - c(vy_1) - c(uy_2))$ to the previous equation, we get

$$\begin{aligned} &c(F_1) - c(F_2) = \\ &\quad (c(ux_1) + c(vx_2) - c(vx_1) - c(ux_2) + c(uy_1) + c(vy_2) - c(vy_1) - c(uy_2)) \cdot (e_H(A_1, A_2) - e_H(A_1, A_3)) \\ &\quad = (g(u) - g(v)) \cdot (e_H(A_1, A_2) - e_H(A_1, A_3)) \neq 0, \end{aligned}$$

using that $g(u) \neq g(v)$ and that $e_H(A_1, A_2) \neq e_H(A_1, A_3)$ by assumption. So $c(F_1) \neq c(F_2)$ and hence $(L_1 \cup L_2, c)$ is a template for H. □

7.6 Graphs violating the C_4-condition

Here we show that if H violates the k-wise C_4-condition and c is a coloring of K_k, then (K_k, c) is a template for H unless c has some very specific structure. This is given by the following definition and lemma.

Definition 7.9. *For some $k \geq 1$, $K_{k,+}$ is a copy of (K_k, c) where c is a 2-edge-coloring of K_k with all edges of color 1. The* $(K_k, +)$-star *is the copy of (K_k, c) where c is a 2-edge-coloring of K_k such that the edges of color 1 induce a star with $k - 1$ leaves. We call the root of this star the* head *of the $(K_k, +)$-star. Define $K_{k,-}$ and the $(K_k, -)$-star analogously.*

Lemma 7.10. *Let $k \geq 4$ and let c be a 2-edge-coloring of K_k such that (K_k, c) is neither monochromatic nor a star. If H does not fulfill the k-wise C_4-condition, then (K_k, c) is a template for H.*

Proof. We need the following claim, which appears implicitly in [4, proof of Claim 6.4]. For completeness, we give a proof.

Claim 7.11. *There exist vertices $a_1, a_2, a_3, a_4 \in V(K_k)$ such that*

$$c(a_1a_2) + c(a_3a_4) \neq c(a_1a_3) + c(a_2a_4).$$

Proof. Let us assume towards a contradiction that for every $a_1, a_2, a_3, a_4 \in V(K_k)$ it holds that

$$c(a_1a_2) + c(a_3a_4) = c(a_1a_3) + c(a_2a_4). \tag{8}$$

Fix an arbitrary vertex $a \in V(K_k)$. If $d_{K_k^+}(a) \geq 3$, let $b, c, d \in N_{K_k^+}(a)$ be three arbitrary vertices. By (8), it follows that $c(bc) = c(cd)$. As this holds for arbitrary $b, c, d \in N_{K_k^+}(a)$, we get that either all the edges in $K_k[N_{K_k^+}(a)]$ have color 1 or they all have color -1. Therefore, $N_{K_k^-}(a)$ is not empty as otherwise, K_k colored by c is monochromatic or a star. By symmetry, if $d_{K_k^-}(a) \geq 3$ then $N_{K_k^+}(a)$ is non-empty.

We have $d_{K_k^+}(a) + d_{K_k^-}(a) = k - 1 \geq 3$. Without loss of generality, let us assume that $d_{K_k^+}(a) \geq 2$. Let $b, c \in N_{K_k^+}(a)$ and $d \in N_{K_k^-}(a)$. By (8), we get that $c(bc) = 1$ and $c(bd) = c(cd) - 1$. Since this holds for arbitrary $b, c \in N_{K_k^+}(a)$ and $d \in N_{K_k^-}(a)$, we get that all the edges in $K_k[N_{K_k^+}(a)]$ have color 1 and all the edges in $K_k[N_{K_k^+}(a), N_{K_k^-}(a)]$ have color -1. Thus, if $d_{K_k^-}(a) = 1$ then K_k colored by c is a copy of a $(K_k, -)$-star (whose head is the unique vertex in $N_{K_k^-}(a)$). So $d_{K_k^-}(a) \geq 2$. Now, by a symmetrical argument to the above, we get that all the edges in $K_k[N_{K_k^+}(a), N_{K_k^-}(a)]$ have color 1, which is a contradiction. □

Let $V(K_k) = \{a_1, a_2, \ldots, a_k\}$ with a_1, a_2, a_3, a_4 as in the statement of the above claim. As H violates the k-wise C_4-condition, there exists a k-coloring $A_1, A_2, \ldots, A_k$ of H with

$$e_H(A_1A_2) + e_H(A_3A_4) \neq e_H(A_1A_3) + e_H(A_2A_4).$$

Let B be a blowup of (K_k, c) with

- $|V_{a_1}| = |V_{a_4}| = |A_1| + |A_4|$,
- $|V_{a_2}| = |V_{a_3}| = |A_2| + |A_3|$, and
- $|V_{a_i}| = 2|A_i|$ for $5 \leq i \leq k$.

We now define two perfect H-factors F_1 and F_2 of H, each consisting of two copies of H. Each H-copy in F_1 and F_2 embeds A_i into V_{a_i} for every $5 \leq i \leq k$.

- F_1 contains a copy of H with A_1 on V_{a_1}, A_2 on V_{a_2}, A_3 on V_{a_3} and A_4 on V_{a_4} and a second copy of H with A_1 on V_{a_4}, A_2 on V_{a_3}, A_3 on V_{a_2} and A_4 on V_{a_1}.
- F_2 contains a copy of H with A_1 on V_{a_1}, A_2 on V_{a_3}, A_3 on V_{a_2} and A_4 on V_{a_4} and a second copy of H with A_1 on V_{a_4}, A_2 on V_{a_2}, A_3 on V_{a_3} and A_4 on V_{a_1}.

We now calculate $c(F_1) - c(F_2)$. Note that for each $5 \leq i < j \leq k$, we have that

$$e_{F_1}(V_{a_i}, V_{a_j}) = 2e_H(A_i, A_j) = e_{F_2}(V_{a_i}, V_{a_j}).$$

For $5 \leq i \leq k$ and $j \in \{1, 4\}$, we get

$$e_{F_1}(V_{a_i}, V_{a_j}) = e_H(A_i, A_1) + e_H(A_i, A_4) = e_{F_2}(V_{a_i}, V_{a_j}),$$

and for $j \in \{2, 3\}$

$$e_{F_1}(V_{a_i}, V_{a_j}) = e_H(A_i, A_2) + e_H(A_i, A_3) = e_{F_2}(V_{a_i}, V_{a_j}).$$

Additionally, we have

$$e_{F_1}(V_{a_1}, V_{a_4}) = 2e_H(A_1, A_4) = e_{F_2}(V_{a_1}, V_{a_4}),$$
$$e_{F_1}(V_{a_2}, V_{a_3}) = 2e_H(A_2, A_3) = e_{F_2}(V_{a_2}, V_{a_3}),$$
$$e_{F_1}(V_{a_1}, V_{a_2}) = e_{F_1}(V_{a_3}, V_{a_4}) = e_{F_2}(V_{a_1}, V_{a_3}) = e_{F_2}(V_{a_2}, V_{a_4}) = e_H(A_1, A_2) + e_H(A_3, A_4),$$

and

$$e_{F_2}(V_{a_1}, V_{a_2}) = e_{F_2}(V_{a_3}, V_{a_4}) = e_{F_1}(V_{a_1}, V_{a_3}) = e_{F_1}(V_{a_2}, V_{a_4}) = e_H(A_1, A_3) + e_H(A_2, A_4).$$

Combining all of the above, we get that

$$c(F_1) - c(F_2) = \sum_{i\in\{1,4\}} \sum_{j\in\{2,3\}} c(a_i a_j) \cdot (e_{F_1}(V_{a_i}, V_{a_j}) - e_{F_2}(V_{a_i}, V_{a_j})) =$$
$$(c(a_1a_2) + c(a_3a_4) - c(a_1a_3) - c(a_2a_4)) \cdot (e_H(A_1, A_2) + e_H(A_3, A_4) - e_H(A_1, A_3) - e_H(A_2, A_4)) \neq 0.$$

Therefore, (K_k, c) is a template for H. □

7.7 Balanced H-tilings with non-uniform edge distribution

In the previous section, we saw that if H violates the r-wise C_4-condition, then (K_r, c) is a template for H for every coloring c of K_r except for some very special cases. In this section we continue investigating when a given coloring of K_r is a template for H, this time assuming that H does satisfy the r-wise C_4-condition. We shall see that if c is a coloring of K_r such that K_r^+ is not regular, then (K_r, c) is a template for H unless every H-factor, i.e. disjoint union of copies of H, has a certain uniformity property. The precise statement is given by the following definition and lemma. Recall that by "H-factor" we simply mean a graph consisting of vertex-disjoint copies of H.

Definition 7.12. *We say that an r-chromatic graph F is* balanced-uniform *if every balanced r-coloring of F with parts $A_1, A_2, ..., A_r$ satisfies that $e_F(A_i, A_j) = e(F)/\binom{r}{2}$ for all $1 \leq i < j \leq r$.*

Lemma 7.13. *Suppose that $r \geq 4$ and $r \neq 5$, and assume that H satisfies the r-wise C_4-condition. Let c be a 2-edge-coloring of K_r such that K_r^+ is non-regular. If there exists a non-balanced-uniform H-factor, then (K_r, c) is a template for H.*

The condition that K_r^+ is non-regular is necessary; indeed, it follows from Lemma 4.2 that (K_r, c) is not a template for H if K_r^+ is regular and H fulfills the r-wise C_4-condition. Also, the assumption that there exists a non-balanced-uniform H-factor is necessary for the proof method. Indeed, in the proof of Lemma 7.13 we show that the *balanced* blowup B of (K_r, c) has two H-factors with different discrepancies, (this showing that (K_r, c) is a template). However, if H-factor were balanced-regular, then every perfect H-factor F of B would have exactly $e(F)/\binom{r}{2}$ edges between any two parts of B (as B is balanced), and so $c(F) = c(K_r) \cdot e(F)/\binom{r}{2}$, meaning that every perfect H-factor of B would have the same discrepancy.

For the proof of Lemma 7.13 we need the following simple claim.

Claim 7.14. *Suppose that $r \geq 4$ and H satisfies the r-wise C_4-condition. Let F be a non-balanced-uniform H-factor. Then there exists a balanced r-coloring $A_1, \dots, A_r$ of F such that $e_F(A_1, A_2) \neq e_F(A_3, A_4)$.*

Proof. By definition, there is a balanced r-coloring $A_1, \dots, A_r$ of F such that $e_F(A_i, A_j)$ are not all equal. By renaming the parts, we may assume that $e_F(A_1, A_2) > e(F)/\binom{r}{2}$. Then there are $1 \leq i < j \leq r$ such that $e_F(A_i, A_j) < e(F)/\binom{r}{2}$. Now, if $i, j \notin \{1, 2\}$ then we are done. Else, suppose without loss of generality that $i = 1$. Let $k \in [r] \setminus \{1, 2, j\}$. We may assume that $e_F(A_1, A_2) = e_F(A_j, A_k)$ and $e_F(A_1, A_j) = e_F(A_2, A_k)$, as otherwise we are done. Now we get that

$$e_F(A_1, A_2) + e_F(A_j, A_k) - e_F(A_1, A_j) - e_F(A_2, A_k) = 2e_F(A_1, A_2) - 2e_F(A_i, A_j) > 0,$$

contradicting the assumption that H fulfills the r-wise C_4-condition. □

Proof of Lemma 7.13. Let c be as in the statement. For a vertex $u \in V(K_r)$, denote

$$c(u, K_r) := \sum_{v\in V(K_r)\setminus\{u\}} c(uv).$$

Let $a, b \in V(K_r)$ be the two vertices maximizing $c(a, K_r)+c(b, K_r)$ and $d, e \in V(K_r)$ the vertices minimizing $c(d, K_r) + c(e, K_r)$. In other words, a, b are the two vertices with highest degree in K_r^+, and d, e are the two vertices with lowest degree. As $r \geq 4$, we may assume that a, b, d, e are distinct. Since K_r^+ is non-regular,

$$c(a, K_r) + c(b, K_r) > c(d, K_r) + c(e, K_r). \tag{9}$$

Let F be a non-balanced-uniform H-factor. By Claim 7.14, there is a balanced r-coloring $A_1, A_2, \dots, A_r$ of F such that $e_F(A_1, A_2) \neq e_F(A_3, A_4)$. Let B be a $4(r-4)!\frac{|F|}{r}$-blowup of (K_r, c). Let F_1 and F_2 be the following two perfect F-factors of B.

- F_1 contains each copy of F with each permutation of A_1, A_2 on V_a, V_b, each permutation of A_3, A_4 on V_d, V_e, and each permutation of $A_5, A_6, ..., A_r$ on the remaining clusters of B.
- F_2 contains each copy of F with each permutation of A_1, A_2 on V_d, V_e, each permutation of A_3, A_4 on V_a, V_b, and each permutation of $A_5, A_6, ..., A_r$ on the remaining clusters of B.

Note that F_1, F_2 each have $4(r-4)!$ copies of F. As $|A_1| = \cdots = |A_r| = \frac{|F|}{r}$, the size of B exactly matches these F-factors. Clearly, every F-factor is also an H-factor, as F is a union of disjoint copies of H. Hence F_1, F_2 are perfect H-factors of B. We now show that $c(F_1) - c(F_2) \neq 0$. Put $X = V(K_r)\backslash\{a, b, d, e\}$. By symmetry, we have for all $v \in X$:

- $e_{F_1}(V_a, V_v) = e_{F_1}(V_b, V_v) = e_{F_2}(V_d, V_v) = e_{F_2}(V_e, V_v) =: y_1$,
- $e_{F_2}(V_a, V_v) = e_{F_2}(V_b, V_v) = e_{F_1}(V_d, V_v) = e_{F_1}(V_e, V_v) =: y_2$.

If $r = 4$ then X is empty, and we set $y_1 = y_2 = 0$. We also have

$$e_{F_1}(V_u, V_v) = e_{F_2}(V_u, V_v)$$

for all $u, v \in X$ and all $u \in \{a, b\}, v \in \{d, e\}$. Finally, observe that

$$e_{F_1}(V_a, V_b) = e_{F_2}(V_d, V_e) = 4(r-4)! \cdot e_F(A_1, A_2), \quad e_{F_1}(V_d, V_e) = e_{F_2}(V_a, V_b) = 4(r-4)! \cdot e_F(A_3, A_4).$$

Therefore, $e_{F_1}(V_a, V_b) \neq e_{F_1}(V_d, V_e)$ holds by our choice of A_1, A_2, A_3, A_4. For convenience, let us denote

$$h(u, v) := c(uv) \cdot (e_{F_1}(V_u, V_v) - e_{F_2}(V_u, V_v))$$

for $uv \in E(K_r)$. From the above, we see that $h(u, v)$ is non-zero only if $uv \in \{ab, cd\}$ or $v \in X, u \in \{a, b, c, d\}$. Also, in the latter case we have $h(u, v) = y_1 - y_2$ if $u \in \{a, b\}$ and $h(u, v) = y_2 - y_1$ if $u \in \{c, d\}$. Finally, $h(a, b) = -h(d, e) = (c(ab) - c(de)) \cdot (e_{F_1}(V_a, V_b) - e_{F_1}(V_d, V_e))$. Using these equations, we get

$$\begin{aligned} &c(F_1) - c(F_2) = \sum_{uv \in E(K_r)} h(u, v) = \\ &(c(ab) - c(de)) \cdot (e_{F_1}(V_a, V_b) - e_{F_1}(V_d, V_e)) + (y_1 - y_2) \sum_{v \in X} (c(av) + c(bv) - c(dv) - c(ev)) \end{aligned} \tag{10}$$

Observe that

$$\sum_{v \in X} (c(av) + c(bv) - c(dv) - c(ev)) = c(a, K_r) + c(b, K_r) - 2c(ab) - c(d, K_r) - c(e, K_r) + 2c(de).$$

Plugging this into (10) and rearranging, we get

$$\begin{aligned} c(F_1) - c(F_2) = (c(ab) - c(de)) \cdot (e_{F_1}(V_a, V_b) - e_{F_1}(V_d, V_e) + 2y_2 - 2y_1) \\ + (y_1 - y_2) \cdot (c(a, K_r) + c(b, K_r) - c(d, K_r) - c(e, K_r)). \end{aligned} \tag{11}$$

Suppose first that $r = 4$. Then $y_1 = y_2 = 0$. Also, $c(ab) \neq c(de)$, because otherwise we would have $c(a, K_r) + c(b, K_r) = c(K_r) = c(d, K_r) + c(e, K_r)$, contradicting (9). Now, using (11), we have

$$c(F_1) - c(F_2) = (c(ab) - c(de)) \cdot (e_{F_1}(V_a, V_b) - e_{F_1}(V_d, V_e)) \neq 0,$$

as required.

As $r \neq 5$, we may assume from now on that $r \geq 6$. Since H satisfies the r-wise C_4-condition, so do F_1, F_2 (as F_1, F_2 are H-factors). Fix arbitrary $u, v \in X$. By the C_4-condition, we have

$$e_{F_1}(V_a, V_b) + e_{F_1}(V_u, V_v) - e_{F_1}(V_a, V_u) - e_{F_1}(V_b, V_v) = 0$$

and

$$-e_{F_1}(V_d, V_e) - e_{F_1}(V_u, V_v) + e_{F_1}(V_d, V_u) + e_{F_1}(V_e, V_v) = 0.$$

Recall that $e_{F_1}(V_a, V_u) = e_{F_1}(V_b, V_v) = y_1$ and $e_{F_1}(V_d, V_u) = e_{F_1}(V_e, V_v) = y_2$. Adding the two above equations, we get

$$0 = e_{F_1}(V_a, V_b) + e_{F_1}(V_u, V_v) - 2y_1 - e_{F_1}(V_d, V_e) - e_{F_1}(V_u, V_v) + 2y_2 = e_{F_1}(V_a, V_b) - 2y_1 - e_{F_1}(V_d, V_e) + 2y_2.$$

As $e_{F_1}(V_a, V_b) \neq e_{F_1}(V_d, V_e)$, we have $y_1 \neq y_2$. Finally, plugging this into (11), we get

$$c(F_1) - c(F_2) = (y_1 - y_2) \cdot (c(a, K_r) + c(b, K_r) - c(d, K_r) - c(e, K_r)) \neq 0,$$

using (9). Therefore, (K_r, c) is a template for H. □

We end this section with the following lemma, showing that if all H-factors are balanced-uniform, then no coloring of K_r is a template for H.

Lemma 7.15. *If every H-factor is balanced-uniform, then for every 2-edge-coloring c of K_r it holds that $c \in \mathcal{K}(H)$.*

Proof. Assume towards a contradiction that there exists a 2-edge-coloring c of K_r such that (K_r, c) is a template for H. Then, by definition, there exists a blowup B of (K_r, c) and two perfect H-factors F_1, F_2 of B such that $c(F_1) \neq c(F_2)$. Let $v_1, v_2, \dots, v_r$ be the vertices of K_r and let B_i be the part of B corresponding to v_i. Take a balanced blowup B' of (K_r, c) with parts $B'_1, \dots, B'_r$ of size $|B| = |B_1| + \dots + |B_r|$ each. For $0 \leq j \leq r-1$, let J_j be a copy of B in B' in which B_i is mapped to $B'_{i+j \pmod r}$ for each $i = 1, \dots, r$. Choose $J_1, \dots, J_r$ to be vertex-disjoint and to partition $V(B')$; this is possible as each part of B' has size $|B|$.

For each $1 \leq j \leq r$, let F_1^j, F_2^j be the H-factors playing the roles of F_1, F_2, respectively, in the copy J_j of B. Note that in J_1, B_i is mapped to B'_i for every $1 \leq i \leq r$. This means that $J \subseteq B'$ is colored the same way as B (i.e., J is a blowup of (K_r, c)). Therefore, $c(F_i^1) = c(F_i)$ for $i = 1, 2$ (with a slight abuse of notation, we denote by c both the coloring of B and that of B'). Let $F'_1 := \bigcup_{j=1}^r F_1^j$, and let F'_2 be obtained from F'_1 by replacing F_1^1 with F_2^1, i.e. $F'_2 := F_2^1 \cup \bigcup_{j=2}^r F_1^j$. Both F'_1, F'_2 are perfect H-factors of B'. Also,

$$c(F'_1) - c(F'_2) = c(F_1) - c(F_2) \neq 0.$$

Note, however, that F'_1, F'_2 are both H-factors with a balanced r-coloring $B'_1, \dots, B'_r$. As F'_1, F'_2 are balanced-uniform by assumption, it follows that $e_{F'_1}(B'_i, B'_j) = e_{F'_2}(B'_i, B'_j) = e(F'_1)/\binom{r}{2}$ for all $1 \leq i < j \leq r$. However, this implies that

$$c(F'_1) = c(F'_2) = c(K_r) \cdot \frac{e(F'_1)}{\binom{r}{2}},$$

a contradiction. □

Remark 7.16. *Given H, it is possible to computationally check whether there exists a non-balanced-uniform H-factor. Like in the proof of Proposition 5.1, for every r-coloring f of H, one defines vectors x_f and y_f corresponding to the sizes of the color classes and the number of edges between color classes, respectively. Then, the task reduces to a certain linear-algebraic statement regarding these vectors.*

7.8 (s,t)-structured graphs

The templates considered in this section consist of two r-cliques L_1, L_2 sharing $r-2$ vertices. We will describe the structure of graphs H for which a particular coloring of $L_1 \cup L_2$ is not a template. This structure is more involved than in previous sections. The precise definition is as follows.

Definition 7.17. *For $s, t, \rho \in \mathbb{R}$, we say that H is* (s,t)-structured *with parameter ρ if for every r-vertex-coloring of H with parts $A_1, \dots, A_r$ and for all $1 \le i < j \le r$, it holds that*

$$\rho(|A_i| + |A_j|) = s \cdot e_H(A_i \cup A_j, V(H) \setminus (A_i \cup A_j)) + t \cdot e_H(A_i, A_j). \tag{12}$$

We say that (s,t)-structured to mean that it is (s,t)-structured with some parameter ρ. Note that if H is (s,t)-structured then it is also $(\alpha \cdot s, \alpha \cdot t)$-structured for every $\alpha \in \mathbb{R}$. Another important fact is that if H is (s,t)-structured with parameter ρ, then so is every H-factor.

The following lemma is the main result of this subsection.

Lemma 7.18. *Let L_1, L_2 be two copies of K_r sharing $r-2$ vertices, and let $e_1 = x_1y_1 = L_1 \setminus L_2$ and $e_2 = x_2y_2 = L_2 \setminus L_1$. Let c be a 2-edge-coloring of $L_1 \cup L_2$. Then, either $(L_1 \cup L_2, c)$ is a template for H or H is (s,t)-structued for $s = \frac{c(L_1)-c(e_1)-c(L_2)+c(e_2)}{2(r-2)}$ and $t = c(e_1) - c(e_2)$.*

Let us give an overview of the proof of Lemma 7.18. We shall take a blowup of $(L_1 \cup L_2, c)$ with $|V_{x_1}| = |V_{y_1}| = |V_{x_2}| = |V_{y_2}|$ and consider a (carefully chosen) perfect H-factor F of B such that each copy of H in F_1 is either on the clusters of L_1 or the clusters of L_2. Since we have chosen the clusters of x_1, y_1, x_2, y_2 to have the same size, we can then find a second H-factor F_2 by swapping the vertices in $V_{x_1} \cup V_{y_1}$ with the vertices in $V_{x_2} \cup V_{y_2}$ in each of the copies of H in F_1. We will show that the only case in which the discrepancy does not change under this operation is that H is (s,t)-structured (with s,t as in the statement of the lemma). We now proceed with the detailed proof.

Proof of Lemma 7.18. For convenience, set

$$q := c(L_1) - c(e_1) - c(L_2) + c(e_2) = \sum_{i=3}^{r} [c(x_1v_i) + c(y_1v_i) - c(x_2v_i) - c(y_2v_i)].$$

Let $A_1, \dots, A_r$ be an r-coloring of H. To ease the notation, put $f_H(A_1, A_2) := e_H(A_1 \cup A_2, V(H) \setminus (A_1 \cup A_2))$ and

$$g(A_1, \dots, A_r) := \frac{\frac{q}{2(r-2)} \cdot f_H(A_1, A_2) + (c(e_1) - c(e_2)) \cdot e_H(A_1, A_2)}{|A_1| + |A_2|}.$$

Our goal is to show that if $(L_1 \cup L_2, c)$ is not a template for H, then $g(A_1, \dots, A_r)$ is the same for every r-coloring $A_1, \dots, A_r$ of H; we then take this common value to be ρ (see Definition 7.17). So let $A_1, A_2, \dots, A_r$ and $B_1, B_2, \dots, B_r$ be two arbitrary r-vertex-colorings of H. We will show that $g(A_1, \dots, A_r) = g(B_1, \dots, B_r)$. Write $V := L_1 \cap L_2 = \{v_3, \dots, v_r\}$. Consider the following blowup B of $(L_1 \cup L_2, c)$:

- $|V_{x_1}|, |V_{y_1}|, |V_{x_2}|, |V_{y_2}| = (r-2)!(|A_1| + |A_2|)(|B_1| + |B_2|)$,
- for $3 \le i \le r$, $|V_{v_i}| = 2(r-3)! \sum_{3 \le i \le r} [(|B_1| + |B_2|)|A_i| + (|A_1| + |A_2|)|B_i|]$.

To calculate $c(F_1) - c(F_2)$, first observe that for every pair $3 \le i < j \le r$,

$$e_{F_1}(V_{v_i}, V_{v_j}) = e_{F_2}(V_{v_i}, V_{v_j}) = 4(r-4)! \cdot \left((|B_1| + |B_2|) \sum_{3 \le i < j \le r} e_H(A_i, A_j) + (|A_1| + |A_2|) \sum_{3 \le i < j \le r} e_H(B_i, B_j) \right).$$

Therefore, $e_{F_1}(V_{v_i}, V_{v_j}) - e_{F_2}(V_{v_i}, V_{v_j}) = 0$. Next, we claim that for each $3 \le i \le r$, it holds that

$$e_{F_1}(V_{x_1}, V_{v_i}) = e_{F_1}(V_{y_1}, V_{v_i}) = e_{F_2}(V_{x_2}, V_{v_i}) = e_{F_2}(V_{y_2}, V_{v_i}) = (r-3)! \cdot (|B_1| + |B_2|) \cdot f_H(A_1, A_2) \tag{13}$$

and

$$e_{F_1}(V_{x_2},V_{v_i}) = e_{F_1}(V_{y_2},V_{v_i}) = e_{F_2}(V_{x_1},V_{v_i}) = e_{F_2}(V_{y_1},V_{v_i}) = (r-3)! \cdot (|A_1|+|A_2|) \cdot f_H(B_1,B_2). \tag{14}$$

Let us prove this for $e_{F_1}(V_{x_1},V_{v_i})$; all other cases are similar. For every $3 \leq j \leq r$, there are $(r-3)!$ permutations that embed A_j into V_{v_i} and A_1 (resp. A_2) into V_{x_1}, and each such permutation contributes $e_H(A_1,A_j)$ (resp. $e_H(A_2,A_j)$) to $e_{F_1}(V_{x_1},V_{v_i})$. Also, each such permutation gives rise to $(|B_1|+|B_2|)$ copies of H in F_1. Summing over all $3 \leq j \leq r$ and all permutations, we get

$$e_{F_1}(V_{x_1},V_{v_i}) = (r-3)! \cdot (|B_1|+|B_2|) \cdot \sum_{k=1}^{2}\sum_{j=3}^{r} e_H(A_k,A_j) = (r-3)! \cdot (|B_1|+|B_2|) \cdot f_H(A_1,A_2),$$

as required.

Lastly, from the definition of F_1, F_2 it follows that

$$\begin{aligned} e_{F_1}(V_{x_1},V_{y_1}) - e_{F_2}(V_{x_1},V_{y_1}) &= -(e_{F_1}(V_{x_2},V_{y_2}) - e_{F_2}(V_{x_2},V_{y_2})) \\ &= 2(r-2)! \cdot [(|B_1|+|B_2|) \cdot e_H(A_1,A_2) - (|A_1|+|A_2|) \cdot e_H(B_1,B_2)]. \end{aligned} \tag{15}$$

We now combine all of the above to calculate $c(F_1) - c(F_2)$. First, we can write

$$c(F_1) - c(F_2) = \sum_{i=3}^{r} \sum_{z \in \{x_1,y_1,x_2,y_2\}} c(zv_i) \cdot (e_{F_1}(V_z,V_{v_i}) - e_{F_2}(V_z,V_{v_i})) + \sum_{i=1}^{2} c(x_iy_i) \cdot (e_{F_1}(V_{x_i},V_{y_i}) - e_{F_2}(V_{x_i},V_{y_i})) \tag{16}$$

By (15), the second term in (16) equals

$$2(r-2)! \cdot (c(e_1) - c(e_2)) \cdot [|B_1|+|B_2|) \cdot e_H(A_1,A_2) - (|A_1|+|A_2|) \cdot e_H(B_1,B_2)]. \tag{17}$$

By (13) and (14), the first term in (16) equals

$$\begin{aligned} &\sum_{i=3}^{r} [c(x_1v_i) + c(y_1v_i) - c(x_2v_i) - c(y_2v_i)] \cdot (r-3)! \cdot [(|B_1|+|B_2|) \cdot f_H(A_1,A_2) - (|A_1|+|A_2|) \cdot f_H(B_1,B_2)] \\ &= (r-3)! \cdot q \cdot [(|B_1|+|B_2|) \cdot f_H(A_1,A_2) - (|A_1|+|A_2|) \cdot f_H(B_1,B_2)]. \end{aligned} \tag{18}$$

If $c(F_1) - c(F_2) \neq 0$ then $(L_1 \cup L_2, c)$ is a template for H and we are done. so suppose that $c(F_1) - c(F_2) = 0$. Then, by plugging (17) and (18) into (16), dividing by $(r-3)!$ and rearranging, we get $g(A_1,\dots,A_r) = g(B_1,\dots,B_r)$, as required. This completes the proof. □

We end this subsection with several properties of (s,t)-structured graphs. The following simple lemma expresses $e(H)$ in terms of $s, t, \rho, |H|$.

Lemma 7.19. *If H is (s,t)-structured with parameter ρ, then*

$$e(H) = \rho \frac{r-1}{(2r-4)s+t}|H|.$$

Proof. Fix an arbitrary r-coloring $A_1,\dots,A_r$ of H. By summing (12) over all pairs $1 \leq i < j \leq r$, we get

$$\begin{aligned} ((2r-4)s+t) \cdot e(H) &= \sum_{1 \leq i < j \leq r} \Big[s \cdot e_H(A_i \cup A_j, V(H) \backslash (A_i \cup A_j)) + t \cdot e_H(A_i,A_j) \Big] \\ &= \rho \sum_{1 \leq i < j \leq r} (|A_i|+|A_j|) = \rho(r-1)\sum_{i=1}^{r}|A_i| = \rho(r-1)|H|. \end{aligned}$$

□

In what follows, we will need the following trivial claim.

Claim 7.20. *Let $r \geq 3$ and let $a_1, \ldots, a_r \in \mathbb{R}$. If there is c such that $a_i + a_j = c$ for all $1 \leq i < j \leq r$, then $a_1 = \cdots = a_r$.*

In the next lemma, we show that if H is (s,t)-structured for two different choices of (s,t), in the sense that the ratio s/t is different, then every r-vertex-coloring $A_1, \ldots, A_r$ of H is balanced, i.e. satisfies $|A_1| = |A_2| = \cdots = |A_r|$. By normalizing, we may and will assume that one of the t's equals 1.

Lemma 7.21. *Let $r \geq 3$ and $s, s', t \in \mathbb{R}$ with $s' - t \cdot s \neq 0$. If H is $(s,1)$- and (s',t)-structured, then every r-coloring of H is balanced.*

Proof. Fix any r-coloring of H with parts $A_1, \ldots, A_r$. By definition (see Definition 7.17), there exist ρ, ρ' such that

$$\rho(|A_1| + |A_2|) = s \cdot e_H(A_1 \cup A_2, V(H) \backslash (A_1 \cup A_2)) + e_H(A_1, A_2),$$

and

$$\rho'(|A_1| + |A_2|) = s' \cdot e_H(A_1 \cup A_2, V(H) \backslash (A_1 \cup A_2)) + t \cdot e_H(A_1, A_2).$$

Combining these two equations, we get

$$(s' \cdot \rho - s \cdot \rho')(|A_1| + |A_2|) = (s' - t \cdot s) \cdot e_H(A_1, A_2),$$

and

$$(\rho' - t \cdot \rho)(|A_1| + |A_2|) = (s' - t \cdot s) \cdot e_H(A_1 \cup A_2, V(H) \backslash (A_1 \cup A_2)).$$

Note that $s' - ts \neq 0$ by assumption. Setting $c = (s'\rho - s\rho')/(s' - ts)$ and $c' = (\rho' - t\rho)/(s' - ts)$, we have

$$\begin{aligned} c(|A_1| + |A_2|) &= e_H(A_1, A_2), \\ c'(|A_1| + |A_2|) &= e_H(A_1 \cup A_2, V(H) \backslash (A_1 \cup A_2)). \end{aligned} \tag{19}$$

Note that $c \neq 0$ because $e_H(A_1, A_2) \neq 0$, as $\chi(H) = r$. By permuting the parts $A_1, \ldots, A_r$, we obtain the analogous equations for every pair of parts A_i, A_j. In particular, for all $1 \leq i < j \leq r$,

$$c(|A_i| + |A_j|) = e_H(A_i, A_j).$$

We now get

$$\begin{aligned} e_H(A_1 \cup A_2, V(H) \backslash (A_1 \cup A_2)) &= \sum_{3 \leq i \leq r} (e_H(A_1, A_i) + e_H(A_2, A_i)) = c \sum_{3 \leq i \leq r} (|A_1| + |A_2| + 2|A_i|) \\ &= c(r-4)(|A_1| + |A_2|) + 2c|H|. \end{aligned}$$

Combining this with (19), we get

$$(c' - c(r-4))(|A_1| + |A_2|) = 2c|H| \neq 0.$$

Applying this argument for A_i, A_j in place of A_1, A_2, we see that $|A_i| + |A_j| = \frac{2c|H|}{c' - c(r-4)}$ for every $1 \leq i < j \leq r$. Using that $r \geq 3$, we get $|A_1| = \cdots = |A_r|$ by Claim 7.20. □

Next, we show that if H is (s,t)-structured with parameter ρ, then for every r-coloring $A_1, \ldots, A_r$ of H, one can express $e_H(A_1, A_2)$ as a function of $|A_1| + |A_2|$. For $s = 0$ this is trivial (recall Definition 7.17), so we assume $s \neq 0$. As it turns out, the cases $s/t \neq \frac{1}{2}$ and $s/t = \frac{1}{2}$ need to be handled separately, and in the latter case we need to additionally assume that H satisfies the r-wise C_4-condition. To avoid repetitions, we handle both cases together. For convenience, we assume that $s = 1$.

Lemma 7.22. *Let $t, \rho \in \mathbb{R}$ and suppose that H is $(1,t)$-structured with parameter ρ. Assume that $t \neq 2$, or $t = 2$ and H satisfies the r-wise C_4-condition. Then for every r-coloring $A_1, \dots, A_r$ of H it holds that*

$$(r-4+t)(2r-4+t) \cdot e_H(A_1, A_2) = \rho(2r-4+t) \cdot (|A_1| + |A_2|) - 2\rho|H|.$$

Proof. Let $A_1, A_2, \dots A_r$ be the parts of a r-coloring of H. By definition, for all $1 \le i < j \le r$,

$$\rho(|A_i| + |A_j|) = e_H(A_i \cup A_j, V(H) \backslash (A_i \cup A_j)) + t \cdot e_H(A_i, A_j). \tag{20}$$

Summing over all pairs $1 \le i < j \le r$, we get

$$\begin{aligned}(r-1)\rho|H| = \sum_{1 \le i < j \le r} \rho(|A_i| + |A_j|) &= \sum_{1 \le i < j \le r} e_H(A_i \cup A_j, V(H) \backslash (A_i \cup A_j)) + t \cdot e_H(A_i, A_j) \\ &= (2r-4+t)e(H). \end{aligned} \tag{21}$$

Let $A = \bigcup_{3 \le i \le r} A_i$. Summing (20) over $3 \le i < j \le r$, we get

$$\begin{aligned}\rho(r-3)(|H| - |A_1| - |A_2|) &= \sum_{3 \le i < j \le r} \rho(|A_i| + |A_j|) \\ &= \sum_{3 \le i < j \le r} e_H(A_i \cup A_j, V(H) \backslash (A_i \cup A_j)) + t \cdot e_H(A_i, A_j) \\ &= (2r-8+t) \cdot e_H(A) + (r-3) \cdot e_H(A_1 \cup A_2, V(H) \backslash (A_1 \cup A_2)). \end{aligned} \tag{22}$$

Now multiply (20) (for $i = 1, j = 2$) by $r - 5 + t$ and add to (22), obtaining:

$$\begin{aligned}&\rho(r-3)|H| + \rho(t-2)(|A_1| + |A_2|) \\ &= (2r-8+t) \cdot e_H(A) + (2r-8+t) \cdot e_H(A_1 \cup A_2, V(H) \backslash (A_1 \cup A_2)) + t(r-5+t) \cdot e_H(A_1, A_2) \\ &= (2r-8+t) \cdot e(H) + [t(r-5+t) - (2r-8+t)] \cdot e_H(A_1, A_2) \\ &= (2r-8+t) \cdot e(H) + (t-2)(r-4+t) \cdot e_H(A_1, A_2), \end{aligned} \tag{23}$$

where the second equality uses $e(H) = e_H(A_1, A_2) + e_H(A) + e_H(A_1 \cup A_2, V(H) \backslash (A_1 \cup A_2))$. Next, we cancel the term $e(H)$ in (23). To this end, multiply (21) by $2r - 8 + t$ and subtract this from (23) multiplied by $2r - 4 + t$, to get:

$$\begin{aligned}&\rho((r-3)(2r-4+t) - (r-1)(2r-8+t)) \cdot |H| + \rho(t-2)(2r-4+t)(|A_1| + |A_2|) \\ &= (t-2)(r-4+t)(2r-4+t) \cdot e_H(A_1, A_2). \end{aligned}$$

Note that $(r-3)(2r-4+t) - (r-1)(2r-8+t) = -2(t-2)$. If $t \neq 2$, then dividing through by $t - 2 \neq 0$ completes the proof. Suppose from now on that $t = 2$. We then need another relation coming from the C_4-condition. For every pair $3 \le i \neq j \le r$, $e_H(A_1, A_2) + e_H(A_i, A_j) - e_H(A_1, A_i) - e_H(A_2, A_j) = 0$. Summing this over all (ordered) pairs i, j, we get

$$\begin{aligned}0 &= \sum_{3 \le i \neq j \le r} e_H(A_1, A_2) + e_H(A_i, A_j) - e_H(A_1, A_i) - e_H(A_2, A_j) \\ &= (r-2)(r-3) \cdot e_H(A_1, A_2) + 2e_H(A) - (r-3) \cdot e_H(A_1 \cup A_2, V(H) \backslash (A_1 \cup A_2)). \end{aligned}$$

Adding the above equation to (22), we get

$$\rho(r-3)(|H| - |A_1| - |A_2|) = (r-2)(r-3) \cdot e_H(A_1, A_2) + (2r-4) \cdot e_H(A). \tag{24}$$

We now continue as before: multiply (20) by $2r-4$ and add this to (24) to get

$$\begin{aligned}&\rho(r-3)|H| + \rho(r-1)(|A_1|+|A_2|) = \\ &= (2r-4)\cdot e_H(A) + (2r-4)\cdot e_H(A_1\cup A_2, V(H)\backslash(A_1\cup A_2)) + ((r-2)(r-3)+2(2r-4))\cdot e_H(A_1,A_2) \\ &= (2r-4)\cdot e(H) + (r-1)(r-2)\cdot e_H(A_1,A_2). \qquad (25)\end{aligned}$$

In the second equality we used $e(H) = e_H(A_1,A_2)+e_H(A)+e_H(A_1\cup A_2, V(H)\backslash(A_1\cup A_2))$. Finally, multiply (21) by $2r-4$ and subtract this from (25) multiplied by $2r-2$, to get

$$-2\rho(r-1)|H| + \rho(r-1)(2r-2)\cdot(|A_1|+|A_2|) = (2r-2)(r-1)(r-2)\cdot e_H(A_1,A_2).$$

Dividing through by $r-1$ completes the proof. □

8 Lower bounds

In this section, we describe some constructions that are used to prove the lower bounds in Theorems 1.4, 1.7 and 1.11. We start with an observation about regular graphs.

Lemma 8.1. *If H is regular then $\delta^*(H) \geq 3/4$.*

Proof. Let c be a 2-edge-coloring of K_4 such that K_4^+ is isomorphic to $K_{1,3}$. Let v_0 be the vertex whose incident edges all have color $+1$. Let $m \in \mathbb{N}$ be divisible by 4 and $|H|$. Let B be an $m/4$-blowup of (K_4, c). If B has no perfect H-factor then $\delta^*(H) \geq \delta(B)/|B| = 3/4$. Let us assume that this is not the case and let F be a perfect H-factor of B. Let $d \in \mathbb{N}$ be such that H is d-regular and note that every H-factor is also d-regular. Therefore, F has $\frac{d}{2}m$ edges. Also, the number of edges of color 1 is exactly $d|V_{v_0}| = dm/4$. Thus, exactly half of the edges have color 1, meaning that F has zero discrepancy. Hence, $\delta^*(H) \geq \frac{\delta(B)}{|B|} = 3/4$. □

Lemma 8.2. *If there exists ρ such that for every connected component U of H it holds that*

$$e_H(U) = \rho|U|,$$

then $\delta^*(H) \geq 1/2$.

Proof. If $\chi^*(H) \geq 2$, then this holds trivially, because $\delta^*(H) \geq 1-1/\chi^*(H)$, as $1-1/\chi^*(H)$ is the threshold for the existence of a perfect H-factor by Theorem 1.2. So let us assume that $\chi^*(H) < 2$. Let $m \in \mathbb{N}$ be divisible by 2 and $|H|$, and let J be an m-vertex graph which is the disjoint union of two-cliques A, B of size $m/2$ each (there are no edges between A and B). Note that $\delta(J) = m/2-1$. Let c be a 2-edge-coloring of J with $c(e) = 1$ for $e \in J[A]$ and $c(e) = -1$ for $e \in J[B]$. Let F be a perfect H-factor of J; if no such F exists then $\delta^*(H) \geq 1/2$ immediately holds. For each copy $H_0 \in F$ of H and for each connected component U of H_0, we have by assumption

$$e_F(U) = \rho|U|.$$

Also, $U \subseteq A$ or $U \subseteq B$, because there are no edges between A and B. So $c(F[U]) = \rho|U|$ if $U \subseteq A$ and $c(F[U]) = -\rho|U|$ if $U \subseteq B$. Summing over all $H_0 \in F$ and all components U of H, we get $c(F) = \rho(|A|-|B|) = 0$. As this holds for any perfect H-factor of J, we get $\delta^*(H) \geq 1/2$. □

Recall the definition of butterfly graphs in the paragraph above Theorem 1.7. In the next Lemma we use the symmetry of butterflies with respect to the coloring to show that if a certain butterfly is not a template for a given graph H, then all H-factors of an appropriate blowup of this butterfly have discrepancy 0, giving a lower bound on $\delta^*(H)$. The construction is depicted on Figure 2.

Lemma 8.3. *If there exists a butterfly which is not a template for H, then $\delta^*(H) \geq 4/7$.*

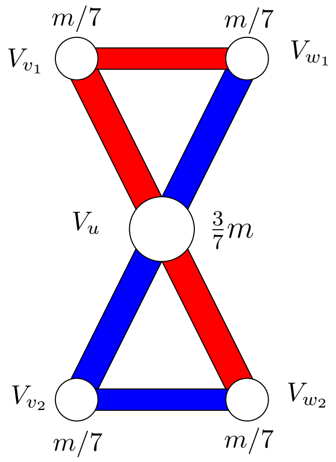

Figure 2: A blowup of a butterfly showing that $\delta^*(H) \geq 4/7$ if the butterfly is not a template for H.

Proof. Let (L, c) be a butterfly with triangles $\{u, v_1, w_1\}$ and $\{u, v_2, w_2\}$, and suppose that (L, c) is not a template for H. By definition, we have

$$c(uv_1) = -c(uv_2),\ c(uw_1) = -c(uw_2)\ c(v_1w_1) = -c(v_2w_2). \tag{26}$$

Let B be a blowup of (L, c) of size m, divisible by 7 and $|H|$, with

- $|V_u| = 3m/7$ and
- $|V_v| = m/7$ for $v \in \{v_1, w_1, v_2, w_2\}$.

We claim that every perfect H-factor of B satisfies $c(F) = 0$. Indeed, consider the automorphism of L which swaps between v_1, v_2 and between w_1, w_2, and let F' be the image of F under this automorphism. Then by (26), we have $c(F') = -c(F)$. On the other hand, as (L, c) is not a template for H, we must have $c(F) = c(F')$. It follows that $c(F) = 0$. This proves that $\delta^*(H) \geq \delta(B)/|B| = 4/7$. □

Next, we prove the lower bound on $\delta^*(H)$ in the first two cases of Theorem 1.11. The constructions use Lemma 4.2 and are depicted in Figure 3.

Lemma 8.4. *If H fulfills the k-wise C_4-condition for some $k \equiv_4 1$, then $\delta^*(H) \geq \frac{k-1}{k}$. Additionally, if $k = \chi(H)$ then $\delta_0(H) = \frac{k-1}{k}$.*

Proof. Let c be a 2-edge-coloring of K_k such that K_k^+ is $(k-1)/2$-regular; such a coloring exists because $k \equiv_4 1$. Since H satisfies the k-wise C_4-condition, so does every H-factor. Hence, Lemma 4.2 with $d = (k-1)/2$ implies that for every blowup B of (K_k, c) and for every perfect H-factor J of B, it holds that

$$c(J) = \frac{2 \cdot (k-1)/2 - k + 1}{k-1} \cdot e(J) = 0.$$

This implies that $c \in \mathcal{K}(H)$. Also, taking B to be the m/k-blowup of (K_k, c), we get that $\delta^*(H) \geq \delta(B)/|B| = \frac{k-1}{k}$. Finally, if $k = \chi(H)$ and we take m to be divisible by $(k-1)!|H|$, then B has a perfect H-factor by Lemma 5.2, and we get that $\delta_0(H) = \frac{k-1}{k}$ by the definition of $\delta_0(H)$. □

Lemma 8.5. *If H is regular and fulfills the k-wise C_4-condition for some $k \not\equiv_4 1$, then $\delta^*(H) \geq \frac{k-1}{k}$.*

Proof. We give three different constructions depending on the residue of k modulo 4, but the general idea is the same for all three. Let $U \subseteq V(K_k)$ with $|U| = k-1$ and $\{v\} = V(K_k) \setminus U$. Let c be any 2-edge-coloring

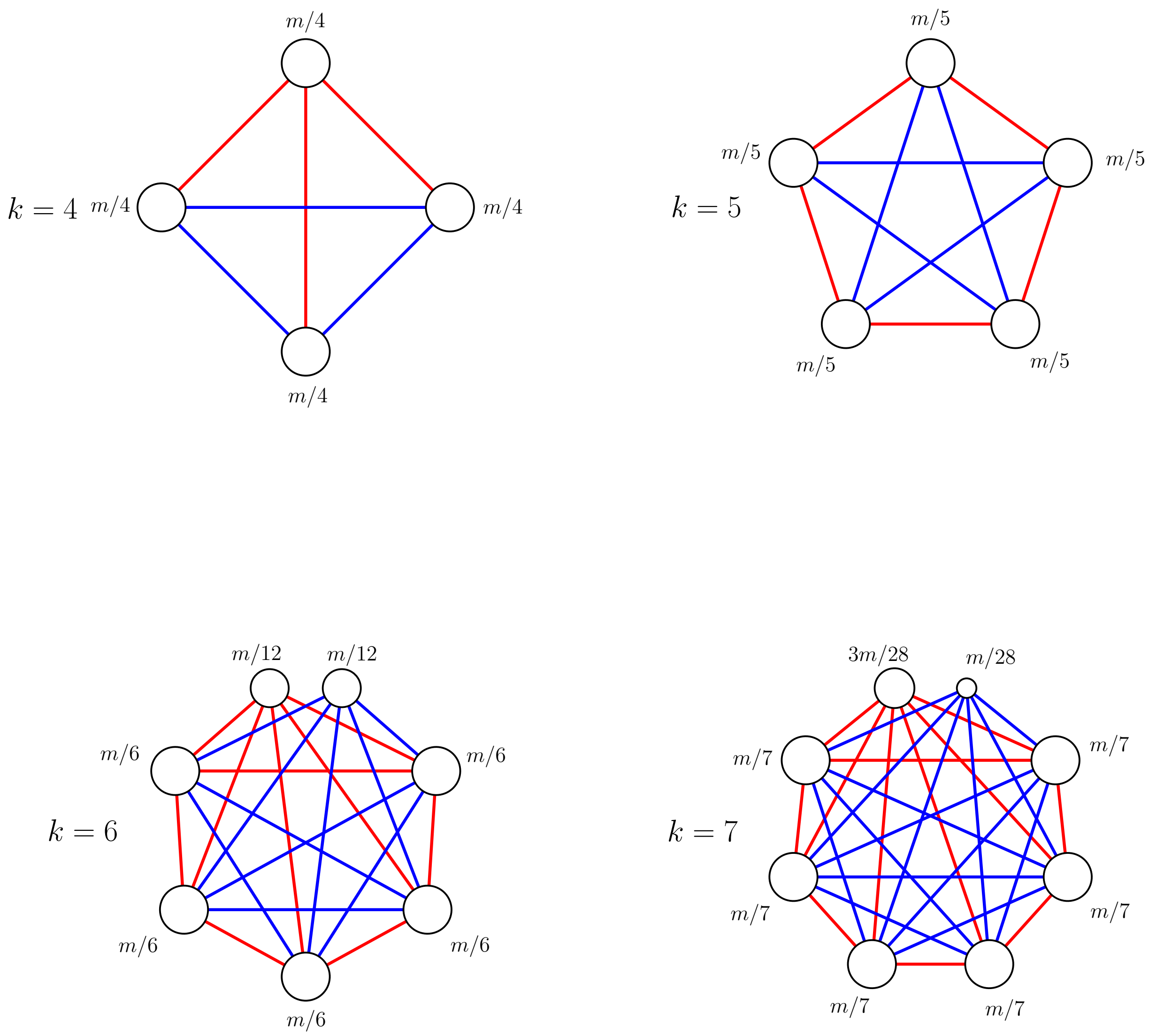

Figure 3: The constructions given in Lemmas 8.4 and 8.5 for $4 \leq k \leq 7$.

of K_k such that $K_k[U]^+$ is ℓ-regular, where ℓ is an even integer to be determined later; such a coloring exists because ℓ is even. Let B be an m/k-blowup of (K_k, c) for some m with m divisible by $4k$. Let $d \in \mathbb{N}$ be such that H is d-regular and let F be an arbitrary perfect H-factor of B. Note that F is also d-regular and satisfies the k-wise C_4-condition. So $e(F) = dm/2$ and V_v is incident to $d|V_v| = \frac{dm}{k}$ edges in F. Hence, $e_F(V_U) = \frac{k-2}{2k}dm$. Additionally, $F[V_U]$ satisfies the $(k-1)$-wise C_4-condition. To see this, observe that every $(k-1)$-coloring of $F[V_U]$ can be extended to a k-coloring of F by adding $V(F) \cap V_v$ as an additional color-class. Thus, by applying Lemma 4.2 with $J = F[V_U]$ (and with $k-1$ in place of k), we get

$$c(F[V_U]) = \frac{2\ell - k + 2}{k-2} \cdot e_F(V_U) = \frac{2\ell - k + 2}{k-2} \cdot \frac{k-2}{2k} dm = \frac{2\ell - k + 2}{2k} dm.$$

We now define a 2-edge-coloring c' of B as follows. First, if e is contained in V_U, then set $c'(e) = c(e)$. Second, to color the edges incident to V_v, split V_v into two sets V_+ and V_- (whose sizes will be determined later), and set $c'(e) = 1$ for all edges incident to V_+ and $c'(e) = -1$ for all edges incident to V_-. Then,

$$c'(F) = c(F[V_U]) + e_F(V_+, V_U) - e_F(V_-, V_U) = \frac{2\ell - k + 2}{2k} dm + d(|V_+| - |V_-|), \tag{27}$$

where the last equality uses that F is d-regular. We now consider the different cases of k modulo 4 and choose ℓ, $|V_+|$ and $|V_-|$ so that $c'(F) = 0$.

- If $k \equiv_4 0$, take $\ell = k/2$, $|V_+| = 0$ and $|V_-| = m/k$.
- If $k \equiv_4 2$, take $\ell = (k-2)/2$ and $|V_+| = |V_-| = \frac{m}{2k}$.
- If $k \equiv_4 3$, take $\ell = (k-3)/2$, $|V_+| = \frac{3m}{4k}$ and $|V_-| = \frac{m}{4k}$.

It is easy to check that ℓ is even and we have $c'(F) = 0$ by (27) in all three cases. □

The final two lemmas of this section provides us with a lower bound on $\max\{\delta_0(H), 1 - 1/\chi^*(H)\}$ (and therefore also on $\delta^*(H)$) for (s,t)-structured graphs (recall Definition 7.17).

Lemma 8.6. *Suppose that $r = 3$ and H is $(1,2)$-structured. Then* $\max\{\delta_0(H), 1 - 1/\chi^*(H)\} \geq 5/8$.

Proof. By the definition of structuredness (see Definition 7.17), there is $\rho \in \mathbb{R}$ such that

$$\rho(|A_i| + |A_j|) = e_H(A_i \cup A_j, V(H) \backslash (A_i \cup A_j)) + 2e_H(A_i, A_j) = e(H) + e_H(A_i, A_j). \tag{28}$$

for every 3-coloring A_1, A_2, A_3 of H and all $1 \leq i < j \leq 3$. By summing the above equation over all $1 \leq i < j \leq 3$, we get

$$2\rho|H| = \sum_{1 \leq i < j \leq r} \rho(|A_i| + |A_j|) = 3e(H) + \sum_{1 \leq i < j \leq r} e_H(A_i, A_j) = 4e(H).$$

So $\rho|H| = 2e(H)$. Subtracting from this the equation $\rho(|A_2| + |A_3|) = e(H) + e_H(A_2, A_3)$, we get

$$\rho|A_1| = e_H(A_1, A_2) + e_H(A_1, A_3). \tag{29}$$

Now consider a triangle T with vertices u_1, u_2, u_3 and let c be the 2-edge-coloring of T where $c(u_1u_2) = c(u_1u_3) = 1$, $c(u_2u_3) = -1$. We first show that $c \in \mathcal{K}(H)$. So let B be an arbitrary blowup of (T, c), and let U_1, U_2, U_3 be the clusters of B (where U_i corresponds to u_i). We need to show that all perfect H-factors of B have the same discrepancy. So let F be a perfect H-factor of B. First note that

$$e(F) = e(H) \cdot |B|/|H| = \rho|B|/2.$$

For each H-copy $H_0 \in F$, consider the 3-coloring A_1, A_2, A_3 of H_0 given by $A_i = V(H_0) \cap U_i$. The number of edges of color 1 in H_0 is precisely $e_{H_0}(A_1, A_2) + e_{H_0}(A_1, A_3)$. By (29), this number is $\rho|A_1|$. Summing over all H-copies $H_0 \in F$, we see that the number of edges of color 1 in F is precisely $\rho|U_1|$. Therefore,

$$c(F) = 2e(F^+) - e(F) = \rho(2|U_1| - |B|/2),$$

which is independent of F. This shows that indeed $c \in \mathcal{K}(H)$.

Let us now consider the specific case where $|U_1| = 2m$ and $|U_2| = |U_3| = 3m$, for an arbitrary integer m. Then $|B| = 8m$ and $\delta(B)/|B| = (|U_1| + |U_2|)/|B| = 5/8$. If B has no perfect H-factor for any choice of m, then $1 - 1/\chi^*(H) \geq \delta(B)/|B| = 5/8$. And otherwise, fix an m for which B has perfect H-factors. Every perfect H-factor F of B satisfies

$$c(F) = \rho(2|U_1| - |B|/2) = 0.$$

By the definition of $\delta_0(H)$, this implies that $\delta_0(H) \geq \delta(B)/|B| = 5/8$, as required. □

Lemma 8.7. *Suppose that $r \geq 6$ and $r \equiv_4 2, 3$. Assume that H is $(0,1)$-structured, or that H is $(1,t)$-structured for some $t \in \{-2,-1,0,1,2\}$ and satisfies the r-wise C_4-condition. Then*

$$\max\{\delta_0(H), 1 - 1/\chi^*(H)\} \geq \frac{3r-5}{3r-2}.$$

The bound of $\frac{3r-5}{3r-2}$ in Lemma 8.7 is particularly interesting because a blowup on m vertices of two copies of K_r sharing $r-2$ vertices has minimum degree at most $\frac{3r-5}{3r-2}m$. The lower bound given by Lemma 8.7 will allow us to assume later on that $\delta(R)/|R| > \frac{3r-5}{3r-2}$ (when proving upper bounds on $\delta^*(H)$). We will then (implicitly) use the fact that R is not a blowup of two r-cliques sharing $r-2$ vertices. See Lemma 11.4.

Proof of Lemma 8.7. We assume that H is (s,t)-structured, where $(s,t)=(0,1)$ or $s=1$ and $t \in \{-2,-1,0,1,2\}$. In particular, $s \in \{0,1\}$ and $(s,t) \neq (0,0)$. First we show that there exist τ, C such that for every r-coloring $A_1, \dots A_r$ of H,

$$e_H(A_i, A_j) = \tau(|A_i| + |A_j|) + \tau \cdot C|H|. \tag{30}$$

It is enough to show this for $i,j=1,2$. Let ρ be such that H is (s,t)-structured with parameter ρ (recall Definition 7.17). If $(s,t)=(0,1)$ then (30) is trivially satisfied for $\tau=\rho$ and $C=0$. Otherwise, we have that $s=1$. Then, by Lemma 7.22, we get that

$$(r-4+t)(2r-4+t) \cdot e_H(A_1, A_2) = \rho(2r-4+t) \cdot (|A_1| + |A_2|) - 2\rho|H|. \tag{31}$$

Observe that in all possible choices of (s,t), we have $r+t \geq 4$, and equality holds only if $r=6$, $s=1$ and $t=-2$. We consider this case first. Then (31) becomes

$$2\rho|H| = 6\rho(|A_1| + |A_2|). \tag{32}$$

We claim that $\rho \neq 0$. To see this, we use that H is $(1,-2)$-structured, summing (12) over all pairs $1 \leq i < j \leq r$. This gives:

$$\begin{aligned}(r-1)\rho|H| = \sum_{1\leq i<j\leq r} \rho(|A_i| + |A_j|) &= \sum_{1\leq i<j\leq r} [e_H(A_i \cup A_j, V(H) \setminus (A_i \cup A_j)) - 2e_H(A_i, A_j)] \\ &= (2r-6) \cdot e(H) = 6e(H) > 0,\end{aligned}$$

Now divide both sides of (32) by ρ to get $|H|/3 = |A_1| + |A_2|$. Since this holds for every r-coloring of H, we get that every r-coloring of H must be balanced by Claim 7.20. This implies that $\chi^*(H) = r$ by the definition of χ^*. Now, $1 - 1/\chi^*(H) = \frac{r-1}{r} \geq \frac{3r-5}{3r-2}$.

Suppose from now on that $r+t > 4$. Then $(r-4+t)(2r-4+t) \neq 0$. Thus, dividing both sides of (31) by $(r-4+t)(2r-4+t)$, we get that (30) is satisfied for $\tau = \frac{\rho}{r-4+t}$ and $C = \frac{-2}{2r-4+t}$. This proves (30). Note also that either $s=0$ and $C=0$, or $s=1$ and $C = \frac{-2}{2r-4+t}$. Since $t \geq -2$, we get in either case that

$$\frac{-1}{r-3} \leq C \leq 0. \tag{33}$$

Next, we use (30) to complete the proof of the lemma. First, we claim that for every 2-coloring c of K_r it holds that $c \in \mathcal{K}(H)$. Indeed, let B be an arbitrary blowup of (K_r, c), and let F be any perfect H-factor of B. Let $B_1, \dots, B_r$ be the parts of B. By summing (30) over all copies of H in F, we get that

$$e_F(B_i, B_j) = \tau(|B_i| + |B_j|) + \frac{|F|}{|H|} \cdot \tau C|H| = \tau(|B_i| + |B_j|) + \tau C|F|.$$

for all $1 \leq i \leq j \leq r$. This implies that

$$c(F) = \sum_{1\leq i<j\leq r} c(ij) \cdot e_F(B_i, B_j) = \tau \sum_{1\leq i<j\leq r} c(ij) \cdot (|B_i| + |B_j| + C|F|), \tag{34}$$

which is independent of F, since $|F| = |B|$. This means that all perfect H-factors of B have the same discrepancy. It now follows, by the definition of $\mathcal{K}(H)$, that $c \in \mathcal{K}(H)$.

Next, we define a certain 2-edge-coloring c of K_r, as follows. Write $V(K_r) = \{v_1, \dots, v_r\}$. We claim that there exists a coloring c such that $c(e) = 1$ for every $e \in E(K_r)$ incident to v_1, and $c(K_r) = 1$. Indeed, there are $r-1$ edges incident to v, and $r - 1 < \binom{r}{2}/2$ for $r \geq 6$. Hence, there is a coloring c such that $c(v_1v_j) = 1$ for all $2 \leq j \leq r$, and c colors exactly $\lceil \binom{r}{2}/2 \rceil = (\binom{r}{2} + 1)/2$ edges with color 1. Here we use that $\binom{r}{2}$ is odd because $r \equiv_4 2, 3$. So the number of edges of color -1 is $(\binom{r}{2} - 1)/2$, and hence $c(K_r) = 1$, as required. We fix such a coloring c from now on.

Fix $C' \in \mathbb{N}$ such that $C \cdot C' \in \mathbb{Z}$, where C is the constant from (30). Fix an integer m divisible by $(r-2)(r-1)(r+1)C'|H|$, and let B be the blowup of (K_r, c) with $|B| = m$ and with parts $B_1, \dots, B_r$ (where B_i corresponds to v_i), such that $|B_1| = x$ and $|B_i| = \frac{m-x}{r-1} =: y$ for all $2 \leq i \leq r$, where

$$x = \frac{(r-3) - (r-1)C}{(r-2)(r+1)} m.$$

By our choice of m, both x and y are integers. Next, we show that

$$\frac{m}{3r-2} \leq x \leq \frac{m}{r}. \tag{35}$$

As $C \leq 0$, to prove the lower bound in (35) it suffices to check that $(r-2)(r+1) \leq (3r-2)(r-3)$, which holds for $r \geq 4$. For the upper bound, we use that $C \geq \frac{-1}{r-3}$ and therefore $x \leq \frac{(r-3)^2+(r-1)}{(r-2)(r+1)(r-3)} m$. Hence, it suffices to verify that $r((r-3)^2 + (r-1)) \leq (r-2)(r+1)(r-3)$, which holds for $r \geq 6$.

The bounds in (35) imply that

$$\frac{m}{r} \leq y \leq \frac{3m}{3r-2}. \tag{36}$$

In particular, B_1 is the smallest part in the blowup B, by (35) and (36). Therefore, $\delta(B) = m - y \geq \frac{3r-5}{3r-2} m$. If for all m, the blowup B has no perfect H-factor, then, using Theorem 1.2, we see that $1 - 1/\chi^*(H) \geq \frac{3r-5}{3r-2}$, as needed. Else, fix m such that B has perfect H-factors, and let F be an arbitrary H-factor of B. For convenience, we use the notation $c(v_i, K_r) := \sum_{j \neq i} c(v_iv_j)$. By (34), we have

$$c(F) = \tau \sum_{1 \leq i < j \leq r} c(v_iv_j) \cdot (|B_i| + |B_j| + C|F|) = \tau \cdot \left(c(K_r) \cdot Cm + \sum_{i=1}^{r} c(v_i, K_r) \cdot |B_i| \right),$$

using that $|F| = |B| = m$. Recall that $c(K_r) = 1$ and $c(v_1, K_r) = r - 1$. Hence, $\sum_{i=1}^r c(v_i, K_r) = 2c(K_r) = 2$ and $\sum_{i=2}^r (v_i, K_r) = 3 - r$. It follows that

$$c(F) = \tau \left((r-1)x - (r-3) \cdot \frac{m-x}{r-1} + mC \right) = 0,$$

by our choice of x. As $c \in \mathcal{K}(H)$, this implies $\delta_0(H) \geq \delta(B)/m \geq \frac{3r-5}{3r-2}$, as needed. □

9 When all r-cliques have the same discrepancy

In this section we consider the situation when all r-cliques in the reduced graph R have the same discrepancy sign (i.e. all have positive discrepancy or all have negative discrepancy). By symmetry, we may assume that all have positive discrepancy (else swap the colors). As usual, we work under the setup described in Section 6.2. The main result of this section is the following:

Lemma 9.1. *Assume the setup of Section 6.2. If H is non-regular and all copies of K_r in R have positive discrepancy, then there exists a perfect H-factor in G' with high discrepancy.*

The proof of Lemma 9.1 is broken into two cases depending on whether or not H is uniform (recall Definition 7.6). These two cases are handled in the following two subsections. Recall the definition of the r-partite graph H^* (see Definition 5.5). Namely, recall that if $r = 2$ then $H^* = H$, and otherwise H^* is a complete r-partite graph satisfying the properties in Lemma 5.4. Recall also that V_0 is the exceptional class in the regular partition of G', and that for a vertex $u \in V(G') \setminus V_0$, we use $V_u \in V(R)$ to denote the part of the partition containing u (so V_u is a vertex of the reduced graph R).

9.1 Proof of Lemma 9.1: H is non-uniform

Here we prove Lemma 9.1 in the case that H is non-uniform.

By Lemma 6.5, G' has a perfect H^*-factor F^*. Suppose first that there exists a copy $J \in F^*$ of H^* disjoint from V_0 with parts $B_1, B_2, \dots B_r \subseteq V(J)$, and vertices $b_1, b_1' \in B_1$, $b_2 \in B_2$, such that $f(b_1b_2) \neq f(b_1'b_2)$. Fix arbitrary $b_i \in B_i$ for $3 \leq i \leq r$. Since H is non-uniform, there exists an r-coloring $A_1, A_2, \dots, A_r$ of H such that $e_H(A_1, A_2) \neq e_H(A_1, A_3)$, by Claim 7.8. This implies that there exists a vertex $a \in A_1$ with $d_H(a, A_2) \neq d_H(a, A_3)$. Consider the r-cliques $L_1 = \{V_{b_1}^R, V_{b_2}^R, \dots V_{b_r}^R\}$ and $L_2 = \{V_{b_1'}^R, V_{b_2}^R, \dots V_{b_r}^R\}$ in R. If L_1, L_2 have different discrepancies (with respect to f_R), then $(L_1 \cup L_2, f_R)$ is a template for H by Lemma 7.2 (as H is non-regular), and then G' has a perfect H-factor with high discrepancy by Lemma 6.7, completing the proof. We may therefore assume that $f_R(L_1) = f_R(L_2)$. Then we can apply Lemma 7.4 with $x = V_{b_1}^R, y = V_{b_1'}^R, z = V_{b_2}^R$, to conclude that $(L_1 \cup L_2, f_R)$ is a template for H. Now we are again done by Lemma 6.7.

So from now on, let us assume that $J \in F^*$ as above does not exists. This means that for every copy $J \in F^*$ of H^* disjoint from V_0, if $B_1, \dots, B_r$ denote the parts of J, then all bipartite graphs (B_i, B_j) are monochromatic with respect to f. In other words, J is a blowup of (K_r, c) for some 2-edge-coloring c of K_r. Fix arbitrary $b_1 \in B_1, b_2 \in B_2, \dots, b_r \in B_r$, and consider the r-clique $L = \{V_{b_1}^R, V_{b_2}^R, \dots V_{b_r}^R\}$ in R. By (6), L is colored by f_R in the same way as K_r by c. Hence, $c(K_r) = f_R(L) > 0$, using our assumption that every r-clique in R has positive discrepancy. If (L, f_R) is a template for H then we are done by Lemma 6.7 as before, and else we have $c \in \mathcal{K}(H)$ by definition. Now, by Lemma 5.6, we see that every perfect H-factor of J has positive discrepancy.

For each $J \in F^*$, let F_J be a perfect H-factor in J. Then $F := \bigcup_{J \in F^*} F_J$ is a perfect H-factor of G'. We saw that if $J \cap V_0 = \emptyset$ then $f(F_J) > 0$, and so $f(F_J) \geq 1$. Using that $|V_0| \leq \varepsilon n \ll \frac{n}{|H^*| e(H^*)}$, we obtain

$$f(F) \geq \frac{n}{|H^*|} - |V_0| - |V_0| \cdot e(H^*) \geq \gamma n.$$

This completes the proof in the case that H is non-uniform.

9.2 Proof of Lemma 9.1: H is uniform

Here we prove Lemma 9.1 in the case that H is uniform. First, by Lemma 6.5, G' has a perfect H^*-factor F^*. The key part of the proof is the following lemma:

Lemma 9.2. *Let $J \in F^*$ be a copy of H^* disjoint from V_0. Then every perfect H-factor F_J of J satisfies*

$$f(F_J) \geq 1,$$

or there exists a template for H in (R, f_R) of size $r + 1$.

Proof. For $r = 2$, the statement follows trivially as each edge in R has positive discrepancy and therefore, R (and hence also J) are monochromatic. Therefore, let us assume that $r \geq 3$. By Lemma 7.2, we may assume that there do not exist two copies of K_r in R sharing $r - 1$ vertices with different discrepancies, as otherwise there is a template for H (since H is non-regular). Let $J \in F^*$ be a copy of H^* disjoint from V_0. Let $A_1, A_2, ..., A_r$ be the clusters of J and fix arbitrary vertices $a_1 \in A_1, a_2 \in A_2, ..., a_r \in A_r$. Let F_J be an arbitrary H-factor in J.

Claim 9.3. *For each $1 \le i < j \le r$, it holds that either for all vertices $u \in A_i$ and $v, v' \in A_j$, $f(uv) = f(uv')$, or for all vertices $u, u' \in A_i$ and $v \in A_j$, $f(uv) = f(u'v)$.*

Proof. Without loss of generality, $i = 1, j = 2$. Observe that if the assertion of the claim does not hold, then there exist $u, u' \in A_1$ and $v, v' \in A_2$ such that $f(uv) \neq f(uv'), f(u'v)$. Without loss of generality, $f(uv) = 1$ and $f(uv') = f(u'v) = -1$. Since J is disjoint from V_0, we get that $L_1 := \{V_u^R, V_v^R, V_{a_3}^R, V_{a_4}^R, \dots, V_{a_r}^R\}$ and $L_2 = \{V_u^R, V_{v'}^R, V_{a_3}^R, V_{a_4}^R, \dots, V_{a_r}^R\}$ are r-cliques in R. We have $f_R(L_1) = f_R(L_2)$ because, by assumption, every two r-cliques in R sharing $r-1$ vertices have the same discrepancy. It follows that

$$f(uv) + \sum_{3 \le i \le r} f(va_i) = f(uv') + \sum_{3 \le i \le r} f(v'a_i).$$

Note that $u'v' \in J$, as J is a complete r-partite graph. By the same argument with u' in place of u, we get

$$f(u'v) + \sum_{3 \le i \le r} f(va_i) = f(u'v') + \sum_{3 \le i \le r} f(v'a_i).$$

By subtracting the second from the first equation, we get

$$f(uv) - f(u'v) = f(uv') - f(u'v').$$

But this is a contradiction since $f(uv) - f(u'v) = 2$ and $f(uv') - f(u'v') \le 0$. □

We continue with the proof of Lemma 9.2. Let us say that (A_i, A_j) is *split* for A_i if there exist vertices $u, u' \in A_i$ and a vertex $v \in A_j$ such that $f(uv) \neq f(u'v)$. Note that if for some i and j, (A_i, A_j) is split for A_i, then all the vertices in A_i have only monochromatic edges to A_j by the above claim. Therefore, each pair (A_i, A_j) can be split at most for one of the two. Recall that by assumption, all r-cliques in R have positive discrepancy. Let

$$x = \sum_{1 \le i < j \le r} f_R(V_{a_i}^R V_{a_j}^R) > 0.$$

First, let us assume that J does not have a split pair. It follows that for all $1 \le i < j \le r$, $G'[A_i, A_j]$ is monochromatic with color $f(a_i a_j) = f_R(V_{a_i}^R V_{a_j}^R)$. As H is uniform, we get that for every H-copy $H_J \in F_J$

$$f(H_J) = \sum_{1 \le i < j \le r} \frac{e(H)}{\binom{r}{2}} f(a_i a_j) = x \frac{e(H)}{\binom{r}{2}}.$$

This implies that

$$f(F_J) = \sum_{H_J \in F_J} f(H_J) = \frac{|H^*|}{|H|} \cdot x \frac{e(H)}{\binom{r}{2}} > 0.$$

Next, let us assume without loss of generality that $A_1 A_2$ is split for A_1. Then, there exist $u, u' \in A_1$ and $v \in A_2$ such that $f(u, v) = 1$ and $f(u', v) = -1$. As the r-cliques $L_1 := \{V_u, V_v, V_{a_3}, \dots, V_{a_r}\}$ and $L_2 := \{V_{u'}, V_v, V_{a_3}, \dots, V_{a_r}\}$ have the same discrepancy and $f_R(V_u V_v) \neq f_R(V_{u'} V_v)$, we may assume by Lemma 7.4 that for every r-vertex-coloring of H with parts $B_1, B_2, \dots, B_r$ it holds for all $b \in B_1$ that

$$d_H(b, B_2) = d_H(b, B_3), \tag{37}$$

as otherwise $(L_1 \cup L_2, f_R)$ is a template for H and we are done. For every $1 \le i \le r$, let $S_i \subseteq \{1, 2, \dots, r\}$ be the set of indices j such that $A_i A_j$ is split for A_i.

Claim 9.4. *For every $1 \le i \le r$ and $u, v \in A_i$ it holds that*

$$\sum_{j \in S_i} f(ua_j) = \sum_{j \in S_i} f(va_j),$$

or there exists a template for H in (R, f_R) of size $r+1$.

Proof. Without loss of generality, suppose that there are $u, v \in A_1$ such that

$$\sum_{j \in S_1} f(ua_j) \neq \sum_{j \in S_1} f(va_j).$$

Consider the r-cliques $L_1 = \{V_u, V_{a_2}, V_{a_3}...V_{a_r}\}$ and $L_2 = \{V_v, V_{a_2}, V_{a_3}...V_{a_r}\}$ in R. Observe that

$$f_R(L_1) - f_R(L_2) = \sum_{j=2}^{r} (f(ua_j) - f(va_j)) = \sum_{j \in S_1}^{r} (f(ua_j) - f(va_j)) \neq 0.$$

Here we used that $f(ua_j) = f(va_j)$ for all $j \notin S_1$. By Lemma 7.2, $(L_1 \cup L_2, f_R)$ is a template for H. $\square$

We now conclude the proof of Lemma 9.2. Fix an H-copy $H_J \in F_J$, and let $B_i = A_i \cap V(H_J)$, $i = 1, \dots, r$. By (37), each vertex $a \in B_i$ has the same number of neighbours in B_j for each $j \in [r] \setminus \{i\}$. So this number is $\frac{d_{H_J}(a)}{r-1}$. Furthermore, if $j \in S_i$ then for each $a \in B_i, a' \in B_j$ we have $f(aa') = f(aa_j)$, as all edges between a and A_j have the same color. Hence, for each $1 \leq i \leq r$,

$$f\left(H_J \cap (B_i \times \bigcup_{j \in S_i} B_j)\right) = \sum_{a \in B_i} \frac{d_{H_J}(a)}{r-1} \sum_{j \in S_i} f(aa_j) = \sum_{a \in B_i} \frac{d_{H_J}(a)}{r-1} \sum_{j \in S_i} f(a_i a_j) = \frac{e(H)}{\binom{r}{2}} \sum_{j \in S_i} f(a_i a_j),$$

where the second equality uses Claim 9.4 with $u = a, v = a_i$, and the last equality uses $\sum_{a \in B_i} d_{H_J}(a) = (r-1)e(H)/\binom{r}{2}$, which holds by the assumption that $e_{H_J}(B_i, B_j) = e(H)/\binom{r}{2}$ for all $i < j$ (H is uniform). Now, we get

$$f(H_J) = \frac{e(H)}{\binom{r}{2}} \left(\sum_{\substack{1 \leq i < j \leq r, \\ i \notin S_j, j \notin S_i}} f(a_i a_j) + \sum_{1 \leq i \leq r} \sum_{j \in S_i} f(a_i a_j) \right).$$

Indeed, using that (A_i, A_j) can not be split for both i and j, we see that each pair $1 \leq i < j \leq r$ appears exactly once in the above two sums. Hence,

$$f(H_J) = \frac{e(H)}{\binom{r}{2}} \sum_{1 \leq i < j \leq r} f(a_i a_j) = x \frac{e(H)}{\binom{r}{2}}.$$

As this holds for every H-copy H_J in F_J, we get that

$$f(F_J) = \frac{|H^*|}{|H|} \cdot x \frac{e(H)|H^*|}{\binom{r}{2}|H|} > 0.$$

So $f(F_J) \geq 1$, as required. This proves Lemma 9.2. $\square$

Using Lemma 9.2, we can now conclude the proof of Lemma 9.1 (for uniform H). If R has a template for H of size $r+1$ then we are done by Lemma 6.7. Else, by Lemma 9.2, for every H^*-copy $J \in F^*$ with $J \cap V_0 = \emptyset$, every H-factor of J has (strictly) positive discrepancy. Let F be a perfect H-factor in G', obtained by taking a perfect H-factor of each $J \in F^*$. Note that at most $|V_0|$ many H^*-copies J contain a vertex of V_0, and each H-factor in H^* contains at most $e(H^*)$ edges. Hence,

$$f(F) \geq \frac{n}{|H^*|} - |V_0| - |V_0| \cdot e(H^*) \geq \gamma n,$$

as required.

10 Violating the C_4-condition

In this section we handle graphs H that violate the k-wise C_4-condition for a certain k. This forms an important part in the proofs of our main results. As always, r denotes the chromatic number of H. The main result is as follows.

Lemma 10.1. *Suppose that H violates the k-wise C_4-condition, where $k \geq \max\{r, 5\}$ or $k = r = 4$. Then*

$$\delta^*(H) \leq \max\{\delta_0(H), 1 - 1/\chi^*(H), 1 - 1/(k-1)\}.$$

Before proving Lemma 10.1, let us prove the following important corollary.

Corollary 10.2. *Let H be an r-chromatic graph. If $r \geq 3$ then $\delta^*(H) \leq 1 - 1/(r+1)$, and if $r = 2$ then $\delta^*(H) \leq 3/4$.*

Proof. The key is to observe that an r-chromatic graph H fails the $(r+2)$-wise C_4-condition. Indeed, take any r-coloring $A_1, \dots, A_r$ of H. Then, considering the $(r+2)$-coloring $A_1, \dots, A_r, A_{r+1}, A_{r+2}$ with $A_{r+1} = A_{r+2} = \emptyset$, we see that $e_H(A_1, A_2) + e_H(A_{r+1}, A_{r+2}) - e_H(A_1, A_{r+1}) - e_H(A_2, A_{r+2}) = e_H(A_1, A_2) > 0$ (as H is r-chromatic). So indeed H violates the $(r+2)$-wise C_4-condition. For $r \geq 3$ (resp. $r = 2$), the corollary now follows by Lemma 10.1 applied with $k = r + 2 \geq 5$ (resp. $k = 5$). □

We now proceed with the proof of Lemma 10.1. As always, we work under the setup introduced in Section 6.2. In particular, we always assume that

$$\delta(R)/|R| \geq \max\{\delta_0(H), 1 - 1/\chi^*(H), 1 - 1/(k-1)\} + \eta/2. \tag{38}$$

Recall the definition of a $(K_k, +)$- and $(K_k, -)$-star, and the head of such a star (see Definition 7.9). Evidently, every 2-edge-colored triangle is either monochromatic or a star. The proof of Lemma 10.1 is split into two cases: $k = r = 4$ and $k \geq 5$. The difference between these cases stems from the fact that the $(K_4, +)$-star has zero discrepancy (while the $(K_k, +)$-star has non-zero discrepancy for $k \geq 5$).

10.1 Proof of Lemma 10.1: $k \geq 5$

Here we prove Lemma 10.1 in the case $k \geq 5$. If there exists a copy of K_k in R which is neither monochromatic nor a star with respect to f_R, then, by Lemma 7.10, this copy of K_k is a template for H, and then by Lemma 6.7, there exists a perfect H-factor in G' with high discrepancy. Therefore, let us assume that all the copies of K_k in R are either monochromatic or a star. In the following argument, we make repeated use of the following three facts:

F.1 Every four vertices in R have at least one common neighbor.

F.2 For all $k' < k$, each copy of $K_{k'} \subseteq R$ is contained in some copy of $K_k \subseteq R$.

F.3 For all $3 \leq k' \leq k$, every copy of $K_{k'}$ which contains a non-monochromatic triangle must be a star with the head of the triangle being the head of the star.

F.1 and **F.2** follow from $\delta(R) > 1 - 1/(k-1) \geq 3/4$, by (38). And **F.3** follows from **F.2**, since otherwise there is a copy of K_k in R which is neither a star nor monochromatic. The following claim is an important step in this proof.

Claim 10.3. *If $v \in V(R)$ is the head of some non-monochromatic triangle $T \subseteq R$. Then, for every triangle $T' \subseteq R$ with $v \in V(T')$, it holds that $f_R(T') = f_R(T)$ (i.e. T' is colored the same way as T) and v is the head of T'.*

Proof. Let $T = \{u, v, w\} \subseteq R$ be as in the statement and let us assume without loss of generality that $f_R(T) = 1$, meaning that $f_R(vu) = f_R(vw) = 1$ and $f_R(uw) = -1$. Let $T' \subseteq R$ be an arbitrary triangle with $v \in T'$, and write $T' = \{u', v, w'\}$. By **F.1**, the vertices u, v, w, u' have a common neighbor $x \in R$. Note that $R[\{u, v, w, x\}]$ is a copy of K_4 in R containing T and thus, by **F.3**, we have $f_R(vx) = -f_R(wx) = 1$. Therefore, $R[\{v, w, x\}]$ is a non-monochromatic triangle with v as its head. By **F.1**, there exists $y \in R$ such that y is a common neighbor of v, w, x, u'. By applying **F.3** to the 4-clique $\{v, w, x, y\}$ we get that $f_R(vy) = -f_R(xy) = 1$, and by applying **F.3** to the 4-clique $\{v, u', x, y\}$ we get that $f_R(vu') = -f_R(u'y) = 1$. Finally, let $z \in R$ be a common neighbor of v, y, u', w'. Using **F.3** as before, we find that $f_R(vz) = -f_R(u'z) = 1$ by considering the 4-clique $\{v, u', y, z\}$, and that $f(vw') = 1 = -f(u'w') = 1$ by considering the 4-clique $\{v, u', w', z\}$. So T' is indeed a non-monochromatic triangle with v as its head and $f_R(T') = 1$. □

Claim 10.3 implies that if v is the head of a non-monochromatic triangle, then v is not contained in any monochromatic triangle and must be the head of any (non-monochromatic) triangle containing it. Also, all edges inside $N_R(v)$ (the neighborhood of v in R) have the same color.

Now, we can make a further statement about the coloring of the copies of K_k in R. Recall that $K_{k,+}$ (resp. $K_{k,-}$) denotes the monochromatic k-clique where all edges have color 1 (resp. -1).

Claim 10.4. *Either every copy of K_k in R is a copy of $K_{k,+}$ or the $(K_k, -)$-star, or every copy of K_k in R is a copy of $K_{k,-}$ or the $(K_k, +)$-star.*

Proof. Observe that $K_{k,+}$ and a $(K_k, -)$-star both contain a monochromatic triangle in color 1, and similarly, $K_{k,-}$ and a $(K_k, +)$-star both contain a monochromatic triangle in color -1. Therefore, if the claim does not hold, then R contains monochromatic triangles L_+ in color 1 and L_- in color -1. By Claim 10.3, none of the vertices in L_+ and L_- are the heads of any stars of size 3, because they belong to a monochromatic triangle. Let $v_1, v_2 \in L_+$ and $v_3, v_4 \in L_-$ and by **F.1**, let u be a common neighbor of v_1, v_2, v_3, v_4. Note that u cannot be the head of any non-monochromatic triangle, since this would imply (by Claim 10.3) that all edges in $N_R(u)$ have the same color, while $f_R(v_1v_2) \neq f_R(v_3v_4)$. It follows that the triangles $\{u, v_1, v_2\}$,$\{u, v_3, v_4\}$ are monochromatic, because none of the vertices $u, v_1, \ldots, v_4$ can be the head of a non-monochromatic triangle. So we have $f_R(uv_1) = f_R(uv_2) = -f_R(uv_3) = -f_R(uv_4) = 1$. Let w be a common neighbor of u, v_1, v_3 (using **F.1**). Without loss of generality, suppose that $f_R(uw) = 1$. Then the triangle w, u, v_3 is not monochromatic, and its head must be w. Now, by Claim 10.3, the triangle w, u, v_1 must also be non-monochromatic with head w. This implies that $f_R(uv_1) = f_R(uv_3)$, a contradiction. □

By Claim 10.4 and without loss of generality, let us assume that every copy of K_k in R is either a copy of $K_{k,+}$ or a $(K_k, -)$-star. Note that both $K_{k,+}$ and a $(K_k, -)$-star have positive discrepancy, because $k \geq 5$. We now consider two sub-cases based on whether H is regular.

Case 1: H is non-regular. If $k = r$, then every copy of K_r in r has positive discrepancy. We then get a perfect H-factor in G' with high discrepancy by Lemma 9.1. So let us assume that $k \geq r + 1$. Suppose first that there is a $(K_k, -)$-star K in R. Clearly, K contains a $(K_{r+1}, -)$-star $K' \subseteq K$. By Corollary 7.3, (K', f_R) is a template for H (as H is non-regular). Now, by Lemma 6.7, G has a perfect H-factor with high discrepancy, completing the proof in this case. Therefore, we may assume that every copy of K_k in R is monochromatic in color 1. Then, by **F.2** with $k' = 2$, all edges in R have color 1. By Lemma 9.1 again, G' has a perfect H-factor with high discrepancy.

Case 2: H is d-regular for some $d \in \mathbb{N}$. Let $U \subseteq V(R)$ be the set of vertices which are the heads of a $(K_3, -)$-star in R. Observe that if $uv \in E(R)$ has color -1 then $u \in U$ or $v \in U$. Indeed, by **F.2**, uv is contained in some triangle in R, and this triangle must be a $(K_3, -)$-star (as every triangle in R is either a $K_{3,+}$ or a $(K_3, -)$-star). The head of this star must be u or v, so one of them is in U. We see that $R[V(R) \backslash U]$ only has edges colored 1. Additionally, U is an independent set in R. To see this, let $u, v \in U$

and assume towards a contradiction that $uv \in E(R)$. By **F.2**, uv is contained in some triangle in R. By Claim 10.3 and the definition of U, both u and v must be the head of this triangle, a contradiction.

By (38) and as $k \geq 5$, we have $\delta(R) \geq (3/4+\eta/2)|R|$. Since U is an independent set in R, we must have $|U| \leq (1/4-\eta/2)|R|$. Therefore, $V_U := \bigcup_{u \in U} V_u$ satisfies

$$|V_U| \leq (1/4-\eta/2)n.$$

Note that all the edges of color -1 in G' are incident to either V_U or V_0, because all edges in R outside U have color 1. By Lemma 6.5, G' has a perfect H-factor F. Since H is d-regular, so is F. Hence, the number of edges of color -1 in F is at most $(|V_U|+|V_0|) \cdot d \leq nd/4$. It follows that

$$f(F) \geq \frac{d}{2}n - nd/4 \geq \gamma n.$$

This concludes the proof.

10.2 Proof of Lemma 10.1: $k = r = 4$

Here we prove Lemma 10.1 in the case $k = r = 4$. If R contains a template for H of size 4, then by Lemma 6.7, there exists a perfect H-factor in G' with high discrepancy, as required. So let us assume that R contains no such template.

We consider two cases. Suppose first that the $(K_4,+)$-star is not a template for H. We claim that in this case, $\delta^*(H) = \delta_0(H) = 3/4$. For the upper bound, recall that if H violates the k-wise C_4-condition, then it also violates the $(k+1)$-wise C_4-condition. Hence, H violates the 5-wise C_4-condition, and by the case $k \geq 5$ of Lemma 10.1, we have $\delta^*(H) \leq 3/4$. For the lower bound, let c be the 2-edge-coloring of K_4 corresponding to the $(K_4,+)$-star. Note that $c(K_4) = 0$. Let B be the $3!|H|$-blowup of (K_4, c). By Lemma 5.2, there is a perfect H-factor of B with discrepancy 0, as $c(K_4) = 0$. As we assumed that the $(K_4,+)$-star is not a template for H, we get that $c \in \mathcal{K}(H)$, and hence

$$\delta_0(H) \geq \delta(B)/|B| = 3/4,$$

as required.

From now on, let us assume that the $(K_4,+)$-star is a template for H, and by symmetry so is the $(K_4,-)$-star. As we assumed that R has no template for H, we get that R contains no $(K_4,+)$-star and no $(K_4,-)$-star. It now follows, by Lemma 7.10, that all copies of K_4 in R are monochromatic. By (38), we have $\delta(R)/|R| > 2/3$. This implies that each triangle in R is contained in a K_4, and hence all triangles in R are monochromatic. We claim that all edges of R have the same color. Suppose not. Then, as R is connected (by $\delta(R)/|R| > 2/3$), there exist vertices $u, v, w \in V(R)$ such that $f_R(uv) = 1$ and $f_R(vw) = -1$. Again using $\delta(R)/|R| > 2/3$, there exists a common neighbor x of u, v, w. It follows that either x, u, v or x, v, w form a non-monochromatic triangle in R, depending on the color of xv with respect to f_R. This gives a contradiction. So we see that R is monochromatic, which means that all edges of G' not touching V_0 have the same color. By Lemma 6.5, G' has an H-factor F. Now we get

$$|f(F)| \geq \left(\frac{n}{|H|} - 2|V_0|\right) e(H) \geq \gamma n.$$

11 Non-regular H

In this section we deal with the case that H is non-regular. This comprises the main part of the proofs of Theorems 1.7 and 1.11. We shall prove three key lemmas (Lemmas 11.1, 11.3 and 11.4) that are used in the proofs of these theorems. The basic idea in the proof of these lemmas is as follows. First, in all cases, the minimum degree assumption implies that the reduced graph R contains r-cliques. Then, by

Lemma 9.1, we may assume that there exists an r-clique L_1 with positive discrepancy, as well as an r-clique L_2 with negative discrepancy. Using Lemma 4.1, we can then connect L_1, L_2 with a sequence of r-cliques $L_1 = L'_1, \dots, L'_\ell = L_2$ with each pair of consecutive cliques L'_i, L'_{i+1} intersecting in at least $r-1$ or at least $r-2$ vertices, depending on the assumed minimum degree of R. We therefore have two r-cliques sharing $r-1$ or $r-2$ vertices, one having positive discrepancy and the other negative. With a slight abuse of notation, we assume that L_1, L_2 are such r-cliques. Then, either $L_1 \cup L_2$ is a template for H (in which case we are done by Lemma 6.7), or the coloring of L_1, L_2 has some specific structure, by the lemmas from Section 7. In more involved cases (mainly Lemma 11.4), we determine properties of the coloring on a large portion of R, under the assumption that R has no small template for H.

In each of the three lemmas we shall make certain assumptions on the residue of r modulo 4, which correspond to different cases in the proof of Theorem 1.11. We also often assume that H satisfies the r-wise C_4-condition. (If H violates the r-wise C_4-condition, then Lemma 10.1 immediately gives the required bounds for Theorem 1.11, as we shall see in Section 12.3.) The first lemma is as follows.

Lemma 11.1. *If $r \not\equiv_4 0$ and H is non-regular, then $\delta^*(H) \leq 1 - 1/r$.*

In the proof of Lemma 11.1, we may assume that $\delta(R)/|R| > 1 - 1/r$. This assumption has two important consequences: First, it guarantees that $|L_1 \cap L_2| = r - 1$, and second, it implies that every r-clique is contained in an $(r+1)$-clique. This allows us to use Lemma 7.2 and Corollary 7.3 to conclude the proof. The details follow.

Proof of Lemma 11.1. As always, we work under the setup described in Section 6.2. In particular, as we are aiming for the bound $\delta^*(H) \leq 1 - 1/r$, we assume that

$$\delta(G')/n,\ \delta(R)/|R| \geq (1 - 1/r + \eta/2).$$

Our goal is to show that G' contains a perfect H-factor with high discrepancy. If R has a template for H of size $r+1$, then we are done by Lemma 6.7. We therefore assume that R has no such template. This implies that for every $(r+1)$-clique M in R, M^+ is regular (with respect to f_R), because otherwise (M, f_R) is a template for H by Corollary 7.3 (as H is non-regular). Next, we need the following very simple claim.

Claim 11.2. *Let M be an $(r+1)$-clique with an edge-coloring c, let d be such that M^+ is d-regular, and let $L \subseteq M$, $|L| = r$. Then $c(L) = (r-1)(d - r/2)$.*

Proof. $e(L^+) = e(M^+) - d = d(r+1)/2 - d = d(r-1)/2$. Hence,

$$c(L) = 2e(L^+) - \binom{r}{2} = 2 \cdot d(r-1)/2 - \binom{r}{2} = d(r-1) - \binom{r}{2} = (r-1)(d - r/2).$$

□

We now continue with the proof of the lemma. Suppose first that there exist two copies $M_1, M_2 \subseteq R$ of K_{r+1} such that M_1^+ is d-regular and M_2^+ is d'-regular for some $d \neq d'$. Let $L_1 \subseteq M_1$ and $L_2 \subseteq M_2$ of size r each. Since $d \neq d'$, it follows by Claim 11.2 that $f_R(L_1) \neq f_R(L_2)$. By Lemma 4.1 there exists a sequence $L'_1, L'_2, \dots L'_\ell \subseteq R$ of copies of K_r with $L'_1 = L_1$ and $L'_\ell = L_2$ and such that L'_i and L'_{i+1} share $r-1$ vertices for each $1 \leq i \leq \ell - 1$. But then, there must exist some $1 \leq i \leq \ell - 1$ such that $f_R(L'_i) \neq f_R(L'_{i+1})$. Now $(L'_i \cup L'_{i+1}, f_R)$ is a template for H by Lemma 7.2, in contradiction to our assumption.

So from now on, we assume that there exists $d \in \mathbb{N}$ such that for every $(r+1)$-clique $M \subseteq R$, M^+ is d-regular. Trivially, $2 \mid d(r+1)$. Note that $d \neq r/2$, because if $r+1$ is even then r is odd and so $d \neq r/2$, and if $r+1$ is odd then d must be even, so $d \neq r/2$ as $r \not\equiv_4 0$. Without loss of generality, let us assume that $d > r/2$ (otherwise consider M^- in place of M^+, replacing d with $r - d$). We claim that every copy L of K_r in R has positive discrepancy. Indeed, as $\delta(R)/|R| > 1 - 1/r$, there exists an $(r+1)$-clique M containing L. Now, Claim 11.2, $f_R(L) = (r-1)(d - r/2) > 0$. Finally, by Lemma 9.1, G' has a perfect H-factor with high discrepancy, as required. □

The following is the second of the three lemmas.

Lemma 11.3. *Suppose that $r \equiv_4 0$. Assume that H is non-regular, fulfills the r-wise C_4-condition, and violates the $(r+1)$-wise C_4-condition. Then*

$$\delta^*(H) \le \max\{\delta_0(H), 1 - 1/\chi^*(H)\}.$$

The proof of Lemma 11.3 proceeds by distinguishing between two cases. The first case is that there exists an H-factor which is not balanced-uniform (recall Definition 7.12). In this case we will show that $\delta_0(H) = 1 - 1/r$ and hence $\delta^*(H) \ge 1 - 1/r$, and this will match the upper bound on $\delta^*(H)$ we get from Lemma 10.1. Here the assumption $r \equiv_4 0$ will play a crucial role. The second case is that there exists a non-balanced-uniform H-factor. Here we will proceed by finding two r-cliques L_1, L_2 with $|L_1 \cap L_2| \ge r-2$ and such that L_1 has positive discrepancy and L_2 has negative discrepancy, as explained above. We will eventually conclude that $L_1 \cup L_2$ is a template for H by Lemma 7.5, finishing the proof. The details follow.

Proof of Lemma 11.3. As always, we work under the setup of Section 6.2. In particular, we assume that

$$\delta(G')/n,\ \delta(R)/|R| \ge \max\{\delta_0(H), 1 - 1/\chi^*(H)\} + \eta/2.$$

As $\chi^*(H) \ge r - 1$ for every r-chromatic graph, we have

$$\delta(R)/|R| > \frac{r-2}{r-1}. \tag{39}$$

Our goal is to show that G' contains a perfect H-factor with high discrepancy. Since H violates the $(r+1)$-wise C_4-condition, we may apply Lemma 10.1 with $k = r+1$ to get

$$\delta^*(H) \le 1 - 1/r.$$

Hence, if $\max\{\delta_0(H), 1 - 1/\chi^*(H)\} = 1 - 1/r$ then we are done. So from now on we assume that

$$\max\{\delta_0(H), 1 - 1/\chi^*(H)\} < 1 - 1/r. \tag{40}$$

In particular, $\chi^*(H) < r$, which implies that H has an unbalanced r-coloring. We now consider two cases. For what follows, recall Definition 7.12.

Case 1: Every H-factor is balanced-uniform. We will show that then $\delta_0(H) = 1 - 1/r$, which would contradict our assumption (40) and hence conclude the proof in this case. Fix a 2-edge-coloring c of K_r such that $c(K_r) = 0$; such a coloring exists because K_r has an even number of edges, as $r \equiv_4 0$. By Lemma 7.15, we have $c \in \mathcal{K}(H)$. Let B be an m-blowup of (K_r, c), where m is divisible by $(r-1)!|H|$. By Lemma 5.2, B has a perfect H-factor. We claim that for every perfect H-factor F_B of B it holds that $c(F_B) = 0$. Indeed, let $B_1, \dots, B_r$ be the parts of B. By assumption, F_B is balanced-uniform. Hence, $e_{F_B}(B_i, B_j) = e(F_B)/\binom{r}{2}$ for all $1 \le i < j \le r$. Therefore, $c(F_B) = c(K_r) \cdot e(F_B)/\binom{r}{2} = 0$, as required. It follows that $\delta_0(H) \ge \delta(B)/|B| = 1 - 1/r$, as claimed.

Case 2: There exists a non-balanced-uniform union F of disjoint copies of H.[1] By Claim 7.14, there exists a balanced r-coloring $A_1, \dots, A_r$ of F such that $e_F(A_1, A_2) \ne e_F(A_3, A_4)$. We may assume that R contains no template for H on at most $r+2$ vertices, as otherwise, by Lemma 6.7, G' contains a perfect H-factor of high discrepancy and we are done.

If R contains an r-clique L such that L^+ is non-regular (with respect to f_R), then by Lemma 7.13, (L, f_R) is a template for H, contradicting our assumption. So suppose from now on that every r-clique L in R is such that L^+ is regular.

[1] It is worth noting that the argument in Case 2 only requires that $r \not\equiv_4 1$ (instead of the stronger assumption $r \equiv_4 0$).

Assume first that there exist two r-cliques $L_1, L_2 \subseteq R$ such that L_1^+ is d-regular and L_2^+ is d'-regular for some $d \neq d'$. Using (39) and Item 2 of Lemma 4.1, we obtain a sequence of r-cliques $L_1', L_2', \dots, L_\ell' \subseteq R$ with $L_1' = L_1$ and $L_\ell' = L_2$, such that each pair of subsequent r-cliques share at least $r-2$ vertices. So there exist two r-cliques L, L' in R sharing at least $r-2$ vertices, such that L^+ is d-regular and L'^+ is d'-regular for some $d \neq d'$. Without loss of generality, let us assume that L_1, L_2 were such r-cliques to begin with. If $|L_1 \cap L_2| = r-1$ then $(L_1 \cup L_2, f_R)$ is a template for H by Lemma 7.2, and if $|L_1 \cap L_2| = r-2$ then $(L_1 \cup L_2, f_R)$ is a template for H by Lemma 7.5. In either case, we get a contradiction to our assumption.

Now assume that r-cliques L in R are d-regular with the same d (in the sense that L^+ is d-regular). Without loss of generality, let us assume that $d \geq (r-1)/2$, as otherwise we may swap the colors, replacing d with $r-1-d$. Note that $d \neq \frac{r-1}{2}$ because $r \not\equiv_4 1$. As $d > \frac{r-1}{2}$, all copies of K_r in R have positive discrepancy. Now, by Lemma 9.1, G' has a perfect H-factor with high discrepancy, completing the proof. □

Finally, we arrive at the last of the three main lemmas, Lemma 11.4. This lemma deals with the case $r \equiv_4 2, 3$. Its proof is by far the most involved part of this section. Recall the definition of a butterfly from the introduction.

Lemma 11.4. *Suppose that $r \geq 3$ and $r \equiv_4 2, 3$. Assume that H satisfies the r-wise C_4-condition and is non-regular. Then*

$$\delta^*(H) \leq \begin{cases} \max\{\delta_0(H), 1 - 1/\chi^*(H), 4/7\} & r = 3 \text{ and some butterfly is not a template for } H, \\ \max\{\delta_0(H), 1 - 1/\chi^*(H)\} & \text{otherwise.} \end{cases}$$

Let us comment on the proof of Lemma 11.4. Similarly to previous proofs, the proof of Lemma 11.4 begins by finding two r-cliques L_1, L_2 with $|L_1 \cap L_2| = r-2$ such that L_1 has positive discrepancy and L_2 has negative discrepancy. As always, we may assume that R contains no small template for H, as otherwise we are done by Lemma 6.7. In particular, $L_1 \cup L_2$ is not a template for H. Then, by Lemma 7.18, H is (s,t)-structured (for s, t given by that lemma). In the case $r \neq 3$ (namely $r \geq 6$), we can use Lemma 8.7 to deduce that $\max\{\delta_0(H), 1 - 1/\chi^*(H)\} \geq \frac{3r-5}{3r-2}$, which allows us to assume that $\delta(R)/|R| > \frac{3r-5}{3r-2}$. This minimum degree assumption is crucial for the proof, allowing us to establish various structural properties of R and f_R. Eventually we show that R is strongly tilted towards one of the colors, which allows us to find a perfect H-factor of high disrepancy.

The case $r = 3$ is somewhat different. Note that $\frac{3r-5}{3r-2} = \frac{4}{7}$ for $r = 3$. Big parts of the proof for $r \neq 3$ carry over to the case $r = 3$, provided we assume that $\delta(R)/|R| > \frac{4}{7}$. However, we may not make this assumption in all cases, because for some 3-chromatic graphs H, the value of $\delta^*(H)$ is smaller than $\frac{4}{7}$. It turns out that making the assumption $\delta(R)/|R| > \frac{4}{7}$ is justified exactly when some butterfly is not a template for H (cf. Lemma 8.3). The proof of Lemma 11.4 is given in the next subsection.

11.1 Proof of Lemma 11.4

By Lemma 11.1, we have

$$\delta^*(H) \leq 1 - 1/r. \tag{41}$$

Therefore, if $\max\{\delta_0(H), 1 - 1/\chi^*(H)\} = 1 - 1/r$ then the assertion of Lemma 11.4 holds. So from now on, we assume that $\max\{\delta_0(H), 1 - 1/\chi^*(H)\} < 1 - 1/r$. In particular, $\chi^*(H) < r$. By the definition of χ^*, this implies that

$$H \text{ has an unbalanced } r\text{-coloring.} \tag{42}$$

As always, we work under the setup of Section 6.2, therefore assuming that

$$\delta(R)/|R| \geq \max\{\delta_0(H), 1 - 1/\chi^*(H)\} + \eta/2, \tag{43}$$

and

$$\delta(R)/|R| \geq 4/7 + \eta/2 \text{ if } r = 3 \text{ and some butterfly is not a template for } H. \tag{44}$$

Since $\chi^*(H) > r - 1$ for every r-chromatic H, (43) implies that

$$\delta(R)/|R| > \frac{r-2}{r-1}. \tag{45}$$

We shall show that G' has a perfect H-factor with high discrepancy. Throughout the proof, we assume that R contains no template for H on at most $r + 2$ vertices, as otherwise we are done by Lemma 6.7. The following claim is used multiple times throughout the proof.

Claim 11.5. *There is no sequence $L'_1, L'_2, \ldots, L'_\ell \subseteq R$ of copies of K_r, such that $|L'_i \cap L'_{i+1}| \geq r - 1$ for every $1 \leq i \leq \ell - 1$ and $f_R(L'_1) \neq f_R(L'_\ell)$.*

Proof. As $f_R(L'_1) \neq f_R(L'_\ell)$, there exists $1 \leq i \leq \ell - 1$ such that $f_R(L'_i) \neq f_R(L'_{i+1})$. Using that H is non-regular, we get that $(L'_i \cup L'_{i+1}, f_R)$ is a template for H by Lemma 7.2. However, we assumed that R has no such template for H, a contradiction. □

By (45), R contains a copy of K_r. Since $r \equiv_4 2, 3$, K_r has an odd number of edges and thus, without loss of generality, let us assume that R contains a copy L_1 of K_r with $f_R(L_1) > 0$. If all copies of K_r in R have positive discrepancy, then we are done by Lemma 9.1. Suppose therefore that R also contains a copy L_2 of K_r with $f_R(L_2) < 0$. By Lemma 4.1 and (45), there exists a sequence $L'_1, L'_2, \ldots, L'_\ell \subseteq R$ of copies of K_r with $L'_1 = L_1$ and $L'_\ell = L_2$ such that every pair of subsequent copies share at least $r - 2$ vertices. Therefore, there exist two copies of K_r sharing at least $r - 2$ vertices with discrepancies of different signs. Without loss of generality, let us assume that L_1 and L_2 are such copies. If $|L_1 \cap L_2| = r - 1$ then this is a contradiction to Claim 11.5. Suppose then that $|L_1 \cap L_2| = r - 2$. Put $V = L_1 \cap L_2$, $L_1 \setminus L_2 = \{x_1, y_1\}$ and $L_2 \setminus L_1 = \{x_2, y_2\}$.

We proceed with the proof of Lemma 11.4. By assumption, $(L_1 \cup L_2, f_R)$ is not a template for H. Hence, by Lemma 7.18, H is (s,t)-structured with

$$s := \frac{f_R(L_1) - f_R(x_1y_1) - f_R(L_2) + f_R(x_2y_2)}{2(r-2)} \tag{46}$$

and

$$t := f_R(x_1y_1) - f_R(x_2y_2). \tag{47}$$

Crucially, note that $(s,t) \neq (0,0)$. Indeed, if $t = 0$ then $s = \frac{f_R(L_1) - f_R(L_2)}{2(r-2)} \neq 0$ because $f_R(L_1) \neq f_R(L_2)$.

By the definition of being (s,t)-structured (see Definition 7.17), there exists $\rho \in \mathbb{R}$ such that for every r-coloring $A_1, \ldots, A_r$ of H and for all $1 \leq i < j \leq r$, it holds that

$$\rho(|A_i| + |A_j|) = s \cdot e_H(A_i \cup A_j, V(H) \backslash (A_i \cup A_j)) + t \cdot e_H(A_i, A_j). \tag{48}$$

First we handle the case that H is uniform (recall Definition 7.6). Then for every r-coloring $A_1, \ldots, A_r$ of H and for all $1 \leq i < j \leq r$, it holds that

$$\begin{aligned}\rho(|A_i| + |A_j|) &= s \cdot e_H(A_i \cup A_j, V(H) \backslash (A_i \cup A_j)) + t \cdot e_H(A_i, A_j) \\ &= (2(r-2)s + t) \cdot \frac{e(H)}{\binom{r}{2}} = \frac{f_R(L_1) - f_R(L_2)}{\binom{r}{2}} \cdot e(H),\end{aligned}$$

where the second equality uses the uniformity of H. It follows that $\rho \neq 0$ because $f_R(L_1) \neq f_R(L_2)$. We get that $|A_i| + |A_j|$ is the same for all $1 \leq i < j \leq r$. By Claim 7.20, $|A_1| = \cdots = |A_r|$. This means that H only has balanced r-colorings, contradicting (42).

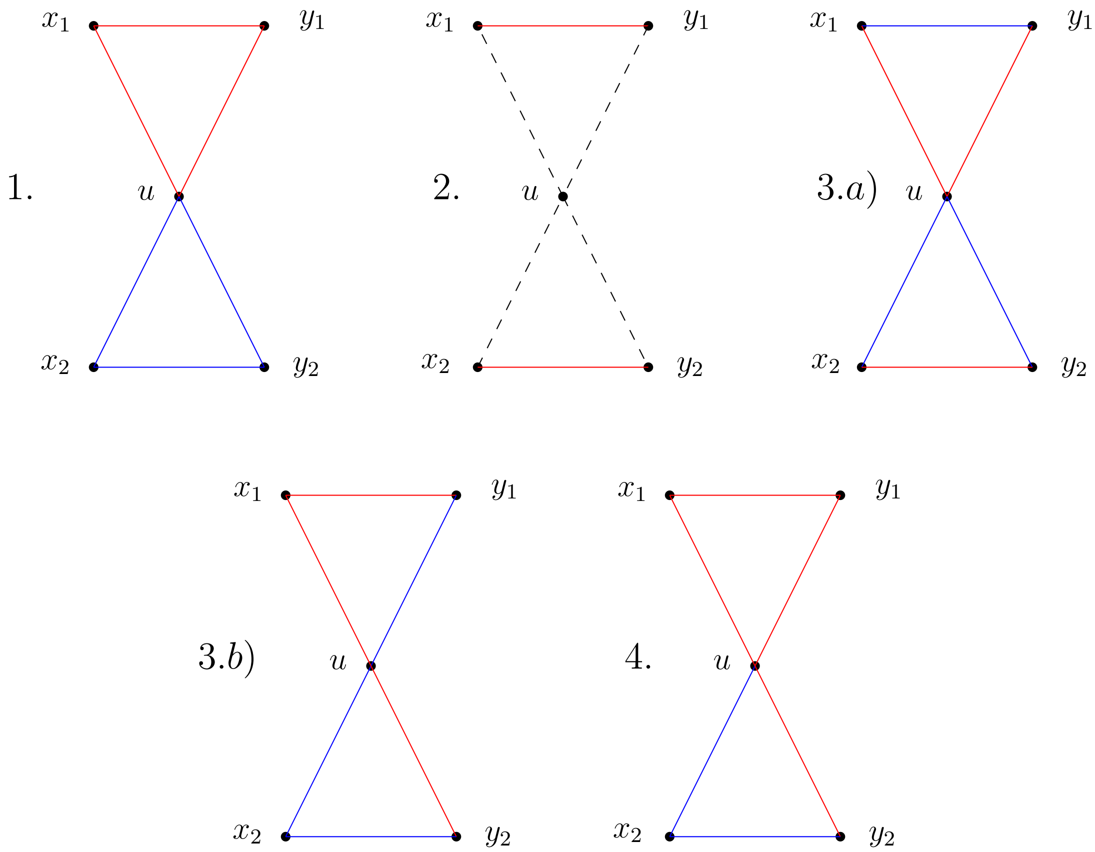

Figure 4: The possible configurations of L_1, L_2 corresponding to different cases in Claim 11.6.

For the rest of the proof, we assume that H is non-uniform. By Lemma 7.7 and as $(L_1 \cup L_2, f_R)$ is not a template for H (by assumption), we have that

$$f_R(L_1) - f_R(x_1y_1) - f_R(L_2) + f_R(x_2y_2) \in \{-4(r-2), -2(r-2), 0, 2(r-2), 4(r-2)\}. \quad (49)$$

In the next claim, for the case $r = 3$, we classify the possible colorings of the triangles L_1, L_2. Also, for each case, we specify the corresponding values of s, t.

Claim 11.6. *Suppose that $r = 3$. Then one of the following holds:*

1. *L_1, L_2 are monochromatic. In this case $(s,t) = (2,2)$.*
2. *$f_R(x_1y_1) = f_R(x_2y_2)$. In this case $t = 0$.*
3. *$(L_1 \cup L_2, f_R)$ is a butterfly and L_1, L_2 are not monochromatic. In this case $(s,t) \in \{(2,-2),(0,2)\}$.*
4. *Exactly one of L_1 and L_2 is monochromatic and $f_R(x_1y_1) \neq f_R(x_2y_2)$. In this case $(s,t) = (1,2)$.*

Proof. Clearly, if L_1 is monochromatic then it has color 1 (as $f_R(L_1) > 0$), and similarly, if L_2 is monochromatic then it has color -1. If $f_R(x_1y_1) = f_R(x_2y_2)$ then we are in Case 2, and $t = 0$ by (47). So let us assume that $f_R(x_1y_1) \neq f_R(x_2y_2)$. If L_1 is monochromatic then $f_R(x_2y_2) = -f_R(x_1y_1) = -1$, so this is covered by Case 1 or Case 4. In both cases, it is immediate to compute s, t from (46) and (47).

Suppose then that L_1 is not monochromatic. By symmetry (with respect to switching the colors), we can also assume that L_2 is not monochromatic.

Write $L_1 \cap L_2 = \{u\}$. Suppose first that $f_R(x_1y_1) = -1$. Then $f_R(x_2y_2) = 1$, and we also must have $f_R(ux_1) = f_R(uy_1) = 1$, $f_R(ux_2) = f_R(uy_2) = -1$ (because $f_R(L_1) > 0$, $f_R(L_2) < 0$). So $(L_1 \cup L_2, f_R)$ is a butterfly (Case 3), and $(s,t) = (2,-2)$ by (46) and (47). Suppose now that that $f_R(x_1y_1) = 1$. Then, $f_R(x_2y_2) = -1$, and without loss of generality (up to switching x_1, y_1 or x_2, y_2), we have $f_R(ux_1) = 1$, $f_R(uy_1) = -1$, $f_R(ux_2) = -1$, $f_R(uy_2) = 1$. So again $(L_1 \cup L_2, f_R)$ is a butterfly, and $(s,t) = (0,2)$. □

Next, we address some subcases of the case $r = 3$, namely, Cases 1-2 in Claim 11.6. These cases need to be handled separately.

11.1.1 Cases 1-2 of Claim 11.6

Throughout this section we assume that $r = 3$. Our goal is to complete the proof of Lemma 11.4 in Cases 1-2 of Claim 11.6. Case 1 is simple, while Case 2 requires considerable work.

Case 1: By (42), H has an unbalanced 3-coloring. Fix such a coloring A_1, A_2, A_3, and suppose without loss of generality that $|A_1| < |A_3|$. By Claim 11.6, $(s,t) = (2,2)$. By (48), we have

$$\rho(|A_i| + |A_j|) = 2e_H(A_i \cup A_j, V(H)\backslash(A_i \cup A_j)) + 2e_H(A_i, A_j) = 2e(H)$$

for every pair $1 \leq i < j \leq 3$. In particular, $\rho \neq 0$. So $|A_i| + |A_j| = 2e(H)/\rho$ for all $1 \leq i < j \leq 3$. But this implies that $|A_1| + |A_2| = |A_2| + |A_3|$, in contradiction to $|A_1| < |A_3|$. This completes Case 1.

Case 2: $f_R(x_1y_1) = f_R(x_2y_2)$. By Claim 11.6, we have $t = 0$. Then $s \neq 0$, because $(s,t) \neq (0,0)$. By normalizing, we get that H is $(1,0)$-structured (recall that if H is (s,t) structured then it is also $(\alpha \cdot s, \alpha \cdot t)$-structured). With a slight abuse of notation, we will set $(s,t) = (1,0)$ for the rest of Case 2. By (48),

$$\rho(|A_i| + |A_j|) = e_H(A_i \cup A_j, V(H)\backslash(A_i \cup A_j)) = e(H) - e_H(A_i, A_j) \tag{50}$$

holds for every 3-coloring A_1, A_2, A_3 of H and for all $1 \leq i < j \leq 3$. Summing this over all i, j, we get

$$2\rho|H| = \rho \sum_{1 \leq i < j \leq 3} (|A_i| + |A_j|) = 3e(H) - \sum_{1 \leq i < j \leq 3} e_H(A_1, A_2) = 2e(H).$$

Hence,

$$e(H) = \rho|H|. \tag{51}$$

This implies that $\rho > 0$. Combining (51) with (50), we get

$$e_H(A_i, A_j) = \rho(|H| - |A_i| - |A_j|). \tag{52}$$

Next, we show that a pair of intersecting triangles in R cannot have opposite edges of different color.

Claim 11.7. *Let L_1', L_2' be two distinct triangles in R with $|L_1' \cap L_2'| \geq 1$. Let e_1, e_2 be edges with $L_1' \backslash L_2' \subseteq e_1$ and $L_2' \setminus L_1' \subseteq e_2$. Then $f_R(e_1) = f_R(e_2)$.*

Proof. Suppose first $|L_1' \cap L_2'| = 1$, so that $e_1 \cap e_2 = \emptyset$. Assume, for the sake of contradiction, that $f_R(e_1) \neq f_R(e_2)$. By assumption, $(L_1' \cup L_2', f_R)$ is not a template for H. Then, by Lemma 7.18, H is (s', t')-structured with $t' = f_R(e_1) - f_R(e_2) \neq 0$ (the value of s' will not be important). By normalizing, H is $(s'', 1)$-structured for some s''. In addition, we saw above that H is $(1,0)$-structured. So by Lemma 7.21, every 3-coloring of H is balanced, a contradiction.

Suppose now $|L_1' \cap L_2'| = 2$, and write $e_1 = vw_1$, $e_2 = vw_2$, $L_1' = \{u, v, w_1\}$, $L_2' = \{u, v, w_2\}$. Recall that $\delta(R)/|R| \geq 1/2 + \eta/2$ by (45). This implies that for each $i = 1, 2$, there exists a common neighbour z_i of u, w_i, $z_i \neq v$. By the previous case for the triangles $\{u, v, w_2\}$ and $\{u, w_1, z_1\}$, it holds that $f_R(w_1z_1) = f_R(vw_2) = f_R(e_2)$. Similarly, $f_R(w_2z_2) = f_R(vw_1) = f_R(e_1)$. Finally, by considering the triangles $\{u, w_1, z_1\}$ and $\{u, w_2, z_2\}$, we get that $f_R(w_1z_1) = f_R(w_2z_2)$, so $f_R(e_1) = f_R(e_2)$. □

Claim 11.7 means that for every vertex $v \in V(R)$, all edges opposite v in triangles containing v have the same color. Let $W_+ \subseteq V(R)$ be the vertices for which these edges have color 1, and let $W_- = V(R)\backslash W_+$ be the vertices for which these edges have color -1. We now consider two cases according to the size of the sets W_+, W_-.

Case 2(a): $|W_+| \geq (1/2+\eta/2)|R|$ or $|W_-| \geq (1/2+\eta/2)|R|$. Without loss of generality, assume that $|W_+| \geq (1/2+\eta/2)|R|$. Recall the definition of the graph H^* (see Definition 5.5). In particular, H^* is a complete 3-partite graph and has a perfect H-factor. Recall the definition of V_0 (i.e., V_0 is the exceptional set in the regular partition of G'). Finally, recall that for a vertex-set $W \subseteq V(R)$, $V_W := \bigcup_{w \in W} V_w \subseteq V(G')$ denotes the union of clusters corresponding to the vertices of W. Put $V_+ = V_{W_+}$ and $V_- = V_{W_-}$. In the following key claim, we show that if J is an H^*-copy in G' disjoint from V_0, then for every perfect H-factor F of J, the discrepancy of F can be expressed in terms of the intersection of $V(J)$ with V_+ and V_-. We will then use this claim to conclude the proof in Case 2(a).

Claim 11.8. *Let J be an H^*-copy in G' with $V(J) \cap V_0 = \emptyset$. Then for every perfect H-factor F of J,*

$$f(F) = \rho\left(|V(F_J) \cap V_+| - |V(F_J) \cap V_-|\right). \tag{53}$$

Proof. Recall that for a vertex $v \in V(G') \setminus V_0$, we use $V_v^R \in V(R)$ to denote the cluster of the regular partition containing v. The assumption $V(J) \cap V_0 = \emptyset$ means that V_v^R is well-defined for every $v \in V(J)$.

Let B_1, B_2, B_3 be the parts of J. We claim that all bipartite graphs (B_i, B_j), $1 \leq i < j \leq 3$, are monochromatic. Indeed, assume by contradiction (and without loss of generality) that there exist $b_1, b_1' \in B_1$ and $b_2 \in B_2$ with $f(b_1b_2) \neq f(b_1'b_2)$. Fix an arbitrary $b_3 \in B_3$. Consider the clusters $V_{b_1}^R, V_{b_1'}^R, V_{b_2}^R, V_{b_3}^R \in V(R)$ corresponding to the vertices b_1, b_1', b_2, b_3, respectively. Then $V_{b_1}^R, V_{b_2}^R, V_{b_3}^R$ and $V_{b_1'}^R, V_{b_2}^R, V_{b_3}^R$ are triangles in R, and the edges $V_{b_1}^R V_{b_2}^R$ and $V_{b_1'}^R V_{b_2}^R$ opposite $V_{b_3}^R$ in these triangles have different colors (as $f(b_1b_2) \neq f(b_1'b_2)$). This is a contradiction, proving our claim that all bipartite graphs (B_i, B_j) are monochromatic. This means that (J, f_R) is a blowup of (K_3, c) for some coloring c of K_3.

Next, we claim that $B_i \subseteq V_+$ or $B_i \subseteq V_-$ for each $i = 1, 2, 3$. Let us prove this for $i = 1$. Fix arbitrary vertices $b_2 \in B_2$ and $b_3 \in B_3$. As J is a complete tripartite graph, every vertex $b_1 \in B_1$ forms a triangle with b_2, b_3. Now, by definition, if $f_R(V_{b_2}^R V_{b_3}^R) = 1$ then $V_{b_1}^R \in W_+$ and hence $b_1 \in V_+$ for all $b_1 \in B_1$, and if $f_R(V_{b_2}^R V_{b_3}^R) = -1$ then $V_{b_1}^R \in W_-$ and hence $b_1 \in V_-$ for all $b_1 \in B_1$. This proves our claim.

Let H' be a copy of H in F, and let $A_i := V(H') \cap B_i$, so that A_1, A_2, A_3 is a 3-coloring of H'. Observe that if $A_3 \subseteq V_+$ (resp. $A_3 \subseteq V_-$) then all edges between A_1, A_2 have color 1 (resp. -1). Moreover, $e_{H'}(A_1, A_2) = \rho|A_3|$ by (52). By using this and the analogous statement for the pairs A_1, A_3 and A_2, A_3, we get the following:

$$f(H') = \sum_{i \in [3]:\, A_i \subseteq V_+} \rho|A_i| - \sum_{i \in [3]:\, A_i \subseteq V_-} \rho|A_i| = \rho\left(\left|V(H') \cap V_+\right| - \left|V(H') \cap V_-\right|\right).$$

Summing the above over all H-copies H' in F gives (53). □

We now conclude Case 2(a). By Lemma 6.5, G' has a perfect H^*-factor F^*. Recall that $e(H) = \rho|H|$ (by (51)) and therefore, any perfect H-factor of H^* has exactly $e(H) \cdot |H^*|/|H| = \rho|H^*|$ edges. Let F be a perfect H-factor of G', obtained by taking a perfect H-factor F_J of each H^*-copy J in F^*. Our assumption $|W_+| \geq (1/2+\eta/2)|R|$ means that $|V_+| \geq (1/2+\eta/2)(n - |V_0|)$ and hence $|V_-| \leq (1/2-\eta/2)(n-|V_0|)$. There are at most $|V_0|$ copies of H^* in F^* which intersect V_0, and for each such copy J, F_J has $\rho|H^*|$ edges. For all other H^*-copies $J \in F^*$, we use (53) with $F = F_J$. Combining all of this, we get:

$$\begin{aligned} f(F) = \sum_{J \in F^*} f(F_J) &\geq \rho \sum_{J \in F^*: V(J) \cap V_0 = \emptyset} \left(|V(F_J) \cap V_+| - |V(F_J) \cap V_-|\right) - \rho|H^*||V_0| \\ &\geq \rho\left(|V_+| - |H^*||V_0| - |V_-|\right) - \rho|H^*||V_0| \geq \rho\eta(n - |V_0|) - 2\rho|H^*||V_0| \geq \rho\eta n/2 \geq \gamma n. \end{aligned} \tag{54}$$

The quantity $|V_+| - |H^*||V_0|$ in the second line of (54) is a lower bound in $\sum_J |V(J) \cap V_+|$ over all J with $V(J) \cap V_0 = \emptyset$. The penultimate inequality in (54) holds because $|V_0| \leq \varepsilon n$, $\varepsilon \ll \frac{1}{|H|}, \eta$, and H^* depends only on H and η. Finally, the last inequality in (54) holds because $\gamma \ll \frac{1}{|H|}, \eta$, and $\rho > 0$ depends only on H. By (54), F has high discrepancy. This concludes Case 2(a).

Case 2(b) $|W_+|, |W_-| < (1/2 + \eta/2)|R|$. We will show that this case is impossible. First, we prove some structural properties of R and f_R.

Claim 11.9. *$R[W_+]$ and $R[W_-]$ are triangle-free.*

Proof. We only prove the assertion for $R[W_+]$; the proof for $R[W_-]$ is analogous. Let $S \subseteq W_+$ be the largest clique in W_+. We need to show that $|S| \leq 2$. Suppose by contradiction that $|S| \geq 3$. Then, all the edges in $R[S]$ must have color 1, because each such edge is contained in a triangle in $R[S]$, hence it is the edge opposite to a vertex from W_+ in a triangle. By (45),

$$\sum_{s \in S} |N_R(s)| \geq (1/2 + \eta/2)|R||S|. \tag{55}$$

Note that no two vertices in S share a neighbor in W_-, because else we get a triangle containing a vertex from W_- whose opposite edge has color 1, contradicting the definition of W_-. Also, each vertex in W_+ can only be connected to at most $|S| - 1$ vertices in S, as otherwise we add this vertex to S and get a larger complete graph inside W_+, contradicting the maximality of S. It follows that

$$\sum_{s \in S} |N_R(s)| \leq |W_-| + (|S| - 1)|W_+| < (1/2 + \eta/2)|R||S|,$$

using $|W_+|, |W_-| < (1/2 + \eta/2)|R|$. This contradicts (55). □

Claim 11.10. *Every vertex in W_+ has an edge of color 1 to some vertex in W_-, and every vertex in W_- has an edge of color -1 to some vertex of W_+.*

Proof. We only prove the assertion for vertices W_+; the other case is symmetrical. So let $v \in W_+$. Since $\delta(R) \geq (1/2 + \eta/2)|R| > |W_+|, |W_-|$ by (45), v must have a neighbour u in W_+. Since $\delta(R) > |R|/2$, there is a common neighbour w of u, v. Then $w \in W_-$ because $R[W_+]$ is triangle-free by Claim 11.9. The edge vw has color 1 because it is the edge opposite $u \in W_+$ in the triangle u, v, w. So v indeed has an edge of color 1 to W_-. □

Claim 11.10 implies that there exists a cycle C in the bipartite graph $R[W_+, W_-]$ with edges of alternating color. To see this, consider an orientation of the edges of $R[W_+, W_-]$ where all edges of color 1 are oriented from W_+ to W_- and all edges of color -1 are oriented from W_- to W_+. In this orientation, every vertex has outdegree at least 1 (by Claim 11.10), and therefore there exists a directed cycle, which corresponds to a cycle whose edges alternate in color.

By averaging, there is a vertex $v \in V(R)$ which is connected to at least $|C| \cdot \delta(R)/|R| > |C|/2$ vertices of C. Note that the edges of C of color 1 as well as the edges of color -1 form a perfect matching of C. Therefore, there exists an edge of color 1 in C such that v is connected to both of its endpoints and similarly, an edge of color -1 in C such that v is connected to both of its endpoints. But then v is contained in a triangle with opposite edge of color 1 and in a triangle with opposite edge of color -1, a contradiction. This completes Case 2(b) and hence Case 2 altogether.

11.1.2 Minimum degree at least $\frac{3r-5}{3r-2}$ and the structure of R

We continue with the proof of Lemma 11.4. From now on, we will assume that if $r = 3$ then Cases 1-2 of Claim 11.6 do not hold. The following key claim provides a lower bound on $\delta(R)/|R|$.

Claim 11.11. *It holds that*

$$\delta(R)/|R| \geq \frac{3r-5}{3r-2} + \eta/2. \tag{56}$$

Proof. By (46) and (49), we have $s \in \{-2,-1,0,1,2\}$. Note also that $t \in \{-2,0,2\}$ by (47), and recall that $(s,t) \neq (0,0)$. We now normalize the parameters s,t in order to apply Lemma 8.7. Recall that if H is (s,t)-structured then it is also $(\alpha \cdot s, \alpha \cdot t)$-structured for every $\alpha \in \mathbb{R}$. If $s = 0$ then by normalizing, we may assume that $t = 1$. And if $s \neq 0$, then by dividing (s,t) by s, we may assume that $s = 1$ and $t \in \{-2,-1,0,1,2\}$. In any case, the normalized parameters fit the assumption of Lemma 8.7. Hence, if $r \geq 6$, then Lemma 8.7 gives

$$\max\{\delta_0(H), 1 - 1/\chi^*(H)\} \geq \frac{3r-5}{3r-2}.$$

So (56) follows from (43).

It remains to prove (56) when $r = 3$. Note that $\frac{3r-5}{3r-2} = \frac{4}{7}$ for $r = 3$. By assumption, we are in Case 3 or 4 of Claim 11.6. Suppose first that we are in Case 3, so $(L_1 \cup L_2, f_R)$ is a butterfly. By assumption, $(L_1 \cup L_2, f_R)$ is not a template for H. Hence, there exists a butterfly which is not a template for H. Now (56) holds by (44). Finally, suppose that we are in Case 4 of Claim 11.6. Then H is $(1,2)$-structured. Now, by Lemma 8.6, $\max\{\delta_0(H), 1 - 1/\chi^*(H)\} \geq \frac{5}{8} > \frac{4}{7}$, so again (56) holds by (43). □

The proof proceeds with a sequence of claims that slowly uncovers the structure of R, deducing the color of various edges from the assumption that R does not have templates for H. For example, in the case $r \neq 3$, we will eventually show that R is strongly tilted towards one of the colors, which will allow us to find a perfect H-factor with high discrepancy. The bound (56) will be crucial.

Recall that $V(L_1) = V \cup \{x_1, y_1\}$ and $V(L_2) = V \cup \{x_2, y_2\}$. Let $X_1 \subseteq V(R)$ be the common neighborhood of y_1 and V, $Y_1 \subseteq V(R)$ the common neighborhood of x_1 and V, $X_2 \subseteq V(R)$ the common neighborhood of y_2 and V, and $Y_2 \subseteq V(R)$ the common neighborhood of x_2 and V. Note that $x_i \in X_i, y_i \in Y_i$ for $i = 1,2$. Using (56), we get

$$|X_1|, |Y_1|, |X_2|, |Y_2| \geq (r-1)\delta(R) - (r-2)|R| \geq \left(\frac{1}{3r-2} + \eta\right)|R|. \tag{57}$$

We now establish some properties of the sets X_1, Y_1, X_2, Y_2.

Claim 11.12. *For every edge $e \in R[X_1 \cup Y_1]$ it holds that $f_R(e) = f_R(x_1y_1)$, and for every edge $e \in R[X_2 \cup Y_2]$ it holds that $f_R(e) = f_R(x_2y_2)$.*

Proof. We only prove the claim for $X_1 \cup Y_1$; the proof for $X_2 \cup Y_2$ is analogous. Let us assume by contradiction that there exist vertices $u, v \in X_1 \cup Y_1$ such that $uv \in E(R)$ and $f_R(uv) = -f_R(x_1y_1)$. By definition, u, v are adjacent to all vertices in V, hence $M_1 := V \cup \{u, v\}$ is an r-clique in R. Without loss of generality, let us assume that $u \in X_1$. Then u is adjacent to y_1, so $M_2 := V \cup \{u, y_1\}$ is also an r-clique. Now, M_1, M_2, L_1 is a sequence of r-cliques with $|M_1 \cap M_2| = |M_2 \cap L_1| = r - 1$, so by Claim 11.5,

$$f_R(M_1) = f_R(L_1).$$

Now, consider the two r-cliques M_1, L_2. We have $V \subseteq M_1 \cap L_2$, so $|M_1 \cap L_2| \geq r-2$. Also, $f_R(M_1) \neq f_R(L_2)$ because $f_R(L_1) \neq f_R(L_2)$. By Claim 11.5 we know that $|M_1 \cap L_2| \neq r-1$, so $|M_1 \cap L_2| = r-2$. As $(M_1 \cup L_2, f_R)$ is not a template for H, we have by Lemma 7.18 that H is (s', t')-structured with

$$s' = \frac{f_R(M_1) - f_R(uv) - f_R(L_2) + f_R(x_2y_2)}{2(r-2)}$$

and

$$t' = f_R(uv) - f_R(x_2y_2).$$

As before, $(s', t') \neq (0,0)$ because $f_R(M_1) \neq f_R(L_2)$. As $f_R(uv) = -f_R(x_1y_1)$, exactly one of t, t' is zero. Also, for the pair among $(s,t), (s',t')$ where the second coordinate is not zero, we can normalize this coordinate to be 1. Now H satisfies the conditions of Lemma 7.21. Hence, by Lemma 7.21, all r-colorings of H are balanced, contradicting (42). □

Claim 11.13. *The sets $X_1 \cup Y_1$ and $X_2 \cup Y_2$ are disjoint, and R has no edges between $X_1 \cup Y_1$ and $X_2 \cup Y_2$.*

Proof. We first prove the second part of the claim. So assume that there exist $u \in X_1 \cup Y_1$ and $v \in X_2 \cup Y_2$ such that $uv \in R$. Without loss of generality, let us assume that $u \in X_1$ and $v \in X_2$ (the other three cases are similar). Then, by definition, u, v are adjacent to all vertices of V, u is adjacent to y_1, and v is adjacent to y_2. So $L_1, V \cup \{u, y_1\}, V \cup \{u, v\}, V \cup \{v, y_2\}, L_2$ is a sequence of copies of K_r with every two consecutive copies sharing at least $r-1$ vertices. This contradicts Claim 11.5 as $f_R(L_1) \neq f_R(L_2)$.

Now assume by contradiction that there is $u \in (X_1 \cup Y_1) \cap (X_2 \cup Y_2)$. Without loss of generality, suppose that $u \in X_1 \cap X_2$. In particular, u is adjacent to y_2. As $y_2 \in Y_2$, the edge uy_2 goes between X_1 and Y_2. This is a contradiction, as we already showed that there are no edges between $X_1 \cup Y_1$ and $X_2 \cup Y_2$. □

Claim 11.14. *Either both X_1 and Y_1 are not independent sets, or both X_2 and Y_2 are not independent sets.*

Proof. Let us assume that one of X_2, Y_2 is independent, and show that then X_1, Y_1 are not independent. So suppose without loss of generality that X_2 is independent. Then X_2 and Y_2 are disjoint, because $x_2 \in X_2$ is connected to all of Y_2. By Claim 11.13, there are no edges between $X_1 \cup Y_1$ and $X_2 \cup Y_2$. Hence, if X_1 were also independent, then every vertex in X_1 would have degree at most

$$|R| - |X_1| - |X_2| - |Y_2| \leq \frac{3r-5}{3r-2}|R|,$$

using (57). This contradicts (56). By the same argument, Y_1 is also not independent. □

Without loss of generality, let us assume that neither X_1 nor Y_1 is an independent set in R. For the rest of the proof, fix an edge $u_1v_1 \in R[X_1]$. By definition, u_1, v_1 are adjacent to y_1 and all vertices of V. Hence, $M := V \cup \{u_1, v_1, y_1\}$ is a clique of size $r+1$ in R. We now show that M is monochromatic.

Claim 11.15. *$R[M]$ is monochromatic with respect to f_R.*

Proof. For convenience, put $c = f_R(x_1y_1)$. By Claim 11.12 and as $x_1, u_1, v_1 \in X_1$, $y_1 \in Y_1$, we have

$$f_R(u_1v_1) = f_R(u_1y_1) = f_R(v_1y_1) = f_R(x_1y_1) = c. \tag{58}$$

By assumption, (M, f_R) is not a template for H. By Corollary 7.3, this means that $R[M]^c$ is d-regular for some d. If $d = r$ then $R[M]$ is monochromatic, so let us assume, by contradiction, that $d < r$.

Suppose first that $r = 3$. By (58), u_1, v_1, y_1 is a monochromatic triangle of color c. A regular graph on 4 vertices containing a triangle must be a complete graph, so $R[M]$ is monochromatic, as required.

From now on, assume that $r \geq 4$. Since $d < r$, u_1 has an edge of color $-c$ to some vertex $z \in M$. By (58), $z \in V$. Now consider the two r-cliques $M_1 := (V \setminus \{z\}) \cup \{y_1, u_1, v_1\}$ and $M_2 := V \cup \{u_1, v_1\}$. Then $|M_1 \cap M_2| = r - 1$, so $f_R(M_1) = f_R(M_2)$ by Claim 11.5. We will now apply Lemma 7.7 to M_1, M_2 with $e_1 = u_1y_1$ and $e_2 = u_1z$ (and with $(V \setminus \{z\}) \cup \{v_1\}$ in the role of V). Note that

$$f_R(M_1) - f_R(e_1) - f_R(M_2) + f_R(e_2) = f_R(e_1) - f_R(e_2) \in \{-2, 2\},$$

using that $f_R(e_2) = -c = -f_R(e_1)$. As $r \geq 4$, it follows that

$$f_R(M_1) - f_R(e_1) - f_R(M_2) + f_R(e_2) \notin \{-4(r-2), -2(r-2), 0, 2(r-2), 4(r-2)\}.$$

Now Lemma 7.7 implies that $(M_1 \cup M_2, f_R)$ is a template for H, contradicting our assumption that R has no such template. □

By Claim 11.15, M is monochromatic. As L_1 intersects the r-clique $V \cup \{u_1, y_1\} \subseteq M$ in $r-1$ vertices, we must have $f_R(L_1) = f_R(V \cup \{u_1, y_1\})$ by Claim 11.5. Since $f_R(L_1) > 0$, it follows that L_1 is monochromatic[2] in color 1, and in particular $f_R(x_1y_1) = 1$.

Recall that H is (s,t)-structured. Using that L_1 is monochromatic in color 1, we now show that only few options for s and t are possible.

Claim 11.16. *The following holds:*

1. *Suppose that $r \neq 3$. Then H is $(1,0)$-structured or $(1,1)$-structured.*
2. *Suppose that $r = 3$. Then $f_R(L_2) = -1$ and H is $(1,2)$-structured.*

Proof. We begin with the case $r \neq 3$, namely $r \geq 6$. If $f_R(x_2y_2) = 1$ then $t = 0$ by definition, recall (47). As $(s,t) \neq (0,0)$, we can normalize to get that H is $(1,0)$-structured.

Suppose now that $f_R(x_2y_2) = -1$. Then $t = 2$. Also, using that $f_R(L_1) = \binom{r}{2}$ and $f_R(L_2) < 0$, we get

$$f_R(L_1) - f_R(x_1y_1) - f_R(L_2) + f_R(x_2y_2) = \binom{r}{2} - f_R(L_2) - 2 \geq \binom{r}{2} - 1 > 2(r-2), \tag{59}$$

where the last inequality holds for $r > 3$. Contrasting this with (49), we see that the LHS of (59) equals $4(r-2)$, which implies that $s = 2$ by (46). So H is $(2,2)$-structured and hence $(1,1)$-structured.

Now suppose that $r = 3$. By assumption, we are in Case 3 or 4 of Claim 11.6. Case 3 is impossible because L_1 is monochromatic, hence we are in Case 4. So H is $(1,2)$-structured and $f_R(L_2) = -1$ (as L_2 is not monochromatic and $f_R(L_2) < 0$). □

Claim 11.17. $X_2 \cap Y_2 = \emptyset$.

Proof. We will show that X_2 is an independent set (the same is true for Y_2). This will imply the claim because $x_2 \in X_2$ is adjacent to all vertices in Y_2 (by the definition of Y_2), so if there existed $u \in X_2 \cap Y_2$, then x_2u would be an edge inside X_2, a contradiction.

So let us assume, by contradiction, that R has an edge u_2v_2 with $u_2, v_2 \in X_2$. By the definition of X_2, u_2, v_2 are adjacent to $V \cup \{y_2\}$.

Suppose first that $r = 3$. By Claim 11.12, the triangle $M_1 := \{u_2, v_2, y_2\}$ is monochromatic (with color $f_R(x_2y_2)$). Write $V = \{v\}$ (recall that $|V| = r-2$). So $M_2 = \{v, v_2, y_2\}$ is a triangle. Now, $M_1, M_2, L_2 = \{v, x_2, y_2\}$ is a sequence of triangles with $|M_1 \cap M_2| = |M_2 \cap L_2| = 2$. Also, M_1 is monochromatic, while L_2 is not monochromatic because $f_R(L_2) = -1$ by Claim 11.16. This contradicts Claim 11.5.

Suppose now that $r \neq 3$, namely $r \geq 6$. Note that $M_0 := V \cup \{y_2, u, v\}$ is a copy of K_{r+1} in R. By assumption, (M_0, f_R) is not a template for H. Hence, by Corollary 7.3, there is $d' \in \mathbb{N}$ such that M_0^- is d'-regular. All edges inside V have color 1, so all edges of M_0 of color -1 must touch $\{y_2, u, v\}$. Considering the edges of color -1 touching V, we get $(r-2)d' = |V|d' \leq 3d'$. As $r \geq 6$, we get that $d' = 0$, i.e. M_0 is monochromatic in color 1. In particular, uv and all edges between y_2 and V have color 1. By Claim 11.12, $f_R(x_2y_2) = f_R(uv) = 1$. Hence, the only edges of L_2 that can have color -1 are edges between x_2 and V. The number of these edges is the same as the number of edges between y_2 and V, which all have color 1. So $f_R(L_2) > 0$, a contradiction. This completes the proof of the claim □

We can now complete the proof in the case $r = 3$. By Claim 11.16, H is $(1,2)$-structured. By Lemma 8.6, we have $\max\{\delta_0(H), 1 - 1/\chi^*(H)\} \geq 5/8$. Thus, by (56),

$$\delta(R)/|R| \geq 5/8. \tag{60}$$

[2]It is worth noting that from this point on, we must have $r \leq 7$; namely, the proof is already complete for all $r \geq 8$. Indeed, since all edges in V have color 1, and L_2 has exactly $2r-3$ edges not contained in V, we have $0 > f_R(L_2) \geq \binom{r-2}{2} - (2r-3)$, which only holds if $r \leq 7$. So the remaining cases are $r \in \{3, 6, 7\}$.

This allows us to improve on (57) as follows:

$$|X_2|, |Y_2| \geq 2\delta(R) - |R| \geq |R|/4.$$

By Claim 11.17, X_2 and Y_2 are disjoint and therefore, $|X_2 \cup Y_2| \geq |R|/2$. Now recall that $x_1 \in X_1$ and by Claim 11.13, x_1 is not adjacent to any vertex in $X_2 \cup Y_2$. This contradicts with (60), completing the proof of Lemma 11.4 for $r = 3$. For the rest of the proof, we assume that $r \neq 3$, namely $r \geq 6$. This case is handled in the following subsection.

11.1.3 Concluding the proof: The case $r \geq 6$

Recall that u_1v_1 is an edge of R with $u_1, v_1 \in X_1$, so that u_1, v_1 are adjacent to all vertices in $V \cup \{y_1\}$. Let $N \subseteq V(R)$ be the common neighborhood of u_1 and v_1, and note that $V \subseteq N$. By Claim 11.13, u_1 and v_1 are not adjacent to any vertex in $X_2 \cup Y_2$. Using that X_2 and Y_2 are disjoint (by Claim 11.17),

$$|N| \geq 2\delta(R) - (|R| - |X_2| - |Y_2|) \geq \left(\frac{3r-6}{3r-2} + \eta\right)|R|, \tag{61}$$

where the last inequality uses (56) and (57). As $\delta(R) > \frac{3r-5}{3r-2}|R|$, each vertex in R is adjacent to all but at most $\frac{3}{3r-2}|R|$ of the vertices, and hence

$$\delta(R[N]) \geq |N| - \frac{3}{3r-2}|R| > \frac{r-3}{r-2}|N|. \tag{62}$$

The minimum degree of $R[N]$ implies the following:

$$\text{For each } k < r-1 \text{, every copy of } K_k \text{ in } R[N] \text{ is contained in a copy of } K_{r-1} \text{ in } R[N]. \tag{63}$$

Claim 11.18. *Every edge inside $N \cup \{u_1, v_1\}$ has color 1.*

Proof. Assume by contradiction that there is an edge e of color -1 inside $N \cup \{u_1, v_1\}$. By (63), there is an $(r-2)$-clique $L \subseteq N$ which contains $e \setminus \{u_1, v_1\}$. Using that u_1, v_1 are adjacent to all vertices in N, we see that $L \cup \{u_1, v_1\}$ is an r-clique containing e and u_1, v_1. As $f_R(e) = -1$, we have $f_R(L \cup \{u_1, v_1\}) < \binom{r}{2}$.

Now consider L and V, which are both cliques of size $r-2$ contained in N. By Item 1 of Lemma 4.1 with $k = r-2$ and $J = R[N]$, using (62), there is a sequence $M_1, M_2, \ldots, M_\ell$ of $(r-2)$-cliques inside $R[N]$, such that $M_1 = L$, $M_\ell = V$, and M_{i-1}, M_i share at least $r-3$ vertices for all $1 \leq i < \ell$. Let $M_i' := M_i \cup \{u_1, v_1\}$. Then M_i' is an r-clique in R, and M_{i-1}', M_i' share at least $r-1$ vertices for all $1 \leq i < \ell$. Also, as $V \cup \{u_1, v_1\}$ is monochromatic in color 1, we have

$$f_R(M_1') = f_R(L \cup \{u_1, v_1\}) \neq \binom{r}{2} = f_R(V \cup \{u_1, v_1\}) = f_R(M_\ell').$$

This contradicts Claim 11.5. □

Let $W = V(R) \backslash N \cup \{u_1, v_1\}$. By Claim 11.18, all edges of color -1 in R are incident to W. Also, by (61), we have

$$|W| \leq \left(\frac{4}{3r-2} - \eta\right)|R| \leq \left(\frac{1}{4} - \eta\right)|R|, \tag{64}$$

using $r \geq 6$. In the following claim we derive some properties of r-cliques which intersect W in only one or two vertices.

Claim 11.19. *The following holds:*

1. *Let L be a copy of K_r in R which has exactly one vertex in W. Then L is monochromatic in color* 1.

2. *Suppose that H is $(1,0)$-structured. Let L be a copy of K_r in R which has exactly two vertices w_1, w_2 in W. Then $f_R(w_1w_2) = 1$.*

Proof. We begin with the first item. Let w be the unique vertex in $L \cap W$. We claim that there is an r-clique L' in R with $L' \subseteq N \cup \{u_1, v_1\}$ and $|L \cap L'| = r - 1$. If $u_1 \notin L$ then $L' = (L \setminus \{w\}) \cup \{u_1\}$ is such an r-clique (here we use the fact that u_1 is adjacent to all vertices in $V(R) \setminus (W \cup \{u_1\})$, and hence to all vertices of $L \setminus \{w\}$). So assume that $u_1 \in L$. By the same argument, we may assume that $v_1 \in L$. Then $L \setminus \{u_1, v_1, w\}$ is a clique of size $r-3$ contained in N. By (63), there is a clique M' of size $r-2$ with $M' \subseteq N$ and $L \setminus \{u_1, v_1, w\} \subseteq M'$. Now $L' = M' \cup \{u_1, v_1\}$ satisfies our requirements. As $|L \cap L'| = r - 1$, we have $f_R(L) = f_R(L')$ by Claim 11.5. By Claim 11.18, L' is monochromatic in color 1, as $L' \subseteq N \cup \{u_1, v_1\}$. So L is also monochromatic in color 1.

We now prove the second item. Put $M' := L \setminus \{w_1, w_2\}$; so M' is a clique of size $r-2$ and $M' \subseteq N \cup \{u_1, v_1\}$. By (63), there is an $(r-2)$-clique $M'' \subseteq N$ with $M' \setminus \{u_1, v_1\} \subseteq M''$. Now, $L' := M'' \cup \{u_1, v_1\}$ is an r-clique with $|L \cap L'| = r - 2$, $L \setminus L' = \{w_1, w_2\}$ and $L' \subseteq N \cup \{u_1, v_1\}$. In particular, the edge $e := L' \setminus L$ has color 1. Suppose by contradiction that $f_R(w_1w_2) = -1$. By assumption, $(L' \cup L, f_R)$ is not a template for H. By Lemma 7.18, H is (s', t')-structured with $t' = f_R(e) - f_R(w_1w_2) = 2$ (the value of s' will not be important). By normalizing, H is $(\frac{s'}{2}, 1)$-structured. Additionally, H is $(1,0)$-structured by assumption. Now, by Lemma 7.21, H has only balanced r-colorings, a contradiction to (42). □

Recall the definition of the graph H^* (see Definition 5.5). In particular, H^* is a complete r-partite graph and has a perfect H-factor. Recall the definition of V_0 in Section 6.2 (namely, V_0 is the exceptional set given by the regularity lemma). The following key claim shows that if an H^*-copy J in G' does not intersect V_0, then every perfect H-factor of J has only few edges of color -1. Using this claim, we then easily complete the proof of the lemma. Recall that H is $(1,0)$- or $(1,1)$-structured by Item 1 of Claim 11.16. Let ρ' be the corresponding parameter (as in Definition 7.17), and note that $\rho' > 0$. Recall that $V_W = \bigcup_{w \in W} V_w \subseteq V(G')$ denotes the union of clusters which correspond to the vertices in $W \subseteq V(R)$.

Claim 11.20. *Let J be an H^*-copy in G' with $V(J) \cap V_0 = \emptyset$, and let $A_1, \dots, A_r$ be the parts of J. Let I be the set of indices $i \in [r]$ such that $A_i \subseteq V_W$. Then for every perfect H-factor F of J,*

$$e(F^-) \leq \rho' \sum_{i \in I} |A_i| \leq \rho' |V(J) \cap V_W|. \tag{65}$$

Proof. The right inequality in (65) is immediate from the definitions. We prove the left inequality. For each $i \in [r]$, choose $a_i \in A_i$ such that $a_i \in A_i \setminus V_W$ if $i \notin I$, and else a_i is arbitrary. Set $L = \{a_1, \dots, a_r\}$, so $L \subseteq J$ is an r-clique in G'. Let $L^R \subseteq R$ be the corresponding r-clique in R, namely $L^R = \{V^R_{a_1}, \dots, V^R_{a_r}\}$. The assumption $V(J) \cap V_0 = \emptyset$ means that the cluster $V_a \in V(R)$ is well-defined for every $a \in V(J)$.

First, suppose that there exist $1 \leq i \neq j \leq r$ and $u, v \in A_i$ and $w \in A_j$ so that $f(uw) \neq f(vw)$. Without loss of generality, let us assume that $i = 1, j = 2$. Recall that we assume that H is non-uniform. Hence, by Claim 7.8, there exists an r-coloring $B_1, B_2, \dots, B_r$ of H such that $e_H(B_1, B_2) \neq e_H(B_1, B_3)$ and thus, there exists $b \in B_1$ such that $d_H(b, B_2) \neq d_H(b, B_3)$. Now consider the two r-cliques $M_1 := \{V^R_u, V^R_w, V^R_{a_3}, \dots, V^R_{a_r}\}$ and $M_2 := \{V^R_v, V^R_w, V^R_{a_3}, \dots, V^R_{a_r}\}$ in R. We have $|M_1 \cap M_2| = r - 1$, so $f_R(M_1) = f_R(M_2)$ by Claim 11.5. Also, $f_R(V^R_u V^R_w) \neq f_R(V^R_v V^R_w)$. Hence, by Lemma 7.4, $(M_1 \cup M_2, f_R)$ is a template for H, contradicting our assumption that R contains no such template.

So from now on, we assume that for all $1 \leq i \neq j \leq r$ and $u, v \in A_i, w \in A_j$, it holds that $f(uw) \neq f(vw)$. This means that all bipartite graphs (A_i, A_j) are monochromatic. In other words, (J, f) is a blowup of (L^R, f_R). As all the edges in R of color -1 are incident to W, all the edges of color -1 in J must be incident to $\bigcup_{i \in I} A_i$. Indeed, if $i, j \notin I$ then $A_i, A_j \not\subseteq V_W$, so there must be an edge of color 1 between A_i, A_j. But as (A_i, A_j) is monochromatic, all edges between A_i, A_j have color 1.

By Claim 11.19, if $|I| \leq 1$ then $e(F^-) = 0$ and the claim holds trivially. Hence, we assume that $|I| \geq 2$. Let us now distinguish two cases. Suppose first that H is $(1,1)$-structured (with parameter ρ'). Then F is also $(1,1)$-structured (with parameter ρ'), as F is an H-factor. Hence (recall Definition 7.17), we have

$$\rho'(|A_i| + |A_j|) = e_F(A_i, A_j) + e_F(A_i \cup A_j, V(F) \backslash (A_i \cup A_j)) \tag{66}$$

for all $1 \leq i < j \leq r$. Now, summing (66) over all pairs i, j with $i, j \in I$, we get

$$\begin{aligned}
\rho'(|I|-1)\sum_{i \in I}|A_i| &= \sum_{i,j \in I} \rho'(|A_i| + |A_j|) = \sum_{i,j \in I} [e_F(A_i, A_j) + e_F(A_i \cup A_j, V(J) \backslash (A_i \cup A_J))] \\
&= (|I|-1) \sum_{k \in I, \ell \in [r] \backslash I} e_F(A_k, A_\ell) + (2|I|-3) \sum_{k,\ell \in I} e_F(A_k, A_\ell) \\
&\geq (|I|-1) \sum_{k,\ell : \{k,\ell\} \cap I \neq \emptyset} e_F(A_k, A_\ell) \\
&\geq (|I|-1) \cdot e(F^-),
\end{aligned}$$

where the last inequality holds because every edge of color -1 in F is incident to $\bigcup_{i \in I} A_i$. Dividing through by $|I| - 1 \geq 1$, we get the left inequality in (65), as required.

Now suppose that H, and hence also F, are $(1,0)$-structured. This means that

$$\rho'(|A_i| + |A_j|) = e_F(A_i \cup A_j, V(J) \backslash (A_i \cup A_J)) \tag{67}$$

for all $1 \leq i < j \leq r$. Summing (67) over all pairs (i,j) with $i, j \in I$, we get

$$\begin{aligned}
\rho'(|I|-1)\sum_{i \in I}|A_i| &= \sum_{i,j \in I} \rho'(|A_i| + |A_j|) = \sum_{i,j \in I} e_F(A_i \cup A_j, V(J) \backslash (A_i \cup A_J)) \\
&= (|I|-1) \sum_{k \in I, \ell \in [r] \backslash I} e_F(A_k, A_\ell) + 2(|I|-2) \sum_{k,\ell \in I} e_F(A_k, A_\ell).
\end{aligned} \tag{68}$$

If $|I| \geq 3$ then $2(|I|-2) \geq |I|-1$, so (68) counts $e_F(A_k, A_\ell)$ at least $|I|-1$ times for every $1 \leq k < \ell \leq r$ with $\{k,\ell\} \cap I \neq \emptyset$. Hence, (68) is an upper bound for $(|I|-1) \cdot e(F^-)$, and the assertion of the claim follows by dividing (68) through by $|I|-1$. Now suppose that $|I| = 2$, say $I = \{1,2\}$ without loss of generality. Then, by Item 2 of Claim 11.19, all edges between A_1 and A_2 have color 1. Therefore,

$$e(F^-) \leq \sum_{k \in I, \ell \in [r] \backslash I} e_F(A_k, A_\ell) \leq \rho' \sum_{i \in I} |A_i|,$$

using (68). So again the left inequality in (65) holds. □

We now complete the proof of Lemma 11.4. By Lemma 6.5, G' has a perfect H^*-factor F^*. For each H^*-copy $J \in F^*$, let F_J be a perfect H-factor of J. Let $F = \bigcup_{J \in F^*} F_J$ be the resulting perfect H-factor of G'. We now use Lemma 7.19 to estimate the number of edges of H, using that H is (s', t')-structured for $s' = 1$ and $t' \in \{0,1\}$. By Lemma 7.19, $e(H) = \rho' \frac{r-1}{(2r-4)s'+t'}|H|$, so

$$\rho'|H|/2 \leq \rho' \frac{r-1}{2r-3}|H| \leq e(H) \leq \rho' \frac{r-1}{2r-4}|H| \leq \rho'|H|.$$

It follows that $e(F) = \frac{n}{|H|} \cdot e(H) \geq \rho' n/2$. There are at most $|V_0|$ copies of H^* in F^* intersecting V_0, and the H-factors of these copies of H^* contain therefore at most $|V_0| \cdot \frac{|H^*|}{|H|} \cdot e(H) \leq \rho' |V_0||H^*|$ edges. Also, if an H^*-copy $J \in F^*$ does not intersect V_0, then F_J contains at most $\rho'|V(J) \cap V_W|$ edges of color -1, by Claim 11.20. It follows that

$$e(F^-) \leq \rho'|V_0||H^*| + \rho' \sum_{J \in F^*} |V(J) \cap V_W| = \rho'|V_0||H^*| + \rho'|V_W| \overset{\text{(a)}}{\leq} \rho' \left(\varepsilon n|H^*| + \left(\frac{1}{4} - \eta\right) n\right) \overset{\text{(b)}}{\leq}$$
$$\rho' \left(\frac{1}{4} - \frac{\eta}{2}\right) n \leq \frac{e(F)}{2} - \frac{\rho'\eta}{2} n \overset{\text{(c)}}{\leq} \frac{e(F)}{2} - \gamma n. \tag{69}$$

Here, inequality (a) uses that $|V_0| \leq \varepsilon n$ and that $|V_W| \leq \left(\frac{1}{4} - \eta\right) n$ by (64). Inequality (b) uses that H^* depends only on H, η and $\varepsilon \ll \frac{1}{|H|}, \eta$. And inequality (c) uses that $\rho' > 0$ depends only on H and $\gamma \ll \frac{1}{|H|}, \eta$. So we got that $f(F) = e(F) - 2e(F^-) \geq \gamma n$, namely F has high discrepancy. This completes the proof.

12 Proof of the main results

12.1 Proof of Theorem 1.4

Proof. Let H be a bipartite graph. By Corollary 10.2, we have $\delta^*(H) \leq 3/4$. By Lemma 8.1, this is tight if H is regular. Therefore, let us assume from now on that H is non-regular.

First, suppose that there exists ρ such that for every connected component U of H it holds that $e_H(U) = \rho|U|$, which corresponds to the second case of Theorem 1.4. By Lemma 8.2, we get that

$$\delta^*(H) \geq 1/2.$$

Now, let us show that $\delta^*(H) \leq 1/2$. We work in the setup described in Section 6.2. In particular, we assume that $\delta(R)/|R| \geq 1/2 + \eta/2$. We need to show that G' has a perfect H-factor with high discrepancy.

If R is monochromatic, then there exists a perfect H-factor in G' with high discrepancy by Lemma 9.1. Therefore, let us assume that R is not monochromatic. This implies that there exist vertices $u, v, w \in V(R)$ such that $f_R(uv) = -f_R(vw) = 1$. Indeed, R is not monochromatic so there are edges xy, st of different colors. By Lemma 4.1 applied with $k = 2$, there is a path in R whose first and last edges are xy and st. On this path, there must be two consecutive edges of different colors, giving the vertices u, v, w as above.

Note that uv, vw are two copies of K_2 with different discrepancies and H is non-regular. By Lemma 7.2, $(\{u, v, w\}, f_R)$ is a template for H, and then by Lemma 6.7 (with $r = 2$), G' has a perfect H-factor with high discrepancy, as required.

Now suppose that we are in the last case of Theorem 1.4, meaning that there are two connected components U, W of H and $\rho \neq \rho'$ such that $e_H(U) = \rho|U|$ and $e_H(W) = \rho'|W|$. In other words, $e_H(U)/|U| \neq e_H(W)/|W|$. Recall that $\delta^*(H) \geq 1 - 1/\chi^*(H)$ trivially holds for every H, so we only need to show that $\delta^*(H) \leq 1 - 1/\chi^*(H)$. Again, we show that G' has a perfect H-factor of high discrepancy under the setting of Section 6.2. As in the previous case, we may assume that R is non-monochromatic as otherwise we are done by Lemma 9.1. Thus, there exist edges e_1, e_2 with $f_R(e_1) \neq f_R(e_2)$. If e_1, e_2 are not disjoint then $(e_1 \cup e_2, f_R)$ is a template for H by Lemma 7.2, and if they are disjoint then it is a template by Lemma 7.1. Either way, we can apply Lemma 6.7 to conclude that G' has a perfect H-factor with high discrepancy. Thus, we get $\delta^*(H) = 1 - 1/\chi^*(H)$. □

12.2 Proof of Theorem 1.7

Let H be a graph with $\chi(H) = 3$. By Corollary 10.2, $\delta^*(H) \leq 3/4$. If H is regular, then $\delta^*(H) = 3/4$ by Lemma 8.1. So suppose from now on that H is non-regular. Recall that every graph H satisfies $\delta^*(H) \geq \max\{\delta_0(H), 1 - 1/\chi^*(H)\}$. If some butterfly is not a template for H, then $\delta^*(H) \geq 4/7$ by Lemma 8.3 and $\delta^*(H) \leq \max\{\delta_0(H), 1-1/\chi^*(H), 4/7\}$ by Lemma 11.4. And if every butterfly is a template for H, then $\delta^*(H) \leq \max\{\delta_0(H), 1 - 1/\chi^*(H)\}$ by Lemma 11.4. This concludes the proof.

12.3 Proof of Theorem 1.11

Let H be an r-chromatic graph, $r \geq 4$. Throughout the proof, we use the fact that

$$1 - 1/(r-1) \leq \max\{\delta_0(H), 1 - 1/\chi^*(H)\} \leq 1 - 1/r, \tag{70}$$

where the first inequality holds because $\chi^*(H) \geq r-1$. We begin with the first case of Theorem 1.11, where we assume that H satisfies Condition 1.9. By Corollary 10.2, $\delta^*(H) \leq 1 - 1/(r+1)$. We now use Condition 1.9 to show that $\delta^*(H) \geq 1 - 1/(r+1)$. Indeed, if $r \equiv_4 0$ then this follows from Lemma 8.4 with $k = r+1 \equiv_4 1$, and if $r \not\equiv_4 0$ then this follows from Lemma 8.5 with $k = r+1 \not\equiv_4 1$ (using that H is regular).

We now move on to the second case of Theorem 1.11. First, we show that if H violates Condition 1.9 then $\delta^*(H) \leq 1 - 1/r$. Indeed, if H violates the $(r+1)$-wise C_4-condition, then $\delta^*(H) \leq 1 - 1/r$ by Lemma 10.1 with $k = r+1$. And if H is non-regular and $r \not\equiv_4 0$, then $\delta^*(H) \leq 1 - 1/r$ by Lemma 11.1. Next, observe that if H satisfies Condition 1.10, then $\delta^*(H) \geq 1 - 1/r$ by Lemmas 8.4-8.5.

Finally, we handle the last case of Theorem 1.11. Here we show that if H violates Conditions 1.9 and 1.10, then $\delta^*(H) \leq \max\{\delta_0(H), 1 - 1/\chi^*(H)\} =: \alpha(H)$. This is tight because $\delta^*(H) \geq \alpha(H)$ for every graph H. If H violates the r-wise C_4-condition, then $\delta^*(H) \leq \alpha(H)$ by Lemma 10.1 with $k = r$, using that $1 - 1/(r-1) \leq \alpha(H)$ by (70). So suppose that H satisfies the r-wise C_4-condition. Then, as H violates Condition 1.10, it must be that H is non-regular. Now, if $r \equiv_4 2, 3$, then $\delta^*(H) \leq \alpha(H)$ by Lemma 11.4. If $r \equiv_4 1$ then $\alpha(H) \geq \delta_0(H) = 1 - 1/r$ by Lemma 8.4 and $\delta^*(H) \leq 1 - 1/r$ by Lemma 11.1, so $\delta^*(H) \leq \alpha(H)$ holds. Suppose now that $r \equiv_4 0$. Then, as H violates Condition 1.9, H must violate the $(r+1)$-wise C_4-condition. Now $\delta^*(H) \leq \alpha(H)$ holds by Lemma 11.3. This completes the proof.

13 Examples

The purpose of this section is to demonstrate that the cases in our theorems are necessary. For the bipartite case, Theorem 1.4, this is fairly easy to see so we only discuss Theorems 1.7 and 1.11. Towards this, we give graphs H as examples for what we consider to be the more interesting cases. The task of finding examples of r-partite graphs becomes much simpler when they have exactly one proper r-coloring (up to permutations of the color-labels). To achieve this, we use the following construction in most of the examples:

C Let H be a graph on vertex-set $V(H)$ with r-partition $A_1, A_2, \dots A_r$ and vertices $a_1 \in A_1, a_2 \in A_2, \dots, a_r \in A_r$. For $1 \leq i \leq r$, a_i is connected to every vertex in $\bigcup_{j \neq i} A_j$.

Then, given an r-coloring of $a_1, a_2, \dots a_r$, we get that for every $1 \leq i \leq r$, all the vertices in A_i must have the same color as a_i and therefore, the coloring is unique up to permutation of the labels. Note that for $1 \leq i < j \leq r$, we can add any edges to $H[A_i, A_j]$ and this property does not change. Additionally such a graph H, is never regular, unless $|A_1| = |A_2| = \cdots = |A_r|$ and H is the complete r-partite graph. One constraint that such graphs H have is that for $1 \leq i < j \leq r$, $e_H(A_i, A_j) \geq |A_i| + |A_j| - 1$, given by the edges incident to a_i and a_j.

- First, we give a tripartite graph H for which

  $$\delta_0(H) < \delta^*(H) = 1 - 1/\chi_{cr}(H).$$

  Towards this, consider H as described in **C** with $|A_1| = 10, |A_2| = 11, |A_3| = 100$. Note that $hcf(H) = 1$, as $|A_2| - |A_1| = 1$. Besides the edges given by a_1, a_2, a_3, let there be arbitrary additional edges such that $e_H(A_1, A_2) = e_H(A_1, A_3) = e_H(A_2, A_3) = 110$. Note that this means that $H[A_1, A_2]$ is complete and $H[A_2, A_3]$ has no extra edges besides the ones touching a_2 or a_3. It is not hard to see that $\delta_0(H) = 0$ and $1 - 1/\chi_{cr}(H) < 1 - 1/r$. Additionally, H can use any butterfly as a template, as otherwise by Lemma 7.18 H is either $(2,2)$-, $(0,2)$- or $(2,-2)$-structured, which it is clearly not.

- Next, we give an example for a tripartite graph H with

$$1 - 1/\chi^*(H) < \delta^*(H) = \delta_0(H).$$

Let H be a tripartite graph as described in **C** with $|A_1| = 5, |A_2| = 20, |A_3| = 21$ and $e_H(A_1, A_2) = 28, e_H(A_1, A_3) = 42$ and $e_H(A_2, A_3) = 252$. It is not hard to check that such H is indeed $(1, 2)$-structured for $\rho = 14$. Additionally, since $|A_3| - |A_2| = 1$, we have $hcf(H) = 1$ and thus

$$1 - 1/\chi^*(H) = 1 - 1/\chi_{cr}(H) = 1 - \frac{41}{92} < 5/8.$$

By Lemma 8.6, it follows that $\delta_0(H) \geq 5/8$. As H is non-regular and since $5/8 > 4/7$, we get by Theorem 1.7 that indeed

$$1 - 1/\chi^*(H) < 5/8 \leq \delta^*(H) = \delta_0(H).$$

- Next, we give an example corresponding to the second case of Theorem 1.7 such that $\delta^*(H) = 4/7 > \max\{1 - 1/\chi^*(H), \delta_0(H)\}$. Towards this, some butterfly should not be a template for H. Let $|A_1| = 5, |A_2| = 20, |A_3| = 21$ and $e_H(A_1, A_2) = 67, e_H(A_1, A_3) = 66$ and $e_H(A_2, A_3) = 51$. We get that H is non-regular and $(1, -1)$-structured for $\rho = 2$. Consider the butterfly given by (L, c) (see the third graph in Figure 1), where L consists of the two triangles $L_1 = \{u, v_1, w_1\}$ and $L_2 = \{u, v_2, w_2\}$ with

$$c(uv_1) = c(uw_1) = -c(uv_2) = -c(uw_2) = -c(v_1w_1) = c(v_2w_2) = -1.$$

We will show that L is not a template for H. Note that by Lemma 7.18, it is necessary that then, H is $(-2, 2)$-structured (or by normalizing $(1, -1)$-structured). Let B be an arbitrary blowup of (L, c). Note that given some H constructed as described in **C**, any copy of H in B is either included in $V_{V(L_1)}$ or in $V_{V(L_2)}$. To see this, consider the placement of the three vertices a_1, a_2, a_3. As they form a triangle, they must be either on $V_{V(L_1)}$ or $V_{V(L_2)}$. Say they are on $V_{V(L_1)}$. But each vertex of H forms a triangle with two of a_1, a_2, a_3 and must therefore also be on $V_{V(L_1)}$. Then, it is not hard to see that since H is $(1, -1)$-structured, L is not a template for H. We have that $\delta_0(H) = 0$, as H is $(1, -1)$-structured with nonzero ρ and also $1 - 1/\chi^*(H) = 1 - 41/92 < 4/7$ as in the previous example. We then get by the second case in Theorem 1.7 that $\delta^*(H) = 4/7$.

- Let us now give an example of an r-partite, regular graph H for some $r \geq 4$ which fulfills the r-wise C_4-condition for some $r \not\equiv_4 0, 1$ and has $1 - 1/r > \max\{\delta_0(H), 1 - 1/\chi^*(H)\}$. Note that Theorem 1.11 shows that $\delta^*(H) = 1 - 1/r$ in that case. To find such a graph, the construction given in (**C**) is not very helpful, as the only regular graph constructed in such a way is the complete, balanced r-partite graph. Thus, let us consider a different construction. For some integer m, let H be the graph obtained from the complete r-partite graph with parts $A_1, \ldots, A_r$ with sizes $|A_1| = (r - 2)m + 1$ and $|A_2| = |A_3| = \cdots = |A_r = (r - 2)m$, by removing a matching of size m between every pair A_i, A_j with $2 \leq i < j \leq r$ such that for any vertex not in A_1, exactly one of its incident edges is removed.

  Counting the edges per vertex, it is not hard to see that for $v \in A_1$, we have that $d_H(v) = (r-1)(r-2)m$ and for $v \in A_i$ for $2 \leq i \leq r$ we have that v is connected to everything but the vertices in A_i and one other vertex. It follows that these vertices are incident to $(r-2)^2m - 1 + (r-2)m + 1 = (r-1)(r-2)m$ edges. Therefore, H is regular. Additionally, we have for $2 \leq i < j \leq r$,

$$e_H(A_i, A_j) = (r - 2)^2m^2 - m$$

  and for all $2 \leq i \leq r$,

$$e_H(A_1, A_i) = (r - 2)m \cdot ((r - 2)m + 1).$$

  Then, it is not hard to see that since $r \geq 4$, H has only one r-coloring (up to permutation of the labelling) and the r-wise C_4-condition holds for this coloring, but the $(r + 1)$-wise C_4 condition does not. To see the latter, consider the natural r-coloring of H and add an additional empty color class A_0. Then the 4-cycle A_0, A_1, A_2, A_3 shows that H violates the $(r+1)$-wise C_4-condition. The coloring $A_1, \ldots, A_r$ shows that $\chi^*(H) < r$. Let us also prove that $\delta_0(H) < 1 - 1/r$. To see this, consider a b-blowup B of (K_r, c) for some $b \in \mathbb{N}$ and 2-edge-coloring c of K_r. As $r \not\equiv_4 0, 1$, $c(K_r) \neq 0$. It is not hard to see that there is at most one way (up to permutations of $A_2, A_3, \ldots, A_r$) to find a perfect H-factor F in B. Note that this H-factor uses the same amount of edges in every bipartite graph $B[V_v, V_u]$, where $u, v \in V(K_r)$. Therefore, $c(F)$ has the same sign as $c(K_r)$ and is non-zero. It follows that $\delta_0(H) < 1 - 1/r$.

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