Difference-Isomorphic Graph Families

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Abstract

Many well-studied problems in extremal combinatorics deal with the maximum possible size of a family of objects in which every pair of objects satisfies a given restriction. One problem of this type was recently raised by Alon, Gujgiczer, Körner, Milojević and Simonyi. They asked to determine the maximum size of a family $\mathcal{G}$ of graphs on $[n]$, such that for every two $G_1, G_2 \in \mathcal{G}$, the graphs $G_1 \setminus G_2$ and $G_2 \setminus G_1$ are isomorphic. We completely resolve this problem for sufficiently large n by showing that this maximum is exactly $2^{\frac{1}{2}\left(\binom{n}{2} - \lfloor \frac{n}{2} \rfloor\right)}$ and characterizing all extremal constructions. We also prove an analogous result for r-uniform hypergraphs.

1 Introduction

Some of the most well-studied problems in extremal combinatorics ask for the maximum size of a collection of objects where every pair of objects satisfies some given property. For example, the celebrated Erdős–Ko–Rado theorem [11] determines the maximum size of a family of subsets of size k of a set of size n in which every two subsets intersect. This is one of the fundamental results of extremal set theory and has many extensions and generalizations. Another example is the famous theorems of Ray-Chaudhuri and Wilson [16] and of Frankl and Wilson [13], bounding the size of a family of subsets of $[n]$ where the intersection of any two sets belongs to a given set of integers L. In addition to their intrinsic interest, these theorems have surprising applications to Ramsey theory and geometry, see [4]. We also refer the reader to [12] for an overview of extremal set theory.

An interesting variant of the above problems is obtained by equipping the ground set with some combinatorial structure, such as being the edge set of a complete graph/hypergraph (see [9] for many examples). This direction of research was initiated by Simonovits and Sós [17] and received considerable attention in recent decades. For example, Ellis, Filmus and Friedgut [10] proved a conjecture of Simonovits and Sós (and improved an earlier bound of [8]) by showing that if $\mathcal{G}$ is a family of graphs on $[n]$ where the intersection of any two graphs contains a triangle, then $|\mathcal{G}| \leq 2^{\binom{n}{3}-3}$. Note that this bound is tight by taking the family of all graphs containing a given triangle. See also [6, 7, 15] for related results. It is worth mentioning that the proofs of this theorem and other results mentioned in the above two paragraphs use various interesting and important techniques, such as probabilistic arguments, linear-algebraic methods, entropy, and discrete Fourier analysis.

Another noteworthy problem on graph families is the following question of Gowers: Is it true that if $\mathcal{G}$ is a family of graphs on $[n]$ with no two graphs $G_1, G_2 \in \mathcal{G}$ satisfying that $G_1 \subseteq G_2$ and $G_2 \setminus G_1$ is a clique, then $|\mathcal{G}| \leq o\left(2^{\binom{n}{2}}\right)$? As explained by Gowers in his blog post [14], this problem is related to the first unknown case of the polynomial density Hales-Jewett theorem.

Recently, Alon, Gujgiczer, Körner, Milojević and Simonyi [3] initiated the study of so-called *graph codes*, which also falls into the general framework of studying graph families with pairwise restrictions.

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Let $\mathcal{F}$ be a family of graphs on $[n]$ which is closed under isomorphism. Using the terminology from [3], a set $\mathcal{G}$ of graphs on $[n]$ is called *$\mathcal{F}$-good* if $G_1 \oplus G_2 \in \mathcal{F}$ for every $G_1, G_2 \in \mathcal{G}$, where $G_1 \oplus G_2$ is the symmetric difference of (the edge sets of) G_1, G_2. Note that if $\mathcal{F}$ is the set of all graphs with at least d edges, then an $\mathcal{F}$-good graph family is simply a binary code (in the sense of coding theory) with distance at least d. Thus, the study of $\mathcal{F}$-good graph families is a variant of the study of codes where the ground set is given the structure of $E(K_n)$. In [3] and in a subsequent work [1], the maximum size of $\mathcal{F}$-good families was estimated for various choices of $\mathcal{F}$, such as the set of all connected graphs, all Hamiltonian graphs, and all $\mathcal{H}$-free graphs for certain graph-families $\mathcal{H}$. See also [2, 5] for related results. As in coding theory, it is also natural to study *linear* $\mathcal{F}$-good graph families, where a graph family is called linear if it is closed under symmetric difference. As a notable example, Alon [1] showed that if $\mathcal{G}$ is a linear graph family such that $G_1 \oplus G_2$ is not a clique for any $G_1, G_2 \in \mathcal{G}$, then $|\mathcal{G}| \leq 2^{\binom{n}{2} - \lfloor \frac{n}{2} \rfloor}$, and this is best possible.

For two (hyper-)graphs G_1, G_2 on $[n]$, we use $G_1 \setminus G_2$ to denote the (hyper-)graph on $[n]$ with edge-set $E(G_1) \setminus E(G_2)$. Some of the constructions of $\mathcal{F}$-good graph families $\mathcal{G}$ in [3] have the property that $G_1 \setminus G_2$ is isomorphic to $G_2 \setminus G_1$ for every $G_1, G_2 \in \mathcal{G}$. This led Alon et al. [3] to ask the question of determining the largest possible size of a graph family with this property. We will call such families *difference-isomorphic*:

Definition 1.1. *A family $\mathcal{G}$ of r-graphs on $[n]$ is called* difference-isomorphic *if for every two r-graphs $G_1, G_2 \in \mathcal{G}$, it holds that $G_1 \setminus G_2$ is isomorphic to $G_2 \setminus G_1$.*

Observe that the family of all perfect matchings is difference-isomorphic and has size $n^{(\frac{1}{2}+o(1))n}$. Alon et al. [3] constructed a slightly larger difference-isomorphic graph family by taking graphs consisting of vertex-disjoint stars. More precisely, fix a partition of $[n]$ into two sets A, B of size $|A| = k, |B| = n - k$, and consider the family of all graphs which consist of k disjoint stars each with $\frac{n}{k} - 1$ leaves, where each star has its center in A and leaves in B. This family is difference-isomorphic and has size $n^{(1+o(1))n}$ for the optimal choice of k, namely $k \approx \frac{n}{\log n}$. Given these constructions, one might expect that difference-isomorphic graph families are quite small. Rather surprisingly, it turns out that this is very far from the truth. Let us now describe a much larger difference-isomorphic graph family. Fix a permutation ψ that consists only of 2-cycles (assuming that n is even); in particular, ψ^2 is the identity, i.e., ψ is an involution. It is easy to see that for every edge $e \in E(K_n)$, there is an edge $f \in E(K_n)$ such that ψ maps e to f and f to e (possibly $e = f$). Now, take $\mathcal{G}$ to be the set of all graphs which contain exactly one of the edges e, f for each such pair (e, f). Then $\mathcal{G}$ is difference-isomorphic (in fact, ψ is the isomorphism witnessing this for every $G_1, G_2 \in \mathcal{G}$), and has size $2^{\frac{1}{2}\left(\binom{n}{2} - \lfloor \frac{n}{2} \rfloor\right)}$; see Proposition 2.6 for the full details. As our main result, we shall show that this construction is in fact the largest possible difference-isomorphic family of graphs on $[n]$. This completely resolves the aforementioned question of Alon et al. [3]. Moreover, our techniques can also be used to establish an analogous result for r-uniform hypergraphs. To state this result, define

$$f_r(n) := \begin{cases} \frac{1}{2}\binom{n}{r} & r \text{ is odd and } n \text{ is even}, \\ \frac{1}{2}\left(\binom{n}{r} - \binom{\lfloor n/2 \rfloor}{\lfloor r/2 \rfloor}\right) & \text{otherwise}. \end{cases} \quad (1)$$

Theorem 1.2. *For every $r \geq 2$, there is $n_0 = n_0(r)$ such that for every $n \geq n_0$, the largest possible size of a difference-isomorphic family of r-graphs on $[n]$ equals $2^{f_r(n)}$.*

As we shall see, the extremal example for Theorem 1.2 satisfies that all isomorphisms $G_1 \setminus G_2 \to G_2 \setminus G_1$ (for $G_1, G_2 \in \mathcal{G}$) are given by the same permutation, which is an involution. We can further strengthen Theorem 1.2 by proving a stability result (see Theorem 2.7), saying that if $\mathcal{G}$ does not have this structure, then it has smaller size.

Paper organization: Section 2 contains the key notions we will use in the proofs, as well as the construction giving the lower bound in Theorem 1.2 and a simple proof of an asymptotic version of the upper bound in Theorem 1.2. The full proof of Theorem 1.2 is then given in Section 3. Section 4 contains some concluding remarks.

2 Definitions, the Construction, and an Approximate Bound

We begin by introducing the key definitions we will use. A permutation $\varphi \in S_n$ on the vertices $[n]$ induces a permutation $\tilde{\varphi} \in S_{\binom{[n]}{r}}$ on the edges in $\binom{[n]}{r}$, namely, $\tilde{\varphi}(e) = \{\varphi(x) : x \in e\}$ for $e \in \binom{[n]}{r}$. We will omit the tilde and simply write $\varphi(e)$ when it is clear from the context that we are considering the action of φ on the edges. Also, note that the uniformity r is not part of the notation, as it will be clear from the context. For a graph G, we write $\varphi(G) = \{\tilde{\varphi}(e) : e \in E(G)\}$ to denote the image of G under $\tilde{\varphi}$.

For r-graphs G, H and a permutation $\varphi \in S_n$, we write $G \xrightarrow{\varphi} H$ to mean that $\varphi(G \setminus H) = H \setminus G$. Let $N_\varphi(G)$ denote the set of all r-graphs H such that $G \xrightarrow{\varphi} H$. Note that $G \in N_\varphi(G)$, and that $G \xrightarrow{\varphi} H$ if and only if $H \xrightarrow{\varphi^{-1}} G$. In the following definition and lemma, we describe the relation $G \xrightarrow{\varphi} H$ by giving an equivalent condition.

Definition 2.1 (Choosable pairs)**.** *Let G be an r-graph and φ be a permutation. A* choosable pair *for (G, φ) is an ordered pair $(e, f) \in \binom{[n]}{r} \times \binom{[n]}{r}$ such that $\varphi(e) = f$, $e \in G$ and $f \notin G$.*

We write $\{e, f\}$ for the unordered pair of the choosable pair (e, f). Observe that every $e \in \binom{[n]}{r}$ can be contained in at most one choosable pair. Indeed, if $e \in E(G)$ then e can only be contained in the choosable pair $(e, \varphi(e))$ (which is a choosable pair if and only if $\varphi(e) \notin G$), and similarly in the case $e \notin E(G)$. Also, if (e, f) is a choosable pair then $e \neq f$, so an edge e satisfying $\varphi(e) = e$ cannot be part of any choosable pair.

Lemma 2.2. *Let $\varphi \in S_n$ and let G, H be r-graphs. Then $G \xrightarrow{\varphi} H$ if and only if the following two conditions hold:*

(i) For every choosable pair (e, f) for (G, φ), the graph H contains exactly one of e and f.

(ii) If $e \in \binom{[n]}{r}$ is not contained in any choosable pair for (G, φ), then G and H agree on e.

In particular, $|N_\varphi(G)| = 2^m$, where m is the number of choosable pairs of (G, φ).

Proof. We first assume that $G \xrightarrow{\varphi} H$ and prove (i)-(ii). For (i), let (e, f) be a choosable pair for (G, φ). So $\varphi(e) = f$ and $e \in G$, $f \notin G$. Suppose by contradiction that H contains neither or both of e, f. If $e, f \notin H$, then $e \in G \setminus H$, and therefore $f = \varphi(e) \in H \setminus G$ (as $G \xrightarrow{\varphi} H$), in contradiction to $f \notin H$. Similarly, if $e, f \in H$ then $f \in H \setminus G$, so $e = \varphi^{-1}(f) \in G \setminus H$, in contradiction to $e \in H$. For (ii), let e be an edge which does not belong to any choosable pair of (G, φ). Suppose by contradiction that $e \in G \setminus H$ or $e \in H \setminus G$. If $e \in G \setminus H$, then $f := \varphi(e) \in H \setminus G$, meaning that (e, f) is a choosable pair. And if $e \in H \setminus G$, then $f := \varphi^{-1}(e) \in G \setminus H$, so (f, e) is a choosable pair. In both cases we got a contradiction.

In the other direction, suppose that (i)-(ii) hold and let us show that $\varphi(G \setminus H) = H \setminus G$. Let $e \in G \setminus H$. By (ii), e must belong to some choosable pair. This pair cannot have the form (f, e) (for some edge f), because this would imply that $e \notin G$. So this pair must have the form (e, f). Then $e \in G$, $f \notin G$, and $f = \varphi(e) \in H \setminus G$, because H contains exactly one of the edges e, f (by (i)) and $e \notin H$. This shows that $\varphi(G \setminus H) \subseteq H \setminus G$. A similar argument shows that $H \setminus G \subseteq \varphi(G \setminus H)$, and thus $\varphi(G \setminus H) = H \setminus G$. ■

2.1 Involutions, the extremal construction, and stability

An *involution* is a permutation $\psi \in S_n$ such that ψ^2 is the identity. In other words, an involution is a permutation whose every cycle has length 1 or 2. As we shall see, involutions play an important role in the proof of Theorem 1.2. We use the following notation:

Definition 2.3 ($\mathcal{C}_1, \mathcal{C}_2$)**.** *For an involution $\psi \in S_n$, denote by $\mathcal{C}_1(\psi)$ the set of fixed points of $\tilde{\psi}$, i.e., the set of edges $e \in \binom{[n]}{r}$ satisfying $\psi(e) = e$. Denote by $\mathcal{C}_2(\psi)$ the set of all 2-cycles of $\tilde{\psi}$, namely, the set of all (unordered) pairs of distinct edges $\{e, f\}$, $e, f \in \binom{[n]}{r}$, with $\psi(e) = f$ and $\psi(f) = e$.*

Observe that if $\psi \in S_n$ is an involution then so is $\tilde{\psi} \in S_{\binom{[n]}{r}}$. Hence, every edge $e \in \binom{[n]}{r}$ belongs to $\mathcal{C}_1(\psi)$ or to some pair in $\mathcal{C}_2(\psi)$. The following lemma gives a relation between involutions and the function $f_r(n)$ defined in (1).

Lemma 2.4. *For $2 \le r \le n$, $f_r(n)$ is the maximum of $|\mathcal{C}_2(\psi)|$ over all involutions $\psi \in S_n$. The maximum is attained by involutions having at most one fixed point.*

Proof. Let ψ be an involution. We have $|\mathcal{C}_1(\psi)| + 2|\mathcal{C}_2(\psi)| = \binom{n}{r}$. Suppose that ψ has a fixed points $x_1, \ldots, x_a$ and b two-cycles $y_1z_1, \ldots, y_bz_b$ so that $a + 2b = n$. Observe that $e \in \binom{[n]}{r}$ satisfies $\psi(e) = e$ if and only if for every $1 \le i \le b$ it holds that $y_i \in e \iff z_i \in e$. The number of such e is

$$|\mathcal{C}_1(\psi)| = \sum_{\substack{0 \le i \le a \\ i \equiv r \bmod 2}} \binom{a}{i}\binom{b}{(r-i)/2} \tag{2}$$

If $a \le 1$ (i.e., $a = 0$ if n is even and $a = 1$ if n is odd), then (2) equals $\binom{n}{r} - 2f_r(n)$. Indeed, if r is odd and n is even then the sum in (2) is empty, because $a = 0$. If r is odd and n is odd then the sum equals $\binom{b}{(r-1)/2} = \binom{\lfloor n/2 \rfloor}{\lfloor r/2 \rfloor}$. Similarly, if r is even then the sum equals $\binom{b}{r/2} = \binom{\lfloor n/2 \rfloor}{\lfloor r/2 \rfloor}$. So we see that when $a \le 1$, we have $|\mathcal{C}_2(\psi)| = \frac{1}{2}\binom{n}{r} - \frac{1}{2}|\mathcal{C}_1(\psi)| = f_r(n)$, as claimed.

Now suppose that $a \ge 2$, and let ψ' be obtained from ψ by making two fixed points of ψ into a 2-cycle. Then $\mathcal{C}_1(\psi') \subseteq \mathcal{C}_1(\psi)$, so $|\mathcal{C}_1(\psi')| \le |\mathcal{C}_1(\psi)|$. Therefore, $|\mathcal{C}_1(\psi)|$ is minimized when ψ has at most one fixed point. Hence, such ψ maximize $|\mathcal{C}_2(\psi)|$. ■

If ψ is an involution, then $G \xrightarrow{\psi} H$ if and only if $H \xrightarrow{\psi} G$ (because $\psi^{-1} = \psi$), so this relation is symmetric. The following lemma gives a simple description of this relation.

Lemma 2.5. *Let $\psi \in S_n$ be an involution and let G_1, G_2 be r-graphs. Then $G_1 \xrightarrow{\psi} G_2$ holds if and only if the following two conditions are satisfied:*

- *G_1, G_2 agree on all edges in $\mathcal{C}_1(\psi)$.*
- *For every $\{e, f\} \in \mathcal{C}_2(\psi)$, G_1, G_2 contain the same number edges from $\{e, f\}$.*

In particular, the relation $G_1 \overset{\psi}{\to} G_2$ is an equivalence relation.

Proof. This follows from Lemma 2.2. Indeed, for every choosable pair (e, f) for (G_1, ψ), we must have $\{e, f\} \in \mathcal{C}_2(\psi)$, because $\psi(e) = f$ and $e \neq f$ (see Definition 2.1). Also, for $\{e, f\} \in \mathcal{C}_2(\psi)$, we have that (e, f) or (f, e) is a choosable pair for (G_1, ψ) if and only if G contains exactly one of the edges e, f. Thus, in the special case where ψ is an involution, Items (i)-(ii) of Lemma 2.2 are equivalent to the two items of Lemma 2.5. ■

By Lemma 2.5, for an involutation ψ, the relation $G_1 \overset{\psi}{\to} G_2$ is an equivalence relation. Therefore, every *ψ-component* of r-graphs, i.e. a set of r-graphs X connected in the relation $G_1 \overset{\psi}{\to} G_2$, is a “clique", namely, $G_1 \overset{\psi}{\to} G_2$ for all $G_1, G_2 \in X$. We refer to such a set X as a *ψ-clique*. When ψ is not specified, we will call X an *involution clique*.

Using the above, we now provide the lower bound in Theorem 1.2. The construction is an involution clique. We also show that every involution cliques has at most the extremal size $2^{f_r(n)}$.

Proposition 2.6. *Let $2 \leq r \leq n$.*

1. *There exists a difference-isomorphic family of r-graphs on $[n]$ of size $2^{f_r(n)}$.*
2. *For every involution ψ, every ψ-clique has size at most $2^{f_r(n)}$.*

Proof. For Item 1, let $\psi \in S_n$ be an involution with at most 1 fixed points. Let $\{e_1, f_1\}, \ldots, \{e_m, f_m\}$ be the 2-cycles of ψ. By Lemma 2.4, we have $m = f_r(n)$. Let $\mathcal{G}$ be the set of all graphs that contain exactly one of the edges e_i, f_i for every $1 \leq i \leq m$, and do not contain any other edges (the remaining edges are the edges in $\mathcal{C}_1(\psi)$). Then $|\mathcal{G}| = 2^m = 2^{f_r(n)}$. Also, Lemma 2.5 implies that $G_1 \overset{\psi}{\to} G_2$ for every $G_1, G_2 \in \mathcal{G}$, meaning that $\mathcal{G}$ is difference-isomorphic.

For Item 2, let $\mathcal{G}$ be a ψ-clique. Then $|\mathcal{G}| \leq 2^{|\mathcal{C}_2(\psi)|}$ by Lemma 2.5 (and equality holds only if for every $(e, f) \in \mathcal{C}_2(\psi)$, all the graphs in $\mathcal{G}$ contain exactly one of the edges e, f). Indeed, the graphs in $\mathcal{G}$ have at most 2 choices for each pair $(e, f) \in \mathcal{C}_2$, and no choice for the remaining edges. By Lemma 2.4, $|\mathcal{C}_2(\psi)| \leq f_r(n)$, so indeed $|\mathcal{G}| \leq 2^{f_r(n)}$. ■

The construction given by Proposition 2.6 is an involution clique. As mentioned in the introduction, we can prove the following stability version of Theorem 1.2, stating that any difference-isomorphic family which is not an involution clique has smaller size.

Theorem 2.7. *For every $r \geq 2$, there is $n_0 = n_0(r)$ such that for every $n \geq n_0$ and every difference-isomophic family $\mathcal{G}$ of r-graphs on $[n]$, if $\mathcal{G}$ is not an involution clique then*

$$|\mathcal{G}| \leq \begin{cases} \left(1 - n^{-O(\sqrt{n})}\right) \cdot 2^{f_2(n)} & r = 2, \\ e^{-\Omega(n^{r-2})} \cdot 2^{f_r(n)} & r \geq 3. \end{cases}$$

Observe that Theorem 1.2 follows by combining Theorem 2.7 and Item 2 in Proposition 2.6.

In the appendix, we will construct a difference-isomorphic r-graph family on $[n]$ which is not an involution clique, but has size $e^{-O(n^{r-2})} \cdot 2^{f_r(n)}$. This shows that when $r \geq 3$, Theorem 2.7 is tight up to the hidden constant in the exponent. We do not know if Theorem 2.7 is also tight for $r = 2$, i.e. if there is a difference-isomorphic graph-family on $[n]$ which is not an involution clique and has size $(1 - o(1))2^{f_2(n)}$.

2.2 An approximate upper bound

Here we give an easy proof of an asymptotic version of Theorem 1.2, namely, we show that a difference-isomorphic family of r-graphs on $[n]$ has size at most $2^{\left(\frac{1}{2}+o(1)\right)\binom{n}{r}}$. For an r-graph G, let G^c denote the complement of G. We start with the following observation.

Lemma 2.8. *If $\mathcal{G}$ is difference-isomorphic, then so is $\{G^c : G \in \mathcal{G}\}$.*

Proof. For every two r-graphs $G_1, G_2 \in \mathcal{G}$, it holds that $G_1^c \setminus G_2^c = G_2 \setminus G_1$ and $G_2^c \setminus G_1^c = G_1 \setminus G_2$. Hence, $G_1^c \setminus G_2^c$ is isomorphic to $G_2^c \setminus G_1^c$. ■

Observe that if $\mathcal{G}$ is difference-isomorphic, then all r-graphs in $\mathcal{G}$ have the same number of edges.

Lemma 2.9. *Let $\mathcal{G}$ be a difference-isomorphic family where all r-graphs have m edges. Then $|\mathcal{G}| \leq n! \cdot 2^m$.*

Proof. Fix any r-graph $G \in \mathcal{G}$. For each subgraph F of G, let $\mathcal{G}_F$ be the set of all $G' \in \mathcal{G}$ with $G \setminus G' = F$. Then $G' \setminus G$ is isomorphic to F. It follows that there are at most $n!$ choices for $G' \setminus G$ (because each isomorphism class has size at most $n!$). As $G' \setminus G$ determines $G' = (G' \setminus G) \cup (G \setminus F)$, we get that $|\mathcal{G}_F| \leq n!$. The lemma follows by summing over all $2^{e(G)} = 2^m$ choices for F. ■

Now let $\mathcal{G}$ be a difference-isomorphic family of r-graphs on $[n]$. By Lemma 2.8, we may assume that all r-graphs in $\mathcal{G}$ have size at most $\left\lfloor \frac{1}{2}\binom{n}{r} \right\rfloor$. Then Lemma 2.9 implies that $|\mathcal{G}| \leq n! \cdot 2^{\frac{1}{2}\binom{n}{r}} = 2^{\left(\frac{1}{2}+o(1)\right)\binom{n}{r}}$, as claimed.

3 Proof of Theorem 1.2

In this section we prove Theorem 2.7, which also implies Theorem 1.2. Throughout the section, we will assume that n is large enough as a function of r, wherever needed.

3.1 Proof overview

For a set of r-graphs X and $\psi \in S_n$, let $e_\psi(X)$ denote the number of (ordered) pairs $(G_1, G_2) \in X \times X$ such that $G_1 \xrightarrow{\psi} G_2$. Note that the pairs (G_1, G_1) are always counted because $G_1 \xrightarrow{\psi} G_1$ holds trivially. Also, if X is a difference-isomorphic family, then $\sum_{\psi \in S_n} e_\psi(X) \geq |X|^2$, because for every pair $G_1, G_2 \in X$ there is $\psi \in S_n$ such that $G_1 \xrightarrow{\psi} G_2$.

Let $\mathcal{G}$ be a difference-isomorphic family of r-graphs. For every $G \in \mathcal{G}$ and every $\varphi, \psi \in S_n$, we will consider $e_\psi(N_\varphi(G))$, i.e. the number of (ordered) pairs $G_1, G_2 \in N_\varphi(G)$ such that $G_1 \xrightarrow{\psi} G_2$. One of the main ingredients in the proof of Theorem 2.7 is upper bounds for $e_\psi(N_\varphi(G))$, and this is the main focus of Section 3.2. Roughly speaking, we will show that $e_\psi(N_\varphi(G))$ is much smaller than $2^{2f_r(n)}$, which is the number of pairs of r-graphs in the extremal construction, unless ψ is an involution and φ is very close to ψ. More precisely, we will show that unless φ, ψ have this specific structure, it holds that $e_\psi(N_\varphi(G)) \leq 2^{2f_r(n)} \cdot e^{-\Omega(n^{r-1})}$; see Lemma 3.8. We note that due to this bound, the proof for the case $r \geq 3$ is simpler than for the case $r = 2$. This is because for $r \geq 3$ we have $e^{\Omega(n^{r-1})} \gg n!$, allowing us to sum the above bound over all permutations $\psi \in S_n$. Thus, let us assume for now that $r \geq 3$. We use our bounds on $e_\psi(N_\varphi(G))$ to bound $|N_\varphi(G) \cap \mathcal{G}|$, using the fact that $|N_\varphi(G) \cap \mathcal{G}|^2 \leq \sum_{\psi \in S_n} e_\psi(N_\varphi(G) \cap \mathcal{G})$ (since $\mathcal{G}$ is difference-isomorphic). As explained above, for a fixed φ, we have a very strong bound on $e_\psi(N_\varphi(G))$ for almost all permutations ψ (i.e., for all ψ which are not very close to φ). For the remaining ψ, we can show that $e_\psi(N_\varphi(G)) \le 2^{f_r(n)} \cdot e^{-\Omega(n)}$ unless $\varphi = \psi$ is an involution (see Lemma 3.9). This bound is weaker, but it suffices because the number of the remaining ψ's is only $e^{o(n)}$. Thus, combining all of the above, we conclude that unless φ is an involution, $|N_\varphi(G) \cap \mathcal{G}|$ is much smaller than the extremal size $2^{f_r(n)}$, i.e. $|N_\varphi(G) \cap \mathcal{G}| \le 2^{f_r(n)} \cdot e^{-\Omega(n)}$. We note that if $r = 2$ then this is no longer true; $N_\varphi(G) \cap \mathcal{G}$ can be of size $\Omega(2^{f_2(n)})$ even if φ is not an involution (see the construction in the proof of Proposition A.1). Consequently, the proof for the case $r = 2$ is considerably more involved than for $r \ge 3$. The proof for $r \ge 3$ is given in Section 3.3 and the proof for $r = 2$ is given in Section 3.4.

We now continue with the overview of the proof for the case $r \ge 3$. A key step in the proof is to show that if $|\mathcal{G}|$ is close to the extremal size $2^{f_r(n)}$, then $\mathcal{G}$ must contain a large involution clique (this is done in Lemma 3.11). To this end, we use the aforementioned fact that $|N_\varphi(G) \cap \mathcal{G}|$ is small whenever φ is not an involution (for every $G \in \mathcal{G}$). Note also that for an involution φ, the set $N_\varphi(G) \cap \mathcal{G}$ is a φ-clique (see Lemma 2.5). Thus, if $\mathcal{G}$ does not contain a large involution clique, then $|N_\varphi(G) \cap \mathcal{G}|$ is small for every permutation $\varphi \in S_n$, i.e. $|N_\varphi(G) \cap \mathcal{G}| \le 2^{f_r(n)} \cdot e^{-\Omega(n)}$. If this is the case, then we can upper-bound $|\mathcal{G}|$, as follows. Fix an arbitrary $G \in \mathcal{G}$, and fix a permutation $\varphi_1 \in S_n$ for which $|N_{\varphi_1}(G) \cap \mathcal{G}| \ge |\mathcal{G}|/n!$ (such a permutation exists by averaging). Letting $\mathcal{G}_1 = N_\varphi(G) \cap \mathcal{G}$ and $\mathcal{G}_2 = \mathcal{G} \setminus \mathcal{G}_1$, we consider, for each permutation φ_2, the number of pairs $(G_1, G_2) \in \mathcal{G}_1 \times \mathcal{G}_2$ with $G_1 \stackrel{\varphi_2}{\to} G_2$. It turns out that the number of such pairs is very small unless φ_2 is close to φ_1 (this is shown in Lemma 3.7). Hence, almost all such pairs (G_1, G_2) correspond to only few (i.e. only $e^{o(n)}$) permutations $\varphi_2 \in S_n$. Thus, by averaging, we can find a permutation φ_2 and a graph $G_1 \in \mathcal{G}_1$ for which $|N_{\varphi_2}(G_1) \cap \mathcal{G}_2| \ge |\mathcal{G}_2| \cdot e^{-o(n)}$. On the other hand, we know that $|N_{\varphi_2}(G_1) \cap \mathcal{G}| \le 2^{f_r(n)} \cdot e^{-\Omega(n)}$, which allows us to deduce that $\mathcal{G}_2$ is small. As $\mathcal{G}_1$ is also small (for the same reason, as $\mathcal{G}_1 = N_{\varphi_1}(G) \cap \mathcal{G}$), we conclude that $|\mathcal{G}| = |\mathcal{G}_1| + |\mathcal{G}_2| \ll 2^{f_r(n)}$.

Summarizing the above discussion, we are able to show that if $|\mathcal{G}|$ is close to the extremal size $2^{f_r(n)}$, then $\mathcal{G}$ must contain a large involution clique. We will use this to conclude the proof of Theorem 2.7 (for $r \ge 3$), as follows. Let $\mathcal{G}'$ be a largest involution clique in $\mathcal{G}$ and let ψ be the corresponding involution. If $\mathcal{G}' = \mathcal{G}$ then $\mathcal{G}$ is an involution clique, so the statement of Theorem 2.7 is vacuous. Otherwise, fix an arbitrary $G \in \mathcal{G} \setminus \mathcal{G}'$. Observe that $N_\psi(G) \cap \mathcal{G}' = \emptyset$, because otherwise $\mathcal{G}' \cup \{G\}$ would be a ψ-clique by Lemma 2.5, contradicting the maximality of $\mathcal{G}'$. Now, for every $\varphi \ne \psi$, we can bound $|N_\varphi(G) \cap \mathcal{G}'|$ by using the fact that $|N_\varphi(G) \cap \mathcal{G}'|^2 \le e_\psi(N_\varphi(G))$, as $\mathcal{G}'$ is a ψ-clique. We then use this to upper bound $|\mathcal{G}'| \le \sum_{\varphi \in S_n} |N_\varphi(G) \cap \mathcal{G}'|$. Here, too, we separate the permutations φ into those which are not very close to ψ, for which we have a very strong bound (this covers almost all permutations), and the remaining few permutations, for which we have a weaker but sufficient bound. This allows us to show that $|\mathcal{G}'|$ is small, which in turn implies that $\mathcal{G}$ is small (as it does not contain large involution cliques), completing the proof.

3.2 Edge counts inside $N_\varphi(G)$

We start with the following important definition.

Definition 3.1 (Good choosable pairs)**.** *Let $\varphi, \psi \in S_n$ and let G be an r-graph. Let (e, f) be a choosable pair for (G, φ). We say that (e, f) is a* good choosable pair *for (G, φ, ψ) if $\psi(e) = f$ and $\psi(f) = e$.*

The following lemma gives an upper bound on $e_\psi(N_\varphi(G))$ in terms of the number of choosable pairs of (G, φ) and the number of good choosable pairs of (G, φ, ψ).

Lemma 3.2. *Let $n \geq r \geq 2$. Let $\varphi, \psi \in S_n$ and let G be an r-graph. Let m be the number of choosable pairs of (G, φ) and let m_g be the number of good choosable pairs of (G, φ, ψ). Then*

$$e_\psi(N_\varphi(G)) \leq 4^{m_g} \cdot 3.9^{m-m_g}. \tag{3}$$

In particular, if $m \leq \frac{1}{2}\binom{n}{r} - x$ and $m_g \leq \frac{1}{2}\binom{n}{r} - y$ for some $y \geq x \geq 0$, then

$$e_\psi(N_\varphi(G)) \leq 2^{\binom{n}{r}-2x} e^{-(y-x)/40}. \tag{4}$$

It is worth comparing the bound in Eq. (3) with the trivial upper bound $e_\psi(N_\varphi(G)) \leq |N_\varphi(G)|^2 = 4^m$, where the equality follows from Lemma 2.2.

Proof of Lemma 3.2. The second part of the lemma follows from Eq. (3) as follows:

$$e_\psi(N_\varphi(G)) \leq \left(\frac{4}{3.9}\right)^{m_g} 3.9^m \leq \left(\frac{4}{3.9}\right)^{\frac{1}{2}\binom{n}{r}-y} 3.9^{\frac{1}{2}\binom{n}{r}-x} = 4^{\frac{1}{2}\binom{n}{r}-x}\left(1-\frac{1}{40}\right)^{y-x} \leq 2^{\binom{n}{r}-2x}e^{-(y-x)/40},$$

where in the last inequality, we used $y \geq x$.

We now prove the first part of the lemma. Let (e_i, f_i), $i = 1, \ldots, m$ be the choosable pairs for (G, φ). Suppose without loss of generality that (e_i, f_i) is a good choosable pair for (G, φ, ψ) if and only if $i \leq m_g$. For each $i \in \{m_g + 1, \ldots, m\}$, (e_i, f_i) is a *bad* (i.e., not good) choosable pair, so $\psi(e_i) \neq f_i$ or $\psi(f_i) \neq e_i$. If $\psi(e_i) \neq f_i$, then there are three options for $\psi(e_i)$: either $\psi(e_i) = e_i$; or $\psi(e_i)$ does not belong to any of the choosable pairs, i.e. $\psi(e_i) \notin \{e_1, f_1, \ldots, e_m, f_m\}$; or $\psi(e_i) \in \{e_j, f_j\}$ for some $j \neq i$. If $\psi(f_i) \neq e_i$ then the analogous statements hold for f_i. We will refer to these three cases as types (i), (ii) and (iii), respectively.

Recall that our goal is to count pairs $(G_1, G_2) \in N_\varphi(G) \times N_\varphi(G)$ with $G_1 \xrightarrow{\psi} G_2$. We shall upper-bound the number of choices for (G_1, G_2) on each of the choosable pairs (e_i, f_i). By Lemma 2.2, G_1, G_2 agree with G on all edges outside $\{e_1, f_1, \ldots, e_m, f_m\}$. Also, for every $1 \leq i \leq m$, each of G_1, G_2 contains exactly one of the edges e_i, f_i. Hence, trivially, (G_1, G_2) has at most 4 options on (e_i, f_i) (i.e., two options for G_1 and two options for G_2). To prove the lemma, we will show that on $\Omega(m - m_g)$ of the bad choosable pairs, there are less than 4 options.

First, suppose there are at least $(m - m_g)/6$ bad choosable pairs of type (i) or (ii). Fix such a pair (e_i, f_i). We assume that $\psi(e_i) = e_i$ or $\psi(e_i) \notin \{e_1, f_1, \ldots, e_m, f_m\}$. The other case, namely that $\psi(f_i) = f_i$ or $\psi(f_i) \notin \{e_1, f_1, \ldots, e_m, f_m\}$, is similar. We claim that it is impossible to have $e_i \in G_1, f_i \notin G_1$ and $e_i \notin G_2, f_i \in G_2$. This would show that one of the 4 possibilities for (G_1, G_2) on (e_i, f_i) is impossible. Indeed, suppose by contradiction that the above holds. Then $e_i \in G_1 \setminus G_2$. Hence, $\psi(e_i) \in G_2 \setminus G_1$, as $G_1 \xrightarrow{\psi} G_2$. This rules out the case $\psi(e_i) = e_i$, so $\psi(e_i) \notin \{e_1, f_1, \ldots, e_m, f_m\}$. But then, by Lemma 2.2, we have that $\psi(e_i) \in G_1$ if and only if $\psi(e_i) \in G_2$, so this again contradicts $\psi(e_i) \in G_2 \setminus G_1$. Summarizing, we see that on at least $(m - m_g)/6$ of the bad choosable pairs, (G_1, G_2) has at most 3 options. As explained before, on every other choosable pair (G_1, G_2) has at most 4 options. Hence, Eq. (3) follows as

$$e_\psi(N_\varphi(G)) \leq 4^{m-\frac{m-m_g}{6}} \cdot 3^{\frac{m-m_g}{6}} = 4^{m_g} \cdot 4^{m-m_g}\left(\frac{3}{4}\right)^{\frac{m-m_g}{6}} \leq 4^{m_g} \cdot 3.9^{m-m_g}.$$

Now suppose that there are at least $5(m - m_g)/6$ bad choosable pairs of type (iii). Let S be the set of indices $i \in \{m_g + 1, \ldots, m\}$ for which (e_i, f_i) is of type (iii). By definition, for each $i \in S$, there exists some $j = j(i) \neq i$ such that $\psi(e_i) \in \{e_j, f_j\}$ or $\psi(f_i) \in \{e_j, f_j\}$. Construct an auxiliary digraph H with vertex set $[m]$ where we add the directed edge $(i, j(i))$ for every $i \in S$. Then every $i \in S$ has out-degree 1 in H (and all other i have out-degree 0), so $e(H) = |S|$. Also, the maximum in-degree in H is at most 2, because the in-neighbors of i correspond to $\psi^{-1}(e_i), \psi^{-1}(f_i)$. Therefore, each edge $(i,j) \in E(H)$ is incident to at most four other edges, corresponding to two in-neighbors of i, one other in-neighbor of j, and one out-neighbor of j. Hence, H contains a matching M of size $|M| \geq |S|/5 \geq (m - m_g)/6$. (We can find such a matching M greedily, by iteratively adding an edge to M and deleting all incident edges.)

Consider any $(i,j) \in M$. By the definition of H, we have $\psi(e_i) \in \{e_j, f_j\}$ or $\psi(f_i) \in \{e_j, f_j\}$. Recall that a priori, (G_1, G_2) has at most $4 \cdot 4 = 16$ options on $\{e_i, f_i, e_j, f_j\}$. We will show that in fact there are at most 13 options. Suppose without loss of generality that $\psi(e_i) = f_j$; the remaining 3 cases (i.e. $\psi(e_i) = e_j$, $\psi(f_i) = e_j$, and $\psi(f_i) = f_j$) are similar. Considering one of the four choices of (G_1, G_2) on (e_i, f_i), observe that if $e_i \in G_1, e_i \notin G_2$, namely $e_i \in G_1 \setminus G_2$, then $f_j = \psi(e_i) \in G_2 \setminus G_1$, which forces the choice $e_j \in G_1, f_j \notin G_1, e_j \notin G_2, f_j \in G_2$ of (G_1, G_2) on (e_j, f_j). Hence, 3 of the 16 choices for (G_1, G_2) on $\{e_i, f_i, e_j, f_j\}$ are impossible, leaving 13 choices.

Recalling that (G_1, G_2) has at most 4 choices for every choosable pair (e_i, f_i), we get the bound

$$e_\psi(N_\varphi(G)) \leq 13^{|M|} \cdot 4^{m-2|M|} = 4^{m_g} \cdot 4^{m-m_g} \left(\frac{13}{16}\right)^{|M|} \leq 4^{m_g} \cdot 4^{m-m_g} \left(\frac{13}{16}\right)^{\frac{m-m_g}{6}} \leq 4^{m_g} \cdot 3.9^{m-m_g}.$$

This completes the proof of Lemma 3.2. ■

In the next lemma we prove the intuitive fact that if a permutation ψ is an "almost-involution" on the edges, in the sense that almost all edges belong to a 2-cycle of $\tilde{\psi}$, then ψ is also an almost-involution on the vertices, in the sense that almost all vertices belong to a 2-cycle of ψ. For the proof, let us recall the Erdős-Ko-Rado theorem [11]:

Theorem 3.3 (Erdős-Ko-Rado, [11])**.** *Let $n \geq 2r$. If $\mathcal{A}$ is a family of distinct r-element subsets of $[n]$ such that each two subsets intersect, then $|\mathcal{A}| \leq \binom{n-1}{r-1}$.*

Lemma 3.4. *Let $r \geq 2$ and suppose that n is sufficiently large in terms of r. Let $\delta \in \mathbb{R}$ satisfy $1 \leq \delta = o(n^r)$. For every $\psi \in S_n$, if $\tilde{\psi} \in S_{\binom{[n]}{r}}$ has at least $\left(\frac{1}{2}\binom{n}{r} - \delta\right)$ 2-cycles, then ψ has at least $\left(\frac{n}{2} - r\delta^{1/r}\right)$ 2-cycles.*

Proof. Let $\mathcal{E}_2$ denote the set of edges $e \in \binom{[n]}{r}$ which belong to some 2-cycle of $\tilde{\psi}$. Fix any vertex $v \in [n]$. First, we claim that if v is incident to more than $\binom{n-2}{r-2}$ distinct edges in $\mathcal{E}_2$, then $\psi^2(v) = v$. Indeed, if $\mathcal{E}_2(v) := \{e \setminus \{v\} : e \in \mathcal{E}_2, v \in e\}$ has size larger than $\binom{n-2}{r-2}$, then by the Erdős-Ko-Rado theorem (Theorem 3.3) with $[n] \setminus \{v\}$ in place of $[n]$ and $r-1$ in place of r, there exist distinct $e_1, e_2 \in \mathcal{E}_2$ such that $v \in e_1, v \in e_2$ and $(e_1 \setminus \{v\}) \cap (e_2 \setminus \{v\}) = \emptyset$, i.e. $e_1 \cap e_2 = \{v\}$. Putting $f_1 = \psi(e_1)$ and $f_2 = \psi(e_2)$, we know that $\psi(f_1) = e_1$ and $\psi(f_2) = e_2$, because $(e_1, f_1), (e_2, f_2)$ are 2-cycles of $\tilde{\psi}$ (as $e_1, e_2 \in \mathcal{E}_2$). Since ψ is a bijection, it holds that $\psi(e_1 \cap e_2) = f_1 \cap f_2$ and $\psi(f_1 \cap f_2) = e_1 \cap e_2$. Since $e_1 \cap e_2 = \{v\}$, we get that $\psi^2(v) = v$, as claimed. This means that v lies in a 1-cycle or a 2-cycle of ψ.

Let X be the set of vertices $v \in [n]$ which are incident to **at most** $\binom{n-2}{r-2}$ edge from $\mathcal{E}_2$. We showed above that every vertex not in X lies in a 1-cycle or a 2-cycle of ψ. Let $x := |X|$. For each vertex $v \in X$, there are at least $\binom{n-1}{r-1} - \binom{n-2}{r-2} = \binom{n-2}{r-1}$ edges $e \in \binom{[n]}{r}$ with $v \in e$ and $e \notin \mathcal{E}_2$. It follows that $\binom{n}{r} - |\mathcal{E}_2| \geq \frac{1}{r} x \binom{n-2}{r-1}$, because each edge is counted at most r times. By assumption, $\tilde{\psi}$ has at least

$\left(\frac{1}{2}\binom{n}{r} - \delta\right)$ 2-cycles, so $|\mathcal{E}_2| \geq \binom{n}{r} - 2\delta$. Thus, $\frac{1}{r}x\binom{n-2}{r-1} \leq 2\delta$, giving $x \leq 2r\delta/\binom{n-2}{r-1} \leq 0.5\delta^{1/r}$ using that $\delta = o(n^r)$.

Let Y be the set of fixed points of ψ, and set $y := |Y|$. If $e \in \binom{[n]}{r}$ is contained in Y then $\psi(e) = e$, so e cannot be in a 2-cycle of $\tilde{\psi}$, i.e. $e \notin \mathcal{E}_2$. It follows that $|\mathcal{E}_2| \leq \binom{n}{r} - \binom{y}{r}$, and hence $2\delta \geq \binom{y}{r}$. So either $y < r$, or $y \geq r$ and then $2\delta \geq \binom{y}{r} \geq (y/r)^r$. In any case, $y \leq r(2\delta)^{1/r} \leq 1.5r\delta^{1/r}$, using $r \geq 2$.

By definition of X and Y, every vertex in $[n] \setminus (X \cup Y)$ lies in a 2-cycle of ψ. There are $n - x - y \geq n - 2r\delta^{1/r}$ such vertices. Hence, the number of 2-cycles in ψ is at least $\frac{n}{2} - r\delta^{1/r}$. ■

Note that good choosable pairs (see Definition 3.1) are closely related to the 2-cycles of $\tilde{\psi}$ (and thus ψ). In the following lemma, we further prove that (G, φ, ψ) has few good choosable pairs unless φ contains most 2-cycles of ψ.

Lemma 3.5. *Let G be an r-graph and $\varphi, \psi \in S_n$. Let t be the number of 2-cycles of ψ which are not 2-cycles of φ. Then $m_g \leq \frac{1}{2}\binom{n}{r} - \binom{t}{r}$, where m_g is the number of good choosable pairs of (G, φ, ψ).*

Proof. Let $u_1v_1, u_2v_2, \ldots, u_rv_r$ be any r distinct 2-cycles of ψ which are not 2-cycles of φ. An edge $e \in \binom{[n]}{r}$ is called a *transversal* if it contains exactly one of u_i, v_i for every $i \in \{1, \ldots, r\}$. It suffices to show that there are at least two transversal edges e, f which are not contained in any good choosable pair for (G, φ, ψ). Indeed, we can then sum over all $\binom{t}{r}$ choices for $u_1v_1, u_2v_2, \ldots, u_rv_r$ to get at least $2\binom{t}{r}$ edges which are not contained in any good choosable pair. This would imply that $m_g \leq \frac{1}{2}\binom{n}{r} - \binom{t}{r}$.

Suppose for contradiction that all transversal edges (of $u_1v_1, u_2v_2, \ldots, u_rv_r$) but at most one are contained in good choosable pairs. Observe that if edge e is a transversal, then $\psi(e)$ is also a transversal, and $e, \psi(e)$ form a 2-cycle of $\tilde{\psi}$. By the definition of good choosable pairs (see Definition 3.1), if e lies in some good choosable pair then this pair must be $\{e, \psi(e)\}$. Thus, our assumption implies that every transversal edge e belongs to a good choosable pair, namely the pair $\{e, \psi(e)\}$.

We first claim that for every distinct $i, j \in \{1, \ldots, r\}$, $\varphi(u_i) \neq v_j$ and $\varphi(v_i) \neq u_j$. Indeed, suppose that $\varphi(u_i) = v_j$ for some $i \neq j$ (the case $\varphi(v_i) = u_j$ is similar). Let $e = \{u_k : k \neq i, j\} \cup \{u_i, v_j\}$ and $f = \psi(e) = \{v_k : k \neq i, j\} \cup \{v_i, u_j\}$. As e, f form a choosable pair, we have $\varphi(e) = f$ or $\varphi(f) = e$ (see Definition 2.1). But $u_i \in e$, $v_j = \varphi(u_i) \notin f$, meaning that $\varphi(e) \neq f$. Similarly, $v_j \in e$ but $u_i = \varphi^{-1}(v_j) \notin f$, so $e \neq \varphi(f)$. In either case, we got a contradiction.

Now, write $U = \{u_1, \ldots, u_r\}$ and $V = \{v_1, \ldots, v_r\}$. As the edges $u_1 \ldots u_r$ and $v_1 \ldots v_r$ form a good choosable pair, it holds that $\varphi(u_1 \ldots u_r) = v_1 \ldots v_r$ or $\varphi(v_1 \ldots v_r) = u_1 \ldots u_r$, i.e. $\varphi(U) = V$ or $\varphi(V) = U$. Without loss of generality, assume that $\varphi(U) = V$. For every $i \in [r]$, we showed above that $\varphi(u_i) \neq v_j$ for every $j \neq i$. As $\varphi(u_i) \in V$, we must have $\varphi(u_i) = v_i$. By assumption, $\varphi(v_i) \neq u_i$ (otherwise u_iv_i would form a common 2-cycle in φ and ψ, in contradiction to the choice of u_iv_i). We also showed that $\varphi(v_i) \neq u_j$ for every $j \neq i$, so $\varphi(v_j) \notin U$. It follows that $\varphi(V) \cap U = \emptyset$. Now, consider the edges $e = \{u_1, v_2, \ldots, v_r\}$ and $f = \psi(e) = \{v_1, u_2, \ldots, u_r\}$. By assumption $\{e, f\}$ forms a good choosable pair, which implies that $\varphi(e) = f$ or $\varphi(f) = e$. In the former case, as $\varphi(u_1) = v_1$, we must have $\varphi(v_2, \ldots, v_r) = u_2 \ldots u_r$; in the latter case, as $\varphi(u_2 \ldots u_r) = v_2 \ldots v_r$, we must have $\varphi(v_1) = u_1$. In either case, $\varphi(V) \cap U \neq \emptyset$, giving a contradiction. We, therefore, conclude that at least two transversal edges do not lie in any good choosable pair for (G, φ, ψ), as required. ■

We now prove the main lemmas of this section, namely Lemma 3.6, Lemma 3.8 and Lemma 3.9. These lemmas establish upper bounds on $e_\psi(N_\varphi(G))$ in various cases. Our goal is to show that $e_\psi(N_\varphi(G))$ is significantly smaller than $2^{2f_r(n)}$, which is the number pairs in the extremal construction (see Proposition 2.6). In the following lemma we show that $e_\psi(N_\varphi(G)))$ is small unless φ, ψ are close to each other and consist mostly of 2-cycles.

Lemma 3.6. *Let $r \geq 2$ and n be sufficiently large in terms of r. Let G be an r-graph and $\varphi, \psi \in S_n$. For every $\delta \geq n^{r/2}$ with $\delta = o(n^r)$, it holds that $e_\psi(N_\varphi(G)) \leq 2^{2f_r(n)} \cdot e^{-\delta}$ unless ψ and φ share at least $\left(\frac{n}{2} - 18r\delta^{1/r}\right)$ 2-cycles.*

Proof. Suppose that $e_\psi(N_\varphi(G)) > 2^{2f_r(n)} \cdot e^{-\delta}$. Denote by m the number of choosable pairs for (G, φ) and by m_g the number of good choosable pairs for (G, φ, ψ). We claim that $m_g > \frac{1}{2}\binom{n}{r} - 80\delta$. So suppose by contradiction that $m_g \leq \frac{1}{2}\binom{n}{r} - 80\delta$. Combining this with the trivial bound $m \leq \frac{1}{2}\binom{n}{r}$, we can apply Eq. (4) with $x = 0$ and $y = 80\delta$ to get

$$e_\psi(N_\varphi(G)) \leq 2^{\binom{n}{r}} e^{-2\delta} \leq 2^{2f_r(n) + n^{r/2}} \cdot e^{-2\delta} \leq 2^{2f_r(n)} \cdot e^{-\delta},$$

where the second inequality uses that $f_r(n) \geq \frac{1}{2}(\binom{n}{r} - n^{r/2})$ (see (1)), and the last inequality uses $\delta \geq n^{r/2}$. The above contradicts our assumption in the beginning of the proof.

So indeed $m_g > \frac{1}{2}\binom{n}{r} - 80\delta$. Recall that the two edges of every good choosable pair form a 2-cycle of $\tilde{\psi}$ (see Definition 3.1). This means $\tilde{\psi}$ has at least $\left(\frac{1}{2}\binom{n}{r} - 80\delta\right)$ 2-cycles. Therefore, by Lemma 3.4 (with 80δ in place of δ), the number of 2-cycles of ψ is at least $\frac{n}{2} - r(80\delta)^{1/r} \geq \frac{n}{2} - 9r\delta^{1/r}$ (using that $r \geq 2$). Let t be the number of 2-cycles of ψ that are not 2-cycles of φ. By Lemma 3.5, it holds that $\frac{1}{2}\binom{n}{r} - \binom{t}{r} \geq m_g > \frac{1}{2}\binom{n}{r} - 80\delta$. This means $t \leq r$ or $80\delta > \binom{t}{r} \geq (t/r)^r$, indicating that $t < r(80\delta)^{1/r} < 9r\delta^{1/r}$. Therefore, the number of common 2-cycles of ψ and φ is at least $\frac{n}{2} - 9r\delta^{1/r} - t > \frac{n}{2} - 18r\delta^{1/r}$, as desired. ∎

Lemma 3.6 allows us to count, for given G, φ, ψ, the number of pairs of graphs G_1, G_2 with $G \xrightarrow{\varphi} G_1, G_2$ and $G_1 \xrightarrow{\psi} G_2$. In some situations, we want to count, for given G_1, G_2 and φ_1, φ_2, the number of graphs G such that $G_1 \xrightarrow{\varphi_1} G$, $G_2 \xrightarrow{\varphi_2} G$. When working inside a difference-isomorphic family $\mathcal{G}$, this can be done by counting pairs of such graphs G, say G, G', using the fact that for every such G, G' there exists some $\psi \in S_n$ with $G \xrightarrow{\psi} G'$ (as $\mathcal{G}$ is difference-isomorphic); thus, we can use Lemma 3.6 and then union-bound over all ψ. This is done in the following lemma.

Lemma 3.7. *Suppose that n is large enough in terms of r. Let $\mathcal{G}$ be a difference-isomorphic family of r-graphs on $[n]$, and let $G_1, G_2 \in \mathcal{G}$ and $\varphi_1, \varphi_2 \in S_n$. For every $\delta \geq \max\{n \ln n, n^{r/2}\}$ with $\delta = o(n^r)$, we have $|N_{\varphi_1}(G_1) \cap N_{\varphi_2}(G_2) \cap \mathcal{G}| \leq 2^{f_r(n)} \cdot e^{-\delta}$ unless φ_1 and φ_2 share at least $\left(\frac{n}{2} - 80r\delta^{1/r}\right)$ 2-cycles.*

Proof. Let $\mathcal{G}' := N_{\varphi_1}(G_1) \cap N_{\varphi_2}(G_2) \cap \mathcal{G}$. Suppose that $|\mathcal{G}'| > 2^{f_r(n)} \cdot e^{-\delta}$. Since $\mathcal{G}$ is difference-isomorphic, we have $\sum_{\psi \in S_n} e_\psi(\mathcal{G}') \geq |\mathcal{G}'|^2$. By averaging, there exists $\psi \in S_n$ such that $e_\psi(\mathcal{G}') \geq |\mathcal{G}'|^2/n!$. We have $e_\psi(N_{\varphi_1}(G_1)) \geq e_\psi(\mathcal{G}') \geq |\mathcal{G}'|^2/n! > 2^{2f_r(n)} \cdot e^{-2\delta}/n! \geq 2^{2f_r(n)} \cdot e^{-3\delta}$, using that $\delta \geq n \ln n$. Now, by Lemma 3.6 with $G = G_1$, $\varphi = \varphi_1$ and with 3δ in place of δ, we conclude that φ_1 and ψ share at least $\left(\frac{n}{2} - 18r(3\delta)^{1/r}\right) \geq \left(\frac{n}{2} - 40r\delta^{1/r}\right)$ 2-cycles. By the same argument for φ_2, we get that φ_2 and ψ also share at least $\left(\frac{n}{2} - 40r\delta^{1/r}\right)$ 2-cycles. It follows that φ_1 and φ_2 share at least $\left(\frac{n}{2} - 80r\delta^{1/r}\right)$ 2-cycles. ∎

In the following lemma, we show that $e_\psi(N_\varphi(G))$ is much smaller than $2^{2f_r(n)}$ unless φ, ψ have some very restrictive structure.

Lemma 3.8. *There exists an absolute constant $c > 0$ such that the following holds. Let $r \geq 2$ and n be sufficiently large in terms of r. Let G be an r-graph and let $\varphi, \psi \in S_n$. Then $e_\psi(N_\varphi(G)) \leq 2^{2f_r(n)} \cdot e^{-\binom{n}{r-1}/100}$ unless the following holds.*

(1) ψ is an involution with at most $\left(2cn^{1-\frac{1}{r}}\right)$ 1-cycles (fixed points).

(2) φ, ψ share at least $\left(\frac{n}{2} - cn^{1-\frac{1}{r}}\right)$ 2-cycles and all 1-cycles.

(3) For every 2-cycle xy of ψ, it holds that $\varphi(x) = y$ or $\varphi(y) = x$.

In particular, if φ, ψ are both involutions, then $e_\psi(N_\varphi(G)) \leq 2^{2f_r(n)} \cdot e^{-\binom{n}{r-1}/100}$ unless $\varphi = \psi$.

Proof. We assume that $e_\psi(N_\varphi(G)) > 2^{2f_r(n)} \cdot e^{-\binom{n}{r-1}/100}$ and show that (1)-(3) hold. Let $u_1v_2, \dots, u_tv_t$ be the common 2-cycles of φ and ψ. By Lemma 3.6 with $\delta = \binom{n}{r-1}$ (note that $\delta \geq n^{r/2}$ for all $r \geq 2$ and sufficiently large n), it holds that $t \geq \frac{n}{2} - 18r\binom{n}{r-1}^{1/r}$. We have $\binom{n}{r-1}^{1/r} \leq (\frac{en}{r-1})^{\frac{r-1}{r}} \leq c'n^{1-\frac{1}{r}}/r$ for some absolute constant c'. Hence, picking $c = 18c'$, we have $t \geq \frac{n}{2} - cn^{1-\frac{1}{r}}$, proving the first part of Item (2).

Put $C = \{u_1, \dots, u_t, v_1, \dots, v_t\}$. If $C = [n]$, then φ and ψ share $\frac{n}{2}$ two-cycles, so $\varphi = \psi$ is an involution, and (1)-(3) are all satisfied. From now on, we assume that $C \neq [n]$. Suppose first that there exists $w \in [n] \setminus C$ such that for every edge $e \in \binom{[n]}{r}$ with $w \in e$ and $e \setminus \{w\} \subseteq C$, e does not belong to any good choosable pair for (G, φ, ψ). As there are $\binom{|C|}{r-1} = \binom{2t}{r-1}$ edges e with $w \in e$ and $e \setminus \{w\} \subset C$, it follows that the number of good choosable pairs m_g (for (G, φ, ψ)) satisfies

$$2m_g \leq \binom{n}{r} - \binom{2t}{r-1} = \binom{n}{r} - \binom{(1-o(1))n}{r-1} = \binom{n}{r} - (1-o(1))\binom{n}{r-1}. \tag{5}$$

Also, letting m be the number of choosable pairs for (G, φ), we have the trivial bound $m \leq \frac{1}{2}\binom{n}{r}$. If $r \geq 3$, then we use Eq. (4) with $x = 0$ and $y = (\frac{1}{2} - o(1))\binom{n}{r-1}$ to get

$$e_\psi(N_\varphi(G)) \leq 2^{\binom{n}{r}} \cdot e^{-(\frac{1}{80}-o(1))\binom{n}{r-1}} \leq 2^{2f_r(n)+n^{r/2}} \cdot e^{-(\frac{1}{80}-o(1))\binom{n}{r-1}} \leq 2^{2f_r(n)} \cdot e^{-\binom{n}{r-1}/100}, \tag{6}$$

where the second equality uses that $f_r(n) \geq \frac{1}{2}\left(\binom{n}{r} - n^{r/2}\right)$ (see Eq. (1)), and the last inequality uses that $r/2 < r-1$ ($r \geq 3$) and that n is sufficiently large. For $r = 2$, we improve the bound by using that $\varphi(u_iv_i) = u_iv_i$ for every $1 \leq i \leq t$, and thus u_iv_i does not belong to any choosable pair of (G, φ). Therefore, we have the improved bounds $2m \leq \binom{n}{2} - t$ and $2m_g \leq \binom{n}{2} - t - (1-o(1))n$ (improving on (5)). So by using Eq. (4) with $x = t/2$ and $y = t/2 + (1/2 - o(1))n$ we get

$$e_\psi(N_\varphi(G)) \leq 2^{\binom{n}{2}-t} \cdot e^{-(\frac{1}{80}-o(1))n} = 2^{2f_2(n)+o(n)} \cdot e^{-(\frac{1}{80}-o(1))n} \leq 2^{2f_2(n)} \cdot e^{-n/100}, \tag{7}$$

where the inequality uses that $2f_2(n) = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor$ and $t = \frac{n}{2} - o(n)$. Now, by (6) and (7) we have $e_\psi(N_\varphi(G)) \leq 2^{2f_r(n)} \cdot e^{-\binom{n}{r-1}/100}$, in contradiction to our assumption.

So from now on, we assume that for every $w \in [n] \setminus C$, there exists $e \in \binom{n}{r}$ such that $w \in e$, $e \setminus \{w\} \subseteq C$, and e lies in some good choosable pair for (G, φ, ψ). By Definition 3.1, the other edge in this pair must be $f := \psi(e)$, and we must have $\psi(f) = e$. Notice that $\psi(C) = C$. Therefore, $\psi(e \setminus C) = f \setminus C$ and $\psi(f \setminus C) = e \setminus C$. As $e \setminus C = \{w\}$, we get that $f \setminus C = \{w'\}$ for some $w' \in [n] \setminus C$ with $\psi(w) = w'$ and $\psi(w') = w$. This means $\psi^2(w) = w$. As this holds for every $w \in [n] \setminus C$, we get that ψ is an involution. Also, since (e, f) or (f, e) is a choosable pair for (G, φ), we have $\varphi(e) = f$ or $\varphi(f) = e$ (see Definition 3.1). As $\varphi(C) = C$, we get that $\varphi(e \setminus C) = f \setminus C$ or $\varphi(f \setminus C) = e \setminus C$, so $\varphi(w) = w'$ or $\varphi(w') = w$. Moreover, if w is a fixed point of ψ then $w' = w$, so w is also a fixed point of φ. Conversely, if w is a fixed point of φ, then also $w' = w$ (because φ is injective and we have $\varphi(w) = w'$ or $\varphi(w') = w$), so w is also a fixed point of ψ. This shows that φ, ψ have the same fixed points. Thus, so far, we proved Items 2 and 3 in the lemma. Finally, since every fixed point of ψ belongs to $[n] \setminus C$, and $|C| = 2t \geq n - 2cn^{1-1/r}$, Item (1) holds as well.

For the “In particular" part of the lemma, observe that if φ, ψ are both involutions and Items (1)-(3) hold, then necessarily $\varphi = \psi$. Indeed, φ, ψ share all the fixed points, and if xy is a 2-cycle of ψ then $\varphi(x) = y$ or $\varphi(y) = x$, meaning xy is also a 2-cycle of φ. ■

If $r \geq 3$, then we can impose even stronger restrictions on φ, ψ.

Lemma 3.9. *Suppose that $r \geq 3$ and n is sufficiently large in terms of r. Let G be an r-graph and let $\varphi, \psi \in S_n$. Then $e_\psi(N_\varphi(G)) \leq 2^{2f_r(n)} \cdot e^{-\binom{n}{r-2}/100}$ unless $\varphi = \psi$ is an involution.*

Proof. Suppose $e_\psi(N_\varphi(G)) > 2^{2f_r(n)} \cdot e^{-\binom{n}{r-2}/100}$. Then (φ, ψ) satisfy (1)-(3) in Lemma 3.8. In particular, ψ is an involution, and φ and ψ share all 1-cycles and $(\frac{n}{2} - o(n))$ 2-cycles. If $\varphi = \psi$ then we are done, so let us assume by contradiction that $\varphi \neq \psi$. Let $u_1v_1, \ldots, u_tv_t$ be the common 2-cycles of φ and ψ. We have $t = \frac{n}{2} - o(n)$. For convenience, denote $C = \{u_1, \ldots, u_t, v_1, \ldots, v_t\}$. Let m be the number of choosable pair for (G, φ) and let m_g be the number of good choosable pairs for (G, φ, ψ). Define an edge-set E by

$$E = \begin{cases} \left\{e \in \binom{C}{r} : |e \cap \{u_i, v_i\}| = 0, 2 \text{ for every } i = 1, \ldots, t\right\} & r \text{ is even}, \\ \left\{e \in \binom{C \cup \{w_0\}}{r} : w_0 \in e, |e \cap \{u_i, v_i\}| = 0, 2 \text{ for every } i = 1, \ldots, t\right\} & r \text{ is odd}, n \text{ is odd}, \\ \emptyset & r \text{ is odd}, n \text{ is even}. \end{cases} \tag{8}$$

where, in the second case, $w_0 \in [n]$ is a fixed point of ψ (which must exist if n is odd). Note that w_0 is also a fixed point of φ, because φ, ψ share all fixed points. Since $\varphi(w_0) = w_0$ and $\varphi(\{u_i, v_i\}) = \{u_i, v_i\}$ for every $1 \leq i \leq t$, we have that $\varphi(e) = e$ for every $e \in E$. Hence, no edge of E is contained in any choosable pair for (G, φ). Therefore, $2m \leq \binom{n}{r} - |E|$. Note also that if r is odd and n is even then $|E| = 0$, and otherwise $|E| = \binom{t}{\lfloor r/2 \rfloor} = \binom{n/2 - o(n)}{\lfloor r/2 \rfloor} = \binom{\lfloor n/2 \rfloor}{\lfloor r/2 \rfloor} - o(n^{\lfloor r/2 \rfloor})$. Using the definition of $f_r(n)$ (see (1)) and that $r \geq 3$, we get that

$$\binom{n}{r} - |E| = 2f_r(n) + o(n^{\lfloor r/2 \rfloor}) = 2f_r(n) + o(n^{r-2}). \tag{9}$$

Next we upper-bound m_g. Since we assumed that $\varphi \neq \psi$, there exist $u, v \in [n]$ that form a 2-cycle in ψ but not in φ. By Item 3 in Lemma 3.8, we have $\varphi(u) = v$ or $\varphi(v) = u$, and we cannot have both because otherwise uv is also a 2-cycle in φ. Without loss of generality, assume that $\varphi(u) = v$ and $\varphi(v) \neq u$. Consider any $e \in \binom{[n]}{r}$ such that $e \setminus C = \{u, v\}$. We claim that e does not lie in any good choosable pair for (G, φ, ψ). Suppose it does. The other edge in this pair must be $f := \psi(e)$ (by the definition of good choosable pairs). As $\psi(C) = C$ and $e \setminus C = \{u, v\}$, we know that $f \setminus C = \{\psi(u), \psi(v)\} = \{u, v\}$. Since (e, f) is a choosable pair, we must have $\varphi(e) = f$ or $\varphi(f) = e$. As $\varphi(C) = C$, we have $\varphi(e \setminus C) = f \setminus C$ or $\varphi(f \setminus C) = e \setminus C$, either of which implies $\varphi(uv) = uv$. But $\varphi(u) = v$, so $\varphi(v) = u$, contradicting our assumption. Therefore, all edges $e \in \binom{[n]}{r}$ with $e \setminus C = \{u, v\}$ do not lie in any good choosable pair for (G, φ, ψ). Note that there at least $\binom{2t}{r-2}$ such edges, and they are not in E. So, $2m_g \leq \binom{n}{r} - |E| - \binom{2t}{r-2}$.

Combining all of the above, we can apply Eq. (4) with $x = \frac{1}{2}|E|$ and $y = \frac{1}{2}|E| + \frac{1}{2}\binom{2t}{r-2}$, to get

$$e_\psi(N_\varphi(G)) \leq 2^{\binom{n}{r} - |E|} \cdot e^{-\binom{2t}{r-2}/80} = 2^{2f_r(n) + o(n^{r-2})} \cdot e^{-\binom{2t}{r-2}/80} \leq 2^{2f_r(n)} \cdot e^{-\binom{n}{r-2}/100},$$

where the equality uses Eq. (9) and the last inequality uses $r \geq 3$ and $2t = n - o(n)$. ■

3.3 Proof of Theorem 2.7 for $r \geq 3$

We start with the following simple lemma, bounding the number of permutations that are close to a given permutation φ_0.

Lemma 3.10. *Let $\varphi_0 \in S_n$ and let $0 \leq A \leq \frac{n}{2}$. The number of permutations $\varphi \in S_n$ that share at least $\left(\frac{n}{2} - A\right)$ 2-cycles with φ_0 is at most n^{2A}.*

Proof. If φ_0 has less than $\left(\frac{n}{2} - A\right)$ 2-cycles then there is nothing to prove. Otherwise, to specify φ as in the statement, it suffices to choose the $\left(\frac{n}{2} - A\right)$ common 2-cycles and then specify a permutation on the remaining $2A$ vertices. The number of choices is at most $\binom{n/2}{n/2-A} \cdot (2A)! = \binom{n/2}{A} \cdot (2A)! \leq \frac{n^A(2A)!}{A!} \leq n^A(2A)^A \leq n^{2A}$, using that $A \leq \frac{n}{2}$. ∎

We now show that if $\mathcal{G}$ is a difference-isomorphic r-graph family of size close to $2^{f_r(n)}$, then $\mathcal{G}$ contains a large involution clique. Note that the following lemma works also for $r = 2$, and will be used in Section 3.4 as well.

Lemma 3.11. *Let $r \geq 2$, n be sufficiently large, and let δ satisfy $n^{1/2}\log^{3/2} n \ll \delta \leq \frac{1}{250}\binom{n}{r-1}$. Let $\mathcal{G}$ be a difference-isomorphic family of r-graphs on $[n]$. Then $|\mathcal{G}| \leq 2^{f_r(n)} \cdot e^{-\delta/4}$ or $\mathcal{G}$ contains an involution clique of size at least $2^{f_r(n)} \cdot e^{-\delta}$.*

Proof. We assume $\mathcal{G}$ has no involution clique of size $2^{f_r(n)} \cdot e^{-\delta}$, and prove that $|\mathcal{G}| \leq 2^{f_r(n)} \cdot e^{-\delta/4}$. For convenience, put $N_\varphi^{\mathcal{G}}(G) := N_\varphi(G) \cap \mathcal{G}$. We first show $N_\varphi^{\mathcal{G}}(G)$ is small for all $G \in \mathcal{G}$ and $\varphi \in S_n$. The intuition is that $e_\psi(N_\varphi^{\mathcal{G}}(G))$ can only be "large" if ψ is an involution (see Lemma 3.8), which in turn implies a "large" ψ-clique inside $N_\varphi^{\mathcal{G}}(G)$ (and this is impossible by our assumption).

Claim 3.12. $\left|N_\varphi^{\mathcal{G}}(G)\right| \leq 2^{f_r(n)} \cdot e^{-\delta/2}$ *for all $G \in \mathcal{G}$ and $\varphi \in S_n$.*

Proof. Since $\mathcal{G}$ is difference-isomorphic, we have

$$\left|N_\varphi^{\mathcal{G}}(G)\right|^2 \leq \sum_{\psi \in S_n} e_\psi(N_\varphi^{\mathcal{G}}(G)) \tag{10}$$

Set $\Delta := 2\delta + n\ln n + n^{r/2}$ and $A := 18r\Delta^{1/r}$. Note that $n^{r/2} \leq \Delta = o(n^r)$ and $A = O(\delta^{1/r} + (n\ln n)^{1/r} + n^{1/2}) = o(\delta/\ln n)$, using $\delta \gg n^{1/2}\log^{3/2} n$ and $r \geq 2$. Partition S_n into parts $P_1 \cup P_2 \cup P_3$ as follows:

$$P_1 := \left\{\psi \in S_n : \varphi \text{ and } \psi \text{ share fewer than } \left(\frac{n}{2} - A\right) \text{ 2-cycles}\right\},$$
$$P_2 := \{\psi \in S_n \setminus P_1 : \psi \text{ is not an involution}\}, \quad P_3 := S_n \setminus (P_1 \cup P_2).$$

Clearly, $|P_1| \leq |S_n| = n!$. By Lemma 3.6 (with Δ in place of δ) and by our choice of A, we have $e_\psi(N_\varphi^{\mathcal{G}}(G)) \leq 2^{2f_r(n)} \cdot e^{-\Delta}$ for every $\psi \in P_1$. Therefore, the contribution of P_1 to the RHS of Eq. (10) is $\sum_{\psi \in P_1} e_\psi(N_\varphi^{\mathcal{G}}(G)) \leq n! \cdot 2^{2f_r(n)} \cdot e^{-\Delta} \leq 2^{2f_r(n)} \cdot e^{-2\delta}$, using that $\Delta \geq n\ln n + 2\delta$.

By Lemma 3.10 with $\varphi_0 = \varphi$ and A, we have $|P_2 \cup P_3| \leq n^{2A} \leq e^{o(\delta)}$ using that $A = o(\delta/\ln n)$. Now, by Lemma 3.8, we have $e_\psi(N_\varphi^{\mathcal{G}}(G)) \leq 2^{2f_r(n)} \cdot e^{-\binom{n}{r-1}/100}$ for every $\psi \in P_2$ (because ψ is not an involution). Hence, the contribution of P_2 to the RHS of Eq. (10) is $\sum_{\psi \in P_2} e_\psi(N_\varphi^{\mathcal{G}}(G)) \leq e^{o(\delta)} \cdot 2^{2f_r(n)} \cdot e^{-\binom{n}{r-1}/100} \leq 2^{2f_r(n)} \cdot e^{-2\delta}$, using the assumption that $\delta \leq \frac{1}{250}\binom{n}{r-1}$.

Finally, we consider P_3. Every $\psi \in P_3$ is an involution. Therefore, for every $H \in \mathcal{G}$, the set $N_\psi^{\mathcal{G}}(H)$ is a ψ-clique by Lemma 2.5, and so $\left|N_\psi^{\mathcal{G}}(H)\right| \le 2^{f_r(n)} \cdot e^{-\delta}$ by our assumption that all involution cliques in $\mathcal{G}$ are small. This means that

$$e_\psi(N_\varphi^{\mathcal{G}}(G)) \le \sum_{H \in N_\varphi^{\mathcal{G}}(G)} \left|N_\psi^{\mathcal{G}}(H)\right| \le \left|N_\varphi^{\mathcal{G}}(G)\right| \cdot 2^{f_r(n)} \cdot e^{-\delta}.$$

Hence, $\sum_{\psi \in P_3} e_\psi(N_\varphi^{\mathcal{G}}(G)) \le |P_3| \cdot \left|N_\varphi^{\mathcal{G}}(G)\right| \cdot 2^{f_r(n)} \cdot e^{-\delta} \le \left|N_\varphi^{\mathcal{G}}(G)\right| \cdot 2^{f_r(n)} \cdot e^{-2\delta/3}$, using $|P_3| \le |P_2 \cup P_3| \le e^{o(\delta)}$. Plugging all of the above into (10), we get

$$\left|N_\varphi^{\mathcal{G}}(G)\right|^2 \le \sum_{\psi \in P_1 \cup P_2 \cup P_3} e_\psi(N_\varphi^{\mathcal{G}}(G)) \le 2 \cdot 2^{2f_r(n)} \cdot e^{-2\delta} + \left|N_\varphi^{\mathcal{G}}(G)\right| \cdot 2^{f_r(n)} \cdot e^{-2\delta/3}. \tag{11}$$

So $\left|N_\varphi^{\mathcal{G}}(G)\right|^2$ is at most twice the first term or at most twice the second term on the RHS of (11). Namely, $\left|N_\varphi^{\mathcal{G}}(G)\right|^2 \le 4 \cdot 2^{2f_r(n)} \cdot e^{-2\delta}$ or $\left|N_\varphi^{\mathcal{G}}(G)\right| \le 2 \cdot 2^{f_r(n)} \cdot e^{-2\delta/3}$. In either case, we get $\left|N_\varphi^{\mathcal{G}}(G)\right| \le 2^{f_r(n)} \cdot e^{-\delta/2}$, as required. ∎

We now continue with the proof of the lemma. Fix an arbitrary $G \in \mathcal{G}$. Since $\mathcal{G}$ is difference-isomorphic, $\mathcal{G} = \bigcup_{\varphi \in S_n} N_\varphi^{\mathcal{G}}(G)$. By averaging, there exists $\varphi_1 \in S_n$ such that $\left|N_{\varphi_1}^{\mathcal{G}}(G)\right| \ge |\mathcal{G}|/n!$. Let $\mathcal{G}_1 = N_{\varphi_1}^{\mathcal{G}}(G)$ and $\mathcal{G}_2 = \mathcal{G} \setminus \mathcal{G}_1$. For $\varphi_2 \in S_n$, let $\mathcal{E}_{\varphi_2} = \{(G_1, G_2) \in \mathcal{G}_1 \times \mathcal{G}_2 : G_2 \overset{\varphi_2}{\to} G_1\}$. Again, using that $\mathcal{G}$ is difference-isomorphic, it holds that $\bigcup_{\varphi_2 \in S_n} \mathcal{E}_{\varphi_2} = \mathcal{G}_1 \times \mathcal{G}_2$. Therefore, $|\mathcal{G}_1|\,|\mathcal{G}_2| \le \sum_{\varphi_2 \in S_n} |\mathcal{E}_{\varphi_2}|$. We will use this to upper bound $|\mathcal{G}_1||\mathcal{G}_2|$. Set $\Delta := \delta + 2n\ln n + n^{r/2}$ and $A = 80r\Delta^{1/r}$. Just as in the proof of Claim 3.12, we have $A = o(\delta/\ln n)$ because $\delta \gg n^{1/2}\log^{3/2} n$ and $r \ge 2$. Define

$$Q = \left\{\varphi_2 \in S_n : \varphi_1 \text{ and } \varphi_2 \text{ share fewer than } \left(\frac{n}{2} - A\right) \text{ 2-cycles}\right\}.$$

We bound $|\mathcal{E}_{\varphi_2}|$ separately for $\varphi_2 \in Q$ and $\varphi_2 \in [n] \setminus Q$. First, fix any $\varphi_2 \in Q$ and $G_2 \in \mathcal{G}_2$. By Lemma 3.7 (with $G_1 := G$ and with Δ in place of δ), and using our choice of A and the assumption $\varphi_2 \in Q$, we have $\left|N_{\varphi_1}^{\mathcal{G}}(G) \cap N_{\varphi_2}^{\mathcal{G}}(G_2)\right| < 2^{f_r(n)} \cdot e^{-\Delta} \le 2^{f_r(n)} \cdot n^{-2n} \cdot e^{-\delta}$, using $\Delta \ge 2n\ln n + \delta$. Note that $\left|N_{\varphi_1}^{\mathcal{G}}(G) \cap N_{\varphi_2}^{\mathcal{G}}(G_2)\right|$ is the number of $G_1 \in \mathcal{G}_1$ with $(G_1, G_2) \in \mathcal{E}_{\varphi_2}$. Hence, by summing over $G_2 \in \mathcal{G}_2$, we get $|\mathcal{E}_{\varphi_2}| \le |\mathcal{G}_2| \cdot 2^{f_r(n)} \cdot n^{-2n} \cdot e^{-\delta}$, and by summing over all at most $n!$ permutations $\varphi_2 \in Q$, we obtain $\sum_{\varphi_2 \in Q} |\mathcal{E}_{\varphi_2}| \le |\mathcal{G}_2| \cdot 2^{f_r(n)} \cdot n^{-n} \cdot e^{-\delta}$.

Next, fix any $\varphi_2 \in S_n \setminus Q$ and $G_1 \in \mathcal{G}_1$. Note that $G_2 \overset{\varphi_2}{\to} G_1$ if and only if $G_1 \overset{\varphi_2^{-1}}{\to} G_2$. Thus, $\mathcal{E}_{\varphi_2}$ is the set of pairs $(G_1, G_2) \in \mathcal{G}_1 \times \mathcal{G}_2$ with $G_1 \overset{\varphi_2^{-1}}{\to} G_2$. By Claim 3.12, $\left|N_{\varphi_2^{-1}}(G_1) \cap \mathcal{G}_2\right| \le \left|N_{\varphi_2^{-1}}^{\mathcal{G}}(G_1)\right| \le 2^{f_r(n)} \cdot e^{-\delta/2}$. By summing over $G_1 \in \mathcal{G}_1$, we get $|\mathcal{E}_{\varphi_2}| \le |\mathcal{G}_1| \cdot 2^{f_r(n)} \cdot e^{-\delta/2}$. By Lemma 3.10 with $\varphi_0 := \varphi_1$ and A as above, it holds that $|S_n \setminus Q| \le n^{2A} \le e^{o(\delta)}$, using $A = o(\delta/\ln n)$. Combining all of the above, we get

$$|\mathcal{G}_1|\,|\mathcal{G}_2| \le \sum_{\varphi_2 \in S_n} |\mathcal{E}_{\varphi_2}| \le |\mathcal{G}_2| \cdot 2^{f_r(n)} \cdot n^{-n} \cdot e^{-\delta} + |\mathcal{G}_1| \cdot 2^{f_r(n)} \cdot e^{-\delta/2 + o(\delta)}. \tag{12}$$

So $|\mathcal{G}_1|\,|\mathcal{G}_2|$ is at most twice the first term or at most twice the second term on the RHS of (12). Namely, $|\mathcal{G}_1| \le 2 \cdot 2^{f_r(n)} \cdot n^{-n} \cdot e^{-\delta}$ or $|\mathcal{G}_2| \le 2 \cdot 2^{f_r(n)} \cdot e^{-\delta/2 + o(\delta)}$. In the former case, using $|\mathcal{G}_1| \ge |\mathcal{G}|/n!$, we get $|\mathcal{G}| \le 2n! \cdot 2^{f_r(n)} \cdot n^{-n} \cdot e^{-\delta} \le 2^{f_r(n)} \cdot e^{-\delta}$, as required. And in the latter case, by combining the bound on $|\mathcal{G}_2|$ with the fact that $|\mathcal{G}_1| = \left|N_{\varphi_1}^{\mathcal{G}}(G)\right| \le 2^{f_r(n)} \cdot e^{-\delta/2}$ (by Claim 3.12), we get that $|\mathcal{G}| = |\mathcal{G}_1| + |\mathcal{G}_2| \le 3 \cdot 2^{f_r(n)} \cdot e^{-\delta/2 + o(\delta)} \le 2^{f_r(n)} \cdot e^{-\delta/4}$, again giving the desired bound. ∎

We are finally ready to prove Theorem 2.7 in the case $r \geq 3$.

Proof of Theorem 2.7 for $r \geq 3$. Let $\mathcal{G}' \subseteq \mathcal{G}$ be a largest involution clique in $\mathcal{G}$; say $\mathcal{G}'$ is a ψ-clique, where $\psi \in S_n$. By assumption, $\mathcal{G} \neq \mathcal{G}'$. It suffices to show that $|\mathcal{G}'| < 2^{f_r(n)} \cdot e^{-\binom{n}{r-2}/250}$ because then, Lemma 3.11 with $\delta = \frac{1}{250}\binom{n}{r-2} \ll n^{r-1}$ implies that $|\mathcal{G}| \leq 2^{f_r(n)} \cdot e^{-\binom{n}{r-2}/1000}$, as desired.

So our goal is to show that $|\mathcal{G}'| \leq 2^{f_r(n)} \cdot n^{-\binom{n}{r-2}/250}$. Fix $G \in \mathcal{G} \setminus \mathcal{G}'$ be arbitrary. As $\mathcal{G}$ is difference-isomorphic, $\mathcal{G}' = \bigcup_{\varphi \in S_n}(N_\varphi(G) \cap \mathcal{G}')$. Note that $N_\psi(G) \cap \mathcal{G}' = \emptyset$, because otherwise $\mathcal{G}' \cup \{G\}$ would be a ψ-clique (by Lemma 2.5), contradicting the maximality of $\mathcal{G}'$. Hence, $|\mathcal{G}'| \leq \sum_{\varphi \in S_n \setminus \{\psi\}} |N_\varphi(G) \cap \mathcal{G}'|$. Set $A := cn^{1-\frac{1}{r}}$, where c is the constant given by Lemma 3.8, and let

$$P_1 = \left\{\varphi \in S_n : \varphi \text{ and } \psi \text{ share fewer than } \left(\frac{n}{2} - A\right) \text{ 2-cycles}\right\}, \quad P_2 = S_n \setminus (P_1 \cup \{\psi\}).$$

By Lemma 3.8, $e_\psi(N_\varphi(G)) \leq 2^{2f_r(n)} \cdot e^{-\binom{n}{r-1}/100}$ for all $\varphi \in P_1$. By Lemma 3.9, $e_\psi(N_\varphi(G)) \leq 2^{2f_r(n)} \cdot e^{-\binom{n}{r-2}/100}$ for all $\varphi \in S_n \setminus \{\psi\}$, and in particular for all $\varphi \in P_2$. Clearly, $N_\varphi(G) \cap \mathcal{G}' \subseteq \mathcal{G}'$ is also a ψ-clique, so $e_\psi(N_\varphi(G)) \geq |N_\varphi(G) \cap \mathcal{G}'|^2$ for all $\varphi \in S_n$. Hence, $|N_\varphi(G) \cap \mathcal{G}'| \leq 2^{f_r(n)} \cdot e^{-\binom{n}{r-1}/200}$ for $\varphi \in P_1$ and $|N_\varphi(G) \cap \mathcal{G}'| \leq 2^{f_r(n)} \cdot e^{-\binom{n}{r-2}/200}$ for $\varphi \in P_2$. Note that $|P_2| \leq n^{2A} \leq e^{o(n)}$ by Lemma 3.10 with $\varphi_0 = \psi$. Combining all of the above and using that $r \geq 3$ and n is sufficiently large, we get:

$$\begin{aligned}|\mathcal{G}'| &\leq \sum_{\varphi \in S_n \setminus \{\psi\}} |N_\varphi(G) \cap \mathcal{G}'| = \sum_{\varphi \in P_1} |N_\varphi(G) \cap \mathcal{G}'| + \sum_{\varphi \in P_2} |N_\varphi(G) \cap \mathcal{G}'| \\ &\leq n! \cdot 2^{f_r(n)} \cdot e^{-\binom{n}{r-1}/200} + e^{o(n)} \cdot 2^{f_r(n)} \cdot e^{-\binom{n}{r-2}/200} \leq 2^{f_r(n)} \cdot e^{-\binom{n}{r-2}/250},\end{aligned}$$

completing the proof. ■

3.4 Proof of Theorem 2.7 for $r = 2$

Throughout this section, we assume that $r = 2$. We begin with some lemmas. First, it will be convenient to introduce the following definition, giving a name to pairs (φ, ψ) which satisfy Items (1)-(3) in Lemma 3.8.

Definition 3.13 (Exceptional). *A pair of permutations (φ, ψ) is* exceptional *if the following holds, where c is the constant given by Lemma 3.8.*

(1) ψ is an involution with at most $2c\sqrt{n}$ 1-cycles (fixed points).

(2) φ, ψ share at least $\left(\frac{n}{2} - c\sqrt{n}\right)$ 2-cycles and all 1-cycles.

(3) For every 2-cycle xy of ψ, it holds that $\varphi(x) = y$ or $\varphi(y) = x$.

Note that (φ, ψ) is exceptional if and only if (φ^{-1}, ψ) is (because φ, φ^{-1} have the same 1- and 2-cycles). Lemma 3.8 states that $e_\psi(N_\varphi(G)) \leq 2^{2f_2(n)} \cdot e^{-n/100}$ unless (φ, ψ) is exceptional. In the following lemma, we strengthen this statement by adding a restriction on the graph G.

Lemma 3.14. *Suppose that $r = 2$ and n is sufficiently large. Let G be a graph and let $\varphi, \psi \in S_n$. Then $e_\psi(N_\varphi(G)) \leq 2^{2f_2(n)} \cdot e^{-n/100}$, unless the following holds:*

1. *(φ, ψ) is exceptional.*

2. *For every 2-cycle xy of ψ, if $\varphi(x) = y$ but $\varphi(y) \neq x$, then $d_G(x) > \frac{n}{2}$ and $d_G(y) < \frac{n}{2}$.*

Proof. Suppose $e_\psi(N_\varphi(G)) > 2^{2f_2(n)} \cdot e^{-n/100}$. That (φ, ψ) is exceptional follows from Lemma 3.8. So it remains to establish Item 2 of the lemma. Let $u_1v_1, \ldots, u_tv_t$ be the common 2-cycles of φ and ψ; so $t = \frac{n}{2} - o(n)$ because (φ, ψ) is exceptional. Write $C = \{u_1, \ldots, u_t, v_1, \ldots, v_t\}$. Let m be the number of choosable pairs of (G, φ) and let m_g be the number of good choosable pairs of (G, φ, ψ). Since $\varphi(u_iv_i) = u_iv_i$, none of the edges u_iv_i belongs to any choosable pair, hence $2m \leq \binom{n}{2} - t$.

Now, let xy by any 2-cycle of ψ such that $\varphi(x) = y$ and $\varphi(y) \neq x$. Clearly, $x, y \notin C$. Denote $C' = \{a \in C : ax \in G \text{ and } \psi(a)y \notin G\}$ and $C'' = C \setminus C'$. We claim that for every $a \in C''$, ax and $\psi(a)y$ do not belong to any good choosable pair of (G, φ, ψ). Indeed, let $a \in C''$, and put $b = \psi(a)$, so that $ab = u_iv_i$ for some $i \in [t]$. By Definitions 2.1 and 3.1, a good choosable pair (e, f) satisfies $\varphi(e) = f$, $e \in G$, $f \notin G$, $\psi(e) = f$ and $\psi(f) = e$. Since $\psi(x) = y$, $\psi(y) = x$, $\psi(a) = b$ and $\psi(b) = a$, a good choosable pair containing ax or by must be (ax, by) or (by, ax). In fact, this pair must be (ax, by), because $\varphi(by) \neq ax$, as $\varphi(b) = a$ and $\varphi(y) \neq x$. However, as $a \notin C'$, we have $ax \notin G$ or $by \in G$, meaning that (ax, by) is not a choosable pair, thus proving our claim that ax and $\psi(a)y$ do not belong to good choosable pairs. This gives $2|C''|$ additional edges which do not belong to any good choosable pair (i.e., in addition to the edges $u_1v_1, \ldots, u_tv_t$). Hence, $2m_g \leq \binom{n}{2} - t - 2|C''|$.

Now we show that $|C'| > n/2$. So suppose by contradiction that $|C'| \leq n/2$. Then $|C''| = |C| - |C'| = 2t - |C'| \geq \frac{n}{2} - o(n)$, and hence $2m_g \leq \binom{n}{2} - t - n + o(n)$. Now, by Eq. (4) with $x = \frac{t}{2}$ and $y = \frac{t}{2} + \frac{n}{2} - o(n)$, we get

$$e_\psi(N_\varphi(G)) \leq 2^{\binom{n}{2} - t} \cdot e^{-\frac{n}{80} + o(n)} \leq 2^{2f_2(n) + o(n)} \cdot e^{-\frac{n}{80} + o(n)} < 2^{2f_2(n)} \cdot e^{-n/100},$$

where we used that $2f_2(n) = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor$ (see Eq. (1)) and $t = \frac{n}{2} - o(n)$. This contradicts our assumption in the beginning of the proof. So indeed $|C'| > n/2$. Now, by the definition of C', we have that $\deg_G(x) \geq |C'|$ and $\deg_G(y) \leq n - 1 - |C'|$. Hence, $\deg_G(x) > \frac{n}{2} > \deg_G(y)$, as required. ■

Lemma 3.11 implies that a large difference-isomorphic graph family must have a large ψ-clique for some involution ψ. In Lemma 3.19 we will show that this ψ-clique must in fact span almost the entire family, meaning that the number of graphs not belonging to the ψ-clique is exponentially smaller than $2^{f_2(n)}$. The proof of Lemma 3.19 relies on the following lemma.

Lemma 3.15. *Let (φ, ψ) be an exceptional pair with $\varphi \neq \psi$, and let $\mathcal{G}$ be a ψ-clique. Then the number of pairs of graphs (G_1, G_2) with $G_1 \in \mathcal{G}$ and $G_1 \xrightarrow{\varphi} G_2$ is at most $2^{2f_2(n)} \cdot 2^{-\Omega(n)}$.*

Proof. Recall that ψ is an involution by Definition 3.13. Fix $y \in [n]$ such that $\varphi(y) \neq \psi(y)$ (we assume that $\varphi \neq \psi$). Put $x = \psi(y)$ and $z = \varphi(y)$, so $x \neq z$. Note that y, z are not 1-cycles in φ or ψ, because φ, ψ share all 1-cycles by Item 2 of Definition 3.13. Hence, xy is a 2-cycle of ψ. By Item 3 in Definition 3.13, we have $\varphi(x) = y$; indeed, $\varphi(y) = x$ is impossible because $\varphi(y) = z \neq x$. Also, setting $w = \psi(z)$, we have $\psi(w) = z$ and $w \neq z$ (because z is not a 1-cycle). Let $u_1v_1, \ldots, u_tv_t$ be the 2-cycles shared by φ and ψ (clearly $u_i, v_i \notin \{x, y, z, w\}$ for every i). By Item 2 of Definition 3.13, we have $t = \frac{n}{2} - o(n)$. For a graph $G \in \mathcal{G}$, let $I(G)$ be the set of all indices $1 \leq i \leq t$ such that $v_iy, u_iz \in G$. Let $\mathcal{G}_0$ be the set of graphs $G \in \mathcal{G}$ such that $|I(G)| \geq \frac{n}{10}$. To prove the lemma, we will show that $\mathcal{G} \setminus \mathcal{G}_0$ is small and that $N_\varphi(G)$ is small for every $G \in \mathcal{G}_0$.

Claim 3.16. $|\mathcal{G} \setminus \mathcal{G}_0| \leq 2^{f_2(n)} \cdot 2^{-\Omega(n)}$.

Proof. Recall the definitions of $\mathcal{C}_1(\psi)$ and $\mathcal{C}_2(\psi)$ (see Definition 2.3). By Lemma 2.5, the graphs in $\mathcal{G}$ agree on all edges in $\mathcal{C}_1(\psi)$, and for every pair $\{e, f\} \in \mathcal{C}_2(\psi)$, there is $a_{e,f} \in \{0, 1, 2\}$ such that every graph in $\mathcal{G}$ contains exactly $a_{e,f}$ of the edges e, f. Hence, letting S be the set of pairs $\{e, f\}$

with $a_{e,f} = 1$, we see that a graph G in $\mathcal{G}$ is determined by deciding if G contains e or f for every pair $\{e, f\} \in S$. In particular, $|\mathcal{G}| \leq 2^{|S|}$. Since $u_i v_i \in \mathcal{C}_1(\psi)$ for every $1 \leq i \leq t$, we have $2|S| \leq \binom{n}{2} - t$.

Let us fix a set $I \subseteq [t]$ and count all $G \in \mathcal{G} \setminus \mathcal{G}_0$ satisfying $I(G) = I$. Note that there are $\binom{t}{k}$ choices for I of a given size k. Fix any $i \in [t]$, and note that $(u_i x, v_i y), (u_i z, v_i w)$ are both 2-cycles of $\tilde{\psi}$. Thus, a priori, a graph $G \in \mathcal{G}$ has (at most) 2 choices for each of these pairs, so 4 choices altogether. If $i \in I(G)$, then by the definition of $I(G)$ we have $v_i y, u_i z \in G$, so G has only one choice on the pairs $(u_i x, v_i y), (u_i z, v_i w)$. Indeed, either $a_{u_i x, v_i y} \in \{0, 2\}$, in which case there is only one choice, or $a_{u_i x, v_i y} = 1$ and the choice $u_i x \notin G, v_i y \in G$ is forced; and the same applies to $(u_i z, v_i w)$. On the other hand, if $i \notin I(G)$, then G has at most 3 choices on $(u_i x, v_i y), (u_i z, v_i w)$, because either one of these two pairs is not in S, in which case G has at most 2 choices on these pairs, or both are in S and the choice $u_i x \notin G$, $v_i y \in G$, $u_i z \in G$, $v_i w \notin G$ is not allowed (because $i \notin I(G)$). It follows that when restricted to the pairs in the set $P := \{(u_i x, v_i y), (u_i z, v_i w) : i = 1, \ldots, t\}$, the number of choices for $G \in \mathcal{G}$ with $I(G) = I$ is $3^{t-|I|}$. Recall that if $G \in \mathcal{G} \setminus \mathcal{G}_0$ then $|I(G)| \leq \frac{n}{10}$. Hence, when restricted to the pairs in P, the number of choices for $G \in \mathcal{G} \setminus \mathcal{G}_0$ is $\sum_{k \leq n/10} \binom{t}{k} 3^{t-k} = 4^t \cdot \mathbb{P}\left[\text{Bin}(t, \frac{1}{4}) \leq \frac{n}{10}\right] \leq 4^{t-\Omega(n)}$, using the Chernoff bound and the fact that $t = \frac{n}{2} - o(n)$. Note also that $2\,|S \setminus P| \leq \binom{n}{2} - t - 4t = \binom{n}{2} - 5t$, because $S \setminus P$ contains neither the edges $u_i v_i$ ($i = 1, \ldots, t$) nor the edges belonging to pairs in P (of which there are $2|P| = 4t$). Now, using that $G \in \mathcal{G}$ has at most 2 choices for each remaining pair $\{e, f\} \in S \setminus P$, we get that $|\mathcal{G} \setminus \mathcal{G}_0| \leq 2^{|S \setminus P|} \cdot 4^{t-\Omega(n)} \leq 2^{\frac{1}{2}\binom{n}{2} - \frac{5t}{2}} \cdot 4^{t-\Omega(n)} = 2^{\frac{1}{2}(\binom{n}{2} - t)} \cdot 2^{-\Omega(n)} = 2^{f_2(n)} \cdot 2^{-\Omega(n)}$, where the last equality uses $f_2(n) = \frac{1}{2}\left(\binom{n}{2} - \lfloor \frac{n}{2} \rfloor\right)$ (see (1)) and $t = \frac{n}{2} - o(n)$. ■

Next, we bound $|N_\varphi(G)|$ for every $G \in \mathcal{G}$. We begin with a simple bound that holds for every $G \in \mathcal{G}$, and then prove a stronger bound for $G \in \mathcal{G}_0$.

Claim 3.17. *For every $G \in \mathcal{G}$, it holds that $|N_\varphi(G)| \leq 2^{f_2(n)} \cdot 2^{o(n)}$.*

Proof. Let m be the number of choosable pairs of (G, φ). As $\varphi(u_i v_i) = u_i v_i$ for every $1 \leq i \leq t$, all these edges do not belong to any choosable pairs. Hence, $m \leq \frac{1}{2}\left(\binom{n}{2} - t\right) = \frac{1}{2}\left(\binom{n}{2} - \frac{n}{2} + o(n)\right) = f_2(n) + o(n)$. Now, Lemma 2.2 implies $|N_\varphi(G)| = 2^m \leq 2^{f_2(n)} \cdot 2^{o(n)}$. ■

Claim 3.18. *For every $G \in \mathcal{G}_0$, it holds that $|N_\varphi(G)| \leq 2^{f_2(n)} \cdot 2^{-\Omega(n)}$.*

Proof. Again, let m be the number of choosable pairs of (G, φ). As in the previous proof, $u_i v_i$ does not belong to any choosable pair (for every $i = 1, \ldots, t$). To improve further, we show that for every $i \in I(G)$, the edge $v_i y$ does not belong to any choosable pair either. Indeed, recall that a choosable pair (e, f) satisfies $\varphi(e) = f$, $e \in G$, $f \notin G$. We have $\varphi(u_i x) = v_i y$ and $\varphi(v_i y) = u_i z$. Hence, if $v_i y$ belongs to a choosable pair (e, f), then either $e = u_i x, f = v_i y$ or $e = v_i y, f = u_i z$. But $v_i y, u_i z \in G$ (because $i \in I(G)$), and this rules out both options. Using that $|I(G)| \geq \frac{n}{10}$ (as $G \in \mathcal{G}_0$), we get $m \leq \frac{1}{2}\left(\binom{n}{2} - t - |I(G)|\right) \leq \frac{1}{2}\left(\binom{n}{2} - t - \frac{n}{10}\right) = \frac{1}{2}\left(\binom{n}{2} - \frac{n}{2}\right) - \Omega(n) = f_2(n) - \Omega(n)$, where we used that $f_2(n) = \frac{1}{2}\left(\binom{n}{2} - \lfloor \frac{n}{2} \rfloor\right)$ and $t = \frac{n}{2} - o(n)$. Now the claim follows from $|N_\varphi(G)| = 2^m$ (see Lemma 2.2). ■

We are now ready to complete the proof of Lemma 3.15. The number of pairs of graphs (G_1, G_2) with $G_1 \in \mathcal{G}$ and $G_1 \xrightarrow{\varphi} G_2$ is $\sum_{G_1 \in \mathcal{G}} |N_\varphi(G_1)| = \sum_{G_1 \in \mathcal{G} \setminus \mathcal{G}_0} |N_\varphi(G_1)| + \sum_{G_1 \in \mathcal{G}_0} |N_\varphi(G_1)|$. For the first sum, using Claim 3.16 and Claim 3.17, it holds that

$$\sum_{G_1 \in \mathcal{G} \setminus \mathcal{G}_0} |N_\varphi(G_1)| \leq |\mathcal{G} \setminus \mathcal{G}_0| \cdot 2^{f_2(n)} \cdot 2^{O(\sqrt{n})} \leq 2^{2f_2(n)} \cdot 2^{-\Omega(n)}.$$

And for the second sum, using Claim 3.18 and $|\mathcal{G}| \le 2^{f_2(n)}$ (see Item 2 of Proposition 2.6), we get

$$\sum_{G_1 \in \mathcal{G}_0} |N_\varphi(G_1)| \le |\mathcal{G}| \cdot 2^{f_2(n)} \cdot 2^{-\Omega(n)} \le 2^{2f_2(n)} \cdot 2^{-\Omega(n)}.$$

This completes the proof of the lemma. ∎

We now show that a large involution clique in a difference-isomorphic family must span almost the entire family. The proof is simply by counting over all $\varphi \in S_n$, similarly to the proof of Lemma 3.11.

Lemma 3.19. *Let $\mathcal{G}$ be a difference-isomorphic graph family, and let $\mathcal{G}' \subseteq \mathcal{G}$ be a maximal involution clique in $\mathcal{G}$. Then $|\mathcal{G}'| < 2^{f_2(n)} \cdot e^{-cn}$ or $|\mathcal{G} \setminus \mathcal{G}'| < 2^{f_2(n)} \cdot e^{-cn}$ for some absolute constant $c > 0$.*

Proof. Let $\psi \in S_n$ be an involution such that $\mathcal{G}'$ is a ψ-clique. For $\varphi \in S_n$, let $\mathcal{E}_\varphi$ be the set of pairs $(G_1, G_2) \in \mathcal{G}' \times (\mathcal{G} \setminus \mathcal{G}')$ such that $G_2 \overset{\varphi}{\to} G_1$. Since $\mathcal{G}$ is difference-isomorphic, we have $|\mathcal{G}'||\mathcal{G} \setminus \mathcal{G}'| \le \sum_{\varphi \in S_n} |\mathcal{E}_\varphi|$. Note that $E_\psi = \emptyset$, because if $G_2 \overset{\psi}{\to} G_1$ for some $G_1 \in \mathcal{G}', G_2 \in \mathcal{G} \setminus \mathcal{G}'$, then $\mathcal{G}' \cup \{G_2\}$ is a larger ψ-clique (see Lemma 2.5 and the following paragraph), contradicting the maximality of $\mathcal{G}'$. As in the proof of Lemma 3.11, we partition S_n into several classes and bound the contribution of each class. Setting $\delta = 2n \ln n$ and $A = 36\sqrt{\delta}$, define:

$$P_1 := \left\{ \varphi \in S_n : \varphi \text{ and } \psi \text{ share fewer than } \left(\frac{n}{2} - A\right) \text{ 2-cycles} \right\},$$
$$P_2 := \{\varphi \in S_n \setminus P_1 : (\varphi, \psi) \text{ is not exceptional}\}, \quad P_3 := S_n \setminus (P_1 \cup P_2 \cup \{\psi\}).$$

Note that as $N_\varphi(G_2) \cap \mathcal{G}' \subseteq \mathcal{G}'$ is a ψ-clique, we have $|N_\varphi(G_2) \cap \mathcal{G}'|^2 \le e_\psi(N_\varphi(G_2))$.

First we consider P_1. Fix $\varphi \in P_1$ and $G_2 \in \mathcal{G} \setminus \mathcal{G}'$. By Lemma 3.6 and by our choice of A, we have $e_\psi(N_\varphi(G_2)) \le 2^{2f_2(n)} \cdot e^{-\delta}$, so $|N_\varphi(G_2) \cap \mathcal{G}'| \le 2^{f_2(n)} \cdot e^{-\delta/2} = 2^{f_2(n)} \cdot n^{-n}$. Summing over all $G_2 \in \mathcal{G} \setminus \mathcal{G}'$ and all (at most $n!$) permutations $\varphi \in P_1$, we get

$$\sum_{\varphi \in P_1} |\mathcal{E}_\varphi| = \sum_{\varphi \in P_1} \sum_{G_2 \in \mathcal{G} \setminus \mathcal{G}'} |N_\varphi(G_2) \cap \mathcal{G}'| \le n! \cdot |\mathcal{G} \setminus \mathcal{G}'| \cdot 2^{f_2(n)} \cdot n^{-n} \le |\mathcal{G} \setminus \mathcal{G}'| \cdot 2^{f_2(n)} \cdot e^{-\Omega(n)}. \tag{13}$$

Next, by Lemma 3.10 with $\varphi_0 = \psi$ and A, we have $|P_2 \cup P_3| \le n^{2A} \le e^{o(n)}$. Also, for each $\varphi \in P_2$ we have $e_\psi(N_\varphi(G_2)) \le 2^{2f_2(n)} \cdot e^{-\Omega(n)}$ by Lemma 3.14, because (φ, ψ) is not exceptional. Therefore, $|N_\varphi(G_2) \cap \mathcal{G}'| \le 2^{f_2(n)} \cdot e^{-\Omega(n)}$. Summing over all $G_2 \in \mathcal{G} \setminus \mathcal{G}'$, we get $|\mathcal{E}_\varphi| \le |\mathcal{G} \setminus \mathcal{G}'| \cdot 2^{f_2(n)} \cdot e^{-\Omega(n)}$. Now fix any $\varphi \in P_3$, and note that $\mathcal{E}_\varphi$ is the number of pairs $(G_1, G_2) \in \mathcal{G}' \times (\mathcal{G} \setminus \mathcal{G}')$ with $G_1 \overset{\varphi^{-1}}{\to} G_2$. Since (φ^{-1}, ψ) is exceptional (as $\varphi \in P_3$), we get by Lemma 3.15 that $|\mathcal{E}_\varphi| \le 2^{2f_2(n)} \cdot 2^{-\Omega(n)}$. Summing over all $e^{o(n)}$ permutations $\varphi \in P_2 \cup P_3$, we obtain

$$\sum_{\varphi \in P_2 \cup P_3} |\mathcal{E}_\varphi| \le |\mathcal{G} \setminus \mathcal{G}'| \cdot 2^{f_2(n)} \cdot e^{-\Omega(n)} + 2^{2f_2(n)} \cdot 2^{-\Omega(n)}. \tag{14}$$

By combining (13) and (14), we get that

$$|\mathcal{G}'| \cdot |\mathcal{G} \setminus \mathcal{G}'| \le \sum_{\varphi \in S_n} |\mathcal{E}_\varphi| \le |\mathcal{G} \setminus \mathcal{G}'| \cdot 2^{f_2(n)} \cdot e^{-\Omega(n)} + 2^{2f_2(n)} \cdot 2^{-\Omega(n)}.$$

Therefore, there exists some absolute constant $c > 0$ such that either $|\mathcal{G}'| < 2^{f_2(n)} \cdot e^{-cn}$, or $|\mathcal{G}'| \ge 2^{f_2(n)} \cdot e^{-cn}$ and $|\mathcal{G}'| \cdot |\mathcal{G} \setminus \mathcal{G}'| < 2^{2f_2(n)} \cdot e^{-2cn}$. In both cases, the assertion of the lemma holds. ∎

The next lemma is the last step towards Theorem 2.7. It shows that if a difference-isomorphic graph family contains an involution clique of size almost $2^{f_2(n)}$, then the family itself is an involution clique.

Lemma 3.20. *There exists a constant $c > 0$ such that the following holds for sufficiently large n. Let $\mathcal{G}$ be a difference-isomorphic family of graphs on n vertices. Let $\mathcal{G}' \subseteq \mathcal{G}$ be a maximum-size involution clique in $\mathcal{G}$, and suppose that $|\mathcal{G}'| \geq \left(1 - n^{-c\sqrt{n}}\right) 2^{f_2(n)}$. Then, $\mathcal{G} = \mathcal{G}'$.*

Proof. Let $\psi \in S_n$ be an involution such that $\mathcal{G}'$ is a ψ-clique. Let c' be the constant given by Lemma 3.8. We will show that $c = 500c'$ suffices. Suppose for contradiction that there exists $G_0 \in \mathcal{G} \setminus \mathcal{G}'$. Recall the definition of $\mathcal{C}_1 := \mathcal{C}_1(\psi)$ and $\mathcal{C}_2 := \mathcal{C}_2(\psi)$ (see Definition 2.3). According to Lemma 2.5, all graphs in $\mathcal{G}'$ agree on the edges in $\mathcal{C}_1$, and for every $\{e, f\} \in \mathcal{C}_2$ there is $a_{e,f} \in \{0, 1, 2\}$ such that all graphs in $\mathcal{G}'$ contain exactly $a_{e,f}$ of the edges e, f. Hence, letting $S = \{\{e, f\} : a_{e,f} = 1\} \subseteq \mathcal{C}_2$, we have $|\mathcal{G}'| \leq 2^{|S|}$. By Lemma 2.4, $|S| \leq |\mathcal{C}_2| \leq f_2(n)$. Hence, it must hold that $|S| = |\mathcal{C}_2| = f_2(n)$, as otherwise, $|\mathcal{G}'| \leq 2^{f_2(n)-1}$, contradicting our assumption on the size of $\mathcal{G}'$ (when n is sufficiently large). Namely, for every $\{e, f\} \in \mathcal{C}_2$, each graph in $\mathcal{G}'$ contains exactly one of e, f.

Since $G_0 \in \mathcal{G} \setminus \mathcal{G}'$, the relation $G_0 \xrightarrow{\psi} G$ does **not** hold for any $G \in \mathcal{G}'$, because otherwise $\mathcal{G}' \cup \{G_0\}$ is a ψ-clique by Lemma 2.5, in contradiction to the maximality of $\mathcal{G}'$. Hence, by Lemma 2.5, G_0 disagrees with the graphs in $\mathcal{G}'$ on one of the edges in $\mathcal{C}_1$, or there is $\{e, f\} \in \mathcal{C}_2$ such that G_0 contains both e, f or none of them. To reduce the number of possibilities to consider, we use Lemma 2.8, which states that $\{G^c : G \in \mathcal{G}\}$ is also difference-isomorphic. So by taking the complements of the graphs in $\mathcal{G}$ (if necessary), we may assume without loss of generality that one of the following holds:

(a) There is $e \in \mathcal{C}_1$ such that $e \in G_0$ and $e \notin G$ for all $G \in \mathcal{G}'$;

(b) G_0 agrees with all $G \in \mathcal{G}$ on all edges in $\mathcal{C}_1$, and there is a pair $\{e, f\} \in \mathcal{C}_2$ such that $e, f \in G_0$.

Since $\mathcal{G}$ is difference isomorphic, we have $\mathcal{G}' \subseteq \bigcup_{\varphi \in S_n} (N_\varphi(G_0) \cap \mathcal{G}')$. We will obtain a contradiction by upper-bounding $\left|\bigcup_{\varphi \in S_n} N_\varphi(G_0) \cap \mathcal{G}'\right|$ (recall that $|\mathcal{G}'| \geq (1 - n^{-c\sqrt{n}})2^{f_2(n)}$). As explained above, we have $N_\psi(G_0) \cap \mathcal{G}' = \emptyset$. Setting $A := 36\sqrt{2n \ln n}$, partition $S_n \setminus \{\psi\}$ into three sets $P_1 \cup P_2 \cup P_3$, as follows:

$$P_1 = \{\varphi \in S_n \setminus \{\psi\} : \varphi \text{ and } \psi \text{ share fewer than } \left(\frac{n}{2} - A\right) \text{2-cycles}\},$$
$$P_2 = \{\varphi \in S_n \setminus P_1 : e_\psi(N_\varphi(G_0)) \leq 2^{2f_2(n)} \cdot e^{-n/100}\}, \quad P_3 = S_n \setminus (P_1 \cup P_2 \cup \{\psi\}).$$

First, for the contribution of permutations $\varphi \in P_1 \cup P_2$, we will use the fact that $|N_\varphi(G_0) \cap \mathcal{G}'|^2 \leq e_\psi(N_\varphi(G_0))$, which holds because $N_\varphi^{\mathcal{G}}(G) \cap \mathcal{G}' \subseteq \mathcal{G}'$ is a ψ-clique. For $\varphi \in P_1$, Lemma 3.6 with $\delta = 2n \ln n$ gives $e_\psi(N_\varphi(G_0)) \leq 2^{2f_2(n)} \cdot n^{-2n}$, and so $|N_\varphi(G_0) \cap \mathcal{G}'| \leq 2^{f_2(n)} \cdot n^{-n}$. Therefore,

$$\sum_{\varphi \in P_1} \left|N_\varphi(G_0) \cap \mathcal{G}'\right| \leq n! \cdot 2^{f_2(n)} \cdot n^{-n} \leq 2^{f_2(n)} \cdot e^{-\Omega(n)}. \tag{15}$$

For $\varphi \in P_2$, we have $e_\psi(N_\varphi(G_0)) \leq 2^{2f_2(n)} \cdot e^{-n/100}$ by the definition of P_2. Therefore, $|N_\varphi(G_0) \cap \mathcal{G}'| \leq 2^{f_2(n)} \cdot e^{-n/200}$. Note that $|P_2| \leq n^{2A} = 2^{o(n)}$ by Lemma 3.10 with $\varphi_0 = \psi$. Therefore,

$$\sum_{\varphi \in P_2} \left|N_\varphi(G_0) \cap \mathcal{G}'\right| \leq 2^{o(n)} \cdot 2^{f_2(n)} \cdot e^{-n/200} \leq 2^{f_2(n)} \cdot e^{-\Omega(n)}.$$

By combining this with Eq. (15), we get

$$\left|\bigcup_{\varphi\in P_3}\left(N_\varphi(G_0)\cap\mathcal{G}'\right)\right| \geq |\mathcal{G}'| - \sum_{\varphi\in P_1\cup P_2}|N_\varphi(G_0)\cap\mathcal{G}'| \geq \left(1-n^{-c\sqrt{n}}\right)2^{f_2(n)} - 2^{f_2(n)}\cdot e^{-\Omega(n)} > \left(1-n^{-c\sqrt{n}/2}\right)2^{f_2(n)}, \tag{16}$$

provided n is sufficiently large.

For the rest of the proof, we will get a contradiction to (16) by upper bounding $\bigcup_{\varphi\in P_3}\left(N_\varphi(G_0)\cap\mathcal{G}'\right)$. We note that here we can no longer achieve this by bounding each term $|N_\varphi(G_0)\cap\mathcal{G}'|$ and summing these terms over $\varphi\in P_3$, because it may be that $|N_\varphi(G_0)\cap\mathcal{G}'| = \Omega(2^{f_2(n)})$ for $\varphi\in P_3$ (see the construction in the proof of Proposition A.1). Instead, we will bound $\bigcup_{\varphi\in P_3}\left(N_\varphi(G_0)\cap\mathcal{G}'\right)$ directly by finding specific choices for the pairs in $\mathcal{C}_2$ which the graphs in $\bigcup_{\varphi\in P_3}\left(N_\varphi(G_0)\cap\mathcal{G}'\right)$ cannot satisfy. More precisely, we will find a set of pairs $P\subseteq\mathcal{C}_2$ (of size $|P| = O(\sqrt{n}\log n)$) such that the graphs in $\bigcup_{\varphi\in P_3}\left(N_\varphi(G_0)\cap\mathcal{G}'\right)$ have only $2^{|P|}-1$ choices on the pairs in P (instead of $2^{|P|}$).

By Lemma 3.14, if $\varphi\in P_3$ then (φ,ψ) is exceptional, and for every 2-cycle xy of ψ, if $\varphi(x)=y$ and $\varphi(y)\neq x$, then $d_{G_0}(x) > d_{G_0}(y)$. In particular, φ shares at least $\left(\frac{n}{2}-c'\sqrt{n}\right)$ 2-cycles with ψ (see Definition 3.13). Hence, $|P_3|\leq n^{2c'\sqrt{n}}$ by Lemma 3.10 with $\varphi_0=\psi$ and $A=c'\sqrt{n}$.

Let $u_1v_1,\dots,u_\ell v_\ell$ be all the 2-cycles of ψ. Then $u_iv_i\in\mathcal{C}_1$ for every $i\in[\ell]$. Write $U:=\{u_1,\dots,u_\ell\}$, $V:=\{v_1,\dots,v_\ell\}$, and note that every point in $[n]\setminus(U\cup V)$ is a fixed point of ψ. Without loss of generality, assume $d_{G_0}(u_i)\geq d_{G_0}(v_i)$ for all $1\leq i\leq\ell$. As (φ,ψ) is exceptional, we know that φ and ψ share all 1-cycles. Observe also that $\varphi(u_i)=v_i$ for all $i\in\{1,\dots,\ell\}$, because by Item 3 in Definition 3.13 we have $\varphi(u_i)=v_i$ or $\varphi(v_i)=u_i$, and it is impossible to have $\varphi(v_i)=u_i$ and $\varphi(u_i)\neq v_i$, since $d_{G_0}(u_i)\geq d_{G_0}(v_i)$. Define the following set, which depends only on the graph G_0 but not on any particular $\varphi\in P_3$, by

$$I := \left\{(i,j)\in\binom{[\ell]}{2} : u_iv_j, v_iu_j\in G_0 \text{ or } u_iv_j, v_iu_j\notin G_0\right\}.$$

Claim 3.21. $|I| < \frac{c}{2}\sqrt{n}\ln n$.

Proof. Fix any $\varphi\in P_3$. Let m be the number of choosable pairs of (G_0,φ) and m_g be the number of good choosable pairs of (G_0,φ,ψ). By Item 2 in Definition 3.13, there are at least $\left(\frac{n}{2}-c'\sqrt{n}\right)$ indices $i\in[\ell]$ such that u_iv_i is a 2-cycle of φ. Note that for such i, we have $\varphi(u_iv_i)=u_iv_i$, and thus u_iv_i is not contained in any choosable pair of (G_0,φ) (see Definition 2.1). Hence, $2m\leq\binom{n}{2}-\frac{n}{2}+c'\sqrt{n}$. In addition, for every $1\leq i<j\leq\ell$, we have $\psi(u_iv_j)=v_iu_j$ and $\psi(v_iu_j)=u_iv_j$. Therefore, if one of the edges u_iv_j, v_iu_j belongs to some good choosable pair of (G_0,φ,ψ), then this pair must be $\{u_iv_j, v_iu_j\}$ (see Definition 3.1). Now, if $(i,j)\in I$, then G_0 contains both or none of the edges u_iv_j, v_iu_j, meaning that $\{u_iv_j, v_iu_j\}$ cannot be a choosable pair (see Definition 2.1). Hence, for every $(i,j)\in I$, the edges u_iv_j and v_iu_j are not contained in any good choosable pair. Therefore, $2m_g\leq\binom{n}{2}-\frac{n}{2}+c'\sqrt{n}-2|I|$. By Eq. (4) with $x=(\frac{n}{2}-c'\sqrt{n})/2$ and $y=(\frac{n}{2}-c'\sqrt{n})/2+|I|$, it holds that

$$e_\psi(N_\varphi(G)) \leq 2^{\binom{n}{2}-\frac{n}{2}+c'\sqrt{n}}\cdot e^{-|I|/40} \leq 2^{2f_2(n)+c'\sqrt{n}}\cdot e^{-|I|/40},$$

where we used $2f_2(n)\geq\binom{n}{2}-\frac{n}{2}$ (see Eq. (1)). Therefore, $|N_\varphi(G_0)\cap\mathcal{G}'|\leq 2^{f_2(n)+c'\sqrt{n}/2}\cdot e^{-|I|/80}$. Now, if $|I|\geq\frac{c}{2}\sqrt{n}\ln n = 250c'\sqrt{n}\ln n$, then, using $|P_3|\leq n^{2c'\sqrt{n}}$, we get:

$$\left|\bigcup_{\varphi\in P_3}\left(N_\varphi(G_0)\cap\mathcal{G}'\right)\right| \leq \sum_{\varphi\in P_3}|N_\varphi(G_0)\cap\mathcal{G}'| \leq n^{2c'\sqrt{n}}\cdot 2^{f_2(n)+c'\sqrt{n}/2}\cdot e^{-|I|/80} < 2^{f_2(n)}/2,$$

when n is sufficiently large. This contradicts Eq. (16), so $|I| < 250c'\sqrt{n}\ln n$. ■

As explained in the beginning, we may assume that (a) or (b) above holds. We now consider these two cases.

Case 1: (a) holds. Namely, there exists $e \in \mathcal{C}_1$ with $e \in G_0$ and $e \notin G$ for all $G \in \mathcal{G}'$. We have $\psi(e) = e$ (because $e \in \mathcal{C}_1$), so either e connects two fixed points of ψ, or $e = u_i v_i$ for some $1 \le i \le \ell$. We first rule out the former case. Indeed, fix any $\varphi \in P_3$ with $N_\varphi(G_0) \cap \mathcal{G}' \neq \emptyset$; such a φ exists by Eq. (16). By Definition 3.13, φ and ψ share all fixed points. Thus, if e connects two fixed points of ψ, then $\varphi(e) = e$. However, taking any $G \in N_\varphi(G_0) \cap \mathcal{G}'$, we get that $e \in G_0 \setminus G$ (by the choice of e) and hence $e = \varphi(e) \in G \setminus G_0$ (as $G_0 \xrightarrow{\varphi} G$), a contradiction. Therefore, $e = u_i v_i$ for some $1 \le i \le \ell$. Note that e is not dependent on any particular choice of $\varphi \in P_3$. We may, without loss of generality, assume that $e = u_1 v_1$. Now, let

$$J = \{1 < i \le \ell : u_1 v_i, v_1 u_i \notin G_0\}, \quad \mathcal{F} = \{G \in \mathcal{G}' : \exists j \in J.\ v_1 u_j \in G\}.$$

By Claim 3.21, $|J| \le |I| \le \frac{c}{2}\sqrt{n}\ln n$, since $(1, j) \in I$ for every $j \in J$. Recall that for every pair of edges $\{e, f\} \in \mathcal{C}_2$, every graph in $\mathcal{G}'$ contains exactly one of e, f. In particular, this holds for $\{e, f\} = \{u_1 v_j, v_1 u_j\} \in \mathcal{C}_2$ for every $j \in J$. Then, when restricting to the edges in $\{u_1 v_j, v_1 u_j : j \in J\}$, the graphs $G \in \mathcal{F}$ have at most $2^{|J|} - 1$ choices, because the choice $u_1 v_j \in G$ (and hence $v_1 u_j \notin G$) for every $j \in J$ is impossible. Hence, using that $|\mathcal{C}_2| \le f_2(n)$ (see Lemma 2.4), we get that $|\mathcal{F}| \le \left(2^{|J|} - 1\right) \cdot 2^{|\mathcal{C}_2| - |J|} \le \left(1 - 2^{-|J|}\right) \cdot 2^{f_2(n)}$.

Now, it suffices to show $\bigcup_{\varphi \in P_3} (N_\varphi(G_0) \cap \mathcal{G}') \subseteq \mathcal{F}$. Indeed, using $|J| \le \frac{c}{2}\sqrt{n}\ln n$, this would imply

$$\left| \bigcup_{\varphi \in P_3} (N_\varphi(G_0) \cap \mathcal{G}') \right| \le |\mathcal{F}| \le \left(1 - 2^{-|J|}\right) \cdot 2^{f_2(n)} \le \left(1 - n^{-c\sqrt{n}/2}\right) 2^{f_2(n)},$$

contradicting Eq. (16).

To show that $\bigcup_{\varphi \in P_3} (N_\varphi(G_0) \cap \mathcal{G}') \subseteq \mathcal{F}$, fix any $\varphi \in P_3$ and suppose, for the sake of contradiction, that there exists some $G \in N_\varphi(G_0) \cap \mathcal{G}'$ with $G \notin \mathcal{F}$. Let S be the cycle of φ containing u_1. Recall that $\varphi(u_i) = v_i \in V$ for all $i \in \{1, \ldots, \ell\}$, and that every $a \in [n]$ is a fixed point of ψ if and only if $a \notin U \cup V$. Since φ and ψ share all fixed points, the same is true for φ, i.e., a is a fixed point of φ if and only if $a \in U \cup V$. It follows that $\varphi(V) = U$, because $\varphi(U) = V$. So S alternates between U and V. To ease notation, let us assume without loss of generality that $S = (u_1, v_1, u_2, v_2, \ldots, u_k, v_k)$ for some k, where $\varphi(v_{i-1}) = u_i$ for $i \in \{2, \ldots, k\}$ and $\varphi(v_k) = u_1$. We will show by induction that $u_1 v_i \in G_0 \setminus G$ for every $i \in \{1, \ldots, k\}$. The base case $i = 1$ holds by the assumption of Case 1. Suppose now that $u_1 v_{i-1} \in G_0 \setminus G$ for some $2 \le i \le k$. Then, $v_1 u_i = \varphi(u_1 v_{i-1}) \in G \setminus G_0$, as $G_0 \xrightarrow{\varphi} G$. Recall that G contains exactly one of the edges $u_1 v_i, v_1 u_i$, as $G \in \mathcal{G}'$ and $\{u_1 v_i, v_1 u_i\} \in \mathcal{C}_2$. Hence, $u_1 v_i \notin G$. Now, if $u_1 v_i \notin G_0$, then $i \in J$, implying that $G \in \mathcal{F}$ because $v_1 u_i \in G$. This is a contradiction, as we assumed $G \notin \mathcal{F}$. This means $u_1 v_i$ must be in G_0, so $u_1 v_i \in G_0 \setminus G$, establishing the induction step. Taking $i = k$, we have $u_1 v_k \in G_0 \setminus G$. So, $u_1 v_1 = \varphi(u_1 v_k) \in G \setminus G_0$, which is impossible as $u_1 v_1 = e \in G_0$ by the assumption of Case 1. This proves $\bigcup_{\varphi \in P_3} (N_\varphi(G_0) \cap \mathcal{G}') \subseteq \mathcal{F}$ and completes the proof of Case 1.

Case 2: (b) holds. Namely, G_0 agrees with the graphs in $\mathcal{G}$ on all edges in $\mathcal{C}_1$, and there is a pair $\{e, f\} \in \mathcal{C}_2$ such that $e, f \in G_0$. Recall that pairs $\{e, f\} \in \mathcal{C}_2$ satisfy $\psi(e) = f$, $\psi(f) = e$ and $e \neq f$.

A pair of edges e, f satisfying this has to be of one of the following three types: (i) $\{e, f\} = \{au_i, av_i\}$ for some fixed point a of ψ and some $i \in \{1, \dots, \ell\}$; (ii) $\{e, f\} = \{u_i u_j, v_i v_j\}$ for some $1 \le i < j \le \ell$; and (iii) $\{e, f\} = \{u_i v_j, v_i u_j\}$ for some $1 \le i < j \le \ell$. We now consider two subcases.

Case 2.1. $\{e, f\}$ is of type (i) or (ii). Without loss of generality, let us assume that if $\{e, f\}$ is of type (i) then $e = au_i, f = av_i$, and if $\{e, f\}$ is of type (ii) then $e = u_i u_j, f = v_i v_j$. Observe that in both cases, we have $\varphi(e) = f$ for every $\varphi \in P_3$, because $\varphi(u_i) = v_i$ and φ, ψ share all fixed points. We claim that $e \in G$ for every $\varphi \in P_3$ and $G \in N_\varphi(G_0)$. Indeed, if $e \notin G$ then $e \in G_0 \setminus G$ and hence $f = \varphi(e) \in G \setminus G_0$ (as $G_0 \overset{\varphi}{\to} G$), contradicting $f \in G_0$. In particular, this means that all graphs in $\bigcup_{\varphi \in P_3} (N_\varphi(G_0) \cap \mathcal{G}')$ must contain e but not f. Recall also that the graphs in $\mathcal{G}'$ have at most two choices for every pair in $\mathcal{C}_2$, and $|\mathcal{C}_2| \le f_2(n)$ (by Lemma 2.4). It follows that $\left|\bigcup_{\varphi \in P_3} (N_\varphi(G_0) \cap \mathcal{G}')\right| \le 2^{f_2(n)-1} = 2^{f_2(n)}/2$, contradicting Eq. (16).

Case 2.2. $\{e, f\}$ is of type (iii). In this case $\{e, f\} = \{u_i v_j, v_i u_j\}$ for some $1 \le i < j \le \ell$. Note that e, f are not dependent on any particular choice of $\varphi \in P_3$. Without loss of generality, let us assume that $i = 1, j = 2$ and $e = u_1 v_2, f = v_1 u_2$. Define

$$J = \{(i, j) : 1 \le i \le 2 < j \le \ell, u_i v_j, v_i u_j \notin G_0\}, \quad \mathcal{F} = \{G \in \mathcal{G}' : \exists (i, j) \in J.\ v_i u_j \in G\}.$$

We proceed similarly to Case 1. First, note that $J \subseteq I$, so $|J| \le |I| \le \frac{c}{2}\sqrt{n}\ln n$ by Claim 3.21. As in Case 1, the graphs in $\mathcal{F}$ have at most $2^{|J|} - 1$ choices on the edges in the set $\{u_i v_j, v_i u_j : (i, j) \in J\}$, because the choice that $v_i u_j \notin G$ for all $(i, j) \in J$ is impossible. Hence, $|\mathcal{F}| \le \left(2^{|J|} - 1\right) \cdot 2^{|\mathcal{C}_2| - |J|} = \left(1 - 2^{-|J|}\right) \cdot 2^{f_2(n)}$. Now, it suffices to show that $\bigcup_{\varphi \in P_3} (N_\varphi(G_0) \cap \mathcal{G}') \subseteq \mathcal{F}$, because this would imply $\left|\bigcup_{\varphi \in P_3} (N_\varphi(G_0) \cap \mathcal{G}')\right| \le |\mathcal{F}| \le (1 - n^{-c\sqrt{n}/2}) \cdot 2^{f_2(n)}$, contradicting Eq. (16).

So let us assume, for the sake of contradiction, that there exists $\varphi \in P_3$ and $G \in N_\varphi(G_0) \cap \mathcal{G}'$ such that $G \notin \mathcal{F}$. We know that every graph in $\mathcal{G}'$ contains exactly one of the edges $e = u_1 v_2$ and $f = v_1 u_2$. Assume that $v_1 u_2 \in G$ and $u_1 v_2 \notin G$; the other case is similar by swapping the roles of $1, 2$ and of e, f. As in Case 1, the cycle in φ containing u_2 alternates between U and V. Let us denote this cycle by $(u_{i_1}, v_{i_1}, \dots, u_{i_k}, v_{i_k})$, where $u_{i_1} = u_2$ and $v_{i_1} = \varphi(u_2) = v_2$ (that is, $i_1 = 2$). Note that we do not make any assumption of whether u_1 is in this cycle or not. We now show by induction that $u_1 v_{i_j} \in G_0 \setminus G$ for every $1 \le j \le k$. The base case $j = 1$ holds by our assumptions that $u_1 v_{i_1} = u_1 v_2 = e \in G_0$ and $u_1 v_2 \notin G$. Suppose now that we proved that $u_1 v_{i_{j-1}} \in G_0 \setminus G$ for some $2 \le j \le k$. Then, $v_1 u_{i_j} = \varphi(u_1 v_{i_{j-1}}) \in G \setminus G_0$, because $G_0 \overset{\varphi}{\to} G$. As $\{v_1 u_{i_j}, u_1 v_{i_j}\} \in \mathcal{C}_2$, every graph in $\mathcal{G}'$ contains exactly one of the edges $v_1 u_{i_j}, u_1 v_{i_j}$. In particular, $u_1 v_{i_j} \notin G$. Observe that $i_j \ne 2$ (because $i_1 = 2$ and $j \ne 1$). Also, $i_j \ne 1$, because otherwise we would have $v_1 u_1 = v_1 u_{i_j} \in G \setminus G_0$, in contradiction to the assumption that G_0 and G agree on all edges in $\mathcal{C}_1$. So $i_j > 2$. Now, if $u_1 v_{i_j} \notin G_0$, then $(1, i_j) \in J$ and hence $G \in \mathcal{F}$ (as $v_1 u_{i_j} \in G$), and this contradicts our assumption $G \notin \mathcal{F}$. This means that $u_1 v_{i_j} \in G_0$ must hold, and therefore $u_1 v_{i_j} \in G_0 \setminus G$, completing the induction step. Now, taking $j = k$, we get $u_1 v_{i_k} \in G_0 \setminus G$. Therefore, $f = v_1 u_2 = v_1 u_{i_1} = \varphi(u_1 v_{i_k}) \in G \setminus G_0$, contradicting our assumption that $f \in G_0$. This proves that indeed $\bigcup_{\varphi \in P_3} (N_\varphi(G_0) \cap \mathcal{G}') \subseteq \mathcal{F}$, completing the proof in Case 2 and hence the entire proof of the lemma. ■

Proof of Theorem 2.7 for $r = 2$. Let $\mathcal{G}' \subseteq \mathcal{G}$ be a largest involution clique in $\mathcal{G}$. By assumption, $\mathcal{G}$ is not an involution clique, hence $\mathcal{G} \ne \mathcal{G}'$. We consider the following three cases depending on the size of $\mathcal{G}'$. Let c_1 and c_2 be the constants from Lemmas 3.19 and 3.20, respectively. Without loss of generality, we may assume that $c_1 < \frac{1}{250}$.

Case 1: $|\mathcal{G}'| \geq \left(1 - n^{-c_2\sqrt{n}}\right) 2^{f_2(n)}$. Then, by Lemma 3.20, $\mathcal{G} = \mathcal{G}'$, contradicting our assumption.

Case 2: $2^{f_2(n)} \cdot e^{-c_1 n} \leq |\mathcal{G}'| \leq \left(1 - n^{-c_2\sqrt{n}}\right) 2^{f_2(n)}$. By Lemma 3.19, $|\mathcal{G} \setminus \mathcal{G}'| \leq 2^{f_2(n)} \cdot e^{-c_1 n}$. When n is sufficiently large, the desired inequality holds as

$$|\mathcal{G}| = |\mathcal{G}'| + |\mathcal{G} \setminus \mathcal{G}'| \leq \left(1 - n^{-c_2\sqrt{n}} + e^{-c_1 n}\right) \cdot 2^{f_2(n)} \leq \left(1 - n^{-2c_2\sqrt{n}}\right) \cdot 2^{f_2(n)}.$$

Case 3: $|\mathcal{G}'| < 2^{f_2(n)} \cdot e^{-c_1 n}$. Then Lemma 3.11 with $\delta = c_1 n < \frac{n}{250}$ implies $|\mathcal{G}| \leq 2^{f_2(n)} \cdot e^{-\Omega(n)}$, as desired.

■

4 Concluding remarks

We showed that if $n \geq n_0(r)$, then $2^{f_r(n)}$ is the largest size of a difference-isomorphic r-graph family on $[n]$. It would be interesting to determine how large $n_0(r)$ should be. Let us observe that for $n = r+1$, our result does not hold. Indeed, let $\mathcal{G}$ be the family consisting of all r-graphs on $[r+1]$ with exactly $\lfloor (r+1)/2 \rfloor$ edges. We claim that $\mathcal{G}$ is difference isomorphic. To this end, for every $G \in \mathcal{G}$, we consider the set $S(G) := \{i : [r+1] \setminus \{i\} \in E(G)\}$, i.e., $S(G)$ is the set of complements of edges in G, which are singletons. Then $S(G)$ is a subset of $[r+1]$ of size $\lceil (r+1)/2 \rceil$. For every $G_1, G_2 \in \mathcal{G}$, put $S_1 = S(G_1), S_2 = S(G_2)$, and take a permutation $\varphi \in S_{[r+1]}$ with $\varphi(S_1 \setminus S_2) = S_2 \setminus S_1$ (we know that $|S_1 \setminus S_2| = |S_2 \setminus S_1|$ because $|S_1| = |S_2|$). Now, let $e \in G_1 \setminus G_2$, so $e = [r+1] \setminus \{i\}$ where $i \in S_1 \setminus S_2$. Then $j := \varphi(i) \in S_2 \setminus S_1$, so $\varphi(e) = [r+1] \setminus \varphi(i) = [r+1] \setminus \{j\} \in G_2 \setminus G_1$. This shows that $\varphi(G_1 \setminus G_2) \subseteq G_2 \setminus G_1$. Using that $|G_1 \setminus G_2| = |G_2 \setminus G_1|$, it holds that $\varphi(G_1 \setminus G_2) = G_2 \setminus G_1$. So we see that $\mathcal{G}$ is a difference-isomorphic family of size $\binom{r+1}{\lfloor (r+1)/2 \rfloor}$. It is easy to check that $\binom{r+1}{\lfloor (r+1)/2 \rfloor} > 2^{f_r(r+1)}$ for all $r \geq 2$, so Theorem 1.2 does not hold when $n = r+1$.

The argument from the previous paragraph, of considering the complements of edges, is more general; it shows that if $\mathcal{G}$ is a difference-isomorphic family of r-uniform hypergraphs on $[n]$, then $\{\{[n] \setminus e : e \in G\} : G \in \mathcal{G}\}$ is a difference isomorphic family of $(n-r)$-uniform hypergraphs on $[n]$. Hence, letting $F_r(n)$ denote the size of the largest difference-isomorphic family of r-graphs on $[n]$, we have that $F_r(n) = F_{n-r}(n)$. Note that $F_1(n) = \binom{n}{\lfloor n/2 \rfloor}$ ("1-uniform" hypergraphs are simply subsets, and two subsets and difference isomorphic if and only if they have the same size), and this recovers the fact that $F_r(r+1) = F_1(r+1) = \binom{r+1}{\lfloor (r+1)/2 \rfloor}$ from the previous paragraph. Moreover, one can check from Eq. (1) that $f_r(n) = f_{n-r}(n)$. Hence, whenever we have $F_r(n) = 2^{f_r(n)}$, we also have $F_{n-r}(n) = 2^{f_{n-r}(n)}$. Therefore, by Theorem 1.2, we have $F_r(r+2) = F_2(r+2) = 2^{f_2(r+2)} = 2^{f_r(r+2)}$ for sufficiently large r. It would be interesting to understand in general for which values of n (depending on r) it holds that $F_r(n) = 2^{f_r(n)}$.

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A Tightness for the stability result

Proposition A.1. *Let $r \geq 2$ and n be sufficiently large in terms of r. Then, there exists a difference-isomorphic family $\mathcal{G}$ of r-graphs on $[n]$ such that $\mathcal{G}$ is not an involution clique, and $|\mathcal{G}| = 2^{f_r(n)-O(n^{r-2})}$.*

Proof. Let $\psi \in S_n$ be the involution such that $\psi(2i-1) = 2i, \psi(2i) = 2i-1$ for all $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$, and that $\psi(i) = i$ for $2\lfloor \frac{n-1}{2} \rfloor < i \leq n$. Let a be the number of fixed points of ψ. We know $a = 1$ if n is odd while $a = 2$ if n is even. By Eq. (2),

$$|\mathcal{C}_1(\psi)| = \sum_{\substack{0 \leq i \leq a \\ i \equiv r \bmod 2}} \binom{a}{i} \binom{(n-a)/2}{(r-i)/2}$$

A similar computation as in Lemma 2.4 shows that $|\mathcal{C}_1(\psi)| = \binom{\lfloor n/2 \rfloor}{\lfloor r/2 \rfloor}$ if n is odd; $|\mathcal{C}_1(\psi)| = 2\binom{n/2-1}{\lfloor r/2 \rfloor}$ if n is even and r is odd; $|\mathcal{C}_1(\psi)| = \binom{n/2-1}{r/2} + \binom{n/2-1}{r/2-1}$ if n and r are even. Hence, $|\mathcal{C}_2(\psi)| = f_2(n)$ if $r = 2$ and $|\mathcal{C}_2(\psi)| = f_r(n) - O(n^{\lfloor r/2 \rfloor})$ if $r \geq 3$.

Let $\varphi \in S_n$ be a permutation such that $\varphi(1) = 2, \varphi(2) = 3, \varphi(3) = 4, \varphi(4) = 1$ and $\varphi(i) = \psi(i)$ for all $5 \leq i \leq n$. Define $e_0 = \{3, 4, \dots, r+2\}$ if r is even and $e_0 = \{3, 4, \dots, r+1, n\}$ if r is odd. By definition, $e_0 \in \mathcal{C}_1(\psi)$. Let $f_0 = \varphi(e_0)$. Then, $f_0 \cap \{1, 2, 3, 4\} = \{1, 4\}$ and f_0 is contained in some 2-cycle of $\tilde{\psi}$. Pick any r-graph G_0 on $[n]$ with the following properties.

- For every $\{e, f\} \in \mathcal{C}_2(\psi)$ with $e \subseteq \{5, \dots, n\}$ (so $f \subseteq \{5, \dots, n\}$), G_0 contains one between them;
- for every $\{e, f\} \in \mathcal{C}_2(\psi)$ with $|e \cap \{1, \dots, 4\}| = 1$ (so $|f \cap \{1, \dots, 4\}| = 1$), G_0 contains e if $e \cap \{1, 3\} \neq \emptyset$ while G_0 contains f if $f \cap \{1, 3\} \neq \emptyset$ (clearly, G_0 contains one of them);
- Among all edges incident to at least 2 vertices in $\{1, 2, 3, 4\}$, only $e_0 \in G_0$;
- G contains no edges in $\mathcal{C}_1(\psi) \setminus \{e_0\}$.

Consider a family $\mathcal{G}'$ of r-graphs on $[n]$, including all r-graphs G such that

- for every $\{e, f\} \in \mathcal{C}_2(\psi)$ with $e \subseteq \{5, \dots, n\}$ (so $f \subseteq \{5, \dots, n\}$), G contains one between them;
- for every $\{e, f\} \in \mathcal{C}_2(\psi)$ with $|e \cap \{1, \dots, 4\}| = 1$ (so $|f \cap \{1, \dots, 4\}| = 1$), G_0 contains e if $e \cap \{1, 3\} \neq \emptyset$ while G contains f if $f \cap \{1, 3\} \neq \emptyset$;
- Among all edges incident to at least 2 vertices in $\{1, 2, 3, 4\}$, only $f_0 \in G$;
- G contains no edges in $\mathcal{C}_1(\psi)$

It is easy to see that every $G \in \mathcal{G}'$ contains the same number of edges in every 2-cycle $\{e, f\} \in \mathcal{C}_2(\psi)$ and contains none of the edges in $\mathcal{C}_1(\psi)$, so Lemma 2.5 implies $\mathcal{G}'$ forms a ψ-clique. In addition, the number of 2-cycles that $G \in \mathcal{G}'$ has a choice is at least $\mathcal{C}_2(\psi) - \binom{4}{2}\binom{n}{r-2}$. If $r = 2$, $|\mathcal{G}'| \geq 2^{f_2(n)-6}$; if $r \geq 3$, $|\mathcal{G}'| \geq 2^{\mathcal{C}_2(\psi) - \binom{4}{2}\binom{n}{r-2}} = 2^{f_r(n) - O(n^{\lfloor r/2 \rfloor}) - O(n^{r-2})} = 2^{f_r(n) - O(n^{r-2})}$. So, $|\mathcal{G}'| \geq 2^{f_r(n) - O(n^{r-2})}$ holds unconditionally.

Now, consider $\mathcal{G} := \mathcal{G}' \cup \{G_0\}$. We will complete the proof by showing that $\mathcal{G} := \mathcal{G}' \cup \{G_0\}$ is a difference-isomorphic graph family of size $2^{f_r(n) - O(n^{r-2})}$ such that $\mathcal{G}$ is not an involution clique. First, we claim that $\mathcal{G}$ is difference-isomorphic. It suffices to show that $G_0 \xrightarrow{\varphi} G$ for all $G \in \mathcal{G}'$. Indeed, $(e, f) \in \binom{[n]}{r} \times \binom{[n]}{r}$ is a choosable pair for (G, φ) if and only if one of the following holds: $\{e, f\} \in \mathcal{C}_2(\psi)$ and $e \subseteq \{5, \dots, n\}$; $\{e, f\} \in \mathcal{C}_2(\psi)$ and $e \cap \{1, 2, 3, 4\} = \{1\}$ or $\{3\}$; $e = e_0$. It is easy to check that (i)(ii) (of Lemma 2.2) are satisfied for all $G \in \mathcal{G}'$. Hence, $G_0 \xrightarrow{\varphi} G$, meaning $\mathcal{G}$ is difference-isomorphic.

Second, we show that $\mathcal{G}$ is not an involution clique. Suppose for contradiction that $\mathcal{G}$ is a ψ'-clique some involution ψ'. We have $e_\psi(N_{\psi'}(G_0)) \geq e_\psi(\mathcal{G}') \geq 2^{2f_r(n) - O(n^{r-2})} \gg 2^{2f_r(n)} \cdot e^{-\binom{n}{r-1}/100}$, when n is sufficiently large in terms of r. Then, Lemma 3.8 implies that $\psi = \psi'$. But since $e_0 \in \mathcal{C}_1(\psi)$, Lemma 2.5 implies that all graphs in $\mathcal{G}'$ must also contain e_0, which is impossible. So, $\mathcal{G}$ is indeed not an involution clique. ■