

Ramsey problems for monotone paths in graphs and hypergraphs

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Abstract

The study of ordered Ramsey numbers of monotone paths for graphs and hypergraphs has a long history, going back to the celebrated work by Erdős and Szekeres in the early days of Ramsey theory. In this paper we obtain several results in this area, establishing two conjectures of Mubayi and Suk and improving bounds due to Balko, Cibulka, Král and Kynčl. We also obtain a color-monotone version of the well-known Canonical Ramsey Theorem of Erdős and Rado, which could be of independent interest.

1 Introduction

An *ordered k -uniform hypergraph* H is a hypergraph whose vertices have a fixed ordering. The *ordered Ramsey number* $R_{<}(G, H)$ of ordered k -uniform hypergraphs G and H is the minimum N such that every red/blue edge-coloring of the complete k -uniform hypergraph on $\{1, \dots, N\}$ has a red copy of G or a blue copy of H whose vertices appear in the correct ordering. One of the most well-studied hypergraphs in this context is the monotone tight path $P_n^{(k)}$, which is the ordered k -uniform hypergraph with vertices $1, \dots, n$ and edges $(i, i+1, \dots, i+k-1)$ for $i = 1, \dots, n-k+1$. We also write $P_n := P_n^{(2)}$ for the monotone graph path with n vertices.

The study of ordered Ramsey numbers dates back to the very beginning of Ramsey theory, as some of the most foundational theorems in the field fall into this framework. A key example is the celebrated Erdős-Szekeres lemma [6], whose proof gives $R_{<}(P_s, P_n) = R_{<}(K_s, P_n) = (s-1)(n-1)+1$.

A central feature of ordered Ramsey problems is that they often originate from and have implications to problems in geometry. For example, the famous Erdős-Szekeres cups-caps theorem [6], which states that $R_{<}(P_s^{(3)}, P_n^{(3)}) = \binom{s+n-4}{s-2} + 1$, was used by Erdős and Szekeres to prove the so-called Happy Ending Theorem in the same paper. This result states that every set of $\binom{2n-4}{n-2} + 1$ points in the plane in general position contains n points in convex position. The happy ending theorem was later extended by several authors by replacing “points” with “convex bodies”. For example, Pach and Tóth [16] showed that there is $N = N(n)$ such that every family of N convex bodies in the plane in general position, with any two bodies having at most two common boundary points, contains n bodies in convex position. Fox, Pach, Sudakov and Suk [7] observed that this problem is related to the 3-color Ramsey number of $P_n^{(3)}$, and used this connection to improve the best known bound for such an $N(n)$. This led to the further study of the multicolor Ramsey number of $P_n^{(k)}$, and this Ramsey number was determined exactly by Moshkovitz and Shapira [11] and Milans, Stolee and West [10].

Another motivation for studying Ramsey numbers of monotone tight paths comes from the work of Mubayi and Suk [13], who observed that the k -uniform Ramsey number $R_{<}(P_s^{(k)}, K_n^{(k)})$ is closely related to the multicolor $(k-1)$ -uniform Ramsey number of cliques. Proving tight bounds for the

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latter is one of the major open problems in Ramsey theory (see [17] and its references for the latest results for the graph case, i.e. $k = 3$).

A systematic study of ordered Ramsey numbers of graphs was initiated by Conlon, Fox, Lee and Sudakov [4] and Balko, Cibulka, Král and Kynčl [1]. One of the main problems considered in these works is the Ramsey problem for powers of paths. Let P_n^t denote the t 'th power of P_n , namely, the ordered graph with vertices $1, \dots, n$ and edge-set $\{(i, j) : i < j \leq i + t\}$. Note that for $P_n^1 = P_n$, the Erdős-Szekeres lemma gives $R_{<}(K_s, P_n^1) = (s - 1)(n - 1) + 1$, which is linear in n for fixed s . Mubayi and Suk [15] recently conjectured that $R_{<}(K_s, P_n^t)$ remains linear in n for all fixed s, t , and proved the bound $R_{<}(K_s, P_n^t) = O_{s,t}(n \log^{s-2} n)$. Here, we confirm their conjecture, which can be seen as an extension of the Erdős-Szekeres lemma.

Theorem 1.1. $R_{<}(K_{s+1}, P_n^t) \leq (24s^3)^{st}n$.

Mubayi and Suk [15] also considered the “diagonal case” of P_n^t versus K_n , and proved the quasipolynomial bound $R_{<}(P_n^t, K_n) \leq 2^{O_t(\log^2 n)}$. Here we improve this to a polynomial bound.

Theorem 1.2. For $t \geq 2$, $R_{<}(P_n^t, K_n) \leq 2^{2t-1}n^{t(2t-1)}$.

It is easy to see that this Ramsey number is at least $R(K_{t+1}, K_n) \geq \tilde{\Omega}(n^{(t+2)/2})$ (see [2, 18]), and therefore the exponent of n should grow in terms of t .

The dependence on t of the exponent of n in the previous result can be further improved from quadratic to linear if instead of K_n one takes the second graph to also be the path-power P_n^t . This problem was first considered by Conlon, Fox, Lee and Sudakov [4], who conjectured that the Ramsey number $R_{<}(P_n^t, P_n^t)$ is polynomial in n . Their actual question concerned graphs with bounded bandwidth. But since every n -vertex ordered graph with bandwidth t is a subgraph of P_n^t , this question reduces to the above conjecture. Balko, Cibulka, Král and Kynčl [1] proved the conjecture by showing that $R_{<}(P_n^t, P_n^t) = O_t(n^{129t})$. In the case $t = 2$, Mubayi [12] improved the bound to $R_{<}(P_n^2, P_n^2) = O(n^{19.5})$. Here we obtain the improved bound $R_{<}(P_n^t, P_n^t) = O_t(n^{4t-2})$. For $t = 2$ this gives $R_{<}(P_n^2, P_n^2) = O(n^6)$. In fact, [Theorem 1.2](#) gives $R_{<}(P_n^2, K_n) \leq 8n^6$.

Theorem 1.3. $R_{<}(P_n^t, P_n^t) \leq (400t^3)^{t^2}n^{4t-2}$.

We now move on to 3-uniform hypergraphs. Mubayi [12] showed that $R_{<}(K_4^{(3)}, P_n^{(3)}) \leq O(n^{21})$ and conjectured that $R_{<}(K_s^{(3)}, P_n^{(3)})$ is polynomial in n for every fixed s . This conjecture was also reiterated by Mubayi and Suk [14]. Very recently, they [15] proved the quasipolynomial bound $R_{<}(K_s^{(3)}, P_n^{(3)}) \leq 2^{O_s(\log^2 n)}$. Here we prove a polynomial bound, establishing Mubayi’s conjecture.

Theorem 1.4. For every $s \geq 3$, there is a constant $C = C(s)$ such that $R_{<}(K_s^{(3)}, P_n^{(3)}) \leq C \cdot n^C$.

We will derive [Theorem 1.4](#) from another result which might be of independent interest. It can be viewed as an ordered version of the celebrated Canonical Ramsey Theorem of Erdős and Rado [5] (here “ordered” refers to an order on the colors, as we shall see). To state our theorem, we need the following definition:

Definition 1.5. Let $\chi : \binom{[N]}{2} \rightarrow \mathbb{N}$. A set $x_1 < \dots < x_s$ is lexicographic if there are colors $c_1, \dots, c_{s-1} \in \mathbb{N}$ such that one of the following holds:

1. $\chi(x_i, x_j) = c_i$ for all $1 \leq i < j \leq s$.
2. $\chi(x_i, x_j) = c_{j-1}$ for all $1 \leq i < j \leq s$.

The canonical Ramsey theorem [5] states that for every s , there is $N = N(s)$ such that in every edge-coloring of K_N (with any number of colors), there is a clique of size s which is either monochromatic, rainbow or lexicographic. It is natural to restrict the number of colors to say n , and ask how large N should be to guarantee a lexicographic s -clique. It is not hard to see that $N = 1 + \sum_{i=0}^{s-2} n^i = O(n^{s-2})$ vertices are enough for n colors. Indeed, the first vertex has degree at least $1 + \sum_{i=0}^{s-3} n^i$ in one of the n colors, and one can then apply induction in its neighbourhood. Observe also that by combining this argument with the Erdős-Szekeres lemma, one can make sure that the sequence of colors c_1, \dots, c_{s-1} (from Definition 1.5) is monotone, namely that $c_1 \geq \dots \geq c_{s-1}$ or $c_1 \leq \dots \leq c_{s-1}$. Indeed, taking $N = O(n^{(s-1)^2})$, one first finds a lexicographic set of size $t = (s-1)^2 + 2$, and then applies the Erdős-Szekeres lemma to c_1, \dots, c_{t-1} . The question becomes much more interesting, however, if one requires that $c_1 \geq \dots \geq c_{s-1}$. In this case we say that the lexicographic set is *non-increasing*. The following theorem shows that polynomially many vertices are still enough to find such a set. This can be viewed as an ordered version of the canonical Ramsey theorem.

Theorem 1.6. *For every $s \geq 2$, there is a constant $C = C(s)$ such that every $\chi : \binom{N}{2} \rightarrow [n]$, $N = Cn^C$, admits a lexicographic non-increasing set of size s .*

The idea of using *non-increasing sets* to bound $R_{<}(K_s^{(3)}, P_n^{(3)})$ is due to Mubayi and Suk [15], see Section 2 for the details.

The rest of this paper is organized as follows. Theorems 1.4 and 1.6 are proved in Section 2. Theorems 1.2 and 1.3 are proved in Section 3. The proof of Theorem 1.1 is given in Section 4, and the last section contains some concluding remarks and open problems.

Notation: For two subsets of vertices A, B in an ordered graph, we write $A < B$ if $a < b$ for all $a \in A, b \in B$.

2 Proof of Theorems 1.4 and 1.6

Throughout this section, we consider functions $\chi : \binom{N}{2} \rightarrow [n]$. We now define the notion of a *non-increasing set*, which plays a key role throughout this section.

Definition 2.1. *Let $\chi : \binom{N}{2} \rightarrow [n]$. A triple $x < y < z$ is called non-increasing if $\chi(x, y) \geq \chi(y, z)$, and moreover $\chi(x, z) = \chi(x, y)$ or $\chi(x, z) = \chi(y, z)$. A set $x_1 < \dots < x_s$ is called non-increasing if every triple x_i, x_j, x_k , $1 \leq i < j < k \leq s$, is non-increasing.*

Let $g(n, s)$ be the minimum N such that in every coloring $\chi : \binom{N}{2} \rightarrow [n]$, there is a non-increasing set of size s . The following was observed by Mubayi and Suk [15] and was their motivation for studying non-increasing sets. For completeness, we include the proof.

Proposition 2.2 ([15]). $R_{<}(K_s^{(3)}, P_n^{(3)}) \leq g(n-2, s)$.

Proof. Fix a red/blue coloring of $K_N^{(3)}$, $N = g(n-2, s)$, and suppose that there is no blue monotone tight path with n vertices. For each pair of vertices $x < y$, let $\chi(x, y)$ be the largest number of vertices in a blue monotone tight path ending at x, y . So $2 \leq \chi(x, y) \leq n-1$. Observe that if $x < y < z$ with xyz blue, then $\chi(x, y) < \chi(y, z)$, because we can extend any longest path ending at x, y with the edge xyz . Hence, a non-increasing set must be a red clique. Observe that there are $n-2$ possible values of $\chi(x, y)$'s. By the definition of $g(n-2, s)$, there is a red clique of size s . ■

Remark 2.3. We note that the proof of [Proposition 2.2](#) only uses that a non-increasing triple x, y, z satisfies $\chi(x, y) \geq \chi(y, z)$, but does not use the full definition of a non-increasing triple. Thus, one can replace $g(n-2, s)$ with the analogous function corresponding to this weaker notion of being non-increasing (only requiring $\chi(x, y) \geq \chi(y, z)$), and potentially get stronger bounds on $R_{<}(K_s^{(3)}, P_n^{(3)})$ via [Proposition 2.2](#). Still, we decided to stick to the stronger notion given in [Definition 2.1](#), because we do not know of better bounds for the weaker notion of g -function (for general s), and also because the stronger notion is needed to prove [Theorem 1.6](#).

We will now prove the following theorem, which together with [Proposition 2.2](#) implies [Theorem 1.4](#).

Theorem 2.4. *For every $s \geq 2$, there is a constant $C = C(s)$ such that $g(n, s) \leq C \cdot n^C$.*

To prove [Theorem 2.4](#), we need to find a non-increasing set of size s . The proof is via induction on s , and to this end, it turns out to be convenient to find the following bigger structure. For $s \geq 2, t \geq 1$, let $H_{s,t}$ be the ordered graph with vertices $x_1 < \dots < x_s = y_1 < \dots < y_t$ such that $\{x_1, \dots, x_s\}$ is non-increasing, (y_1, \dots, y_t) is a path, and $\chi(x_{s-1}, x_s) \geq \chi(y_1, y_2) \geq \dots \geq \chi(y_{t-1}, y_t)$. Let $f(n; s, t)$ be the minimum N such that every n -coloring χ of the edges of K_N admits a copy of $H_{s,t}$. As $H_{s,1}$ is just a non-increasing s -set, it holds that $f(n; s, 1) = g(n, s)$. The following gives a recursive bound on $f(n; s, t)$.

Lemma 2.5. *For $s \geq 3, t \geq 1$, it holds that $f(n; s, t) \leq f(n; s-1, t+1)^{s+t} \cdot n^{s-1}$.*

Proof. Put $M = f(n; s-1, t+1)$ and $N = M^{s+t} n^{s-1}$, and fix a coloring $\chi : \binom{N}{2} \rightarrow [n]$. Put $h = s+t-1 = |V(H_{s-1, t+1})|$. By definition, every M vertices contain a copy of $H_{s-1, t+1}$. By double counting, there are at least $\frac{\binom{N}{M}}{\binom{N-h}{M-h}} = \frac{\binom{N}{h}}{\binom{M}{h}} \geq (N/M)^h$ copies of $H_{s-1, t+1}$. By the pigeonhole principle, there is a choice of colors $\alpha_1, \dots, \alpha_{s-2}, \beta \in [n]$, and a set \mathcal{H} of at least $\frac{N^h}{M^h n^{s-1}}$ copies $(x_1, \dots, x_{s-1} = y_1, \dots, y_{t+1})$ of $H_{s-1, t+1}$, with the property that $\chi(x_i, x_{s-1}) = \alpha_i$ for every $1 \leq i \leq s-2$, and $\chi(y_1, y_2) = \beta$. By the definition of $H_{s-1, t+1}$, $\chi(x_{s-2}, x_{s-1}) \geq \chi(y_1, y_2)$ and the triple x_i, x_{i+1}, x_{s-1} is non-increasing for all $1 \leq i \leq s-3$. Thus, $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{s-2} \geq \beta$. By averaging, there is a choice of $x_1, \dots, x_{s-2}, y_2, \dots, y_{t+1}$ and a set X of size $|X| \geq \frac{N}{M^h n^{s-1}} = M$, such that $(x_1, \dots, x_{s-2}, x, y_2, \dots, y_{t+1}) \in \mathcal{H}$ for all $x \in X$. By the definition of M , X contains a copy of $H_{s-1, t+1}$, say on the vertices $u_1, \dots, u_{s-1} = v_1, \dots, v_{t+1}$. We now consider two cases:

Case 1: $\chi(v_1, v_2) \leq \beta$. Then, we have $\alpha_{s-2} \geq \beta \geq \chi(v_1, v_2) \geq \dots \geq \chi(v_t, v_{t+1})$. We claim that $x_1, \dots, x_{s-2}, v_1, v_2, v_3, \dots, v_{t+1}$ form a copy of $H_{s,t}$ with s -clique $\{x_1, \dots, x_{s-2}, v_1, v_2\}$ and path v_2, \dots, v_{t+1} . It suffices to show that $\{x_1, \dots, x_{s-2}, v_1, v_2\}$ is non-increasing. For convenience, write $x_{s-1} := v_1$ and $x_s := v_2$. Let $1 \leq i < j < k \leq s$. If $(j, k) \neq (s-1, s)$, then x_i, x_j, x_k is non-increasing because $\{x_1, \dots, x_{s-2}, v_1\}$ and $\{x_1, \dots, x_{s-2}, v_2\}$ are non-increasing, as $v_1, v_2 \in X$. And for $(j, k) = (s-1, s)$, we have $\chi(x_i, x_{s-1}) = \chi(x_i, x_s) = \alpha_i$ and $\chi(x_{s-1}, x_s) = \chi(v_1, v_2) \leq \beta \leq \alpha_{s-2} \leq \alpha_i$, meaning that x_i, x_{s-1}, x_s is non-increasing.

Case 2: $\chi(v_1, v_2) \geq \beta$. Then $\chi(u_{s-2}, u_{s-1}) \geq \chi(v_1, v_2) \geq \beta$, where the first inequality is by the definition of $H_{s-1, t+1}$. Also, $\chi(u_i, y_2) = \beta$ for all $1 \leq i \leq s-1$ because $u_1, \dots, u_{s-1} \in X$. We claim that $u_1, \dots, u_{s-1}, y_2, y_3, \dots, y_{t+1}$ make a copy of $H_{s,t}$ with s -clique $\{u_1, \dots, u_{s-1}, y_2\}$ and path y_2, \dots, y_{t+1} . First, note that $\chi(u_{s-1}, y_2) = \beta \geq \chi(y_2, y_3) \geq \dots \geq \chi(y_t, y_{t+1})$. So it remains to check that $\{u_1, \dots, u_{s-1}, y_2\}$ is non-increasing. For convenience, write $u_s := y_2$. Let $1 \leq i < j < k \leq s$.

If $k \leq s - 1$, then the triple u_i, u_j, u_k is non-increasing because $\{u_1, \dots, u_{s-1}\}$ is non-increasing (by the definition of $H_{s-1, t+1}$). And for $k = s$, we have $\chi(u_i, u_s) = \chi(u_j, u_s) = \beta$ and $\chi(u_i, u_j) \geq \chi(u_i, u_{s-1}) \geq \chi(u_{s-2}, u_{s-1}) \geq \beta$, where the first two inequalities use that the triples $\{u_i, u_j, u_{s-1}\}$ and $\{u_i, u_{s-2}, u_{s-1}\}$ are non-increasing. This completes the proof of the lemma. \blacksquare

The following theorem implies [Theorem 2.4](#), as $f(n; s, 1) = g(n, s)$.

Theorem 2.6. *For every $s \geq 2, t \geq 1$, there is a constant $C = C(s, t)$ such that $f(n; s, t) \leq O_{s,t}(n^C)$. Moreover, one can take $C(s, t) = (s + t)^{s-1} - 3(s + t)^{s-2} + \sum_{i=0}^{s-3} (s - 1 - i)(s + t)^i$.*

Proof. The proof is by induction on s , starting with the base case $s = 2$. Observe that $H_{2,t}$ is just a monotone path of length t with non-increasing χ -labels. By a result of Chvatal and Komlós [3], in every edge-coloring χ of an ordered K_N , $N > \binom{p+q-2}{p-1}$, there are vertices $y_1 < \dots < y_{p+1}$ with $\chi(y_1, y_2) \geq \chi(y_2, y_3) \geq \dots \geq \chi(y_p, y_{p+1})$ or vertices $y_1 < \dots < y_{q+1}$ with $\chi(y_1, y_2) < \chi(y_2, y_3) < \dots < \chi(y_q, y_{q+1})$. Apply this with $p = t, q = n + 1$, assuming $N > \binom{n+t-1}{t-1}$. The second outcome is impossible because there are only n colors. And the first outcome gives a copy of $H_{2,t}$. Hence, we can take $C(2, t) = t - 1$ for all $t \geq 1$. It is easy to see that this coincides with the choice of $C(s, t)$ in the statement of the theorem.

Now let $s \geq 3$. By [Lemma 2.5](#) and the induction hypothesis, we have

$$f(n; s, t) \leq f(n; s - 1, t + 1)^{s+t} \cdot n^{s-1} \leq O_{s,t} \left(n^{(s+t) \cdot C(s-1, t+1) + s-1} \right),$$

so one can take $C(s, t) = (s + t) \cdot C(s - 1, t + 1) + s - 1$. It is easy to check that the choice of $C(s, t)$ in the statement of the theorem also satisfies this recursion. \blacksquare

Next we prove [Theorem 1.6](#). This theorem follows by combining [Theorem 2.4](#) with the following proposition:

Proposition 2.7. *For $s \geq 3$, every non-increasing set of size 2^{2s-3} contains a lexicographic set of size s .*

It remains to prove [Proposition 2.7](#). To this end, we need the following recursive definition.

Definition 2.8 (weakly lexicographic). *A set of size two is weakly lexicographic. For $s \geq 3$, a set $x_1 < \dots < x_s$ is weakly lexicographic if one of the following holds:*

1. *There is a color $c \in [n]$ such that $\chi(x_1, x_i) = c$ for all $2 \leq i \leq s$, and x_2, \dots, x_s is weakly lexicographic.*
2. *There is a color $c \in [n]$ such that $\chi(x_i, x_s) = c$ for all $1 \leq i \leq s - 1$, and x_1, \dots, x_{s-1} is weakly lexicographic.*

Let us say that a set x_1, \dots, x_s is *forward* (resp. *backward*) lexicographic if it satisfies Item 1 (resp. Item 2) in [Definition 1.5](#).

Lemma 2.9. *Let $s, t \geq 2$. Every weakly lexicographic set of size $s + t - 2$ contains a forward lexicographic set of size s or a backward lexicographic set of size t .*

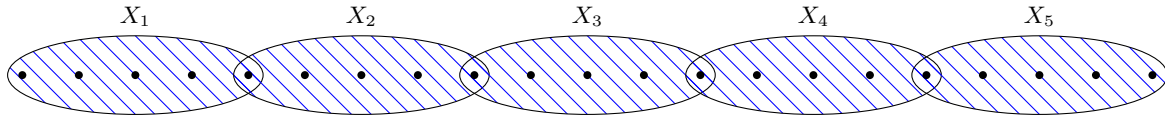


Figure 1: An example of a 5-clique chain X_1, \dots, X_5 .

Proof. By induction on $s + t$. The base case is $s = 2$ or $t = 2$. This case is evident because every set of size 2 is both forward and backward lexicographic. Suppose now that $s, t \geq 3$. Let $x_1 < \dots < x_{s+t-2}$ be a weakly lexicographic set. Suppose without loss of generality that Item 1 in Definition 2.8 holds. By induction, x_2, \dots, x_{s+t-2} contains a forward lexicographic set of size $s - 1$ or a backward lexicographic set of size t . In the latter case, we are done; in the former case, by adding x_1 we get a forward lexicographic set of size s (so again we are done). ■

Lemma 2.10. *Every non-increasing set of size 2^{s-1} contains a weakly lexicographic set of size s .*

Proof. By induction on s . For $s = 2$ this is clear. Let $s \geq 3$ and let $x_1 < \dots < x_k$ be a non-increasing set with $k = 2^{s-1}$. For each $2 \leq i \leq k - 1$, it holds that $\chi(x_1, x_i) = \chi(x_1, x_k)$ or $\chi(x_i, x_k) = \chi(x_1, x_k)$ because the triple x_1, x_i, x_k is non-increasing. Suppose, without loss of generality, that at least $(k - 2)/2 = 2^{s-2} - 1$ of the $2 \leq i \leq k - 1$ satisfy $\chi(x_1, x_i) = \chi(x_1, x_k) =: c$. Let I be the set consisting of these $2 \leq i \leq k - 1$ and the element k . Then $\chi(x_1, x_i) = c$ for all $i \in I$, and $|I| \geq 2^{s-2}$. By induction, $\{x_i : i \in I\}$ contains a weakly lexicographic set of size $s - 1$, which together with x_1 forms a weakly lexicographic sets of size s . ■

Proposition 2.7 follows by combining Lemmas 2.9 and 2.10 (where we apply Lemma 2.9 with $s = t$).

3 Proof of Theorems 1.2 and 1.3

Definition 3.1 (t -clique chain). *In an ordered graph, a t -clique chain consists of t -cliques (cliques of size t) X_1, \dots, X_m such that for every $1 \leq i \leq m - 1$, $|X_i \cap X_{i+1}| = 1$ and the last element of X_i is the first element of X_{i+1} .*

See Figure 1 for an example of a clique chain.

Lemma 3.2. *For $n, m, t \geq 1$, the following holds:*

1. *Every red/blue edge-coloring of K_N , $N = (R(K_t, K_t) - 1) \cdot m^2 + 1$, contains a monochromatic t -clique chain with m cliques.*
2. *Every red/blue edge-coloring of K_N , $N = (R(K_t, K_n) - 1) \cdot m + 1$, contains a blue K_n or a red t -clique chain with m cliques.*

Proof. We first prove Item 1. Suppose that the assertion does not hold. For each vertex v , let $\chi_r(v)$ (resp. $\chi_b(v)$) be the largest number of t -cliques in a red (resp. blue) t -clique chain ending at v . Then $0 \leq \chi_r(v), \chi_b(v) \leq m - 1$ for all v . By the pigeonhole principle, there are values $0 \leq c_r, c_b \leq m - 1$ and a set U with $|U| \geq N/m^2 > R(K_t, K_t) - 1$, such that $\chi_r(v) = c_r$ and $\chi_b(v) = c_b$ for all $v \in U$. As $|U| \geq R(K_t, K_t)$, there is a monochromatic t -clique $x_1 < \dots < x_t$ in U . Suppose without loss of generality that this clique is red. Then $\chi_r(x_t) > \chi_r(x_1)$, because any longest red t -clique chain ending at x_1 can be extended using the t -clique $\{x_1, \dots, x_t\}$. This is a contradiction to $\chi_r(x_1) = \chi_r(x_t) = c_r$.

The proof of Item 2 is similar to that of Item 1. Suppose that the statement does not hold, and let $0 \leq \chi_r(v) \leq m-1$ be defined as above. By the pigeonhole principle, there is $0 \leq c_r \leq m-1$ and a vertex-set U , $|U| \geq N/m$, such that $\chi_r(v) = c_r$ for all $v \in U$. We have $|U| \geq R(K_t, K_n)$ by the choice of N . If U contains a blue K_n then we are done, and else U contains a red t -clique $x_1 < \dots < x_t$. As in the previous item, we have $\chi_r(x_t) > \chi_r(x_1)$, in contradiction to $\chi_r(x_1) = \chi_r(x_t) = c_r$. ■

Proof of Theorem 1.3. Put $R = R(K_t, K_t)$ and $N = 2R^{2t-1}n^{4t-3} \cdot R_{<}(K_t, P_n^t)$, and note that $N < (400t^3)^{t^2}n^{4t-2}$, using Theorem 1.1 and $R(K_t, K_t) \leq 4^t$. Fix a red/blue coloring of K_N , and suppose by contradiction that there is no monochromatic copy of P_n^t . For each monochromatic t -clique x_1, \dots, x_t , let $\chi(x_1, \dots, x_t)$ be the largest ℓ such that there is a monochromatic P_ℓ^t which ends at x_1, \dots, x_t . Then $t \leq \chi(x_1, \dots, x_t) \leq n-1$. A *good pair* is a pair of monochromatic t -cliques X, Y in the same color, such that the last element of X is the first element of Y , and $\chi(X) \geq \chi(Y)$.

Claim 3.3. *There are at least $\frac{N^{2t-1}}{R^{2t-1}n^{4t-4}}$ good pairs.*

Proof. First, we observe that every set of $(R-1)n^2 + 1$ vertices contains a good pair. Indeed, by Item 1 of Lemma 3.2, every $(R-1)n^2 + 1$ vertices contain a monochromatic t -clique chain with n cliques X_1, \dots, X_n . There must exist an $1 \leq i \leq n-1$ such that $\chi(X_i) \geq \chi(X_{i+1})$, because $\chi(X_i) \in \{1, 2, \dots, n-1\}$ for every i . Then (X_i, X_{i+1}) is a good pair.

Now, every set of Rn^2 vertices contains at least n^2 good pairs (by repeatedly finding a good pair and deleting one of its vertices). By double counting, there are at least

$$n^2 \cdot \frac{\binom{N}{Rn^2}}{\binom{N-2t+1}{Rn^2-2t+1}} = n^2 \cdot \frac{\binom{N}{2t-1}}{\binom{Rn^2}{2t-1}} \geq \frac{N^{2t-1}}{R^{2t-1}n^{4t-4}}$$

good pairs. ■

By Claim 3.3 and averaging, there are vertices $x_1 < \dots < x_{t-1} < z_1 < \dots < z_{t-1}$ and a set Y' of vertices $x_{t-1} < y < z_1$ such that $|Y'| \geq \frac{N}{R^{2t-1}n^{4t-4}} \geq 2n \cdot R_{<}(K_t, P_n^t)$, and such that for every $y \in Y'$, $\{x_1, \dots, x_{t-1}, y\}$ and $\{y, z_1, \dots, z_{t-1}\}$ form a good pair. Without loss of generality, for at least half of the vertices $y \in Y'$, the t -cliques $\{x_1, \dots, x_{t-1}, y\}, \{y, z_1, \dots, z_{t-1}\}$ are red. Also, by the pigeonhole principle over the value of χ , there exists a set $Y \subseteq Y'$, $|Y| \geq |Y'|/(2n) = R_{<}(K_t, P_n^t)$, and there exists a value $t \leq c \leq n-1$, such that $\chi(x_1, \dots, x_{t-1}, y) = c$ and $\{x_1, \dots, x_{t-1}, y\}, \{y, z_1, \dots, z_{t-1}\}$ are red for all $y \in Y$. Note that $\chi(y, z_1, \dots, z_{t-1}) \leq c$ for all $y \in Y$, by the definition of a good pair. As $|Y| = R_{<}(K_t, P_n^t)$ and Y contains no blue P_n^t , it must contain a red clique $y_1 < \dots < y_t$. Now take a red P_c^t ending at x_1, \dots, x_{t-1}, y_1 , and extend it by adding the vertices $y_2, \dots, y_t, z_1, \dots, z_{t-1}$, using that $x_1, \dots, x_{t-1}, z_1, \dots, z_{t-1}$ are connected to y_1, \dots, y_t in red, and that y_1, \dots, y_t is a red clique. It follows that $\chi(y_t, z_1, \dots, z_{t-1}) > \chi(x_1, \dots, x_{t-1}, y_1) = c$, in contradiction to $\chi(y_t, z_1, \dots, z_{t-1}) \leq c$. ■

Proof of Theorem 1.2. The proof is similar to that of Theorem 1.3. Put $M = R(K_t, K_n)$ and $N = (2Mn)^{2t-1}$. As $R(K_t, K_n) \leq \binom{n+t-2}{t-1} \leq n^{t-1}$ (by the Erdős-Szekeres bound [6]), we have $N \leq 2^{2t-1}n^{t(2t-1)}$. Fix a red/blue coloring of K_N , and suppose by contradiction that there is no red P_n^t and no blue K_n . For each red t -clique x_1, \dots, x_t , let $\chi(x_1, \dots, x_t)$ be the largest ℓ such that there is a red P_ℓ^t that ends at x_1, \dots, x_t ; so $t \leq \chi(x_1, \dots, x_t) \leq n-1$. A *good pair* is a pair of red t -cliques X, Y such that the last element of X is the first element of Y , and $\chi(X) \geq \chi(Y)$.

Claim 3.4. *There are at least $\frac{N^{2t-1}}{2^{2t-1}M^{2t-2}n^{2t-2}}$ good pairs.*

Proof. First, we observe that every set of $(M - 1)n + 1$ vertices contains a good pair. Indeed, by Item 2 of [Lemma 3.2](#), every set of $(M - 1)n + 1$ vertices contains a blue K_n or a red t -clique chain with n cliques X_1, \dots, X_n . In the latter case, there must be $1 \leq i \leq n - 1$ such that $\chi(X_i) \geq \chi(X_{i+1})$, which gives a good pair.

Now, every set of $2Mn$ vertices contains at least Mn good pairs (by repeatedly finding a good pair and deleting one of its vertices). By double counting, there are at least

$$Mn \cdot \frac{\binom{N}{2Mn}}{\binom{N-2t+1}{2Mn-2t+1}} = Mn \cdot \frac{\binom{N}{2t-1}}{\binom{2Mn}{2t-1}} \geq \frac{N^{2t-1}}{2^{2t-1} M^{2t-2} n^{2t-2}}$$

good pairs. ■

By [Claim 3.4](#) and averaging, there are vertices $x_1 < \dots < x_{t-1} < z_1 < \dots < z_{t-1}$ and a set Y' of vertices $x_{t-1} < y < z_1$ such that $|Y'| \geq \frac{N}{2^{2t-1} M^{2t-2} n^{2t-2}} \geq Mn$, and such that for all $y \in Y'$, $\{x_1, \dots, x_{t-1}, y\}$ and $\{y, z_1, \dots, z_{t-1}\}$ form a good pair. By the pigeonhole principle over the value of χ , there exists a set $Y \subseteq Y'$, $|Y| \geq |Y'|/n = M$, and a value $t \leq c \leq n - 1$, such that $\chi(x_1, \dots, x_{t-1}, y) = c$ for all $y \in Y$. Then $\chi(y, z_1, \dots, z_{t-1}) \leq c$ for all $y \in Y$, by the definition of a good pair. As $|Y| = M = R(K_t, K_n)$ and Y contains no blue K_n , it must contain a red clique $y_1 < \dots < y_t$. As in the proof of [Theorem 1.3](#), we get $c \geq \chi(y_t, z_1, \dots, z_{t-1}) > \chi(x_1, \dots, x_{t-1}, y_1) = c$, a contradiction. ■

4 Proof of [Theorem 1.1](#)

We begin with a brief sketch. Our strategy for upper bounding $R(K_{s+1}, P_n^t)$ is to find a certain structure which we call an *s-red-net* (see [Definition 4.1](#)). We will show (see [Lemma 4.10](#)) that this structure implies the existence of a red K_{s+1} or a blue P_n^t . To find an *s-red-net*, we will first find blue cliques $V_1 < \dots < V_M$, each of large constant size, and partition each V_i into s consecutive equal-sized parts. For each i and $0 \leq j \leq s - 2$, we will define $\chi_j(i)$ as the largest ℓ such that there exists a blue P_ℓ^t whose last t vertices belong to the first $(s - 1 - j)$ parts in the partition of V_i . The key point is that if $i_1 < i_2$ and $\chi_j(i_1) \geq \chi_j(i_2)$, then the bipartite graph between certain parts of V_{i_1} and certain parts of V_{i_2} must be almost red. Using this, we can reduce the task of finding an *s-red-net* to the task of finding a certain structure in the functions $\chi_0, \dots, \chi_{s-2}$. We call this structure a $(\chi_0, \dots, \chi_{s-2})$ -forest, see [Definition 4.2](#) and [Lemma 4.3](#).

We now introduce some definitions. A *rooted forest* is a collection of rooted trees. The *depth* of a vertex is the distance from the root (of the corresponding tree). A rooted forest is *balanced* if all leaves have the same depth. We always consider forests F with $V(F) \subseteq [M]$ for some integer M , so that there is a natural ordering on $V(F)$. We say that a rooted forest is *well-ordered* if each vertex comes before all of its descendants, and for every two vertices $y < y'$ that are the same depth, the descendants of y come before y' , and thus before all the descendants of y' . In particular, if T_1, T_2 are two components (trees) in the forest, then $T_1 < T_2$ or $T_2 < T_1$. Also, if x is the root of a component T , $y_1 < \dots < y_k$ are the children of x , and S_i is the subtree rooted at y_i ($1 \leq i \leq k$), then $x < S_1 < \dots < S_k$, and each S_i is well-ordered. Observe that if F is balanced well-ordered forest of depth d , and x_1, \dots, x_m are the roots of the components (trees) of F , then after deleting x_1, \dots, x_m we get a balanced well-ordered forest of depth $d - 1$ (the trees in this forest are the subtrees rooted at the children of x_1, \dots, x_m). We write $|F|$ for the number of vertices in a forest F .

We are now ready to define the notion of an *s-red-net*, which will play a key role in the proof.

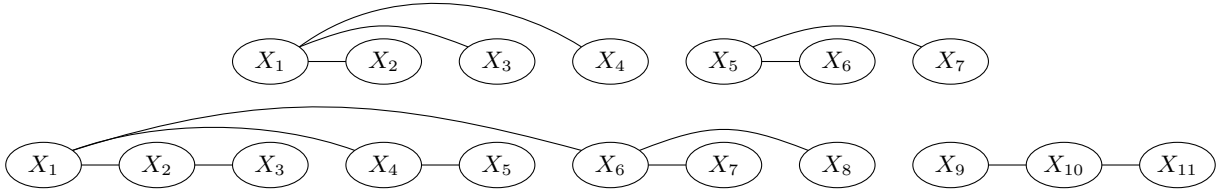


Figure 2: An example of a 2-red-net (top) and a 3-red-net (bottom). In the top example $V(F) = \{1, \dots, 7\}$ and in the bottom example $V(F) = \{1, \dots, 11\}$. An edge between X_i and X_j indicates that there is an edge between i and j in F .

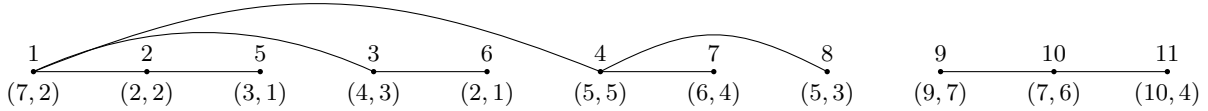


Figure 3: An example of a (χ_0, χ_1) -forest on vertex set $\{1, 2, \dots, 11\}$: the number above each node is its index, and the pair below node i is $(\chi_0(i), \chi_1(i))$.

Definition 4.1. Consider a red/blue edge-coloring of K_N . Let $s \geq 1$. An s -red-net of order r is a pair $(F, (X_v)_{v \in V(F)})$, where F is a balanced well-ordered forest of depth $s - 1$, and for each $v \in V(F)$, $X_v \subseteq [N]$ is a blue clique (in K_N) of size r , such that the following holds:

1. For $v, u \in V(F)$, if $v < u$ then $X_v < X_u$.
2. For every $v \in V(F)$ and every descendant u of v , there is no blue $K_{t,t}$ with one part in X_v and the other in X_u .

See Figure 2 for an example of an s -red-net for $s = 2, 3$. We note that since there is a total order of the vertices of F , there is also a corresponding total order of $(X_v)_{v \in V(F)}$.

4.1 Finding s -red-nets

As mentioned above, we will find an s -red-net by finding a certain structure in a family of functions $\chi_0, \dots, \chi_{q-1} : [M] \rightarrow [n]$. We now define this structure.

Definition 4.2. Let $\chi_0, \dots, \chi_{q-1} : [M] \rightarrow [n]$ be q functions. A $(\chi_0, \dots, \chi_{q-1})$ -forest is a well-ordered balanced forest F of depth q with $V(F) \subseteq [M]$, such that the following holds: For every $0 \leq d \leq q - 1$ and $a \in V(F)$ at depth d , it holds that $\chi_d(a) \geq \chi_d(a')$ for every child a' of a . We denote by $L(F)$ the set of leaves of F .

Note that in the case $q = 1$, a (χ_0) -forest simply consists of elements x_1, \dots, x_m (the roots of the trees in the forest) and sets Y_1, \dots, Y_m (the sets of leaves of the trees) such that $x_1 < Y_1 < \dots < x_m < Y_m$ and $\chi_0(x_i) \geq \chi_0(y)$ for all $y \in Y_i$ and $1 \leq i \leq m$. See Figure 3 for an example when $q = 2$.

Lemma 4.3. For every $q \geq 1$ and functions $\chi_0, \dots, \chi_{q-1} : [M] \rightarrow [n]$, there is a $(\chi_0, \dots, \chi_{q-1})$ -forest F with $|L(F)| \geq M/2^{q-1} - n$.

Proof. The proof is by induction on q . First, define a sequence x_1, x_2, \dots as follows. Set $x_1 = 1$, and for each $i \geq 2$, let x_i be the smallest $x > x_{i-1}$ with $\chi_0(x) > \chi_0(x_{i-1})$. Let $x_1 < \dots < x_k$ be the resulting sequence. As $\chi_0(x_1) < \dots < \chi_0(x_k)$, we have $k \leq n$.

First, we handle the base case $q = 1$. In this case, put $Y_i = \{x_i + 1, \dots, x_{i+1} - 1\}$ for $1 \leq i \leq k - 1$, and $Y_k = \{x_k + 1, \dots, M\}$. By the definition of the sequence $(x_i)_i$, we have that $\chi_0(x_i) \geq \chi_0(y)$ for every $y \in Y_i$ and $1 \leq i \leq m$. Now, write I for the set of $i \in [k]$ with $Y_i \neq \emptyset$. For $i \in I$, let T_i be the tree of depth 1 with root x_i and leaf-set Y_i . Let F be the forest with components T_i , $i \in I$. Then F is well-ordered and balanced of depth 1. Also, $L(F) = |Y_1| + \dots + |Y_k| = M - k \geq M - n$. So F is the required (χ_0) -forest.

Suppose now that $q \geq 2$. Apply the induction hypothesis for $q - 1$ to the functions $\chi_1, \dots, \chi_{q-1}$ and to $[M] \setminus \{x_1, \dots, x_k\}$ in place of $[M]$. This gives a $(\chi_1, \dots, \chi_{q-1})$ -forest F' with $V(F') \subseteq [M] \setminus \{x_1, \dots, x_k\}$ and

$$|L(F')| \geq \frac{M - n}{2^{q-2}} - n \geq \frac{M}{2^{q-2}} - 2n.$$

Let $S_1 < \dots < S_\ell$ be the components of F' . For each $1 \leq i \leq \ell$, let y_i be the root of S_i , and let j_i be the maximum $1 \leq j \leq k$ with $x_j < y_i$ (this is well-defined because $x_1 = 1$). Then $x_{j_i} < y_i < x_{j_i+1}$ if $j_i < k$, and $y_i \geq x_k$ if $j_i = k$. It follows that $\chi_0(x_{j_i}) \geq \chi_0(y_i)$ by the definition of the sequence $(x_j)_j$. (This means that y_i is a potential child of x_{j_i} in the $(\chi_0, \dots, \chi_{q-1})$ -forest we are going to construct.) For $1 \leq j \leq k$, let $I_j = \{1 \leq i \leq \ell : j_i = j\}$, the set of potential children of x_j . Let J_0 be the set of all $1 \leq j \leq k$ such that $I_j \neq \emptyset$. For each $j \in J_0$, let Z_j be the minimum interval (in $[M]$) which contains the set $\{x_j\} \cup \bigcup_{i \in I_j} V(S_i)$; namely, the left endpoint of Z_j is x_j , and the right endpoint of Z_j is the rightmost element of $\bigcup_{i \in I_j} V(S_i)$. See Figure 4 for an example. Recall that $L(S_i)$ denotes the set of leaves of S_i .

Claim 4.4. *There is $J \subseteq J_0$ such that $Z_j < Z_{j'}$ for every pair $j, j' \in J$ with $j < j'$, and*

$$\sum_{j \in J} \sum_{i \in I_j} |L(S_i)| \geq \frac{1}{2} (|L(S_1)| + \dots + |L(S_\ell)|) \geq \frac{M}{2^{q-1}} - n. \quad (1)$$

Proof. Let H be the interval graph of the intervals Z_j , $j \in J_0$; namely, $V(H) = J_0$ and j, j' are adjacent if $Z_j, Z_{j'}$ intersect. We claim that H is triangle-free. Indeed, suppose that $j < j' < j''$ make a triangle in H . Since $Z_j, Z_{j''}$ intersect, there is $v \in \bigcup_{i \in I_j} V(S_i)$ with $v > x_{j''}$. Let $i \in I_j$ such that $v \in V(S_i)$. We know $y_i < x_{j'}$ because $j_i = j < j'$. Now, take an arbitrary $i' \in I_{j'}$ (the set $I_{j'}$ is non-empty because $j' \in J_0$). If $i' \leq i$, then $y_{i'} \leq y_i$, which means $j_{i'} \leq j_i = j < j'$. And if $i' > i$, then $S_i < S_{i'}$. As $v \in V(S_i)$, this implies that $y_{i'} > v > x_{j''}$, so $j_{i'} \geq j'' > j'$. In either case, $j_{i'} \neq j'$, contradicting the fact that $i' \in I_{j'}$. This proves our claim that H is triangle-free.

Interval graphs are perfect (see [8, Chapter 8]), so H is 2-colorable. Take $J \subseteq J_0$ to be the color class that maximizes $\sum_{j \in J} \sum_{i \in I_j} |L(S_i)|$. Then (1) holds. Also, for every pair $j, j' \in J$ with $j < j'$, we have $Z_j \cap Z_{j'} = \emptyset$ (because J is independent in H) and hence $Z_j < Z_{j'}$ (because $Z_j, Z_{j'}$ are intervals and $x_j \in Z_j$ is to the left of $x_{j'} \in Z_{j'}$). ■

We now complete the proof of the lemma. For each $j \in J$, form a balanced tree T_j by taking x_j as the root and attaching S_i as a subtree of the root for every $i \in I_j$. Then T_j is well-ordered and balanced of depth q , because the S_i 's are well-ordered and have depth $q - 1$. The children of x_j are all the y_i with $i \in I_j$, and we already saw that $\chi_0(x_j) \geq \chi_0(y_i)$ for each such y_i . Also, by the claim, we have $Z_j < Z_{j'}$ for all $j, j' \in J$ with $j < j'$. Hence, writing $J = \{j_1, \dots, j_m\}$, we have $V(T_{j_1}) < \dots < V(T_{j_m})$. Let F be the forest with components T_{j_1}, \dots, T_{j_m} . Then F is a $(\chi_0, \dots, \chi_{q-1})$ -forest; indeed, the requirement in Definition 4.2 holds for $d = 0$ (as we just saw), and also holds for $1 \leq d \leq q - 1$ because (S_1, \dots, S_ℓ) is a $(\chi_1, \dots, \chi_{q-1})$ -forest. Also, $|L(F)| \geq M/2^{q-1} - n$ by (1). This completes the proof. ■

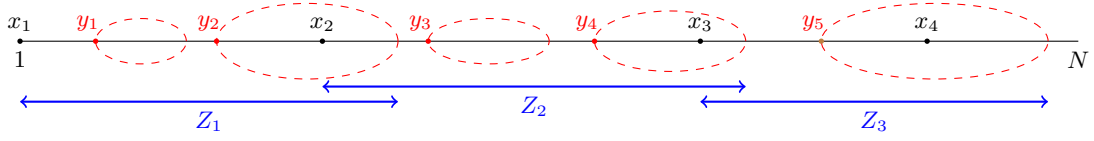


Figure 4: An example for the proof of [Lemma 4.3](#) where $q = 2$, $k = 4$ and $\ell = 5$. Every red circle stands for a component of F' rooted at some y_i . Also, $J_0 = \{1, 2, 3\}$, so Z_j is defined for $j = 1, 2, 3$.

4.2 Using s -red-nets

In this section, we show in [Lemma 4.10](#) that a large enough s -red-net implies the existence of a red K_{s+1} or a blue P_n^t . This is done via a stronger statement (see [Lemma 4.7](#)), which has the advantage of allowing an inductive proof. The following definition will play an important role.

Definition 4.5. Let $\mathcal{C} = (F, (X_v)_v)$ be an s -red-net. Let x be the root of the leftmost component (tree) of F and x' be the root of the rightmost component (tree) of F . The head of \mathcal{C} , denoted $\mathcal{H}(\mathcal{C})$, consists of the s vertices of F on the leftmost path from x to a leaf (i.e., a path that always goes to the leftmost child). Similarly, the tail of \mathcal{C} , denoted $\mathcal{T}(\mathcal{C})$, consists of the s vertices of F on the rightmost path from x' to a leaf.

As an example, if \mathcal{C} is as depicted in the top part of [Figure 2](#), then $\mathcal{H}(\mathcal{C}) = \{1, 2\}$ and $\mathcal{T}(\mathcal{C}) = \{5, 7\}$, and if \mathcal{C} is as depicted in the bottom part of [Figure 2](#), then $\mathcal{H}(\mathcal{C}) = \{1, 2, 3\}$ and $\mathcal{T}(\mathcal{C}) = \{9, 10, 11\}$.

Definition 4.6. For each subset $S = \{s_1, \dots, s_r\}$ of $[N]$, write $S^U := \{s_1, \dots, s_{r/3}\}$, $S^M := \{s_{r/3+1}, \dots, s_{2r/3}\}$, $S^D := \{s_{2r/3+1}, \dots, s_r\}$.¹

Our key lemma is the following.

Lemma 4.7. Let $s \geq 1$, $n \geq t \geq 1$ and $r \geq 3t$. Suppose $\mathcal{C} = (F, (X_v)_v)$ is an s -red-net of order r . Then, at least one of the following holds.

- (a) There exist $s + 1$ distinct vertices $v_1, \dots, v_{s+1} \in V(F)$, along with sets $A_i \subseteq X_{v_i} (\subseteq [N])$, $|A_i| \geq r/3$, such that for every $1 \leq i < j \leq s + 1$, there is no blue $K_{t,t}$ with one part in A_i and the other in A_j .
- (b) There exist s integers $\ell_1, \dots, \ell_s \geq t$ with $\sum_{i=1}^s \ell_i \geq |F|r/3$, and there exist vertex-disjoint blue copies P_1, \dots, P_s of $P_{\ell_1}^t, \dots, P_{\ell_s}^t$, respectively, such that the following holds: there exist two bijections $\sigma : [s] \rightarrow \mathcal{H}(\mathcal{C})$ and $\pi : [s] \rightarrow \mathcal{T}(\mathcal{C})$ such that for each $i \in [s]$, the first t vertices of P_i lie in $X_{\sigma(i)}^M$ and the last t vertices of P_i lie in $X_{\pi(i)}^M$.

Before proving [Lemma 4.7](#), let us sketch the proof when $s = 1, 2$, as these cases are easy to describe and already contain the main ideas. We will also explain how [Lemma 4.7](#) is used to show that there exists a red K_{s+1} or a blue P_n^t .

In the case $s = 1$, the forest F consists of isolated vertices, say $x_1 < \dots < x_{|F|}$. The key observation is that if for every $1 \leq i \leq |F| - 1$, there is a blue $K_{t,t}$ with parts $L_i \subseteq X_{x_i}^D$ and $R_i \subseteq X_{x_{i+1}}^U$, then we can construct a copy of P_ℓ^t (for some ℓ) from these $K_{t,t}$'s by connecting R_i and L_{i+1} inside $X_{x_{i+1}}$, using that the sets X_{x_i} are all blue cliques; see [Figure 5](#). Note that this P_ℓ^t contains the middle part $X_{x_i}^M$ for every i , so $\ell \geq \frac{1}{3} \sum_i |X_{x_i}| = |F|r/3$. This case corresponds to Item (b) in the lemma.

¹To avoid floor and ceiling signs, we will assume that r is divisible by 3.

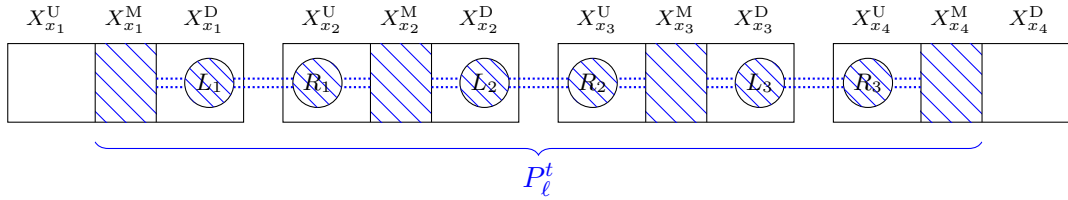


Figure 5: Proof of [Lemma 4.7](#) for $s = 1$: Every part with a blue shadow induces a blue clique, and every two parts connected by two blue dashed lines induce a blue complete bipartite graph.

On the other hand, if, for some $1 \leq i \leq |F| - 1$, there is no blue $K_{t,t}$ with one part in $X_{x_i}^D$ and one part in $X_{x_{i+1}}^U$, then Item (a) in the lemma holds with $v_1 = x_i, v_2 = x_{i+1}$. Note that $K_{t,t}$ -free graphs are sparse, so here we get a bipartite graph which is very dense in red.

Now let us consider the case $s = 2$. In this case, the trees in F have depth 1; namely, each tree consists of a root and leaves connected to the root. Let $x_1 < \dots < x_m$ be the roots, and let Y_k be the set of children of x_k . For each $1 \leq k \leq m$, we would like to apply the case $s = 1$ to the set of vertices Y_k . If Item (a) in the lemma holds, then we have vertices $v_1, v_2 \in Y_k$ and sets $A_i \subseteq X_{v_i}$ with no blue $K_{t,t}$ between A_1, A_2 . Together with X_{x_k} , this gives three sets with no blue $K_{t,t}$ between any two, meaning that Item (a) in the lemma holds (for $s = 2$). (Here we use the definition of an s -red-net, which implies that there is no blue $K_{t,t}$ between X_{x_k} and X_y for any $y \in Y_k$.)

So we may assume that when applying [Lemma 4.7](#) to Y_k (with $s = 1$), Item (b) holds. This gives a copy P_k of P_ℓ^t inside $\bigcup_{y \in Y_k} X_y$, where $\ell \geq |Y_k|/r/3$. We now take another “trivial” t ’th-power of a path which just consists of all the vertices in $X_{x_k}^M$; let us denote it by P'_k , so $|P'_k| = |X_{x_k}^M|$. Our goal now is to connect the ends of P_k and P'_k to the beginnings of P_{k+1} and P'_{k+1} . This connection forms a matching, i.e., we want to connect P_k to P_{k+1} and P'_k to P'_{k+1} , or P_k to P'_{k+1} and P'_k to P_{k+1} ; see [Figure 6](#). Doing this for every $1 \leq k \leq m - 1$ gives disjoint blue copies of $P_{\ell_1}^t, P_{\ell_2}^t$ (for some ℓ_1, ℓ_2) which together cover X_x^M for every $x \in V(F)$. This corresponds to Item (b) in the lemma. As $P_{\ell_1}^t, P_{\ell_2}^t$ together cover many vertices, one of them must be long, and this will give the desired blue P_n^t .

To achieve the aforementioned connection for a specific $1 \leq k \leq m - 1$, we do as follows. Let u_k be the rightmost vertex in $Y_k \subseteq V(F)$ and let v_{k+1} be the leftmost vertex in $Y_{k+1} \subseteq V(F)$; then the last t vertices of P_k are in $X_{u_k}^M$, and the first t vertices of P_{k+1} are in $X_{v_{k+1}}^M$. We now define an auxiliary bipartite graph with parts $\{x_k, u_k\}$ and $\{x_{k+1}, v_{k+1}\}$, where $u \in \{x_k, u_k\}$ is connected to $v \in \{x_{k+1}, v_{k+1}\}$ if there is a blue $K_{t,t}$ with one part in X_u^D and one part in X_v^U . A perfect matching in this bipartite graph gives the desired connection (see [Figure 6](#)). On the other hand, if there is no perfect matching, then there is an isolated vertex. So suppose, for example, that x_{k+1} is adjacent to neither x_k nor u_k . Then $X_{x_k}^D, X_{u_k}^D, X_{x_{k+1}}^U$ are three sets such that between any two there is no blue $K_{t,t}$ (recall that there is no blue $K_{t,t}$ between X_{x_k} and X_{u_k} by the definition of an s -red-net). So Item (a) in the lemma holds. Finally, note that a bipartite graph with no blue $K_{t,t}$ is sparse in blue, and hence very dense in red. Thus, a (large enough) tripartite graph with no blue $K_{t,t}$ between any two parts must contain a red triangle (we will argue this in [Lemma 4.10](#)). We now proceed with the full proof of [Lemma 4.7](#).

Proof of [Lemma 4.7](#). The proof is by induction on s . First, for the base case $s = 1$, F consists of $|F|$ isolated vertices $x_1 < \dots < x_{|F|}$. Note that $\mathcal{H}(\mathcal{C}) = \{x_1\}$ and $\mathcal{T}(\mathcal{C}) = \{x_{|F|}\}$. If, for some $1 \leq i \leq |F| - 1$, there is no blue $K_{t,t}$ with one part in $X_{x_i}^D$ and the other in $X_{x_{i+1}}^U$, then (a) is satisfied with $v_1 := x_i$ and $v_2 := x_{i+1}$ with $A_1 := X_{x_i}^D$ and $A_2 := X_{x_{i+1}}^U$. Otherwise, for every $1 \leq i \leq |F| - 1$,

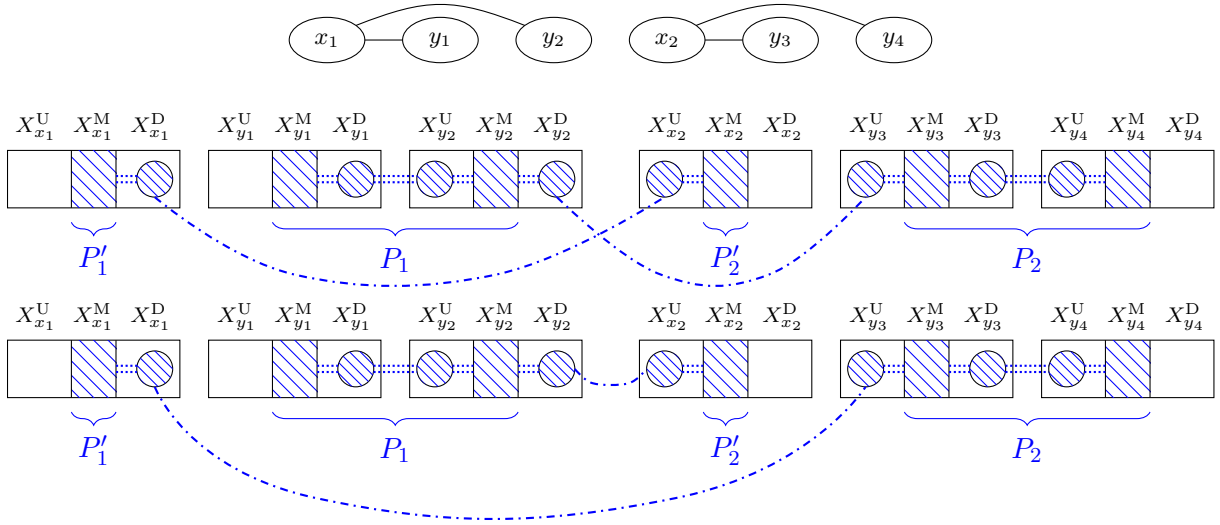


Figure 6: Proof of Lemma 4.7 for $s = 2$: The top picture shows two trees (of depth one) which we want to connect. The middle and bottom show the two ways of connecting P_1, P'_1 to P_2, P'_2 (the connections are the blue dotted curves). Every part with a blue shadow induces a blue clique and every two parts connected by a single blue dash-dotted line induce a blue complete bipartite graph.

there exists a blue $K_{t,t}$ with one part $L_i \subset X_{x_i}^D$ and the other part $R_i \subset X_{x_{i+1}}^U$. As each X_{x_i} forms a blue clique in K_N , we can connect these $|F| - 1$ blue $K_{t,t}$'s by

$$X_{x_1}^M \rightarrow L_1 \rightarrow R_1 \rightarrow X_{x_2}^M \rightarrow L_2 \rightarrow R_2 \rightarrow X_{x_3}^M \rightarrow \cdots \rightarrow X_{x_{|F|-1}}^M \rightarrow L_{|F|-1} \rightarrow R_{|F|-1} \rightarrow X_{x_{|F|}}^M,$$

which forms a blue P_ℓ^t , for $\ell \geq |F|r/3$, with first t vertices in $X_{x_1}^M$ and last t vertices in $X_{x_{|F|}}^M$. Thus, (b) is satisfied. This completes the proof of the base case $s = 1$.

For the inductive step, let $s \geq 2$, and suppose that the lemma holds for $s - 1$. We assume that (a) does not hold and show that (b) must hold. Let $x_1 < \cdots < x_m$ be the roots of the components (trees) in F . For each $k \in [m]$, let F_k denote the forest obtained from the tree rooted at x_k by deleting x_k (so the components of F_k are the trees rooted at the children of x_k). Then F_k is a well-ordered balanced forest of depth $s - 2$. Let \mathcal{C}_k denote the $(s - 1)$ -red-net $(F_k, (X_v)_{v \in F_k})$. Note that $\mathcal{H}(\mathcal{C}) = \{x_1\} \cup \mathcal{H}(\mathcal{C}_1)$ and $\mathcal{T}(\mathcal{C}) = \{x_m\} \cup \mathcal{T}(\mathcal{C}_m)$. We will apply the induction hypothesis to \mathcal{C}_k with $s - 1$ in place of s . We now show that Item (b) of the lemma must hold:

Claim 4.8. *For each $k \in [m]$, (b) holds for \mathcal{C}_k in terms of $s - 1$.*

Proof. Suppose not. Then by the induction hypothesis for \mathcal{C}_k (with $s - 1$ in place of s), (a) must hold for \mathcal{C}_k , i.e. there exist $v_1, \dots, v_s \in V(F_k)$ and A_1, \dots, A_s with $A_i \subset X_{v_i}$ and $|A_i| \geq |X_{v_i}|/3$, such that for every $1 \leq i < j \leq s$, there is no blue $K_{t,t}$ with one part in A_i and the other in A_j . Note that for every $1 \leq i \leq s$, there is no blue $K_{t,t}$ with one part in X_{x_k} and the other in A_i , by the definition of an s -red-net (as v_i is a descendant of x_k). So the $s + 1$ vertices x_k, v_1, \dots, v_s , along with the sets X_{x_k}, A_1, \dots, A_s , satisfy (a), contradicting our assumption that (a) does not hold for \mathcal{C} . ■

By the above claim, for each $k \in [m]$, there exist integers $\ell_{k,1}, \dots, \ell_{k,s-1} \geq t$ with $\sum_{i=1}^{s-1} \ell_{k,i} \geq |F_k|r/3$, there exist vertex-disjoint blue copies $P_{k,1}, \dots, P_{k,s-1}$ of $P_{\ell_{k,1}}^t, \dots, P_{\ell_{k,s-1}}^t$, respectively, and

there exist two bijections $\sigma_k : [s-1] \rightarrow \mathcal{H}(\mathcal{C}_k)$ and $\pi_k : [s-1] \rightarrow \mathcal{T}(\mathcal{C}_k)$ such that for each $i \in [s-1]$, the first t vertices of $P_{k,i}$ lie in $X_{\sigma_k(i)}^M$ and the last t vertices of $P_{k,i}$ lie in $X_{\pi_k(i)}^M$.

We now add an additional blue copy of $P_{\ell_{k,s}}^t$ with $\ell_{k,s} := |X_{x_k}^M| = r/3 \geq t$, as follows: let $P_{k,s}$ be the copy of $P_{\ell_{k,s}}^t$ on all the vertices of $X_{x_k}^M$ (recall that X_{x_k} induces a blue clique by [Definition 4.1](#)). It will be convenient to set $\mathcal{H}'(\mathcal{C}_k) := \{x_k\} \cup \mathcal{H}(\mathcal{C}_k)$ and $\mathcal{T}'(\mathcal{C}_k) := \{x_k\} \cup \mathcal{T}(\mathcal{C}_k)$. Note that $\mathcal{H}(\mathcal{C}) = \mathcal{H}'(\mathcal{C}_1)$ and $\mathcal{T}(\mathcal{C}) = \mathcal{T}'(\mathcal{C}_m)$. Also, with a slight abuse of notation, we extend σ_k and π_k by setting $\sigma_k(s) = x_k$ and $\pi_k(s) = x_k$, to get bijections $\sigma_k : [s] \rightarrow \mathcal{H}'(\mathcal{C}_k)$ and $\pi_k : [s] \rightarrow \mathcal{T}'(\mathcal{C}_k)$. Note that $\sum_{i=1}^s \ell_{k,i} \geq (|F_k| + 1)r/3$ and that $P_{k,1}, \dots, P_{k,s}$ are vertex-disjoint blue copies of $P_{\ell_{k,1}}^t, \dots, P_{\ell_{k,s}}^t$, respectively, such that for each $i \in [s]$, the first t vertices of $P_{k,i}$ are in $X_{\sigma_k(i)}^M$ and the last t vertices of $P_{k,i}$ are in $X_{\pi_k(i)}^M$.

To prove that Item (b) holds, we need to “connect” $P_{k,1}, \dots, P_{k,s}$ to $P_{k+1,1}, \dots, P_{k+1,s}$ for every $1 \leq k \leq m-1$. More precisely, we will prove the following by induction on k :

- (b') For every $1 \leq k \leq m$, there exist integers $\ell_1, \dots, \ell_s \geq t$ with $\sum_{i=1}^s \ell_i \geq \sum_{j=1}^k (|F_j| + 1)r/3$, and there exist vertex-disjoint copies P_1, \dots, P_s of $P_{\ell_1}^t, \dots, P_{\ell_s}^t$, respectively, such that the following holds: there exist two bijections $\sigma : [s] \rightarrow \mathcal{H}'(\mathcal{C}_1)$ and $\pi : [s] \rightarrow \mathcal{T}'(\mathcal{C}_k)$ such that for each $i \in [s]$, the first t vertices of P_i lie in $X_{\sigma(i)}^M$ and the last t vertices of P_i lie in $X_{\pi(i)}^M$.

Note that $\sum_{j=1}^m (|F_j| + 1) = |F|$. Hence, by setting $k = m$ in (b'), we get (b).

So it remains to prove (b'). In the base case $k = 1$, we take P_1, \dots, P_s to be $P_{1,1}, \dots, P_{1,s}$ and $\sigma := \sigma_1, \pi := \pi_1$. For the inductive step, suppose $k \geq 2$ and we have found P_1, \dots, P_s and σ, π satisfying (b') for $k-1$. Our goal is to extend P_1, \dots, P_s by $P_{k,1}, \dots, P_{k,s}$ to obtain P'_1, \dots, P'_s satisfying (b') for k . We will also define appropriate σ', π' . To this end, we define an auxiliary bipartite graph H with sides $\mathcal{T}'(\mathcal{C}_{k-1})$ and $\mathcal{H}'(\mathcal{C}_k)$ such that $u \in \mathcal{T}'(\mathcal{C}_{k-1})$ is adjacent to $v \in \mathcal{H}'(\mathcal{C}_k)$ in H if there exists a blue $K_{t,t}$ with one part in X_u^D and the other in X_v^U .

Claim 4.9. *H has a perfect matching.*

Proof. Suppose not. By Hall's marriage theorem, there exists a subset $S \subseteq \mathcal{T}'(\mathcal{C}_{k-1})$ with $|N_H(S)| < |S|$, where $N_H(S)$ is the neighborhood of S in H . Take $T = \mathcal{H}'(\mathcal{C}_k) \setminus N_H(S)$. Then there is no edge in H between S and T . Hence, for each $u \in S$ and each $v \in T$, there is no blue $K_{t,t}$ with one part in X_u^D and the other in X_v^U . Observe also that for distinct $u, v \in \mathcal{T}'(\mathcal{C}_{k-1})$ or $u, v \in \mathcal{H}'(\mathcal{C}_k)$, there is no blue $K_{t,t}$ with one part in X_u and the other in X_v . This follows from [Definition 4.1](#), since $\mathcal{T}'(\mathcal{C}_{k-1})$ and $\mathcal{H}'(\mathcal{C}_k)$ are paths from a root to a leaf in F , and so, for every two $u, v \in \mathcal{T}'(\mathcal{C}_{k-1})$ or $u, v \in \mathcal{H}'(\mathcal{C}_k)$, it holds that u is a descendant of v or vice versa. Also, note that $|S| + |T| = |S| + s - |N_H(S)| \geq s + 1$. So we see that the (at least $s+1$) vertices in $S \cup T$, along with the sets $(X_u^D)_{u \in S}$ and $(X_v^U)_{v \in T}$, satisfy Item (a) in the lemma. This contradicts our assumption that (a) does not hold for \mathcal{C} . \blacksquare

By the above claim, there exists a bijection $\tau : \mathcal{T}'(\mathcal{C}_{k-1}) \rightarrow \mathcal{H}'(\mathcal{C}_k)$ that specifies the perfect matching of H . Fix $i \in [s]$. Due to (b') for P_1, \dots, P_s (with $k-1$ in place of k), the last t vertices of P_i lie in X_u^M for $u = \pi(i) \in \mathcal{T}'(\mathcal{C}_{k-1})$ (and this is the only u with this property). Set $v := \tau(u) \in \mathcal{H}'(\mathcal{C}_k)$. We know that the first t vertices of $P_{k,j}$ lie in X_v^M for $j = \sigma_k^{-1}(v)$ (and this is the only j with this property). We have $x_{k-1} < x_k$, and hence $V(F_{k-1}) < V(F_k)$, as F is well-ordered. Therefore, $u < v$, since $u \in V(F_{k-1}) \cup \{x_{k-1}\}$ and $v \in V(F_k) \cup \{x_k\}$. Now, by [Definition 4.1](#), we have $X_u < X_v$. As $(u, v) \in E(H)$, there is some blue $K_{t,t}$ with parts L and R such that $L \subseteq X_u^D$ and $R \subseteq X_v^U$. Also, both X_u and X_v are blue cliques. Thus, we can extend P_i by $P_i \rightarrow L \rightarrow R \rightarrow P_{k,j}$,

giving a blue copy of $P_{\ell'_i}^t$ with $\ell'_i > \ell_i + \ell_{k,j}$. We denote this copy by P'_i . Doing the above for all $i \in [s]$, we obtain blue copies P'_1, \dots, P'_s of $P_{\ell'_1}^t, \dots, P_{\ell'_s}^t$, respectively, where

$$\sum_{i=1}^s \ell'_i \geq \sum_{i=1}^s \ell_i + \sum_{i=1}^s \ell_{k,i} \geq \sum_{j=1}^{k-1} (|F_j| + 1)r/3 + (|F_k| + 1)r/3 = \sum_{j=1}^k (|F_j| + 1)r/3.$$

As τ specifies a perfect matching in H , P'_1, \dots, P'_s are vertex-disjoint. Clearly, for every $1 \leq i \leq s$, the first t vertices of P'_i are the same as those of P_i , and thus lie in $X_{\sigma(i)}^M$. The last t vertices of P'_i are the same as those of $P_{k,j}$ for $j = \sigma_k^{-1}(\tau(\pi(i)))$, and thus lie in $X_{\pi_k(j)}^M$. Then P'_1, \dots, P'_s , $\sigma' := \sigma$ and $\pi' := \pi_k \circ \sigma_k^{-1} \circ \tau \circ \pi$ satisfy (b') for k . This completes the inductive step, thus proving the lemma. ■

Lemma 4.10. *Let $s \geq 1$, $n \geq t \geq 1$. Suppose $\mathcal{C} = (F, (X_v)_v)$ is an s -red-net of order $r \geq 3(4s^2)^t$ with $|F| \geq 3sn/r$. Then, K_N contains a red K_{s+1} or a blue P_n^t .*

Proof. We apply Lemma 4.7 to \mathcal{C} , and consider the following two cases.

Case 1: (a) holds for \mathcal{C} , i.e. there exist distinct subsets $A_1, A_2, \dots, A_{s+1} \subseteq [N]$, each of size $a = r/3$, such that for all $1 \leq i < j \leq s+1$, there is no blue $K_{t,t}$ with one part in A_i and the other in A_j . Fix any $1 \leq i < j \leq s+1$. Let $z(a \times a, K_{t,t})$ denote the maximum number of edges in a $K_{t,t}$ -free $a \times a$ bipartite graph. By the Kővari-Sós-Turán theorem [9], we have the following bound for the number of blue edges between A_i and A_j :

$$e_{\text{blue}}(A_i, A_j) \leq z(a \times a, K_{t,t}) < t^{1/t} a^{2-1/t} + ta \leq a^2 / \binom{s+1}{2}.$$

In the last inequality, we used the fact that $a = r/3 \geq (4s^2)^t \geq (2s^2)^{2t}$. Now, sample vertices $v_1 \in A_1, \dots, v_{s+1} \in A_{s+1}$ independently and uniformly at random. For every $1 \leq i < j \leq s+1$, the probability that (v_i, v_j) is blue is smaller than $1/\binom{s+1}{2}$. Taking a union bound over all i, j , we get that with positive probability, v_1, \dots, v_{s+1} form a red K_{s+1} .

Case 2: (b) holds for \mathcal{C} , i.e. there exist P_1, \dots, P_s such that for each $i \in [s]$, P_i is a blue copy of $P_{\ell_i}^t$ for some $\ell_i \geq t$, and $\sum_{i=1}^s \ell_i \geq |F|r/3$. By averaging, there is $i \in [s]$ such that $\ell_i \geq |F|r/(3s) \geq n$. Thus, P_i contains a blue copy of P_n^t . ■

4.3 Putting it all together

Proof of Theorem 1.1. Set $r = 3(4s^2)^t$, $M = 2^{s-1}n$, and $N = M \cdot R(K_{s+1}, K_{sr})$. Note that $R(K_{s+1}, K_m) \leq \binom{m+s-1}{s} \leq m^s$ (by the Erdős-Szekeres bound [6]). Hence, $N \leq 2^{s-1}n \cdot (rs)^s \leq (24s^3)^{st}n$. Fix a red/blue edge-coloring of K_N and suppose by contradiction that there is no red K_{s+1} and no blue P_n^t . Then every $R(K_{s+1}, K_{sr})$ vertices contain a blue clique of size sr . As $N = M \cdot R(K_{s+1}, K_{sr})$, this means that we can find blue cliques $V_1 < \dots < V_M$ of size sr each. Partition $V_i = X_i^{(0)} \cup \dots \cup X_i^{(s-1)}$ with $|X_i^{(j)}| = r$ for all $0 \leq j \leq s-1$, and $X_i^{(s-1)} < X_i^{(s-2)} < \dots < X_i^{(0)}$.

For each $1 \leq i \leq M$ and $0 \leq j \leq s-2$, let $\chi_j(i)$ be the maximum ℓ such that there exists a blue copy of P_ℓ^t whose last t vertices belong to $X_i^{(j+1)} \cup \dots \cup X_i^{(s-1)}$. By definition, for every $1 \leq i \leq M$,

$$\chi_0(i) \geq \chi_1(i) \geq \dots \geq \chi_{s-2}(i) \geq t. \quad (2)$$

Also, $\chi_j(i) < n$ for all j because there is no blue P_n^t . By Lemma 4.3 (with $q = s-1$), there is a $(\chi_0, \dots, \chi_{s-2})$ -forest F with $|L(F)| \geq M/2^{s-2} - n \geq n$. The following is the key property we need:

Claim 4.11. *Let $0 \leq d < d' \leq s - 1$, let $a \in F$ at depth d and let a' be a descendant of a at depth d' . Then there is no blue $K_{t,t}$ with one part in $X_a^{(d)}$ and the other part in $X_{a'}^{(d')}$.*

Proof. Let $a = b_0, b_1, \dots, b_{d'-d} = a'$ be the unique path from a to a' in F . Then b_i is at depth $d + i$. We have $\chi_{d+i}(b_i) \geq \chi_{d+i}(b_{i+1})$ for every $0 \leq i < d' - d$, by the definition of a $(\chi_0, \dots, \chi_{s-2})$ -forest (Definition 4.2). Also, $\chi_{d+i}(b_{i+1}) \geq \chi_{d+i+1}(b_{i+1})$ for every $0 \leq i \leq d' - d - 2$, by (2). So $\chi_{d+i}(b_i) \geq \chi_{d+i+1}(b_{i+1})$ for $0 \leq i \leq d' - d - 2$, meaning that the sequence $(\chi_{d+i}(b_i))_{i=0}^{d'-d-1}$ is non-increasing. Setting $i = 0$ and $i = d' - d - 1$, we get $\chi_d(a) = \chi_d(b_0) \geq \chi_{d'-1}(b_{d'-d-1}) \geq \chi_{d'-1}(b_{d'-d}) = \chi_{d'-1}(a')$, where the last inequality again uses the definition of a $(\chi_0, \dots, \chi_{s-2})$ -forest.

Now suppose by contradiction that there is a blue $K_{t,t}$ with sides $A \subseteq X_a^{(d)}$ and $A' \subseteq X_{a'}^{(d')}$. By the definition of $\chi_d(\cdot)$, there exists a blue copy P of $P_{\chi_d(a)}^t$ whose last t vertices belong to $X_a^{(d+1)} \cup \dots \cup X_a^{(s-1)}$. We can extend P by adding the sets A and A' , using that both A and A' induce blue cliques, and all edges between A and the last t vertices of P are blue (because $V_a, V_{a'}$ are blue cliques). Extending P in this way gives a blue copy Q of $P_{\chi_d(a)+2t}^t$, whose last t vertices are $A' \subseteq X_{a'}^{(d')}$. Hence, $\chi_{d'-1}(a') \geq \chi_d(a) + 2t > \chi_d(a) \geq \chi_{d'-1}(a')$, a contradiction to $\chi_d(a) \geq \chi_{d'-1}(a')$. ■

We now use Claim 4.11 to find an s -red-net. For every $0 \leq d \leq s - 1$ and every $v \in V(F)$ at depth d in F , define $X_v := X_v^{(d)} \subseteq V_v$. Then X_v is a blue clique of size r . For every $v, u \in V(F)$ with $v < u$, we have $V_v < V_u$ and hence $X_v < X_u$. Also, by Claim 4.11, for every $v \in V(F)$ and every descendant u of v , there is no blue $K_{t,t}$ with one part in X_v and one part in X_u . Hence, $(F, (X_v)_v)$ is an s -red-net of order r . Also, $|F| \geq n \geq 3sn/r$. By Lemma 4.10, there is a red K_{s+1} or a blue P_n^t . This completes the proof. ■

5 Concluding remarks and open problems

In Theorem 1.1, we showed that $R_{<}(K_{s+1}, P_n^t) \leq s^{O(st)}n$. As for lower bounds, it holds that $R_{<}(K_{s+1}, P_n^t) > (R(K_{s+1}, K_{t+1}) - 1) \cdot (n - 1)/t$. Indeed, partition the vertices into $(n - 1)/t$ intervals of size $R(K_{s+1}, K_{t+1}) - 1$, and on each of the intervals, put a red/blue coloring with no red K_{s+1} and no blue K_{t+1} . All edges between the intervals are blue. Then there is no red K_{s+1} and no blue P_n^t , because a blue P_n^t would have to contain $t + 1$ vertices from one of the intervals which appear consecutively in the P_n^t , and hence must form a blue K_{t+1} . Combining our upper and lower bounds, we obtain $s^{\Omega(t)} \cdot n \leq R_{<}(K_{s+1}, P_n^t) \leq s^{O(st)} \cdot n$ (when say $t \ll s$). It would be interesting to determine the correct dependence of the exponent on s and t .

In Theorem 1.2 we proved that $R_{<}(P_n^t, K_n) = O_t(n^{t(2t-1)})$. It would be interesting to improve the exponent further.

Conjecture 5.1. $R_{<}(P_n^t, K_n) \leq n^{O(t)}$.

This conjecture, if true, would be tight (up to the implied constant in the exponent) because $R_{<}(P_n^t, K_n) \geq R(K_{t+1}, K_n) \geq \tilde{\Omega}(n^{(t+2)/2})$, see [2, 18]. We note that our proof method for Theorem 1.2 can be used to prove $R_{<}(P_n[t], K_n) \leq n^{O(t)}$, where $P_n[t]$ is the t -blowup of the monotone path P_n (i.e., $P_n[t]$ is obtained by replacing each vertex of P_n with t vertices and replacing edges with complete bipartite graphs). This bound is tight because $R_{<}(P_n[t], K_n) \geq R(K_{t,t}, K_n) \geq n^{\Omega(t)}$.

Proposition 5.2. $R_{<}(P_n[t], K_n) \leq (2tn^3)^{2t-1}$.

Proof sketch. The proof is similar to that of [Theorem 1.2](#). A *semi-red t -clique* is a clique $x_1 < \dots < x_t$ such that all edges x_1x_i ($2 \leq i \leq t$) and x_ix_t ($1 \leq i \leq t-1$) are red. It can be shown that every set of $N := (t-2)n^2$ vertices contains a blue K_n or semi-red t -clique. Indeed, if there is no vertex with forward red degree at least $(t-2)n$, then one can greedily find a blue clique of size $\frac{N}{(t-2)n} \geq n$. Thus, there is a vertex x_1 with forward red degree at least $(t-2)n$. By the same argument inside the forward red neighbourhood of x_1 , we can find a vertex x_t with at least $t-2$ backward red neighbours x_2, \dots, x_{t-1} . Now $x_1 < x_2 < \dots < x_t$ is a semi-red t -clique.

One can then show, using the argument from [Lemma 3.2](#), that every set of $((t-2)n^2 - 1)n + 1$ vertices contains a blue K_n or a chain of n semi-red t -cliques. Then, we essentially repeat the proof of [Theorem 1.2](#): consider a red/blue coloring of K_N with $N = (2tn^3)^{2t-1}$. We assume that there is no red $P_n[t]$ or blue K_n . For a semi-red clique $x_1 < \dots < x_t$, define $\chi(x_1, \dots, x_t)$ to be the largest number ℓ such that x_1, \dots, x_t are the last t vertices of a red $P_\ell[t]$. Then $1 \leq \chi(x_1, \dots, x_t) \leq n-1$. Using the choice of N , one can obtain vertices $x_1 < \dots < x_{t-1} < z_1 < \dots < z_{t-1}$, a set Y with $|Y| \geq tn^2$, and a value $1 \leq c \leq n-1$, such that for every $y \in Y$, $\{x_1, \dots, x_{t-1}, y\}, \{y, z_1, \dots, z_{t-1}\}$ are semi-red, $\chi(x_1, \dots, x_{t-1}, y) = c$, and $\chi(y, z_1, \dots, z_{t-1}) \leq c$. Then, by finding a semi-red $(t+2)$ -clique $y_1 < \dots < y_{t+2}$ inside Y , we can extend a longest $P_c[t]$ ending at x_1, \dots, x_{t-1}, y_1 by adding the vertices $y_2, \dots, y_{t+1}, y_{t+2}, z_1, \dots, z_{t-1}$. This gives $\chi(y_{t+2}, z_1, \dots, z_{t-1}) > \chi(x_1, \dots, x_{t-1}, y_1)$, a contradiction. ■

In [Theorem 1.3](#) we obtain a new bound on $R_{<}(P_n^t, P_n^t)$, whose exponent grows linearly with t . Since we do not have a corresponding lower bound, we wonder whether the following might be true.

Problem 5.3. *Is there a constant C independent of t , such that $R_{<}(P_n^t, P_n^t) = O_t(n^C)$?*

Even improving the exponent to $o(t)$ would be interesting.

In [Theorem 2.4](#) we proved that $g(n, s) = O_s(n^C)$ for a constant $C = C(s)$. It may be interesting to determine the order of growth of (the optimal such) $C(s)$. Our proof of [Theorem 2.4](#) gives an upper bound on $C(s)$ of the order s^s , and this is likely far from optimal. Does $C(s)$ grow polynomially in s ? Also, for $s = 3$, is it true that $g(n, 3) = O(n)$?

One could also consider the analogous extremal functions for weaker notions of a non-increasing triple (see [Definition 2.1](#)). There are two such notions: one is to require that $\chi(x, y) \geq \chi(x, z) \geq \chi(y, z)$, and one is only to require that $\chi(x, y) \geq \chi(y, z)$. For each of these notions, how large should N be to guarantee a non-increasing set of size s in an n -coloring of K_N ?

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