Transition de phase du modèle d'Ising sur triangulations aléatoires

Linxiao Chen
University of Helsinki

travail en common avec Joonas Turunen

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Introduction

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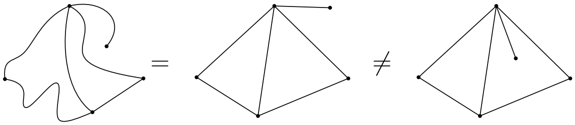

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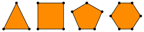
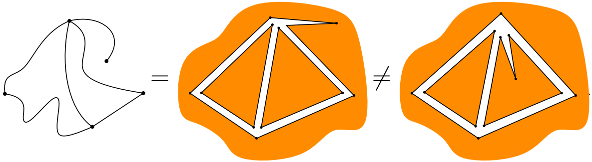
 : faces



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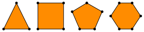
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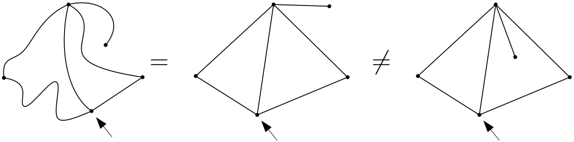

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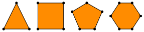

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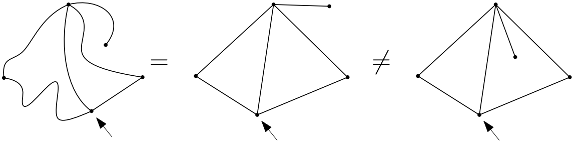

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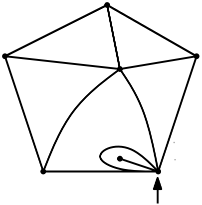

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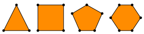

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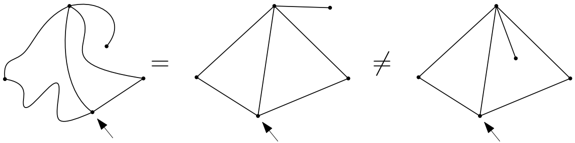

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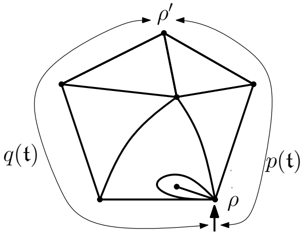

Let $\mathcal{T}_2$ be the set of triangulations of the disk, endowed with a partition of its boundary into 2 intervals. Denote by $p(\mathfrak{t})$ and $q(\mathfrak{t})$ the lengths of these intervals.

Ising-triangulation

Let $F(\mathfrak{t})$ denote the set of internal faces of a triangulation $\mathfrak{t}$. We consider the following set of Ising-decorated triangulations (or *Ising-triangulation* for short):

$$\mathcal{IT}_{+-} = \{(\mathfrak{t}, \sigma) \mid \mathfrak{t} \in \mathcal{T}_2 \text{ and } \sigma \in \{+, -\}^{F(\mathfrak{t})}\}$$

The elements of $\mathcal{IT}_{+-}$ are endowed with the Dobrushin boundary condition which assign the spin + (resp. –) outside the boundary interval of length $p(\mathfrak{t})$ (resp. $q(\mathfrak{t})$). Let $\mathcal{E}(\mathfrak{t}, \sigma)$ be the set of monochromatic edges in $(\mathfrak{t}, \sigma)$.

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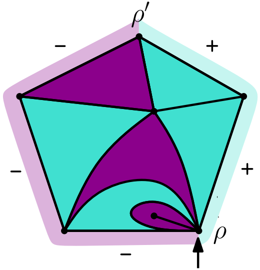
Example: $(\mathfrak{t}, \sigma) \in \mathcal{IT}_{+-}$



$$p(\mathfrak{t}) = 3, \quad q(\mathfrak{t}) = 2,$$
$$|F(\mathfrak{t})| = 7, \quad |\mathcal{E}(\mathfrak{t}, \sigma)| = 5.$$

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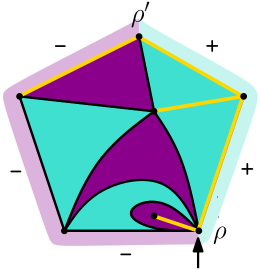

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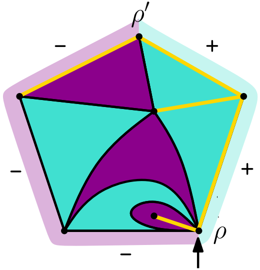

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Generating functions

$$Z(u, v; t, \nu) := \sum_{(\mathfrak{t},\sigma)\in\mathcal{IT}_{+-}} u^{p(\mathfrak{t})} v^{q(\mathfrak{t})} t^{|F(\mathfrak{t})|} \nu^{|\mathcal{E}(\mathfrak{t},\sigma)|}$$

$$Z_q(u; t, \nu) := [v^q] Z(u, v; t, \nu)$$

$$z_{p,q}(t, \nu) := [u^p v^q] Z(u, v; t, \nu) = [u^p] Z_q(u; t, \nu)$$

By convention $z_{0,0}(t, \nu) = Z(0, 0; t, \nu) = 1$.

Boltzmann-Ising triangulation

For all $p, q \geq 0$ such that $(p, q) \neq (0, 0)$, and $t, \nu > 0$ such that $z_{p,q}(t, \nu) < \infty$, we define a probability measure on the set $\{(\mathfrak{t}, \sigma) \in \mathcal{IT}_{+-} \mid p(\mathfrak{t}) = p \text{ and } q(\mathfrak{t}) = q\}$ by

$$\mathbb{P}^{t,\nu}_{p,q}(\mathfrak{t}, \sigma) = \frac{t^{|F(\mathfrak{t})|} \nu^{|\mathcal{E}(\mathfrak{t},\sigma)|}}{z_{p,q}(t, \nu)} .$$

We will call a random variable of law $\mathbb{P}^{t,\nu}_{p,q}$ *Boltzmann Ising-triangulation of* (p, q)*-gon*.

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Let $t_c(\nu)$ be the radius of convergence of the series $t \mapsto z_{1,0}(t, \nu)$.
Today we focus on the case $t = t_c(\nu)$ ("maximal volume") and $\nu > 1$ (ferromagnetic).

Previous results

Let $\nu_c = 1 + 2\sqrt{7}$.

Theorem (Bernardi-Bousquet-Mélou 11, Albenque-Laurent-Schaeffer 18)

For all $\nu > 1$ and $(p, q) \neq (0, 0)$, we have

$$[t^n]z_{p,q}(t, \nu) \underset{n\to\infty}{\sim} \begin{cases} \kappa_{p,q}(\nu) \cdot t_c(\nu)^{-n} \cdot n^{-5/2} & (\nu \neq \nu_c) \\ \kappa_{p,q}(\nu_c) \cdot t_c(\nu_c)^{-n} \cdot n^{-7/3} & (\nu = \nu_c) \end{cases}$$

Moreover, $t_c(\nu)$ is continuous on $(1, \infty)$ and analytic on $(1, \nu_c) \cup (\nu_c, \infty)$.

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Corollaries

- $t_c(\nu)$ is also the radius of convergence of $t \mapsto z_{p,q}(t,\nu)$ and $z_{p,q}(t_c(\nu),\nu) < \infty$, for all $\nu > 1$ and $(p,q) \neq (0,0)$.
- $-\lim_{n\to\infty} \frac{1}{n} \log[t^n]z_{p,q}(t,\nu) = \log t_c(\nu)$. Thus $\log t_c(\nu)$ is the free energy per volume.

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From now on, we assume that $t = t_c(\nu)$ and omit the parameter t from the notations.

Previous results (C.-Turunen, arXiv:1806.06668)

Theorem (Critical asymptotics)

In the limit where $p \to \infty$ and q is fixed:

$$z_{p,q}(\nu_c) \sim a_q(\nu_c) \cdot u_c(\nu_c)^{-p} \cdot p^{-7/3} \qquad \text{and} \qquad a_p(\nu_c) \sim b(\nu_c) \cdot u_c(\nu_c)^{-p} \cdot p^{-4/3}$$

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Theorem (Critical local limit)

One can construct probability distributions $(\mathbb{P}_p^{\nu_c})_{p\geq 0}$ and $\mathbb{P}_\infty^{\nu_c}$ supported on the set of infinite Ising-triangulations of the half plane, such that $\mathbb{P}_{p,q}^{\nu_c} \xrightarrow[q\to\infty]{} \mathbb{P}_p^{\nu_c} \xrightarrow[p\to\infty]{} \mathbb{P}_\infty^{\nu_c}$ weakly with respect to the local topology.

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Theorem (Scaling limit of interface length)

Let $L_p^{\nu_c}$ be the length of the left-most Ising interface from ρ to ρ' in a Boltzmann Ising-triangulation of law $\mathbb{P}_p$. Then $L_p^{\nu_c}/p$ converges in distribution to the random variable of density $\frac{1}{\mathcal{Z}}(1+\mu x)^{-7/3}\mathbb{1}_{x>0}$, where $\mu > 0$ is an absolute constant.

Main results

Off-critical and near-critical asymptotics

Let $u_c(\nu)$ be the radius of convergence of $Z_0(u;\nu)$.

Theorem (Continuous phase transition in the "surface tension")

$u_c(\nu)$ is positive and continuous on $(1,\infty)$, and is analytic everywhere except at ν_c.

Theorem (Off-critical asymptotics)

Fix $\nu \neq \nu_c$. In the limit where $p \to \infty$ and q is fixed:

$$z_{p,q}(\nu) \sim a_q(\nu) \cdot u_c(\nu)^{-p} \cdot p^{-5/2} \quad , \quad a_p(\nu) \sim \begin{cases} b(\nu) \cdot u_c(\nu)^{-p} \cdot p^{-5/2} & (\nu < \nu_c) \\ b(\nu) \cdot u_c(\nu)^{-p} & (\nu > \nu_c) \end{cases}$$

In the limit where $p, q \to \infty$ and $q/p \to \lambda$ for some fixed $\lambda \in (0,\infty)$:

$$z_{p,q}(\nu) \sim \begin{cases} c(\lambda;\nu) \cdot u_c(\nu)^{-(p+q)} \cdot p^{-5/2} & (\nu < \nu_c) \\ c(\lambda;\nu) \cdot u_c(\nu)^{-(p+q)} \cdot p^{-5} & (\nu > \nu_c) \end{cases}$$

Remark: in both of the two limits above, we have $-\lim \frac{1}{p+q} \log\left(z_{p,q}(\nu)\right) = \log u_c(\nu)$.

Theorem (Near-critical asymptotics)

Fix $\beta \in (0, \infty]$ and assume $|\nu - \nu_c| = p^{-\beta}$. In the limit where $p \to \infty$ and q fixed:

$$z_{p,q}(\nu) \sim \tilde{a}_q(\beta) \cdot u_c(\nu)^{-p} \cdot p^{-\alpha_0(\beta)} \qquad \text{and} \qquad a_p(\nu) \sim \tilde{b}(\beta) \cdot u_c(\nu)^{-p} \cdot p^{-\alpha_1(\beta)}$$

When $p, q \to \infty$ and $q/p \to \lambda$ for some fixed $\lambda \in (0, \infty)$:

$$z_{p,q}(\nu) \sim \tilde{c}(\lambda; \beta) \cdot u_c(\nu)^{-(p+q)} \cdot p^{-\alpha(\beta)}$$

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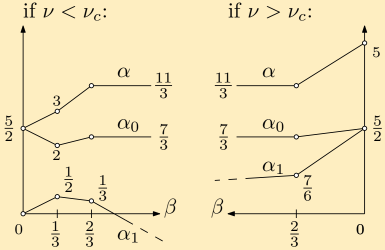

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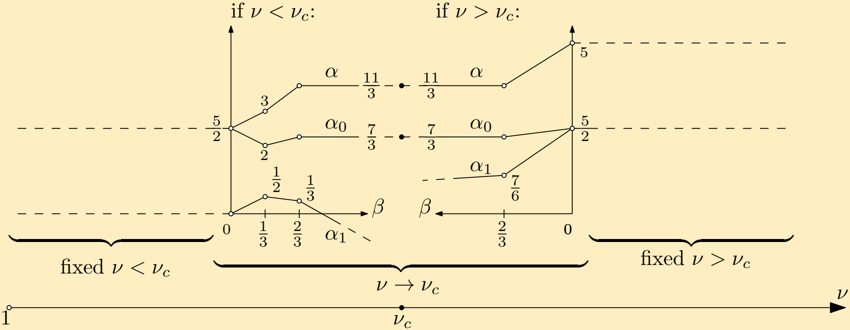

Local limit

The *local distance* between two Ising-triangulations $(\mathfrak{t}, \sigma)$ and $(\mathfrak{t}', \sigma')$ is defined by

$$d_{\texttt{loc}}((\mathfrak{t}, \sigma), (\mathfrak{t}', \sigma')) = 2^{-\sup\{r \in \mathbb{N} :\ B_r(\mathfrak{t},\sigma) = B_r(\mathfrak{t}',\sigma')\}}$$

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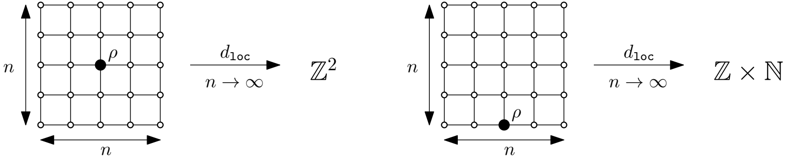

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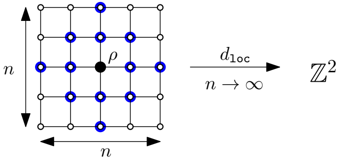

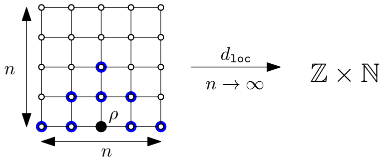

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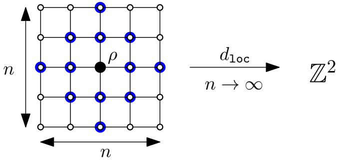
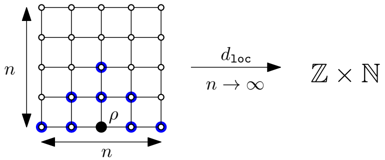
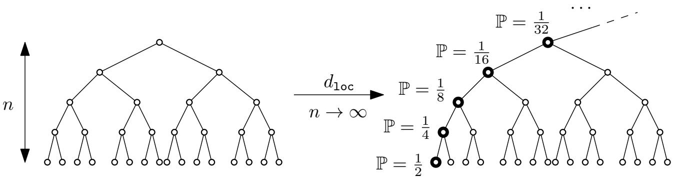




G_n = complete binary tree of height n,
ρ_n = a vertex chosen uniformly in the tree.



Local limit

Theorem (Critical and off-critical local limits)

For each $\nu > 1$*, one can construct probability distributions* $(\mathbb{P}_p^\nu)_{p\geq 0}$ *and* $\mathbb{P}_\infty^\nu$ *such that* $\mathbb{P}_{p,q}^\nu \xrightarrow[q\to\infty]{} \mathbb{P}_p^\nu \xrightarrow[p\to\infty]{} \mathbb{P}_\infty^\nu$ *weakly with respect to the local distance.*

In the limit $p, q \to \infty$ *and* $q/p \to \lambda \in (0, \infty)$*, the convergence becomes* $\mathbb{P}_{p,q}^\nu \to \mathbb{P}_\infty^\nu$.

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For each $\nu > 1$, one can construct probability distributions $(\mathbb{P}_p^\nu)_{p\geq 0}$ and $\mathbb{P}_\infty^\nu$ such that $\mathbb{P}_{p,q}^\nu \xrightarrow[q\to\infty]{} \mathbb{P}_p^\nu \xrightarrow[p\to\infty]{} \mathbb{P}_\infty^\nu$ weakly with respect to the local distance.
In the limit $p, q \to \infty$ and $q/p \to \lambda \in (0, \infty)$, the convergence becomes $\mathbb{P}_{p,q}^\nu \to \mathbb{P}_\infty^\nu$.
Moreover

- *For all $\nu > 1$ and $p \geq 0$, $\mathbb{P}_p^\nu$ is supported on the set of one-ended triangulations with one infinite boundary (i.e. triangulations of the half plane).*
- *For $\nu \geq \nu_c$, the distribution $\mathbb{P}_\infty^\nu$ is also supported on the above set.*
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Theorem (Near-critical local limit)

When $\nu \to \nu_c$ at the same time as $p, q \to \infty$, we have $\mathbb{P}_{p,q}^\nu \xrightarrow[q\to\infty]{} \mathbb{P}_p^{\nu_c}$,
$\mathbb{P}_p^\nu \xrightarrow[p\to\infty]{} \mathbb{P}_\infty^{\nu_c}$ and $\mathbb{P}_{p,q}^\nu \xrightarrow[p,q\to\infty]{} \mathbb{P}_\infty^{\nu_c}$ weakly with respect to the local distance.

Scaling limit of the main interface (work in progress)

Let $L^{\nu}_{p,q}$ be the length* of the left-most Ising interface going from ρ to ρ' in a Boltzmann Ising-triangulation of law $\mathbb{P}^{\nu}_{p,q}$.
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Conjecture (off-critical and critical limit)

Fix $\nu > 1$ and $\lambda \in (0,\infty)$. In the limit $p,q \to \infty$ and $q/p \to \lambda$, the random variable $L_{p,q}^{\nu}/p$ converges in law to 0 if $\nu > \nu_c$, to a deterministic value $\ell(\lambda;\nu) > 0$ if $\nu < \nu_c$, and to the random variable of density $\frac{1}{Z}(1+\mu x)^{-7/3}(\lambda+\mu x)^{-7/3}\mathbb{1}_{\{x>0\}}$ if $\nu = \nu_c$.

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Conjecture (near-critical limit)

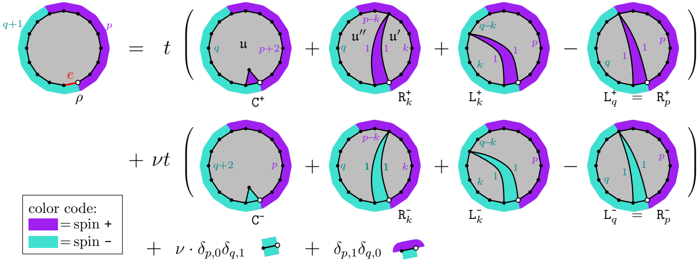
Fix $\beta > 0$ and $\lambda \in (0, \infty)$. In the limit $p, q \to \infty$, $q/p \to \lambda$ and $|\nu - \nu_c| = p^{-\beta}$,

1. if $\nu > \nu_c$, or $\nu < \nu_c$ and $\beta > 1/3$ then $L/p^{\delta(\beta)}$ converges in distribution to a non-trivial random variable on $(0, \infty)$, where $\delta(\beta) = 2\alpha_0(\beta) - \alpha(\beta) \in (0, 1]$.
2. if $\nu < \nu_c$ and $\beta < \frac{1}{3}$, then L/p converges to a deterministic value $\ell(\lambda;\beta) > 0$.

Method

Peeling process and Tutte's equation

We start with an Ising-triangulation $(\mathfrak{t}, \sigma) \in \mathcal{IT}_{+-}$ with unknown interior, and explore the internal face adjacent to the – boundary edge next to the root.



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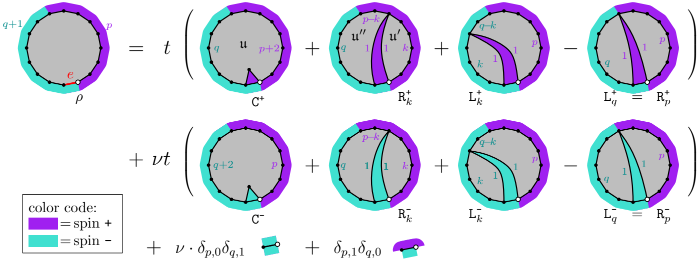

Tutte's equation:

$$z_{p,q+1} = t\Big(z_{p+2,q} + \sum_{p_1+p_2=p} z_{p_1+1,0}\, z_{p_2+1,q} + \sum_{q_1+q_2=q} z_{1,q_1}\, z_{p+1,q_2} - z_{p+1,0}\, z_{1,q}\Big) + \delta_{p,1}\,\delta_{q,0}$$

$$+\, \nu t\Big(z_{p,q+2} + \sum_{q_1+q_2=q} z_{0,q_1+1}\, z_{p,q_2+1} + \sum_{p_1+p_2=p} z_{p_1,1}\, z_{p_2,q+1} - z_{p,1}\, z_{0,q+1}\Big) + \nu\,\delta_{p,0}\,\delta_{q,1}\,,$$

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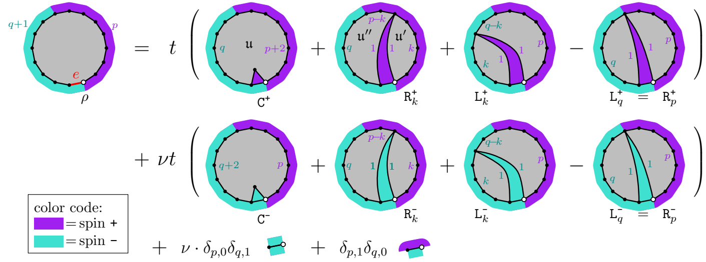

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$$\Delta_u Z = \nu t \cdot \Big(\Delta_u^2 Z + \big(\Delta_u Z_0(u) + Z_1(v)\big)\Delta_u Z - \Delta_u Z_0(u) Z_1(v) \Big) + \nu u$$

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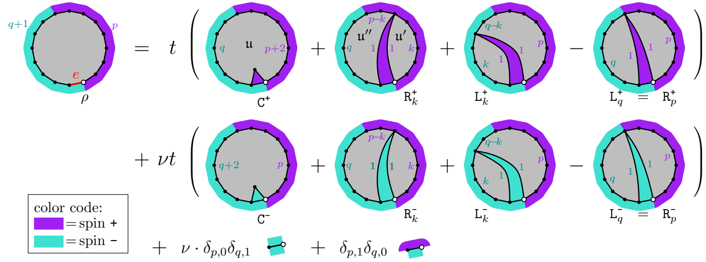

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$$\Longrightarrow \mathcal{P}\Big(Z(u,v);\ Z_0(u), Z_0(v), Z_1(u), Z_1(v);\ u, v, t, \nu \Big) = 0.$$

Solution of Tutte's equation

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$\mathcal{P}_0$ is an algebraic equation with one catalytic variable for $Z_0(u)$. There is an algorithm that eliminates $z_{1,0}$ and $z_{3,0}$ from it (Bernardi-Bousquet-Mélou '11).*
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The algebraic curve in the plane (u, Z_0) defined by $\tilde{\mathcal{P}}_0 = 0$ happens to be of genus zero. One can find algorithmically a rational parametrization of the function $Z_0(u)$.*

$\Longrightarrow \quad u = \tilde{u}(H; t, \nu) \qquad$ and $\qquad Z_0 = \tilde{Z}_0(H; t, \nu)$

* : In practice, the general method is too complex and a trick is required.

Solution of Tutte's equation

Lemma

The function $Z(u, v; t, \nu)$ has an explicit rational parametrization of the form

$$t^2 = \hat{T}(S, \nu), \quad tu = \hat{U}(H, S, \nu), \quad tv = \hat{U}(K, S, \nu) \quad \textit{and} \quad Z = \hat{Z}(H, K, S, \nu)$$

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Lemma (Parametrization of the critical line $t = t_c(\nu)$)

The solution of the equation $t_c(\nu)^2 = \hat{T}(S, \nu)$ can be rationally parametrized by

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Consequence: $Z(u, v; t_c(\nu), \nu)$ has a parametrization of the form

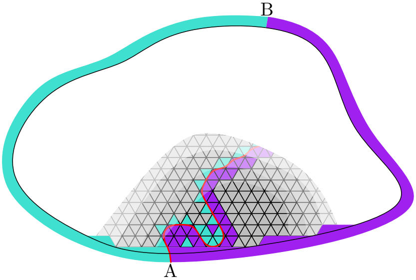
$$\nu = \hat{\nu}(R), \quad u = \hat{u}(H, R), \quad v = \hat{u}(K, R) \quad \text{and} \quad Z = \tilde{Z}(H, K, R)$$

Let ϕ_R be the inverse of $H \mapsto \hat{u}(H, R)$, for a fixed R, then we have

$$Z(u, v; \hat{\nu}(R)) = \tilde{Z}(\phi_R(u), \phi_R(v), R)$$

Peeling process (continued)

Consider an exploration process of the triangular lattice e.g. along the left-most interface.



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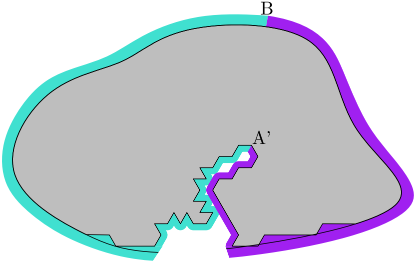

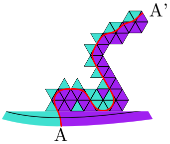

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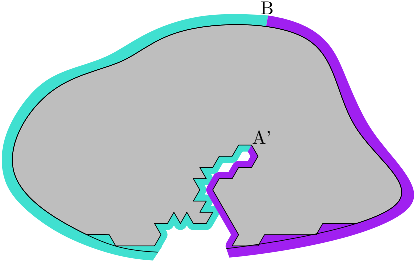

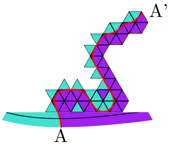

Spatial Markov property

The law of $\sigma|_{\mathfrak{t}(\text{unexplored})}$ only depends on $\sigma|_{\mathfrak{t}(\text{explored})}$ via its boundary.

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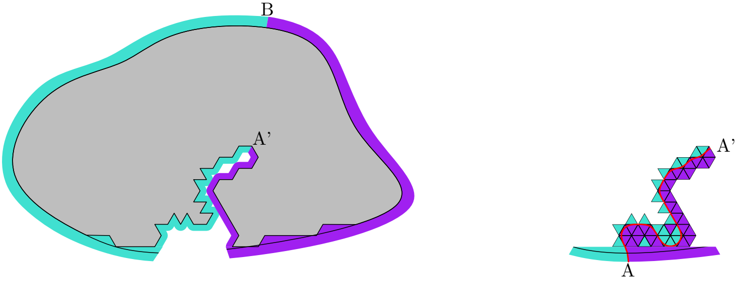

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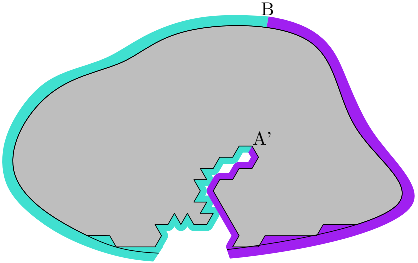

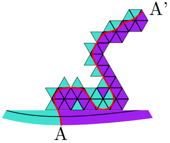

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This problem does not exist for the Boltzmann Ising-triangulation !
$\rightsquigarrow$ Markovian process on the space Dobrushin boundary conditions $(p, q) \in \mathbb{N}^2$.

Perimeter process

Let (P_n, Q_n) be the boundary condition of the unexplored region after n peeling steps.

Lemma

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Let $(X_n, Y_n) = (P_n, Q_n) - (P_0, Q_0)$.

Lemma

When $p, q \to \infty$, *the law of* $(X_n, Y_n)_{0\leq n\leq N}$ *converges to a well-defined random walk.*

One can construct the local limit of $\mathbb{P}^{\nu}_{p,q}$ from the limit law of $(X_n, Y_n)_{0\leq n\leq N}$.

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For $m \gg 1$, consider the event $J_m = \{P_1 \leq m \text{ or } Q_1 \leq m\}$.

Claim

The local limit of $\mathbb{P}^{\nu}_{p,q}$ is an infinite Ising-triangulation of the half plane if $\mathbb{P}^{\nu}_{p,q}(J_m) \underset{p,q\to\infty}{\longrightarrow} 0$, and is two-ended if $\lim_{p,q\to\infty} \mathbb{P}^{\nu}_{p,q}(J_m) > 0$.

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The scaling limit of the interface length depends on the rate of convergence of $\mathbb{P}^{\nu}_{p,q}(J_m) \underset{p,q\to\infty}{\longrightarrow} 0$.

Merci pour votre attention !